Economic Modelling 61 (2017) 235–247

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/econmod

A simple dynastic economy with parental time investment in children's patience☆

Tetsugen Haruyama[a], Hyun Park[b,*]

[a] Graduate School of Economics, Kobe University, Rokkodai, Nada-Ku, Kobe 657-8501, Japan
[b] Department of Economics, Kyung Hee University, 001 Hoegi-Dong, Dongdaemoon-Gu, Seoul 130-701, Republic of Korea

ARTICLE INFO

JEL classification:
C62
D91
I31
O15

Keywords:
Bifurcations
Dynastic economy
Endogenous time preferences
Indeterminacy
Instability

ABSTRACT

This article explores the dynamics of a general equilibrium when an individual's rate of time preferences is endogenous in a dynastic competitive economy. We postulate that altruistic parents allocate time to make their children patient to improve their lifetime welfare. The paper shows multiplicity and instability of the competitive equilibrium. Local and global indeterminacy emerges due to complementarity between a balanced growth rate and parental time allocation. Indeterminacy implies income and growth disparity among generations. In contrast, a balanced growth path is unique and determinate in the corresponding social optimum. A unique social optimum introduces a potential policy instrument for stabilizing a cyclical competitive equilibrium.

1. Introduction

Time preference of the present vis-à-vis the future is a pivotal component of human decisions on a finite lifetime horizon. Future orientation is an important component of noncognitive abilities influenced by the members of a family as well as society. An individual makes a decision by balancing her own utility and her family's welfare, given that her time preferences are affected by intentional investment within the family and by an external habit formation in society. In particular, when parents make a conscious decision to influence the economic success of their children (Mulligan, 1997), they are altruistically motivated to invest their time and efforts to improve their children's cognitive and noncognitive abilities including knowledge, problem-solving, human capital, ingenuity, perseverance, discipline, patience, persistence, self-esteem, and other positive attributes.

While the conventional literature focuses on cognitive abilities for intergenerational correlations, recent empirical studies consider noncognitive abilities to explain human capital accumulation and skill formation (see the excellent survey in Carneiro and Heckman, 2003) and intergenerational transmission from parents to their offspring (Gouskova et al., 2010). However, the mechanism for observed intergenerational correlations across generations is not well understood in the literature. To fill this gap, we postulate that parents influence children's tastes or preferences and thereby shape children's attitudinal and personal traits toward future decisions.[1] The main purpose of this paper is to provide an underlying mechanism for intergenerational differences and similarities when parents devote their time and resources so that their children delay gratification of future outcomes and welfare.

The vehicle of our analysis is a dynamic general equilibrium model for a dynastic competitive equilibrium and its corresponding social optimum in an overlapping-generations economy. This paper, however, departures from the conventional overlapping-generations economy by introducing the formation of time discounting in a dynasty competitive

☆ We are indebted to John Conlon, George Economides, Anne Lee, Kyuyub Lee, Shane Oh, Apostolis Philippopoulos, and Makoto Yano for insightful comments and criticism. We also thank Co-Editor, Sushanta Mallick, and the anonymous referees for their valuable comments and suggestions on our first manuscript. Haruyama gratefully acknowledges financial support from the Kobe University Center for Social Systems Innovation. Park acknowledges the SSK (2014S1A3A2044459) financial support in Korea Research Foundation.

* Corresponding author.
E-mail addresses: haruyama@econ.kobe-u.ac.jp (T. Haruyama), econhyun@khu.ac.kr (H. Park).

[1] Cognitive abilities that combine innate intelligence and schooling with income and wealth constraints account for an estimated 23% of the intergenerational correlation in income difference (Bowles and Gintis, 2002). This finding suggests that noncognitive abilities play an important role in intergenerational interactions on economic outcome including income, savings, education, and on.

http://dx.doi.org/10.1016/j.econmod.2016.12.012
Received 16 June 2016; Received in revised form 5 December 2016; Accepted 13 December 2016
Available online 30 December 2016

economy. In the spirit of Becker and Mulligan (1997) and Doepke and Zilibotti (2005a, 2005b),[2] we explore how parents instill children's future orientation and study the property of endogenous time discounting to delay gratification in a dynastic overlapping-generations economy. That is, our intention is to explain how parents shape the time-discounting characteristic of their children's attitudinal and personal traits and thereby determine the dynamics of macroeconomic allocations. Assuming parental influence on children's time preferences, children make their own future-oriented decisions on physical capital accumulation as well as their own consumption and leisure–parental choices and future generations' welfare. Investment on children's time preferences involves time-variant cost in the sense that parents must forgo their own leisure to rear and parent their children. We posit that parents allocate a flow of their time by forgoing leisure to instill patience into their children. That is, an altruistic individual's decision involves the balance between felicity from her own leisure and her children's well-being.[3] Hence, our analysis helps us to understand a dynastic competitive economy by taking into account individual incentives and market structures on intergenerational correlations including consumption, leisure, income and wealth, capital accumulation, and economic growth.[4]

Our main results are as follows. Extending our analysis to an overlapping-generations economy with endogenous time discounting, we first define the convex value functions for the decentralized competitive equilibrium and its corresponding social optimum and then derive their associated equilibrium and optimum policy functions on two periods of parental time allocations. These policy functions characterize the evolution of endogenous time discounting and thereby pin down consumption, leisure, parenting, and capital over dynastic overlapping generations. We derive the conditions under which the dynastic competitive equilibrium exhibits a unique transitional path and a unique balanced growth path. Furthermore, under the standard conditions on the time discount function, we show the possibility of multiple competitive balanced growth paths. Multiplicity arises when (i) the marginal intensity of the time-discounting formation is strong; (ii) the marginal disutility of forgoing leisure is low for parental time spent with children given a constant marginal utility of consumption; and (iii) the parental time allocation in equilibrium is small enough, and thus children is relatively impatient. Therefore, the amount of parental time spent and the endogenous rate of time preferences are the primal sources of the multiplicity of balanced growth paths in a dynastic competitive economy.[5]

Indeterminacy in this paper is intuitive from the fact that the stationary equilibrium growth rate is non-monotonic with parental time spent with children in a dynastic competitive economy. Non-monotonicity arises because the immediate disutility of parenting is compensated by the altruistic reward from the future utility gain when children's patience stimulates saving and capital and thus increases the children's lifetime welfare. Non-monotonicity, therefore, induces the complementarity between savings and investment in patience. Thus, each balanced growth path enjoys a different long-run growth rate among rational expectations competitive equilibria. That is, global indeterminacy emerges, and the sunspot equilibrium is self-fulfilled in a dynastic competitive economy (see Benhabib and Farmer, 1999; Cazzavilan and Pintus, 2004). An important implication is that indeterminacy can be used as an instrument to explain income and growth disparities within the same generation or across different generations in dynastic competitive economies with the same fundamentals, including preferences and technology (Galor and Zeira, 1993). Hence, the self-fulfilling equilibrium exhibits income and growth disparities in a dynastic competitive equilibrium with endogenous children's time preferences. This finding therefore implies that, with the endogenous time preference formation, parental time spent with children can account for income and growth disparity within and/or across generations.

Notwithstanding global indeterminacy in a dynastic competitive economy, we establish global determinacy in the corresponding social planning economy. The balanced growth path is unique because a social planner takes into account the intertemporal external effects of parental time allocation over dynastic generations. That is, the social optimum allocation completely internalizes the spillover effects of the parental effort to shape children's time preferences and thus restores the uniqueness of the social optimum allocation under parental altruism to children within the same family. Consequently, the balanced growth rate of the social optimum is determined under the standard joint conditions on felicity and technology, including the intensity of altruistic preferences, and is independent of the parental time allocation for children's time preferences. Thus, the social planning mechanism is properly designed for Pareto improvement in the long run even though time discounting is endogenously shaped over generations in dynastic competitive economies. This invites a potential policy instrument under which a Pareto efficient allocation can be implemented in the decentralized competitive economy.

We, however, discern that local indeterminacy, if exists, is likely to arise in the corresponding social optimum allocation when the parental time allocation is small so that a propositional rate of time discounting is large with respect to the inverse of the leisure allocation. Hence, both the level and marginal changes of the time discount function are critical to the possibility of local indeterminacy in the transitional social optimum. The increasing marginal impatience in this paper is consistent with the indeterminacy condition in the literature including (e.g., Jafarey and Park, 1998). We also demonstrate that endogenous time preferences generate non-monotonicity of the balanced growth path with respect the parental time allocation so that complementarity arises between saving and investment in patience. The conditions on the local indeterminacy in the social optimum are compatible to those in the dynastic competitive equilibrium. Hence, a social planning mechanism in the short run, if indeterminacy exists, fails to select the unique transitional dynamic path around the unique balanced growth path in an overlapping-generations economy with endogenous time preferences.

To verify our analytic results, we demonstrate numerical examples for theoretically and empirically plausible parameter values in a dynastic competitive economy and its corresponding social optimum allocation. We use a set of parameter values that verifies the uniqueness or multiplicity of a balanced growth path and a transition dynamic path in both the decentralized competitive equilibrium and its corresponding social optimum allocation. Furthermore, a set of parameter values illustrates the local stability of the balanced growth path and thus the local indeterminacy in the competitive transitional paths. The numerical exercises also allow us to pin down the bifurcation parameter values on time preferences of parental time allocation for Hopf and Flip bifurcation in the decentralized competitive economy.

Finally, the paper exercises a simple comparative dynamic analysis to provide theoretical justification for a few empirical observations on

[2] See Doepke and Zilibotti (2005a, 2005b) for an overlapping-generations economy in which each family chooses occupation and invests future-enhancing capital. Becker and Mulligan (1997) considered a finite lifetime model in which each individual invests her own future-enhancing capital throughout her lifetime horizon.

[3] We assume intergenerational altruism (which is also called "pure altruism"). The literature widely uses this modeling approach to capture inter vivos transfer or bequest motives (e.g., Barro 1974; de la Croix and Michel, 2002). However, empirical studies challenge these altruistic models and, in general, point out their poor empirical performance (see Altonji et al., 1997). Nonetheless, as an effective working hypothesis to catch a clear-cut alternative to self-interest motives, we use the altruistic approach to create the parental incentive to invest in children's patience.

[4] Guryan et al. (2008) pointed out that parental time spent with children is different from the patterns of leisure or home production. Following their view of time spent with children, we treat parental time allocation as investment for children.

[5] In an infinite horizon model with the patient capital stock, Strulik (2012) found the conditions for the unique transitional and balanced growth path, whereas Kawagishi (2014) showed, in the presence of consumption and investment externalities, indeterminate transitional paths in conjunction with multiple balanced growth paths.

intergeneration correlations in the existing literature. First, the paper shows that parental time spent with children makes children patient, which increases children's savings and future consumption but reduces leisure for the adult and thus also reduces the adult's own savings and capital investment for future consumption. These two conflicting effects determine the economic growth in a dynastic competitive economy (also see Haaparanta and Puhakka, 2004). Second, we find the consumption ratio between the young period and old period, whose slope becomes steeper when an individual becomes patient. This finding implies that time discounting is not exogenously constant but endogenously changes over an individual's lifetime. This result justifies the consumption pattern over the life cycle (Gourinchas and Parker, 2002). Third, we demonstrate that the time discount function affects an adult's consumption ratio over generations (Carroll and Summer, 1991). This finding suggests that the intergenerational correlation due to the time that parents allocate to their children has noncognitive personal traits including patience and future orientation (Carneiro and Heckman, 2003). Finally, the high intensity of altruism and the low marginal utility of leisure–parenting time increase the parental time allocation for shaping children's patience for the future orientation. The social optimum path grows fast when parents are altruistic,[6] the marginal utility is high with respect to leisure, and total factor productivity is large. These comparative dynamic relations are a testable hypothesis in the literature.

The remainder of the paper is organized as follows. Section 2 provides a brief review of the literature. Section 3 introduces a dynastic competitive economy with endogenous time discounting and characterizes a dynamic competitive equilibrium allocation. Section 4 studies the global multiplicity of balanced growth paths. Section 5 shows the local indeterminacy of transitional dynamic paths in a decentralized competitive economy. Section 6 examines the property of a corresponding social optimum allocation for a dynastic competitive economy. Section 7 discusses uniqueness, stability, and determinacy of a social optimum allocation. Section 8 provides a few numerical examples to illustrate the properties established in the dynastic competitive and social optimum allocations. Section 9 provides the concluding remarks.

2. Related literature on endogenous time preferences and overlapping generations

The economy in this paper considers endogenous growth and time preferences in an otherwise standard overlapping-generations model. Previous studies, including Galor and Ryder (1989) and de la Croix and Michel (2002), establish the existence, uniqueness, and stability of the competitive equilibrium and social planning optimal path with a neoclassical technology.[7] However, the economy herein allows perpetual economic growth and permits the intergenerational interaction in preferences in a dynastic competitive economy. For complex and cyclical dynamics, de la Croix and Michel (1999) introduced consumption externalities (e.g., external habit stocks) to generate the cyclical dynamic path in nonseparable lifetime utility. On the other hand, we posit that an intergenerational connection in time preferences causes endogenous business cycles though the parental time spent with children under altruistic motivations in a dynastic economy. Unlike the finite lifetime economy we propose, Becker and Boyd (1992) and Park (2000) showed that the infinite time horizon model exhibits uniqueness and stability when the time discount function is endogenously determined in both a growing and non-growing economy.[8] The literature, including Nourry (2001), clearly shows that more than one steady state emerge in overlapping-generations models when the economy extends to one with consumption and production externalities, many goods and sectors, an endogenous labor supply, and increasing returns technology. However, we keep the usual features of a standard overlapping-generations economy, including its usual dimensions of goods and sectors, where time discounting is a function of a flow, rather than a stock (e.g., Becker and Mulligan, 1997; Kawagishi, 2014), of the parental time allocation.

Doepke and Zilibotti (2005b) examined the effect of the parental efforts of the endogenous rate of children's time preferences based on intergenerational income correlations by occupation choices with intergenerational impatience. Doepke and Zilibotti (2005a, 2005b) introduced future capital stocks to determine time discounting and emphasized the intergenerational correlations between parents and their children. Although they acknowledged the existence of multiple equilibria and instability over generations, Doepke and Zilibotti (2005b) did not advance the discussion. In fact, unlike our complementarity condition on a parental time allocation and children's welfare, their potential multiplicity is due to a simultaneous choice of future capital stocks and occupations with an associated constant income profile. Moreover, the present paper also keeps the usual dimensions of the standard overlapping-generations model by assuming that our time discount function is not a stock but a flow variable in a sense that time discounting is formed by a period-by-period time allocation. This feature of our model strengthens the indeterminacy results.

Our multiplicity and instability properties of the dynastic competitive equilibrium are complimentary to those in the overlapping-generations literature. Galor and Ryder (1989) showed the uniqueness of the (nontrivial) steady state equilibrium under a strengthen Inada condition on preference and technology. However, unlike our growing economy with endogenous time preferences, Galor and Ryder demonstrated that the self-fulfilling expectation is unique under the restrictive monotonicity conditions on the capital policy function and also recognized that indeterminacy is possible when the Inada conditions are weakened in the neoclassical models of technology and preferences. Contrary to our results with AK–technology, Reichlin (1986) found equilibrium cycles and bifurcations when a few input factors in production are strongly complementary with a unique stationary equilibrium. On the other hand, Michel and Venditti (1997) demonstrated endogenous fluctuations when the felicity function is not separable from current and future consumption and the rate of time preferences is small, which contradicts our finding of high time discounting in a dynastic competitive economy. Sarkar (2007) also demonstrated that an increase in the time discount function in income, consumption, or investment—unlike a decrease in time discounting in terms of parental time allocation in our economy—leads to endogenous equilibrium cycles in an infinite horizon economy with recursive preferences in an infinite lifetime model.[9]

3. Dynastic competitive economy

The paper sets up a dynastic general equilibrium model. We consider both a decentralized competitive economy and its corresponding planning economy. The dynastic competitive economy consists of infinitely many identical households and producers and three types of goods: labor–leisure–parenting, private and public capital stocks, and a final output and consumption good. The economy is populated by

[6] Charles and Hurst (2003) provided evidence of the wealth correlation across generation, which leads to economic growth in our model because all savings are consumed in the adult period in a dynastic economy.

[7] Galor and Ryder's (1989) conditions for uniqueness and stability for the steady state is not applicable to our model because their condition is in a non-growing economy with neoclassical technology, whereas we focus on a persistently growing economy with linear AK–technology.

[8] Many studies examine endogenous time preferences in an infinitely lived agent model, whose time discount function is determined by consumption or income (e.g., see Epstein and Hynes, 1983; Becker and Boyd, 1992). Alternatively, de la Croix and Michel (1999) introduced internal and external habits to influence children's preferences and showed the possibility of complex dynamic paths of equilibrium.

[9] Hirose and Ikeda (2015) extended the instability property to multicountry economies in the presence of more than two decreasing-marginal-impatient countries. Also see Kawagishi (2014).

overlapping generations of altruistic individuals who live for the two periods, the young and the old in a lifecycle.[10] Preferences of every individual in the dynastic economy are identical except time preferences. The utility of each individual depends on consumption when young and old, leisure and parental time spent with his children when old, and his children's utility. Individual's time preferences are endogenously formed during his young period. An altruistic parent devotes his time and efforts to influence his children's noncognitive abilities, in particular, children's time discounting in preferences. There is no human capital accumulation, bequest, and pension.

The production sector is standard in a perpetually growing economy. Each producer produces the final output by using private capital and labor. The final output is either consumed or invested on the future capital stocks. The private capital stocks generate positive spillovers to other producers and thus creates public capital stocks as productive externalities. This technology leads to a linear AK–technology in aggregate capital stocks, thereby permitting long-run economic growth in a dynastic competitive economy. All agents are endowed with perfect foresight. Time is discrete on the infinite horizon. There is no uncertainty or population growth.

Formally, in each period t, each producer employs two input factors, capital stocks K_t and labor L_t, to produce the final output Y_t by taking advantage of productive public capital stocks $\widetilde{K}_t$ in the decentralized competitive market. At a given rate of interest and wages in competitive input markets, the representative producer pays the wage for labor and rent for private capital. The production function is assumed to be the constant returns to scale of the private labor and capital. As previously mentioned, each producer's capital investment generates positive externalities as public capital stocks so that aggregate technology yields increasing returns to scales in a decentralized competitive economy. Endogenous growth models (e.g., Romer, 1986) clearly define the property of this aggregated AK–technology.[11] In particular, this technology allows the aggregate competitive economy to be perpetually growing without a finite limit.

More specifically, the production function $F(K_t, L_t; \widetilde{K}_t)$ is defined as

$$Y_t = F(K_t, L_t; \widetilde{K}_t) = \widetilde{A}_t K_t^{\alpha} L_t^{1-\alpha}, \tag{1}$$

where $0 \le \alpha \le 1$, and $\widetilde{A}_t$ consists of the total factor productivity A and production externalities $\widetilde{K}_t$ in a decentralized competitive market. That is, $\widetilde{A}_t \equiv A \widetilde{K}_t^{1-\alpha}$. As previously mentioned, production externalities $\widetilde{K}_t$ as public capital stocks are assumed to be the sum (i.e., $\sum K_t$) of the individual producer's private capital stocks K_t.[12] Assuming that capital fully depreciates after one period in production, the representative producer maximizes its profits at real interest rate $1+r_t$ and real wage rate w_t in the competitive factor market. For each period $t \ge 1$, the necessary conditions for the producer's problem are

$$1+r_t = \frac{\partial F}{\partial K_t} = \alpha A, \tag{2}$$

$$w_t = \frac{\partial F}{\partial L_t} = (1-\alpha) A K_t, \tag{3}$$

where $L_t=1$ and $\widetilde{K}_t = K_t$ at the market equilibrium. For given public capital stocks $\widetilde{K}_t$, the constant returns to scale of private inputs ensure the existence of a dynastic competitive equilibrium. Unlike a usual AK–growth model (e.g., Rebelo, 1991), the non-zero wage income avoids a trivial dynastic competitive equilibrium. The modified AK–production technology exhibits positive constant returns to aggregate capital stocks K_t, and thus the dynastic competitive economy enjoys persistent long-run economic growth.[13]

In each period, new individuals are born, and each individual lives two periods. Every individual has one child at the beginning of her adult life. All individuals are identical within and across generations. An individual in the representative household is endowed with one unit of time for labor–leisure-parenting in each period. When young, she consumes from wage income, supplies labor for production, and saves for the next period consumption; when old, she consumes from the return from savings and allocates one unit of time for leisure for her own felicity and parental time spent for shaping her children's patience. For our purposes, we abstract away from transmitting cognitive abilities, for example, facilitating education and human capital accumulation.[14] This assumption allows us to focus on the effect of noncognitive abilities in the intergenerational correlation (Cunha et al., 2010).

Each individual has the same felicity function and time preference function. A particular aspect of the individual's lifetime utility is that individual time preference is not exogenous and the corresponding time discounting is formed during her childhood. We postulate that dynastic altruism motivates a parent concerning her children's welfare who then allocates parental time with her children to shape their preferences, in particular, their time discount rates.[15] Hence, each individual's time discount factor depends on the amount of time that the parent decides to allocate to instill patience in her children. To focus on intergenerational interactions of time preferences, we assume that there is neither bequest nor income transfer to the next generations in a dynastic economy. Unlike future-oriented capital stocks as in Becker and Mulligan (1997) and Doepke and Zilibotti (2005a, 2005b), parental devotion to children's time preferences is a flow of forgoing leisure so that no intergenerational formation of time preferences exists on the infinite horizon in the dynastic competitive economy.[16] That is, the rate of time discounting is not directly inherited from the previous generations and therefore is not a state variable over generations.

Formally, an individual's lifetime utility function U_t for generation t is from the consumption of c_t when young, consumption d_{t+1} and leisure l_{t+1} when old, and his children's utility U_{t+1} in the next generation $t+1$. The individual's lifetime utility is the time-separable discount sum of felicities when young and old, plus his children's lifetime utility. In the first period of his life, a young individual inelastically supplies one unit of labor $L_t=1$ for production and earns the labor income at competitive market wage w_t.[17] In the second period, an old individual spends all of his income on his own consumption, transfers no income to his children out of his savings s_t from the previous young period, and allocates one unit of endowed time between enjoying his own leisure l_{t+1} and spending parental time p_{t+1} with his children.

[10] Our setting can be extended to a three-period model including a child period in addition to the young and old periods. Our main results prevail depending on what choice variables are available in a child period (see, e.g., de la Croix and Michel, 2002). However, the two-period model that we employ in this paper provides a full mechanism to depict the role of parental time allocation for children's impatience on dynamics of a dynastic competitive economy.

[11] See Romer (1986) for a role of productive externalities in the infinite horizon model. Doepke and Zilibotti (2005b) also adopted AK–technology with future capital stocks in an overlapping-generations model.

[12] We are aware that the congestion effect of production externalities is not present in an endogenous growth model. However, it is easy to deal with this scale effect on long-run growth when we demonstrate a numerical analysis on our property of a dynastic economy (Li, 2003).

[13] We use a model of endogenous growth because it greatly simplifies analysis. Our tentative calculation suggests that even if the neo-classical technology is assumed, complicated dynamics similar to those reported here are likely to arise. We believe that our model is sufficient to illustrate the key results without introducing unnecessary complications.

[14] A more realistic model may involve allocating parental resources on children's education and human capital accumulation in addition to shaping children's noncognitive abilities (herein, patience in preferences).

[15] In the previous studies, "future-oriented capital" is regarded as a form of "human capital." Furthermore, this literature debates whether future-oriented capital is a tangible or a non-tangible good and includes cognitive or noncognitive skills (Becker and Mulligan, 1997; Mulligan, 1997). Also see Loewenstein (1987).

[16] See Gouskova et al. (2010).

[17] Even though labor–leisure–parenting is endogenously determined in the competitive economy, we abstract from endogenously choosing labor supply thus maintaining its dynamics as in a two-sector overlapping-generations economy. This assumption strengthens rather than weakens our indeterminacy results (see the detail discussion in Benhabib and Farmer (1999)).

The lifetime utility U_t of the representative individual born in generation t, t≥1, is defined as[18]

$$U_t = u(c_t) + \beta(p_t)v(d_{t+1}, l_{t+1}) + \gamma U_{t+1}, \tag{4}$$

where $u(c_t)$ and $v(d_{t+1}, l_{t+1})$ are felicity functions when young and old, respectively; c_t and d_{t+1} are consumption when young and old, respectively; l_{t+1} is allocated for leisure when old.[19] Every parent is altruistic to the degree γ on her children's utility. Following Doepke and Zilibotti (2005a, 2005b), each parent spends her time to instill patience in her children. Hence, the time discount factor $\beta(p_t)$ for generation t is a function of parental time spent p_t by the parent of generation t within the same family. We assume that the felicity function $u(c_t)$ for the young is increasing, twice differentiable, and concave in c_t.[20] We also impose the Inada condition on $u(c_t)$; that is, $\lim_{c_t\to 0} u_1(c_t)=\infty$, where $u_1(c_t)$ is a marginal utility of c_t. The felicity function $v(d_{t+1}, l_{t+1})$ for the old is also increasing, twice differentiable, and concave in $\{d_{t+1}, l_{t+1}\}$ and satisfies the Inada conditions; that is, $\lim_{d_{t+1}\to 0} v_1(d_{t+1}, l_{t+1})=\infty$; $\lim_{l_{t+1}\to 0} v_2(d_{t+1}, l_{t+1})=\infty$, where v_1 and v_2 are the marginal utility of d_{t+1} and l_{t+1}, respectively.[21] These assumptions on $u(c_t)$ and $v(d_{t+1}, l_{t+1})$ are standard in the literature. Recall that the time discount factor $\beta(p_t)$is endogenous. It is natural to assume that the time discount function is increasing in p_t with $0 < \beta(0)\leq\beta(1)<1$.[22] An increasing time-discount function shows that the more parental time spent with children, the more patient and future-oriented children become within the same family. We also assume that the time discount function $\beta(p_t)$ is concave and twice differentiable in p_t. The concavity of the discount function, along with the concave felicity functions, makes the individual's optimization problem a convex problem. The Inada and boundary conditions on the lifetime utility ensure an interior nontrivial dynamic path, and we impose the differentiability condition on the felicity and discount function for the dynamic comparative and stability analyses.

The budget constraints for the representative individual of generation t are

$$c_t + s_t = w_t, \tag{5a}$$

$$d_{t+1} = \frac{1}{R_{t+1}}s_t, \tag{5b}$$

$$l_{t+1} + p_{t+1}=1, \tag{5c}$$

where w_t is the wage rate and $R_{t+1} = \frac{1}{1+r_{t+1}}$ is the inverse of the rate of interest to the capital, $1+r_{t+1}$ in (2). Given w_t and R_{t+1} in the dynastic competitive economy, the individual chooses $\{c_t, d_{t+1}, l_{t+1}, p_{t+1}\}$ to maximize her lifetime utility subject to the lifetime budget constraints (5a) and (5b) and leisure–parenting time constraints (5c).

Now, we formally define the value function $U_t(p_t)$ for the representative individual's optimization problem as

$$U_t(p_t) = max_{[c_t, d_{t+1}, l_{t+1}, p_{t+1}]}\{u(c_t) + \beta(p_t)v(d_{t+1}, l_{t+1}) + \gamma U_{t+1}(p_{t+1})\}, \tag{6}$$

where $d_{t+1} = \frac{w_t - c_t}{R_{t+1}}$ and $l_{t+1}=1-p_{t+1}$ from (5a), (5b) and (5c). For simplicity, we consider a solution such that these constraints are binding in each period $t \geq 1$.[23] In the dynastic competitive economy, including the individual budget constraints, the dynamic Euler–Lagrangian conditions for the individual's problem are summarized as

$$u_1(c_t) - \frac{\beta(p_t)}{R_{t+1}}v_1(d_{t+1},1-p_{t+1})=0, \tag{7a}$$

$$\beta(p_t)v_2(d_{t+1},1-p_{t+1}) - \gamma U'_{t+1}(p_{t+1}) = 0, \tag{7b}$$

where U'_{t+1} is the derivative of $U_{t+1}(p_{t+1})$ with respect to p_{t+1}. Eq. (7a) depicts the intertemporal non-arbitrage condition for consumption within generation t. That is, the marginal utility of consumption of the young is equal to the time-discounted marginal utility of consumption of the old, along with the marginal returns of savings from the foregoing consumption of the young. Eq. (7b) is the intergeneration non-arbitrage condition for parental time allocation between generation t and $t + 1$. That is, the present value of the marginal utility of the leisure of the old is equal to the discounted marginal lifetime utility of her child. In addition, the envelope condition of the value function $U_t(p_{t+1})$ yields

$$U'_{t+1}(p_{t+1}) = \beta'(p_{t+1})v(d_{t+2},1-p_{t+2}). \tag{7c}$$

This shows that along the optimal path the marginal lifetime utility of parental time allocation is equivalent to the marginal increase in the discounted second-period utility of the child when the discount factor increases marginally due to a one-unit increase in parenting time. Finally, combining (7b) and (7c) implies that the marginal utility of leisure is equal to the marginal increase in the child's welfare due to the marginal increment of patience weighting on the children's lifetime utility. Finally, the present value of the capital stocks in terms of the marginal utility of consumption approaches zero, so that generation t's lifetime utility $U_t(p_t)$ is bounded as time goes to infinity. That is, we have a transversality condition:[24]

$$\lim_{t\to\infty}\beta(p_t)u_1(c_{t+1})K_{t+1}=0. \tag{7d}$$

In summary, we have the following theorem:

Theorem 1. Under the assumptions on preferences including time preference and technology, the dynastic competitive equilibrium $\{c_t, d_{t+1}, K_{t+1}, l_{t+1}, p_{t+1}, 1+r_{t+1}, w_t\}$ with the initial condition $\{K_1,p_1\}$ is characterized by the set of necessary and sufficient conditions (2), (3), (7a), (7b), (7c), and (7d) including the market clearing conditions:

$$Y_t = c_t + s_t + d_t, \tag{8a}$$

$$l_{t+1} + p_{t+1} = L_{t+1}, \tag{8b}$$

$$s_t = K_{t+1}, \tag{8c}$$

$$\widetilde{K}_{t+1} = K_{t+1}, \tag{8d}$$

$$L_{t+1}=1, \tag{8d}$$

for $t\geq 1$ in the dynastic overlapping-generations economy with endogenous time preferences.

The dynastic competitive equilibrium is characterized by the set of

[18] Chakraborty and Das (2005) developed a two-period overlapping generations model where the probability of surviving at the old period depends on health investment from the young's wage income. In the model, the "effective" discounting for the old's felicity is endogenized through own health expenditure. Bhattacharya and Qiao (2007) used a similar setup to endogenize a rate of longevity. Both approaches, however, differ from ours in the sense that a time discount factor (herein, the rate of children's time preferences) is endogenized due to intergenerational investment (herein, parental time allocation) in a dynamic general equilibrium model.

[19] Alternatively, we can apply the discount factor $\beta(p_t)$ not only on the adult's consumption, $v(d_{t+1},1-p_{t+1})$, but also on her children's lifetime utility, $U_{t+1}(p_{t+1})$. However, this alternative definition of the utility function does not alter any of our main results as long as we properly adjust the level effect of the discount factor. Moreover, we can formulate the lifetime utility as weighted time-discounting felicities such that $U_t=(1-\beta(p_t))u(c_t)+\beta(p_t)v(d_{t+1}, l_{t+1})+\gamma U_{t+1}$. Given a proper normalization, this lifetime utility is equivalent to one in the present model and thus supports the main results in our analysis.

[20] The felicity of $u(c_t)$ is thought of as a reduced form of $v(c_t, l_t)$ with $l_t=0$, for all $t \geq 1$. That is, $(c_t)\equiv v(c_t,0)$.

[21] $u_1(c_t)$ denotes the derivative with respect to c_t;$v_1(d_{t+1}, l_{t+1})$ and $v_2(d_{t+1}, l_{t+1})$ are the derivatives with respect to d_{t+1} and l_{t+1}, respectively.

[22] Strictly speaking, the upper bound of the time discount function can be greater than 1 as long as it is finite, when an individual places more weight on future felicity than present felicity. Furthermore, $p_t=1$ and thus $\beta(1)$ cannot be a solution because it violates the Inada condition on $v(d_{t+1}, l_{t+1})$.

[23] It is clear that these constraints are binding under the assumptions on the felicity, production, and discount function.

[24] We can impose an alternative transversality condition: $\lim_{t\to\infty}\beta(p_t)U_{t+1}(p_{t+1})=0$. This condition also ensures boundedness of the lifetime utility so that the individual's maximization problem is well defined over generations.

conditions in Theorem 1. Since the utility and profit maximization is a convex problem, the associated first-order conditions are not only necessary and but also sufficient for the competitive equilibrium. These equilibrium conditions portray an important extension of the standard overlapping-generations economy (e.g., de la Croix and Michel, 2002; Galor and Ryder, 1989) to a dynamic general equilibrium in the presence of production externalities and the endogenous formation of time preferences. The former leads to a perpetually growing economy; the latter justifies the intergenerational similarity and correlation in the presence of noncognitive abilities in the dynastic economy.

For further analysis on the role of endogenous time preferences of the dynastic competitive equilibrium, we specify the felicity functions for the young and the old, respectively, as

$$u(c_t) = c_t^a,$$

$$v(d_{t+1},\ l_{t+1}) = d_{t+1}^a l_{t+1}^b,$$

where $0 < a < 1, 0 < b < 1$. Then, by substituting (7c), the corresponding first-order conditions to (7a) and (7b) are, respectively,

$$\frac{1}{c_t^{1-a}} - \frac{\beta(p_t)}{R_{t+1}}\left(\frac{1}{d_{t+1}}\right)^{1-a}(1-p_{t+1})^b=0, \tag{9a}$$

$$b\beta(p_t)d_{t+1}^a(1-p_{t+1})^{b-1} - \gamma\beta'(p_{t+1})d_{t+2}^a(1-p_{t+2})^b=0. \tag{9b}$$

With these conditions, we analytically pin down the rate of growth and derive the dynamic equation of the parental time allocation in the dynastic competitive economy. First, by modifying (9a) and then plugging it into the household's budget set $c_t + R_{t+1}d_{t+1} = w_t$ in (5a) with $R_{t+1} = \frac{1}{\alpha A}$ in (9a), we have

$$c_t = \frac{w_t}{1+\Phi(p_t,\ p_{t+1})}, \tag{10a}$$

where $\Phi(p_t,\ p_{t+1})\equiv[\beta(p_t)(\alpha A)^a(1-p_{t+1})^b]^{\frac{1}{1-a}}$. Second, because $K_{t+1} \equiv s_t = w_t - c_t$ from (5a), we obtain[25]

$$K_{t+1} = w_t\frac{\Phi(p_t,\ p_{t+1})}{1 + \Phi(p_t,\ p_{t+1})}.$$

Third, by substituting $w_t = (1-\alpha)AK_t$ into the previous equation, we obtain the following capital policy function:

$$K_{t+1} = (1-\alpha)AK_t\frac{\Phi(p_t,\ p_{t+1})}{1 + \Phi(p_t,\ p_{t+1})}. \tag{10b}$$

From now on, we call (10b) "the k-policy function" for the dynastic competitive equilibrium. The k-policy function dictates the dynamic properties of the dynastic competitive equilibrium in terms of the parental time allocations over generations.

The next lemma reports the rate of economic growth Γ_{t+1}, which is obtained by rewriting the k-policy function in (10b):

Lemma 1. Under the assumptions in the dynastic competitive economy, the following first-order difference equation depicts the growth rate Γ_{t+1} of the dynastic competitive equilibrium in Theorem 1:

$$\Gamma_{t+1} \equiv \frac{K_{t+1}}{K_t} = (1-\alpha)A\frac{\Phi(p_t,\ p_{t+1})}{1 + \Phi(p_t,\ p_{t+1})}, \tag{11}$$

where $\Phi(p_t,\ p_{t+1})\equiv[\beta(p_t)(\alpha A)^a(1-p_{t+1})^b]^{\frac{1}{1-a}}$.

Eq. (11) shows that the rate of economic growth Γ_{t+1} is a function of equilibrium parental time allocations p_t and p_{t+1}. To understand the channels through which they affect growth, first consider p_t. It is the old's parental time, which instills patience in the child by increasing her discount factor $\beta(p_t)$. Its increase boosts savings and capital stocks at $t+1$ (i.e., a higher K_{t+1}). The higher value of p_{t+1} comes at the cost of a lower leisure time l_{t+1}, which in turn reduces the marginal utility of consumption of the old $v_1(d_{t+1}, l_{t+1})$. It lowers the saving incentive, thus discouraging growth (i.e., a lower K_{t+1}). These opposing effects of p_t and p_{t+1} induce non-monotonic transitional economic growth in a dynastic competitive economy and thus suggest the possibility of dynastic growth cycles over generations along with the dynamics of the time discount formation.

We now consider the special case that parents devote no time for the children, that is, $p_t = p_{t+1}=0$, and thus $\beta(0) \equiv \beta^0, 0<\beta^0<1$. This economy collapses to a usual dynastic competitive economy with constant exogenous time preferences. Then, from (11), $\Gamma_{t+1} \equiv \frac{K_{t+1}}{K_t} = (1-\alpha)A\frac{\Phi(0,0)}{1+\Phi(0,0)}$, where $\Phi(0,0)\equiv[\beta^0(\alpha A)^a]^{\frac{1}{1-a}}$. The economic growth rate becomes time invariant with no parental time allocation. This economic growth rate is equivalent to that of the standard AK–growth model. This confirms that endogenous time preferences play a crucial role for nontrivial transitional dynamics in the dynastic competitive economy.

We also find that the growth rate is zero when no limiting spillover effect of the private investment exist (i.e., $\alpha \to 1$) such as in a standard AK–technology in an endogenous growth model. This rate of returns to capital limits wage income close to zero (i.e., $w_t = (1-\alpha)AK_t \cong 0$) and thus limits consumption to zero due to the lack of positive saving in the young period and thus no income in the old period.[26] This finding suggests that, in contrast to an infinite horizon AK–growth model (e.g. Rebelo, 1991), the standard AK–technology supports no positive long-run growth in a competitive overlapping-generations economy. This result justifies our technology with production externalities in production to ensure strictly positive wage income and capital accumulation for a persistently growing dynastic competitive economy.

Now we characterize the pattern of consumption in the lifecycle and the evolution of consumption over generations by combining the competitive equilibrium conditions in Theorem 1. First, (5a), (5b), and (10a) yield that $R_{t+1}d_{t+1}=s_t = w_t - c_t = w_t\frac{\Phi(p_t,p_{t+1})}{1+\Phi(p_t,p_{t+1})}$. Then, using this expression with (10a), the consumption ratio for the young and the old is $\frac{d_{t+1}}{c_t} = \alpha A\Phi(p_t, p_{t+1})$ for generation t. This consumption ratio is useful for understanding how the pattern of consumption changes over an individual's lifecycle. Note that the ratio becomes steeper for a higher value of A, which is interpreted as the total factor productivity. That is, as (11) shows, a fast-growing economy is associated with a steep increase in lifetime consumption (Carroll and Summer, 1991). Second, the slope of the lifecycle consumption path becomes steeper when a person becomes more patient; that is, $\beta'(p_t)>0$, ceteris paribus. This result is consistent with the findings of Carneiro and Heckman (2003) and Bowles and Gintis (2002), which pointed out that a steep consumption profile empirically indicates the presence of intergenerational correlation in a life-cycle model.[27]

Moreover, by using (8a) with $s_t = R_{t+1}d_{t+1}$, we find that the adult's consumption ratio $\frac{d_{t+1}}{d_t}$ between generation $t-1$ and t is identical to the economic growth rate Γ_{t+1}, i.e., $\frac{d_{t+1}}{d_t}=(1-\alpha)A\frac{\Phi(p_t,p_{t+1})}{1+\Phi(p_t,p_{t+1})}$. This result predicts that, regardless the young's consumption, the consumption schedule over the adult periods resembles the economic growth path when time discounting is endogenous. As a result, the intergenerational correlation over consumption mainly depends on the altruistic parental time allocations to parents' own children under the endogeneity of time

[25] This model is capable of introducing population growth by rewriting the individual budget constraints accordingly; for example, $K_{t+1} = N_t s_t$, $N_t=(1+n)N_{t-1}$, where n is a population growth rate. However, this extension holds the main results as long as fertility is exogenous. A complete analysis of endogenous fertility is our interest in this paper so that we will leave the issue of population growth to future research (Barro and Becker, 1989).

[26] Strictly speaking, a zero wage income in equilibrium permits no consumption for the old adult so that it violates the Inada condition on felicity functions; therefore, dynastic competitive equilibrium does not exist.

[27] However, the literature is not conclusive on whether a steep consumption profile in the life cycle results from cognitive or noncognitive abilities including personality traits. Doepke and Zillibotti (2005a, 2005b) pointed out the same property in the lifecycle consumption pattern along with endogenous time preferences.

discounting as noncognitive personal traits. The preference heterogeneity of impatience leads to a different consumption and income profile among families within and over dynastic generations. Therefore, this analysis provides a mechanism that shows parental investment in children's time preferences has an important effect on persistent consumption disparity among generations.

Finally, we obtain the second-order difference equation of parental time spent p_t and p_{t+1}. By substituting the consumption ratio $\frac{d_{t+1}}{d_t}$ for the old over generations into (9b), we summarize the dynamics of the dynastic competitive equilibrium in terms of a second-order difference equation of the parental time allocation path $\{p_t, p_{t+1}, p_{t+2}\}$ in the following lemma. In the remainder of this paper, we take advantage of the second-order difference equation, called "the p-policy function," which can acts as a workhorse to analyze the property of the dynastic competitive equilibrium.

Lemma 2. Under the assumptions of the dynastic competitive economy, the following second-order difference equation of the p-policy function of $\{p_t, \quad p_{t+1}, \quad p_{t+2}\}$ dictates the dynastic competitive equilibrium in Theorem 1:

$$\frac{b}{\gamma} = (1-p_{t+1})^{1-b}(1-p_{t+2})^{b}\frac{\beta'(p_{t+1})}{\beta(p_t)}\left[(1-\alpha)A\frac{\Phi(p_{t+1},\ p_{t+2})}{1+\Phi(p_{t+1},\ p_{t+2})}\right]^{a}, \tag{12}$$

where $\Phi(p_t,\ p_{t+1}) \equiv [\beta(p_t)(\alpha A)^{a}(1-p_{t+1})^{b}]^{\frac{1}{1-a}}$.

The p-policy function in the previous lemma is an implicit function of the path of parental time spent $\{p_t, p_{t+1}, p_{t+2}\}$. In addition to the non-monotonic dynamics of the economic growth component $(1-\alpha)A\frac{\Phi(p_t, p_{t+1})}{1+\Phi(p_t, p_{t+1})}$ as in (11), the other components of the p-policy function also complicate the dynamics of the competitive equilibrium path: the weighted marginal rates of time discounting $\frac{\beta'(p_{t+1})}{\beta(p_t)}$ and the intensity on felicities of leisure associated with p_{t+1} and p_{t+2} in the first two terms in (12). The next two sections investigate in detail these joint effects on the dynamics of the dynastic competitive equilibrium.

4. Balanced growth path in the dynastic competitive economy

This section characterizes the dynastic competitive equilibrium by investigating the property of the p-policy function. Before examining the transitional dynamic equilibrium path, we focus on a balanced growth equilibrium path in the dynastic competitive economy. The balanced growth equilibrium path satisfies the long-run stationary condition such that consumption paths $\bar{c}_t$ and $\bar{d}_t$ and the capital stocks $\bar{k}_t$ grow at a strictly positive constant rate Γ_t, and leisure $\bar{l}_t$ and the parental time allocation $\bar{p}_t$ are constant over periods; that is, there exists $\bar{t}\geq 1$ such that $\bar{l}_t = \bar{l} > 0$, $\bar{p}_t = \bar{p} > 0$ for $t \geq \bar{t}$. Then, the p-policy function of (12) in the balanced growth path satisfies

$$\frac{b}{\gamma} = (1-\bar{p})\frac{\beta'(\bar{p})}{\beta(\bar{p})}\left[(1-\alpha)A\frac{\Phi(\bar{p},\ \bar{p})}{1+\Phi(\bar{p},\ \bar{p})}\right]^{a}, \tag{13}$$

where $\Phi(\bar{p},\ \bar{p}) = [\beta(\bar{p})(\alpha A)^{a}(1-\bar{p})^{b}]^{\frac{1}{1-a}}$.

The steady state p-policy function provides the existence and uniqueness conditions for the competitive balanced growth path. More specifically, first, because $\beta(\bar{p})$ is concave, the term $\frac{\beta'(\bar{p})}{\beta(\bar{p})}$ is nonincreasing in $\bar{p}$.[28] Second, $\Phi(\bar{p}, \bar{p})$ can be either increasing or decreasing in $\bar{p}$, depending on the value of $\bar{p}$. Third, when $\Phi(\bar{p}, \bar{p})$ is nonincreasing, that is, $\frac{\mathrm{d}\Phi(\bar{p},\bar{p})}{\mathrm{d}\bar{p}} \leq 0$, along with concave $\beta(\bar{p})$, the right-hand side of (13) monotonically decreases in $\bar{p}$. Fourth, by a simple calculation, $\frac{\mathrm{d}\Phi(\bar{p},\bar{p})}{\mathrm{d}\bar{p}} \leq 0$ if and only if $\frac{\beta'(\bar{p})}{\beta(\bar{p})} \leq \frac{b}{1-\bar{p}}$. Hence, the unique steady state $\bar{p}$ exists in the case of $\frac{\beta'(\bar{p})}{\beta(\bar{p})} \leq \frac{b}{1-\bar{p}}$, and thus the balanced growth path $\{\bar{c}_t, \bar{d}_t, \bar{K}_t, \bar{l}, \bar{p}, 1+\bar{r}, \bar{w}\}$ is unique in the dynastic competitive economy.

On the other hand, when $\Phi(\bar{p}, \bar{p})$ is nondecreasing in $\bar{p}$, that is, $\frac{\beta'(\bar{p})}{\beta(\bar{p})} \geq \frac{b}{1-\bar{p}}$, the right-hand side of (13) is not monotonic in $\bar{p}$. Consequently, the mixed effects of $\bar{p}$ suggest that more than one steady state solution $\bar{p}$ for (13) can exist in balanced growth paths.[29] Therefore, multiple balanced growth paths can exist in a dynastic competitive economy. We, however, cannot show the result analytically, and thus we numerically demonstrate the possibility of its multiplicity in Section 8. The multiplicity of the balanced growth paths also invites the complex transitional dynamics in the dynastic competitive economy. In the following sections we demonstrate, along with various bifurcations, the aggregate instability and indeterminacy of the transitional dynamic equilibrium paths.

The following theorem summarizes the previous argument:

Theorem 2. Under the assumptions in the dynastic competitive economy, suppose that $\frac{\beta'(\bar{p})}{\beta(\bar{p})} \leq \frac{b}{1-\bar{p}}$, the steady state p-policy function in (13) has a unique parental time allocation and thus the dynastic competitive equilibrium allocation $\{\bar{c}_t, \bar{d}_t, \bar{K}_t, \bar{l}, \bar{p}, 1+\bar{r}, \bar{w}\}$ is globally determinate. On the other hand, suppose that $\frac{\beta'(\bar{p})}{\beta(\bar{p})} \geq \frac{b}{1-\bar{p}}$, the steady state p-policy function in (13) can yield multiple parental time allocations and thus dynastic competitive equilibrium allocation $\{\bar{c}_t, \bar{d}_t, \bar{K}_t, \bar{l}, \bar{p}, 1+\bar{r}, \bar{w}\}$ can be globally indeterminate.

The global indeterminacy is based on the joint effect from parental time allocation $\bar{p}$ and the amount of leisure $1-\bar{p}$. Theorem 2 shows that the global indeterminacy can arise when the weighted marginal time discounting $\frac{\beta'(\bar{p})}{\beta(\bar{p})}$ is greater than the weighted marginal utility of leisure or the weighted marginal disutility of the parental time allocation $\frac{v_2(\bar{d}_{t+1}, 1-\bar{p})}{v(\bar{d}_{t+1}, 1-\bar{p})} = \frac{b}{1-\bar{p}}$. Notice that an increase in $\bar{p}$ decreases $\frac{\beta'(\bar{p})}{\beta(\bar{p})}$ and increases $\frac{b}{1-\bar{p}}$. This relation implies that a small $\bar{p}$ is most likely to satisfy the global indeterminacy condition. Therefore, global indeterminacy arises when the parental investment does not make children patient enough in the family.

Now we closely examine the properties for the growth rate of the balanced growth path. From (11) in Lemma 1, the balanced growth rate with the constant parental time allocation $\bar{p}$ is

$$\bar{\Gamma} \equiv \frac{\bar{K}_{t+1}}{\bar{K}_t} = (1-\alpha)A\frac{\Phi(\bar{p},\quad \bar{p})}{1+\Phi(\bar{p},\quad \bar{p})}, \tag{14}$$

where $\Phi(\bar{p}, \bar{p}) = [\beta(\bar{p})(\alpha A)^{a}(1-\bar{p})^{b}]^{\frac{1}{1-a}}$. First, positive long-run balanced growth requires that A is large enough that $(1-\alpha)A\Phi(\bar{p}, \bar{p}) \geq 1+\Phi(\bar{p}, \bar{p})$ at the steady state value of $\bar{p}$. Second, the positive long-run growth condition is again a joint condition of technology and preferences including the time preferences factor $\beta(\bar{p})$. However, unlike the marginal changes of time preferences in the p-policy function, the long-run growth rate depends on the level, not the marginal rate, of time discounting. Hence, the short-run property prevails in the long run.[30] Third, the simple observation shows that the long-run growth rate $\bar{\Gamma}$ is also not monotonic because $\Phi(\bar{p}, \bar{p})$ is not monotonic in parental time spent $\bar{p}$. That is, the long-run growth rate $\bar{\Gamma}$ increases when $\Phi(\bar{p}, \bar{p})$ increases in $\bar{p}$, whereas long-run growth rate $\bar{\Gamma}$ decreases when $\Phi(\bar{p}, \bar{p})$ decreases in $\bar{p}$. In sum, the non-monotonicity of the balanced growth rates affects the nonlinear balanced growth condition in (14) and the nonlinear steady state p-policy function in (13).

Then, we have the following lemma.

Lemma 3. In the dynastic competitive equilibrium, the balanced growth rate $\bar{\Gamma} \equiv \frac{\bar{K}_{t+1}}{\bar{K}_t}$ for all $t \geq \bar{t}$ for a large $\bar{t} \geq 1$ in (14) is non-monotonic in $\bar{p}$ as is the balanced competitive equilibrium in (13).

More closely examining the balanced growth paths, we further

[28] Note that $d\left[\frac{\beta'(\bar{p})}{\beta(\bar{p})}\right]/dp \leq 0$ is because $\beta(\bar{p})$ is concave.

[29] In fact, both the sign of $\frac{\mathrm{d}\Phi(\bar{p},\bar{p})}{\mathrm{d}\bar{p}}$ and $\frac{\mathrm{d}^2\Phi(\bar{p},\bar{p})}{(\mathrm{d}\bar{p})^2}$ are ambiguous.

[30] Refer to (11) and (12).

specify the time-discount function as in Doepke and Zilibotti (2005a):

$$\beta(p) = \beta^0 + \eta p^\theta,$$

where $1>\beta^0>0$ and $\eta > 0$ is the intensity of the time-discounting factor with the concavity parameter, $0 < \theta < 1$. Then, the p-policy function in (13) becomes in a balanced growth path:

$$\frac{b}{\gamma} = (1-\bar{p})\frac{\theta\eta\bar{p}^{\theta-1}}{\beta^0 + \eta\bar{p}^\theta}\left[(1-\alpha)A\frac{\Phi(\bar{p},\bar{p})}{1+\Phi(\bar{p},\bar{p})}\right]^a, \tag{15}$$

where $\Phi(\bar{p},\bar{p}) = [(\beta^0+\eta\bar{p}^\theta)(\alpha A)^a(1-\bar{p})^b]^{\frac{1}{1-a}}$. A few observations help us to depict the shape of the right-hand side of the steady state p-equation (15). First, the first two terms, $(1-\bar{p})$ and $\frac{\theta}{\bar{p}}$, decrease the p-policy function. Second, we also confirm that $\Phi(\bar{p},\bar{p})$ is decreasing in $\bar{p}$ in the case that $\frac{\eta\bar{p}^\theta}{\beta^0+\eta\bar{p}^\theta}\frac{\theta}{\bar{p}} \le \frac{b}{1-\bar{p}}$. Hence, combining these conditions, the right-hand side of the steady state p-policy function is monotonically deceasing in $\bar{p}$. Hence, the steady state parental time spent $\bar{p}$ is unique, and thus the dynastic competitive equilibrium is also unique in the dynastic competitive equilibrium. That is, the specification of the time discount function shows that, ceteris paribus, the strong concavity parameter θ of shaping offspring's time discounting determines the uniqueness of the steady state competitive equilibrium. This condition is also compatible to the standard uniqueness condition in the overlapping-generations economy under which the felicity function is concave with a large exogenous time-discount factor β^0 (see, e.g., de la Croix and Michel, 1999).

On the other hand, we also confirm that $\Phi(\bar{p},\bar{p})$ is increasing in $\bar{p}$ when $\frac{\eta\bar{p}^\theta}{\beta^0+\eta\bar{p}^\theta}\frac{\theta}{\bar{p}} \ge \frac{b}{1-\bar{p}}$. Hence, given that the first two terms of the right hand side of (15) is decreasing in $\bar{p}$, an increasing $\Phi(\bar{p},\bar{p})$ in $\bar{p}$ implies that the p-policy function is not necessarily monotonic in $\bar{p}$, and thus more than one balanced growth path is plausible in the long run.[31] Thus, it verifies the possibility of multiple balanced growth paths along with the multiple parental time spent $\bar{p}$ for shaping the children's rates of time preferences. In detail, the multiplicity of balanced growth paths is most likely to arise whenever—with respect to a small marginal utility of leisure i.e., $\frac{v_2(\bar{d}_{t+1},1-\bar{p})}{v(\bar{d}_{t+1},1-\bar{p})} = \frac{b}{1-\bar{p}}$,—the intensity η of time discount formation is large; the marginal increment θ of the discount function is large (i.e., the weak concavity of the time discount function); and the exogenous time discounting β^0 is small. This analysis thus verifies that endogenous time preferences affect the long-run equilibrium path in the dynastic competitive economy.

5. Uniqueness and indeterminacy of a transitive competitive equilibrium

In this section, we extend our analysis to examine the uniqueness and stability property of a transitional dynamic equilibrium in the dynastic competitive economy. We introduce a new variable $z_t \equiv p_{t+1}$ for all $t \ge 1$. Then, we redefine a second-order difference equation of the p-policy function in (12) as a two-dimensional set of the first-order difference equations of $\{p_{t+1}, z_{t+1}\}$:

$$p_{t+1} - z_t = 0, \tag{16a}$$

$$\Omega(p_t, z_t, z_{t+1}) - \frac{b}{\gamma} = 0, \tag{16b}$$

where

$$\Omega(p_t,z_t,z_{t+1}) \equiv (1-z_t)^{1-b}(1-z_{t+1})^b\frac{\beta'(z_t)}{\beta(p_t)}\left[(1-\alpha)A\frac{\Phi(z_t,\ z_{t+1})}{1+\Phi(z_t,\ z_{t+1})}\right]^a,$$

$$\Phi(z_t,z_{t+1}) \equiv [\beta(z_t)(\alpha A)^a(1-z_{t+1})^b]^{\frac{1}{1-a}}.$$

[31] A few numerical examples in Section 8 illustrate that the right-hand side takes an inverted U-shape in $\bar{p}$.

Then, in the balanced growth path, we have $p_t = \bar{p}$ and $z_t = \bar{z}$ for all $t>\bar{t}$ for a large $\bar{t}\ge 1$. By applying the usual method, we linearize the first-order difference Eqs. (16a) and (16b) and then evaluate them in the balanced growth path $\{\bar{p},\bar{z}\}$ in (13). The two-dimensional system of the difference equations for the dynastic competitive economy is summarized as

$$\begin{bmatrix} p_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mathrm{M} & \mathrm{N} \end{bmatrix}\begin{bmatrix} p_t - \bar{p} \\ z_t - \bar{z} \end{bmatrix}, \tag{17}$$

where $\mathrm{M} = -\frac{\Omega_1}{\Omega_3}$ and $\mathrm{N} = -\frac{\Omega_2}{\Omega_3}$. Here, Ω_i denotes a partial derivative of $\Omega(p_t, z_t, z_{t+1})$ with respect to the ith variable. Some tedious computations yield

$$\Omega_1 = -\frac{\beta'(\bar{p})}{\beta(\bar{p})}\Omega(\bar{p},\bar{z},\bar{z}),$$

$$\Omega_2 = \left[-\frac{1-b}{1-\bar{p}} + \frac{\beta''(\bar{p})}{\beta'(\bar{p})} + \frac{a}{1-a}\frac{\beta'(\bar{p})}{\beta(\bar{p})}\frac{1}{1+\Phi(\bar{z},\bar{z})}\right]\Omega(\bar{p},\bar{z},\bar{z}),$$

$$\Omega_3 = -\left[\frac{b}{1-\bar{p}} + \frac{ab}{1-a}\frac{1}{1-\bar{p}}\frac{1}{1+\Phi(\bar{z},\bar{z})}\right]\Omega(\bar{p},\bar{z},\bar{z}).$$

To examine uniqueness and stability for the balanced growth path of $\{p_t, z_t\}$, we take advantage of the property of the 2×2 matrix $J \equiv \begin{bmatrix} 0 & 1 \\ \mathrm{M} & \mathrm{N} \end{bmatrix}$ in (17). First, the determinant $D(J)$ and the trace $T(J)$ of the matrix J are $D(J) = -\mathrm{M}$ and $T(J) = \mathrm{N}$, respectively. Second, both variables $\{p_t, z_t\}$ are non-predetermined jump variables in the two-dimensional difference equations. Third, the two distinct eigenvalues of the matrix J provide the existence of stable or unstable manifolds. We consider the three cases to determine whether each eigenvalue of the matrix$J$ is in the stable or unstable manifold.

In the first case that $|1-\mathrm{M}| > |\mathrm{N}|$ and $|-\mathrm{M}|<1$, both eigenvalues are inside the unit interval, and thus the two associated manifolds are stable around a balanced growth path. Hence, the balanced growth path is absolutely stable, and thus local indeterminacy emerges. There exist infinitely many transitional paths, each of which converges to the same balanced growth path. In the second case that $|1-\mathrm{M}| > |\mathrm{N}|$ and $|-\mathrm{M}|>1$, both eigenvalues are outside the unit interval, and thus the two associated manifolds are unstable around a balanced growth path. Hence, the balanced growth path is absolutely unstable, and thus every transitional path, except the balanced growth path, explodes and thus violates the transversality condition. Therefore, the transitional competitive equilibrium path is trivially stationary. That is, the transitional dynamic path is unique, and the balanced growth path is locally determinate. In the third case that $|1-\mathrm{M}| < |\mathrm{N}|$, one eigenvalue is inside the unit interval and the other eigenvalue is outside the unit interval; one stable and one unstable manifold exist around the balanced growth path. Hence, the balanced growth path is (saddle) unstable; a continuum of the transitional paths near the balanced growth path exists because the number of non-predetermined jump variables of $\{p_t, z_t\}$ exceeds the number of stable manifolds. Therefore, the balanced equilibrium path is locally indeterminate in the dynastic competitive economy.

We conclude that the first and third case exhibit local indeterminacy and the existence of a continuum of transitional dynamic paths in the dynastic competitive economy. Hence, local indeterminacy depends on the marginal change $\beta''(\bar{p})$ as well as the level change $\beta'(\bar{p})$ in the rate $\beta(\bar{p})$ of time preferences. The non-monotonicity of $\Phi(\bar{p},\bar{p})$ also plays an important role in the local indeterminacy property of the balanced equilibrium path. Therefore, the endogeneity of the time preferences influences local indeterminacy in the dynastic competitive economy. In Section 8, we use a set of plausible parameter values of the dynastic competitive economy to provide a few numerical examples that illustrate uniqueness and indeterminacy.

6. Social optimum allocations

In this section, we examine the property of a socially efficient allocation corresponding to the dynastic competitive economy in the presence of two externalities: the first one for the productive public capital in technology and the second one for the parental time allocation for shaping time preferences of the future offspring within the same family. The benevolent social planner internalizes those externalities to maximize the social welfare in an overlapping-generations economy. Formally, the social planner optimizes its objective by taking into account the following labor–leisure–parental time and the usual resource constraints given the initial level of parenting p_1 and of capital K_1:

$$p_t + l_t = L_t, \tag{18a}$$

$$Y_t = c_t + d_t + K_{t+1}, \tag{18b}$$

where L_t=1 in period t≥1 in the aggregate dynastic economy. The last resource constraints are obtained by internalizing the capital externalities $\widetilde{K}_t = K_t$ in the planning economy, so that the aggregated technology is realized as Y_t=$F(K_t$,1;K_t)=AK_t. Then, the social interest rate is $1+r_t^* \equiv \frac{\partial F}{\partial K_t}=A$. Hence, the social planning economy is reduced to an endogenous growth model of AK–technology.

We now define the benevolent planner's objective function as

$$W_t \equiv \lim_{T\to\infty}\sum_{s=t}^{T}\gamma^s[u(c_s) + \beta(p_s)v(d_{s+1}, l_{s+1})] + \gamma^{T+1}W_{T+1},$$

for all $t \geq 1$. Suppose that $\lim_{T\to\infty}\gamma^{T+1}W_{T+1}=0$,[32] the social welfare function W_t is well defined under the concavity of the felicity function $u(c_t)$ and $v(d_{t+1}, l_{t+1})$ and the time discount function $\beta(p_t)$. Then, subject to the resource feasibility constraints (18a) and (18b), the social planer maximizes

$$\sum_{t=1}^{\infty}\gamma^t[u(c_t)+\beta(p_t)v(d_{t+1},1-p_{t+1})],$$

given the initial level of parenting and capital stocks $\{p_1, K_1\}$. It is clear that the social planning problem is a convex program under the assumptions on the felicity functions, time discount function, and production function. Then, the dynamic Euler–Lagrangian condition for the planner's problem with respect to $\{c_{t+1}, K_{t+1}, p_{t+1}\}$ are

$$\gamma u_1(c_{t+1}) - \beta(p_t)v_1(d_{t+1},1-p_{t+1})=0, \tag{19a}$$

$$\beta(p_t)v_1(d_{t+1},1-p_{t+1}) - \gamma A\beta(p_{t+1})v_1(d_{t+2}, 1-p_{t+2})=0, \tag{19b}$$

$$\beta(p_t)v_2(d_{t+1},1-p_{t+1}) - \gamma\beta'(p_{t+1})v(d_{t+2},1-p_{t+2}) = 0. \tag{19c}$$

The following transversality condition ensures the bounded W_t under the condition that the present value of capital becomes zero as the period goes to the infinity:

$$\lim_{t\to\infty}\beta(p_t)u_1(c_{t+1})K_{t+1}=0. \tag{19d}$$

Note that the social efficiency conditions (19a), (19b), (19c), and (19d) for social optimization correspond to the competitive equilibrium conditions (7a), (7b), (7c) and (7d) in the dynastic competitive equilibrium. However, in contrast to the dynastic competitive equilibrium, the social rate of interest is $1+r_t^*=A > 1+r_t = \alpha A$ for $t \geq 1$, by internalizing production externalities. In addition, internalizing the time preferences externalities over the generations increases in the periods of the non-arbitrage condition in $\{p_t, p_{t+1}, p_{t+2}\}$.

In the next theorem, we characterize the corresponding social optimum equilibrium to the dynastic competitive economy:

Theorem 3. In the dynastic competitive economy with endogenous time preferences, the corresponding dynastic social optimum allocation $\{c_t, d_{t+1}, K_{t+1}, l_{t+1}, p_{t+1}\}$ with the initial condition $\{p_1, K_1\}$ is characterized by the set of necessary and sufficient conditions: (i) the economic feasibility conditions (18a) and (18b) and (ii) the dynamic Euler-Lagrangian condition (19a), (19b), and (19c) including (iii) the transversality condition (19d).

For further analysis on the social optimum allocation, we also specify the felicity functions as in the dynastic competitive economy: $u(c_t) = c_t^a$, $v(d_{t+1}, l_{t+1}) = d_{t+1}^a l_{t+1}^b$. By applying the same method as the dynastic competitive economy, we derive the social optimal growth rate Γ_{t+1} and the social p-policy function from the conditions (19a), (19b), and (19c) for the social optimum.

The following lemma summarizes the result.

Lemma 4. In the dynastic competitive economy with endogenous time preferences, the corresponding social optimum dynamics are governed by the following second-order difference equation of the p-policy function:

$$\frac{b}{\gamma} = (1-p_{t+1})^{1-b}(1-p_{t+2})^b\frac{\beta'(p_{t+1})}{\beta(p_t)}\left[\gamma A\frac{\beta(p_{t+1})}{\beta(p_t)}\frac{(1-p_{t+2})^b}{(1-p_{t+1})^b}\right]^{\frac{a}{1-a}}, \tag{20}$$

with the following economic growth rate:

$$\Gamma_{t+1} \equiv \frac{K_{t+1}}{K_t} = \left[\gamma A\frac{\beta(p_{t+1})}{\beta(p_t)}\frac{(1-p_{t+2})^b}{(1-p_{t+1})^b}\right]^{\frac{1}{1-a}}. \tag{21}$$

In the remainder of this section, we focus on the balanced growth condition for a social optimum allocation where $p_t = \widetilde{p}$ for all $t>\tilde{t}$ for some large $\tilde{t}\geq 1$ on the second-order difference equation of the p-policy function of $\{p_t, p_{t+1}, p_{t+2}\}$ in (20) with the balanced growth rate in (21). First, from (21), the social optimum balanced growth rate $\widetilde{\Gamma}$ is

$$\widetilde{\Gamma} = (\gamma A)^{\frac{1}{1-a}}. \tag{22}$$

It is clear that the social optimum balanced growth rate is generically unique. Moreover, the balanced growth rate is a joint condition of preferences and technology, but it is independent of the time discount function. This result is because the social planner completely internalizes the formation of time preferences across the generations. This long-run growth property with endogenous preferences is an extension of a social optimum with exogenous time preferences in a standard dynastic economy and depends on the intensity γ of altruism, the felicity parameter a in preferences, and the total factor productivity A in technology. In detail, (22) shows that the faster the social optimum path grows when parents are more altruistic, the higher marginal utility of consumption is with respect to leisure, and the higher total factor productive is.

Second, from (20), the balanced growth path $\{\tilde{c}_t, \tilde{d}_t, \widetilde{K}_{t+1}, \tilde{l}, \widetilde{p}\}$ satisfies the following steady state p-policy function for the social optimum path:

$$\frac{b}{\gamma}=(1-\widetilde{p})\frac{\beta'(\widetilde{p})}{\beta(\widetilde{p})}(\gamma A)^{\frac{a}{1-a}}, \tag{23}$$

where $\widetilde{p}$ is the parental time allocation in the social optimum balanced growth path. As in the competitive equilibrium, this steady state p-policy function is dictated by the three components: the path of parental time allocation, the weighted marginal change of the time discount function, and the social optimum balanced growth rate in (22). The right-hand side of the expression is monotonically decreasing in $\widetilde{p}$, since the time discount function $\beta(\widetilde{p})$ is increasing and concave. As result, the social optimum balanced growth path is unique and thus globally determinate.

The results are summarized as follows.

Lemma 5. In the dynastic competitive economy with endogenous time preference, the corresponding social optimum allocation $\{c_t, d_{t+1}, K_{t+1}, l_{t+1}, p_{t+1}\}$ is depicted by the balanced p-policy function in (23) with the balanced growth rate in (22). Moreover, the social

[32] This condition is considered to be an alternative transversality condition. Also see argument in (19d).

optimum balanced growth allocation $\{\tilde{c}_t, \tilde{d}_t, \tilde{K}_{t+1}, \tilde{l}, \tilde{p}\}$ in the long run is unique and thus is globally determinate.

In contrast to the multiplicity of balanced growth paths in the dynastic competitive economy, the social planning program restores the uniqueness of the balanced growth path by internalizing both the production externalities and the generational preference externalities in the leisure–parenting time allocation for intergeneration welfare. Again, a simple observation of (23) confirms that a high intensity γ of altruism, a low marginal utility b of leisure–parenting time, and a high marginal utility a of consumption increase the parenting time allocation for shaping children's time preferences in the long run.

7. Social optimum transitional dynamic path

Next we examine the local stability of the social optimum balanced growth path and the indeterminacy of the social optimum transitional dynamics. We adopt the same method as in the dynastic competitive economy. We, first, define a new auxiliary variable $z_t \equiv p_{t+1}$. Second, from (21), we have a two-dimensional dynamic system of the first-order difference equations of $\{p_t, z_t\}$ for the social optimum:

$$p_{t+1} - z_t = 0, \tag{24a}$$

$$\Theta(p_t, z_t, z_{t+1}) - \frac{b}{\gamma} = 0, \tag{24b}$$

where $\Theta(p_t, z_t, z_{t+1}) \equiv (1-z_t)^{1-b}(1-z_{t+1})^b \frac{\beta'(z_t)}{\beta(p_t)}\left[\gamma(1+A)\frac{\beta(z_t)(1-z_{t+1})^b}{\beta(p_t)(1-z_t)^b}\right]^{\frac{a}{1-a}}$. Third, we linearize the system of the first-order difference Eqs. (24a) and (24b) and evaluate them at the balanced growth path $\{\tilde{p}, \tilde{z}\}$, where $p_t = \tilde{p}$, and $z_t = \tilde{z}$ for all $t > \tilde{t}$ for some large $\tilde{t} \geq 1$. As result, we obtain a two-dimensional linear system of the first-order difference equations:

$$\begin{bmatrix} p_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mathrm{X} & \mathrm{Y} \end{bmatrix}\begin{bmatrix} p_t - \tilde{p} \\ z_t - \tilde{z} \end{bmatrix}, \tag{25}$$

where $\mathrm{X} \equiv -\frac{\Theta_1}{\Theta_3}$ and $\mathrm{Y} \equiv -\frac{\Theta_2}{\Theta_3}$. Here, Θ_i denotes the partial derivative of $\Theta(p_t, z_t, z_{t+1})$ with respect to the ith variable. Fourth, some tedious computations give us

$$\Theta_1 = -\frac{1}{1-a}\frac{\beta'(\tilde{p})}{\beta(\tilde{p})}\Theta(\tilde{p}, \tilde{z}, \tilde{z}),$$

$$\Theta_2 = \left[-\frac{1-a-b}{1-a}\frac{1}{1-\tilde{p}} + \frac{\beta''(\tilde{p})}{\beta'(\tilde{p})} + \frac{a}{1-a}\frac{\beta'(\tilde{p})}{\beta(\tilde{p})}\right]\Theta(\tilde{p}, \tilde{z}, \tilde{z}),$$

$$\Theta_3 = -\frac{b}{1-a}\frac{1}{1-\tilde{p}}\Theta(\tilde{p}, \tilde{z}, \tilde{z}).$$

Fifth, we take advantage of the determinant $D(Q)$ and the trace $T(Q)$ of the 2×2 matrix $Q \equiv \begin{bmatrix} 0 & 1 \\ \mathrm{X} & Y \end{bmatrix}$ in (25). Clearly, we have that $D(Q) = -\mathrm{X}$ and $T(Q) = \mathrm{Y}$.

Finally, to examine the uniqueness and stability property of social optimum, we recall that $\{p_t, z_t\}$ are non-predetermined jump variables. As in the competitive equilibrium path, three cases of the two dimensional dynamic system for social optimum are possible. In the first case that $|1-\mathrm{X}| > |\mathrm{Y}|$ and $|-\mathrm{X}| < 1$ $\left(\text{equivalently, } \frac{b}{1-p} > \frac{\beta'(p)}{\beta(p)} > \left|-\frac{1}{1-p} + \frac{\beta''(p)}{\beta'(p)}\right|\right)$, the two eigenvalues of the matrix Q are inside of the unit interval and thus exhibit two stable manifolds. Therefore, the balanced growth path is absolutely stable, and thus, the local indeterminacy emerges in social optimum; a continuum of the transitional social optimum paths exists. In the second case that $|1-\mathrm{X}| > |\mathrm{Y}|$ and $|-\mathrm{X}| > 1$ $\left(\text{equivalently, } \frac{\beta'(p)}{\beta(p)} > max\left\{\frac{b}{1-p}, \left|-\frac{1}{1-p} + \frac{\beta''(p)}{\beta'(p)}\right|\right\}\right)$, both eigenvalues are outside of the unit interval. Hence, every transitional dynamic path, except the balanced growth path, explores and thus violates the transversality condition. Therefore, the balanced growth path is only the social optimum transitional path. Hence, the social optimum transitional path is unique, and the balanced growth path is locally determinate. In the third case that $|1-\mathrm{X}| < |\mathrm{Y}|$ $\left(\text{equivalently, } \frac{\beta'(p)}{\beta(p)} < \left|-\frac{1}{1-p} + \frac{\beta''(p)}{\beta'(p)}\right|\right)$, one eigenvalue is inside of the unit interval and the other eigenvalue is outside of the unit interval; thus, one stable and one unstable manifold exist. That is, a continuum of the transitional paths exists near the balanced growth path because the number of non-predetermined variables of $\{p_t, z_t\}$ exceeds the number of the unstable manifolds.

We thus conclude that the first and third case yield the locally indeterminate balanced growth path and the continuum of transitional dynamic paths. Hence, local indeterminacy, if exists, is most likely to arise when the parental time allocation is small so that the weighted marginal change in the time discount function is enough with respect to the joint effect of the inverse of allocation of leisure and the degree of the concavity—the weighted change in the marginal changes—of the time discount function.[33] Again, as in the dynastic competitive equilibrium, both the level and the marginal changes of the time discount function are critical to the possibility of local indeterminacy. We also cannot analytically solve for the eigenvalues so in the following section we numerically examine the stability property of the corresponding social optimum under a set of plausible parameter values for the dynastic competitive economy.

8. Numerical examples of uniqueness, indeterminacy, cycles, and bifurcations

We now consider numerical examples to illustrate the uniqueness and stability property of the dynastic competitive equilibrium and its corresponding social optimum allocation. Example I sets the following parameter values: $a = 0.7$; $b = 0.3$; $\gamma = \left(\frac{1}{1+0.07}\right)^{35}$; $\beta^0 = \left(\frac{1}{1+0.1}\right)^{35}$; $\eta = \left(\frac{1}{1+0.035}\right)^{35} - \beta^0$; $\theta = 0.5$; $A = 25$; and $\alpha = 0.33$.[34] We find that both the competitive equilibrium and the corresponding social optimum allocation are unique.[35] More specifically, by imposing these parameter values into (13) and (14), we find that the steady state parental time spent $\overline{p}$ is 0.213 and the long-run rate of growth $\overline{\Gamma}$ is 1.033 in the dynastic competitive equilibrium whereas from (22) and (23) in the corresponding social optimum allocation the steady state parenting time spent $\tilde{p}$ is 0.488 and the balanced growth rate $\tilde{\Gamma}$ is 1.084. Hence, in the presence of endogenous time preferences, both the competitive equilibrium and its social optimum path are globally determinate in the dynastic overlapping-generations economy.

Example I also demonstrates the unique transitional dynamic path for both the competitive equilibrium and the corresponding social optimum. Given the set of parameter values in Example I, the matrix J in (17) yields that the determinant $D(J) = 1.649$ and the trace $T(J) = 0.180$ in the balanced growth path competitive equilibrium. Hence, $D(J)$ and $T(J)$ satisfy that $|1 + D(J)| > |T(J)|$ and $|D(J)| > 1$. Therefore, every competitive balanced growth path, except the balanced growth path, is locally unstable, and thus the balanced growth path is the only transitional competitive dynamic path that satisfies the transversality condition. Therefore, the transitional dynamic path is unique and the balanced growth path is locally determinate. In the corresponding social optimum allocation, we also show the unique transitional dynamic path and the determinate balanced growth path. In detail, first, the matrix Q in (25) has the determinant $D(Q) = 1.466$ and the trace $T(Q) = 0.502$ and satisfies $|1 + D(Q)| > |T(Q)|$ and $|D(Q)| > 1$.

[33] This statement is drawn from the complementary range of the second case of uniqueness and determinacy.

[34] We believe that these numerical parameter values are reasonably plausible in a dynastic economy in the literature, even though our intention is not to justify these simulation results to fit the real economy.

[35] Mathematica computes all numerical solutions.

Therefore, $D(Q)$ and $T(Q)$ verify the local instability of the social optimum in the long run, and the transitional social optimum path is unique. Thus, the social optimum balanced growth path is locally determinant.

Next, Example II demonstrates that steady state is unique both in the competitive equilibrium and the social optimum, although it is locally indeterminate in the former and determinate in the latter. In this example, the parameter values are $a = 0.7$; $b = 0.3$; $\gamma = \left(\frac{1}{1+0.1}\right)^{35}$; $\beta^0 = \left(\frac{1}{1+0.1}\right)^{35}$; $\eta = \left(\frac{1}{1+0.001}\right)^{35} - \beta^0$; $\theta = 0.3$; $A = 30$; and $\alpha = 0.33$.[36] We find that the competitive balanced growth path is unique with parental time spent $\overline{p} = 0.211$ and the rate of growth $\overline{\Gamma} = 1.088$ in the decentralized competitive economy. We also verify that the corresponding social optimum balanced growth allocation is unique with parental time spent $\widetilde{p} = 0.036$ and rate of growth $\widetilde{\Gamma} = 1.006$. Interestingly, this example provides additional information that the dynastic competitive economy overspends parental time for instilling children's patience and, in turn, grows too fast in a decentralized competitive equilibrium.

To demonstrate the existence of a continuum of transitional paths in the competitive equilibrium, we consider the matrix J in (17), which yields determinant $D(J) = 1.468$ and trace $T(J) = -3.954$. Hence, $D(J)$ and $T(J)$ satisfy $|1 + D(J)| < |T(J)|$. Therefore, one stable and one unstable manifold exist, and the competitive balanced growth path is saddle unstable. Under the two-dimensional system of the non-predetermined jump variables $\{p_t, z_t\}$, a continuum of transitional dynamic paths exist, and the balanced growth path is locally indeterminate in the competitive equilibrium. This example shows that local indeterminacy coexists with global determinacy. As discussed in the previous sections, endogenous time discounting plays a major role on the coexistence of local and global indeterminacy. Turning to the corresponding social optimum, the matrix Q in (25) gives $D(Q) = 24.13$ and $T(Q) = -1.749$, that is, $|1 + D(Q)| > |T(Q)|$ and $|D(Q)| > 1$, confirming local determinacy.

Next, Example III shows the case of multiple steady states. The parameter values in this example are $a = 0.7$; $b = 0.3$; $\gamma = \left(\frac{1}{1+0.12}\right)^{35}$; $\beta^0 = \left(\frac{1}{1+0.15}\right)^{35}$; $\eta = \left(\frac{1}{1+0.01}\right)^{40} - \beta^0$; $\theta = 0.98$; $A = 55$; and $\alpha = 0.33$. Parameter θ is noticeably different from those in Examples I and II. That is, we use a weak concave time discount function in this example. We then show that this dynastic competitive economy has two competitive balanced growth equilibrium paths associated with the two equilibrium parental time allocations $\overline{p}^*$ and $\overline{p}^{**}$. The first balanced competitive allocation satisfies $\overline{p}^* = 0.053$ and the second steady state satisfies $\overline{p}^{**} = 0.412$. We also find that the associated competitive balanced growth rates are $\overline{\Gamma}^* = 1.0034$ and $\overline{\Gamma}^{**} = 1.105$, respectively.[37] Hence, global indeterminacy arises in this example. The multiplicity of the competitive balanced growth paths means that the economic fundamentals with the same technology and preferences including a time discount function cannot determine the unique expectation balanced growth allocation. Therefore, either $\overline{p}^*$ or $\overline{p}^{**}$ for the competitive equilibrium is self-fulfilling in the dynastic competitive economy. The global indeterminacy property can explain the persistent difference in economic growth and income disparity within an intrageneration and/or across intergenerations in dynastic competitive economies (Benhabib and Farmer, 1999; Galor and Zeira, 1993).[38] More important to our context, the growth and income inequality are due to how a parent influences the future orientation of her children thus affecting the consumption, leisure, savings, and parental time allocation of their offspring in future generations. Global indeterminacy therefore justifies some empirical evidence on intergenerational correlations on, for example, income, education, personal attitudes and traits, and habits (Mulligan, 1997).

Furthermore, in the first balanced growth path $\overline{p}^*$ in Example III, the associated matrix J^* in (17) yields determinant $D(J^*) = 10.58$ and trace $T(J^*) = 23.45$ and so satisfies $|1 + D(J^*)| < |T(J^*)|$. Hence, one stable and one unstable manifold exist, and thus the competitive balanced growth path is saddle unstable. Therefore, a continuum of transitional dynamic paths exists, and the balanced growth path is locally indeterminate in the dynastic competitive equilibrium. On the other hand, matrix J^{**} in (17) associated with the second balanced growth path $\overline{p}^{**}$ has the determinant $D(J^{**}) = 1.781$ and the trace $T(J^{**}) = 0.0304$ and thus satisfies $|1 + D(J^{**})| > |T(J^{**})|$ and $|D(J^{**})| > 1$. Hence, the second balanced growth path is locally unstable so that every transitional path, except the balanced growth path, explores and thus violates the transversality condition. Therefore, this competitive balanced growth path is determinate and the associated transitional dynamic path is unique in the dynastic competitive economy. This example confirms our theoretical result that local indeterminacy is more likely to arise when parental time spent with children is small (i.e., $\overline{p}^* < \overline{p}^{**}$), and thus children are less patient in the future (i.e., $\beta(\overline{p}^*) < \beta(\overline{p}^{**})$). Regarding the corresponding social optimum, steady state is unique at $\widetilde{p} = 0.054$ with $\widetilde{\Gamma} = 1.0039$, $D(Q) = 48.00$ and $T(Q) = 33.25$, confirming local determinacy. Table 1 summarizes the results of those three examples.

So far, we have examined whether a dynastic competitive equilibrium and the corresponding social optimum are determinate or indeterminate, using different sets of parameter values. We now extend this analysis to illustrate various bifurcations in the dynastic competitive equilibrium. The following three examples consider a bifurcation parameter in the felicity function. Example IV takes as a bifurcation parameter a in felicity function of the young and the old,[39] assuming the other parameter values as follows: $b = 0.19$; $\gamma = \left(\frac{1}{1+0.075}\right)^{35}$; $\beta^0 = \left(\frac{1}{1+0.1}\right)^{35}$; $\eta = \left(\frac{1}{1+0.033}\right)^{35} - \beta^0$; $\theta = 0.5$; $A = 26$; and $\alpha = 0.33$. Fig. 1 illustrates the result of the simulation exercise. The red curve shows the combinations of the determinant $D(J)$ and the trace $T(J)$ as a preference parameter a changes. First, local indeterminacy arises approximately in the range $a \in (0.747, 0.783)$.[40] The result is from the dynastic competitive equilibrium, which satisfies $|1 + D(J)| > |T(J)|$ and $|D(J)| < 1$ around a balanced growth path. On the other hand, for $a \in (0, 0.747)$, ceteris paribus, the competitive balanced growth path is absolutely unstable thus showing the local uniqueness in the balanced growth path associated with each parameter a. Moreover, we find that Flip bifurcation arises at $a \simeq 0.783$ at which $|1 + D(J)| = |T(J)|$ holds, whereas, Hopf bifurcation arises at $a \simeq 0.747$ with $|D(J)| = 1$.

Example V demonstrates the robustness of the emergence of Hopf and Flip bifurcations in Example IV. Given the same set of the parameter values as in Example IV, but with $b = 1 - a$, we find local indeterminacy approximately in the range $a \in (0.753, 0.799)$, as Fig. 2 shows. We find Hopf bifurcation at $a \simeq 0.753$ and Flip bifurcation at $a \simeq 0.799$. Again, as a robustness test of the bifurcation property in Example IV and V, the final Example VI (see Fig. 3) maintains the same parameter values except we choose b satisfying $a + b \leq 1$ given $a = 0.78$. With the bifurcation parameter b, we find the absolute stability and thus local indeterminacy when $b \in (0.1885, 0.22)$. We also

[36] Example II is different from Example I mainly in the parameter values for the time discount function η and θ, altruistic parameter γ and productivity A.

[37] From (22), the balanced growth rate is 0.11 for the corresponding social optimum in the dynastic competitive economy.

[38] The income disparity within an intergeneration is from the fact that more than a self-fulfilling equilibrium exist in two different economies with the same fundamentals including the same initial conditions.

[39] This result is consistent with the condition for multiplicity where a high discounting factor violates the turnpike property in the literature (McKenzie, 1986; Michel and Venditti, 1997).

[40] Note that the numbers reported here are approximate due to the nature of simulation.

find Filp bifurcation at b=0. 1885. Examples IV, V, and VI bolster the argument for the emergence of bifurcations accompanied with uniqueness, indeterminacy, and instability with parental time allocations on endogenous time preferences over the generations in the dynastic competitive economy.

9. Concluding remarks

This paper examines an overlapping-generations economy when time discounting in preferences is shaped by parental time spent with children in the same family. The standard results, such as existence, uniqueness, stability, and many other properties, do not necessarily hold in the dynastic competitive economy in the presence of endogenous time preference formation. For the dynastic competitive economy with preference externalities over generations (i.e., an intergenerational correlation in altruistic preferences), we find that the long-run growth rate of a balanced growth path in the competitive economy is not monotonic in terms of the equilibrium time allocation with children. The non-monotonicity is due to the joint effect of the parental time allocation, namely, the welfare loss from reducing leisure and the altruistic award from patient children's welfare gain. Multiple balanced growth paths arise when the time discount function increases fast enough in the parental time spent with children to shape their future orientation. Indeterminacy of the balanced growth path and thus the continuum of transitional dynamic paths results. Hence, the dynastic competitive economy is locally and globally indeterminate. This finding suggests a different income distribution within a generation and/or across generations in the dynastic competitive growing economy with endogenous children's time preferences.

In contrast, the balanced growth path in the corresponding social optimum is generically unique in the long run when the social planning program eventually internalizes preference externalities over generations in the dynastic competitive economy. However, the social planner can fail to ensure that the transitional dynamic path is unique in the short-run social optimum. This result is mainly because preference externalities are not completely internalized, and the parental time allocation with children has a strong effect on not only the level change but also the marginal changes in time discounting in the formation for time preferences. This effect occurs over the dynastic overlapping generations in an infinite horizon model with the multiple effects of externalities. That is, we show that the social optimum dynamic path is globally determinate in conjunction with either locally determinate or indeterminate transitional paths in social optimum.

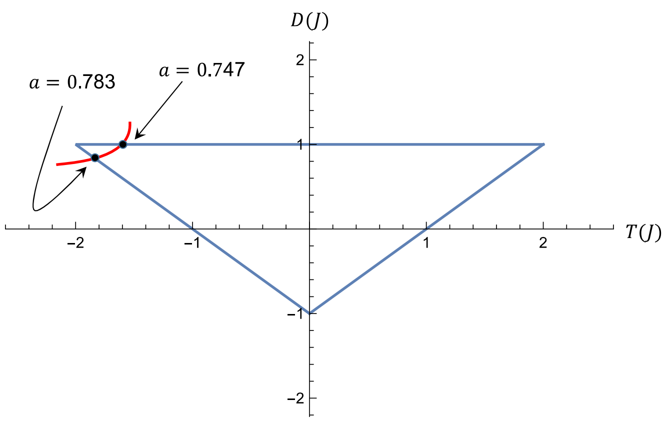

Fig. 1. Example IV. *Note*: The red curve shows the combination of the determinant and the trace as a changes from 0.71 to 0.81 in Example IV. The blue triangle shows the area of local indeterminacy.

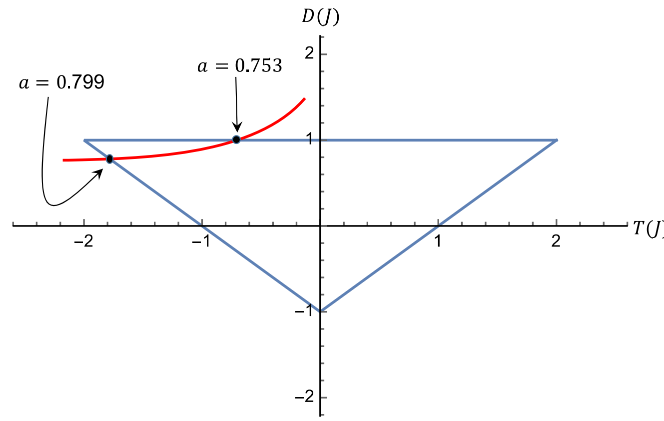

Fig. 2. Example V. *Note*: The red curve shows the combination of the determinant and the trace as a changes from 0.71 to 0.81 with $a = 1 - b$ imposed in Example V. The blue triangle shows the area of local indeterminacy.

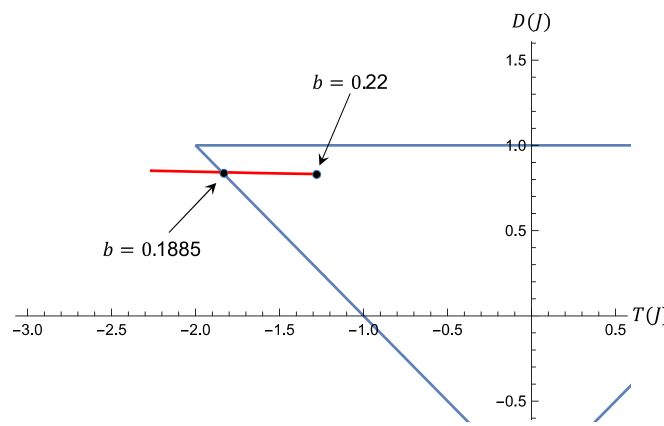

Fig. 3. Example VI. *Note*: The red curve shows the combination of the determinant and the trace as b changes from 0.171 to 0.22 in Example VI. The blue triangle (partly shown) indicates the area of local indeterminacy.

Table 1
Example I, II, and III: Parental time, long-run growth rate, and determinacy/indeterminacy.

Examples	Competitive Economy		Social Optimum	
I	$\bar{p}$=0.213 $\bar{\Gamma}$=1.033	Locally determinate	$\tilde{p}$=0.488 $\tilde{\Gamma}$=1.084	Locally determinate
II	$\bar{p}$=0.211 $\bar{\Gamma}$=1.088	Locally indeterminate	$\tilde{p}$=0.036 $\tilde{\Gamma}$=1.006	Locally determinate
III	$\bar{p}^{*}$=0.053 $\bar{\Gamma}^{*}$=1.0034	Locally indeterminate	$\tilde{p}$=0.054 $\tilde{\Gamma}$=1.0039	Locally determinate
	$\bar{p}^{**}$=0.412 $\bar{\Gamma}^{**}$=1.105	Locally determinate		

The competitive dynamics in the paper also provide the mechanism of noncognitive abilities on empirically observed intergenerational correlations. Patience is regarded as a component of noncognitive abilities determining how well people focus on long-term tasks and exert self-restrict. We find that the consumption pattern between the young and old periods implies that the slope of the consumption path becomes steeper when an individual becomes more patient during his lifecycle. We also show that the consumption pattern over generations is identical to the path of economic growth when the dynastic economy is perpetually growing. The characteristics of the dynamic consumption are an evidence of the endogenous formation of time discounting over generations. Therefore, the parent's time allocation for his own children dictates the intergenerational correlation over consumption, saving, investment, and income distribution under the formation of time preferences as noncognitive personal traits.

In future works, we may incorporate into a model of the endogenous time preferences bequest motives so that altruistic parents transfer a part of their income to children as well as parental time to influence their children's time preferences. We may also examine the dynamics of human capital accumulation when a trade-off exists between parental time allocations to labor supply for production and children's formal and informal education. To evaluate a public policy on formal education, future analysis should combine public education policy with parental time allocation for future generations. Those extensions will provide insights into the effects of public policies on the allocation of the private and public education over multiple generations.

References

Altonji, J.G., Hayshi, F., Kotlikoff, L.J., 1997. Parental altruism and intra vivos transfers: theory and evidence. J. Political Econ. 105, 1121–1166.

Barro, R., 1974. Are government bonds net wealth? J. Political Econ. 82, 1095–1117.

Barro, R., Becker, G., 1989. Fertility choice in a model of economic growth. Econometrica 57, 481–501.

Becker, G., Mulligan, C.B., 1997. The endogenous determination of time preference. Q. J. Econ. 112, 729–758.

Becker, R., Boyd, J.H., III, 1992. Recursive utility and optimal capital accumulation II: sensitivity and duality theory. Econ. Theory 2, 547–563.

Benhabib, J., Farmer, R.E.A., 1999. Indeterminacy and sunspots in macroeconomics. In: Taylor, J.B., Woodford, M. (Eds.), Handbook of Macroeconomics 1. Elsevier, Amsterdam, 387–448.

Bhattacharya, J., Qiao, X., 2007. Public and private expenditures on health in a growth model. J. Econ. Dyn. Control 31, 2519–2535.

Bowles, S., Gintis, H., 2002. The inheritance of inequality. J. Econ. Perspect. 16, 3–30.

Carneiro, P., Heckman, J., 2003. Human capital policy. In: Heckman, J., Krueger, B., Friedman, M. (Eds.), Inequality in America: What Role for Human Capital Policies?. MIT Press, Cambridge, MA.

Carroll, C., Summer, L., 1991. Consumption growth parallels income growth: some evidence. In: Bernheim, D., Shaven, J. (Eds.), National Saving and Economic Performance. University Chicago Press, Chicago.

Cazzavilan, G., Pintus, P., 2004. Robustness of multiple equilibria in OLG economies. Rev. Econ. Dyn. 7, 458–475.

Chakraborty, S., Das, M., 2005. Mortality, fertility, and child labor. Econ. Lett. 86, 273–278.

Charles, K., Hurst, E., 2003. The correlation of wealth across generations. J. Political Econ. 111, 1155–1182.

Cunha, F., Hechman, J., Schennach, S., 2010. Estimating the technology of cognitive and noncognitive skill formation. Econometrica 78, 883–931.

de la Croix, D., Michel, P., 1999. Optimal growth when tastes are inherited. J. Econ. Dyn. Control 23, 519–537.

de la Croix, D., Michel, P., 2002. A theory of economic growth: dynamics and policy in overlapping generations. Cambridge University Press, Cambridge, UK.

Doepke, M., Zilibotti, F., 2005a. Patience capital and the demise of the aristocracy (Working paper no. 735). Stockholm University.

Doepke, M., Zilibotti, F., 2005b. Social class and the spirit of capitalism. J. Eur. Econ. Assoc. 3, 516–524.

Epstein, L.G., Hynes, J.A., 1983. The rate of time preference and dynamic economic analysis. J. Political Econ. 91, 611–635.

Galor, O., Ryder, H.E., 1989. Existence, uniqueness, and stability of equilibrium in an overlapping-generations model with productive capital. J. Econ. Theory 49, 360–375.

Galor, O., Zeira, J., 1993. Income distribution and macroeconomics. Rev. Econ. Stud. 60, 35–52.

Gourinchas, P.-O., Parker, J., 2002. Consumption over the life cycle. Econometrica 70, 47–89.

Gouskova, E., Chiteji, N., Stafford, F., 2010. Pension participation: do parents transmit time preference? J. Fam. Econ. Issues 31, 138–150.

Guryan, J., Hurst, E., Kearney, M., 2008. Parental education and parental time with children. J. Econ. Perspect. 22, 23–46.

Haaparanta, P., Puhakka, M., 2004. Endogenous time preference, investment and development traps. Institute for Economies in Transition, Bank of Finland, Discussion Papers no. 4/2004.

Hirose, K., Ikeda, S., 2015. Decreasing marginal impatience destabilizes multi-country economies. Econ. Model. 50, 237–244.

Jafarey, S., Park, H., 1998. The dynamics of optimal wealth distributions with recursive utility. Econ. Lett. 61, 149–158.

Kawagishi, T., 2014. Investment for patience in an endogenous growth model. Econ. Model. 42, 508–515.

Li, C.W., 2003. Endogenous growth without scale effects: a comment. Am. Econ. Rev. 93, 1009–1018.

Loewenstein, G., 1987. Anticipation and the valuation of delayed consumption. Econ. J. 97, 666–684.

McKenzie, L.W., 1986. Optimal economic growth, turnpike theorems and comparative dynamics. In: Arrow, K., Intrigator, M. (Eds.), Handbook of Mathematical Economics 3. North-Holland, Amsterdam, 1281–1355.

Michel, P., Venditti, A., 1997. Optimal growth and cycles in overlapping generations models. Econ. Theory 9, 511–528.

Mulligan, C.B., 1997. Parental priorities and economic inequality. University of Chicago Press, Chicago.

Nourry, C., 2001. Stability of equilibria in the overlapping generations model with endogenous labor supply. J. Econ. Dyn. Control 25, 1647–1663.

Reichlin, P., 1986. Equilibrium cycles in an overlapping-generations economy with production. J. Econ. Theory 40, 89–102.

Park, H., 2000. Global asymptotic stability of a competitive economy with recursive preferences. Econ. Theory 15, 565–587.

Rebelo, S., 1991. Long-run policy analysis and long-run growth. J. Political Econ. 99, 500–521.

Romer, P., 1986. Increasing returns and economic growth. J. Political Econ. 94, 1002–1037.

Sarkar, J., 2007. Growth dynamics in a model of endogenous time preferences. Int. Rev. Econ. Financ. 16, 528–542.

Strulik, H., 2012. Patience and prosperity. J. Econ. Theory 147, 336–352.