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Mean-variance portfolio selection with only risky assets under regime switching

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ABSTRACT

This paper explores a portfolio selection model of multiple risky assets with regime switching. There are $n + 1$ risky assets in the financial market available to the mean-variance investors. The feasibility issue is solved by constructing an equivalent condition. We derive the analytical expressions of the efficient frontier and efficient feedback portfolio via three systems of ordinary differential equations that admit unique solutions. The mutual fund theorem is also proved. Several numerical examples are provided to demonstrate how the efficient frontier is affected by the market regime movement and the investor's time horizon.

1. Introduction

Portfolio selection is concerned with the allocation of the investor's assets amongst different types of financial securities so as to optimize the total return of the portfolio. Along with a desirable investment return, however, investors are also seeking to control the future uncertainties of their portfolio. Thus, a measure needs to be defined as a quantifiable indicator of the portfolio risk. Markowitz (1952) firstly gave an accurate definition of investment risk by applying the mathematical terminology of "variance" in the probability theory. Considering the trade-off between the mean and variance of a portfolio, an optimal investment strategy was achieved, known as "efficient portfolio". In a single period setting, nevertheless, Markowitz's mean-variance model failed to capture the dynamic process of portfolio selection facing investors in the real world. A number of literatures have been devoted to the extension of the original single period model to the multi-period case. For more details, the reader is referred to Pliska (1997) and Li and Ng (2000).

Since continuous-time finance theory was pioneered by Robert C. Merton in the 1970s, financial modelling in continuous-time setting has been thriving and employed to deal with a range of theoretical and practical problems. The continuous-time mean-variance portfolio selection model was originally formulated and solved by Zhou and Li (2000), which obtained both the efficient portfolio and efficient frontier in closed form by applying the stochastic linear-quadratic (LQ) control theory. Thereafter, their model has been extensively studied by numerous literatures. Lim and Zhou (2002) considered a complete market with bounded random coefficients in a general framework and

obtained the efficient frontier by solving two backward stochastic differential equations. Chiu and Wong (2011) applied their technique to solve a mean-variance portfolio selection problem with cointegrated risky assets. The constant elasticity of variance (CEV) model was employed to characterize the evolution of a risky asset price in Shen et al. (2014). No bankruptcy constraint was explored in Bielecki et al. (2005) by using the martingale approach. Besides, a few papers also took market frictions into account. Li et al. (2002) imposed shorting prohibition on the trading of stocks while borrowing from the bank account was still permitted. Fu et al. (2010) supposed a spread between the interest rates for lending and borrowing. Furthermore, asset-liability management was studied under the mean-variance framework. Chiu and Li (2006) considered a dynamic liability process driven by a geometric Brownian motion. Xie et al. (2008) modelled an uncontrollable liability with a drifted Brownian motion. Leippold et al. (2011) introduced endogenous liabilities and obtained efficient portfolio and efficient frontier in a multi-period setting. Their model was paralleled to a continuous-time asset-liability management problem by Yao et al. (2013). Both the CEV process and geometric Brownian motions were used by Zhang and Chen (2016) to model the multiple asset processes and exogenous liability respectively in a complete market.

To better capture the random environment of the financial market, regime-switching models have been applied to some of the key financial parameters, such as interest rate, equity risk premium and stock volatility. The basic idea is that these financial parameters are supposed to move along with the underlying market state. For example, investors would anticipate a higher appreciation rate and a lower volatility when the stock market is believed to be bullish. In the previous literatures,

the market regime is usually characterized by a Markov chain the value of which switches within a finite state space. Numerous models have been developed to solve some of the fundamental financial problems, such as asset pricing. See Buffington and Elliott (2002); Guo (2001) and Elliott et al. (2005). Analytical solutions were derived for a general investment-consumption model with regime switching in Sotomayor and Cadenillas (2009). The associated value function was solved explicitly with different types of consumption utility functions. A regime-switching model was originally formulated to solve the mean-variance portfolio selection problem by Zhou and Yin (2003). They obtained the explicit expressions of the efficient portfolio and efficient frontier via the solutions of two systems of linear ordinary differential equations (ODEs). Chen et al. (2008) extended their work by introducing a Markov-modulated geometric Brownian motion to model the insurance company's uncontrollable liability process. Similarly, the investor's exogenous liability was assumed to be a Markov-modulated Brownian motion in Xie (2009).

In this paper, we follow the work of Yao et al. (2014) as the first attempt to address a financial market without risk free assets. This hypothesis could reflect the stochastic nature of the interest rate over a long time horizon. Moreover, Markowitz (1952) proposed the mean-variance principle with the purpose of addressing the diversification problem of various stocks. The efficient frontier and global minimum variance were derived with the absence of risk free assets. Along this line, Yao et al. (2014) formulated a dynamic portfolio selection problem of only risky assets as a direct extension of Markowitz's single period model. In their paper, a different conclusion has been drawn regarding the efficient frontiers. In contrast with the static case, the capital market line in the continuous time model is strictly above the efficient frontier of a hyperbolic shape that corresponds to the case of only risky assets. This is due to the fact that investors continuously adjust its allocation to risk free assets to maintain an optimal strategy.

Our paper extends Yao et al. (2014) by considering a regime-switching financial market where all relevant parameters are driven by a continuous time Markov chain. We employ the Lagrange multiplier and "completion of square" technique in the LQ stochastic control theory. This commonly used approach is well applied because the investor's wealth process, although in a more general form, is still governed by a linear stochastic differential equation (SDE). By solving three systems of linear ODEs, we derive the efficient frontier and efficient feedback portfolio in closed form. Unsurprisingly, the efficient frontier is no longer a straight line, and the global minimum variance is strictly greater than zero, since there is no risk-free portfolio, that is, the investor cannot construct a dynamic portfolio so as to achieve a pre-specified investment return with zero variance at the terminal time.

The remaining part of the paper is outlined as follows. Section 2 formulates a continuous time mean-variance portfolio selection model of only risky assets under regime switching. One equivalent condition is proved for the problem feasibility, and the Lagrange multiplier is introduced in Section 3. In Section 4, the unconstrained dual problem is analytically solved via three systems of linear ODEs. Section 5 derives the efficient feedback portfolio, efficient frontier, global minimum variance and mutual fund theorem. Several numerical examples are provided to illustrate our results in Section 6. Section 7 gives a brief conclusion.

2. Problem formulation

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, on which are defined an m-dimensional standard Brownian motion $W(t) := (W_1(t), \ldots, W_m(t))'$ and a continuous time stationary Markov chain $\alpha(t)$ with a finite state space $\mathcal{M} = \{1, 2, \ldots, l\}$ and a generator matrix $Q = (q_{ij})_{l \times l}$. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $W(t)$ and $\alpha(t)$ augmented by the null sets contained in $\mathcal{F}$. We assume the independence of $W(t)$ and $\alpha(t)$ to ensure that $W(t)$ is a standard Brownian

motion with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. All the vectors are supposed to be column vectors. The transpose of any matrix A is denoted by A'. The norm $\|\cdot\|$ is defined as $\left\| A \right\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$, where $A = (a_{ij})_{m \times n}$.

We consider a financial market composed of $n + 1$ risky assets price processes of which, denoted by $S_i(\cdot)$, $i = 0, 1, 2, \ldots, n$, are characterized by the following Markov-modulated geometric Brownian motions

$$
\begin{cases}
dS_i(t) = S_i(t)\left[b_i(t, \alpha(t))dt + \sum_{j=1}^{m} \sigma_{ij}(t, \alpha(t))dW_j(t) \right], \\
S_i(0) = S_i^0,
\end{cases}
$$

where $b_i(t, k)$ and $\sigma_i(t, k) := (\sigma_{i1}(t, k), \ldots, \sigma_{im}(t, k))'$, $k = 1, 2, \ldots, l$, are the appreciation rate and volatility vector of the ith risky asset respectively, corresponding to the market regime k.

Suppose that an agent, with an initial wealth x_0, is investing in the $n + 1$ assets and adjusting his portfolio weights continuously within a finite time horizon $T > 0$. Both long and short positions are permitted without any transaction cost. The agent's wealth process $x(\cdot)$ would evolve as a linear SDE

$$
\begin{cases}
dx(t) = [b_0(t, \alpha(t))x(t) + B(t, \alpha(t))'u(t)]dt \\
\qquad\quad + [x(t)\sigma_0(t, \alpha(t))' + u(t)'\sigma(t, \alpha(t))]dW(t), \\
x(0) = x_0, \quad \alpha(0) = i_0,
\end{cases}
\tag{2.1}
$$

where i_0 is the initial market mode and $u(\cdot) := (u_1(\cdot), \ldots, u_n(\cdot))'$ is defined as the agent's portfolio vector, the kth element of which represents the market value of the kth risky asset held by the agent. The remaining part $x(t) - \sum_{k=1}^{n} u_k(t)$ is allocated to the 0th asset. Both $B(t, \alpha(t))$ and $\sigma(t, \alpha(t))$ are defined as below

$$
B(t, \alpha(t)) := [b_1(t, \alpha(t)) - b_0(t, \alpha(t)), \ldots, b_n(t, \alpha(t)) - b_0(t, \alpha(t))]', \sigma(t, \alpha(t))
$$

$$
:= [\sigma_1(t, \alpha(t)) - \sigma_0(t, \alpha(t)), \ldots, \sigma_n(t, \alpha(t)) - \sigma_0(t, \alpha(t))]'.
$$

Note that $\sigma(t, \alpha(t))$ is a matrix of order $n \times m$.

Remark 2.1. If $\sigma_0(t, \alpha(t)) \equiv 0$, the 0th risky asset could be taken as a risk free bank account which yields a predictable future return regardless of the market randomness modelled by the Brownian motion $W(\cdot)$. In this particular scenario, the agent's wealth process would reduce to (2.6) in Zhou and Yin (2003).

Before we formulate the mean-variance portfolio optimization problem, several assumptions need to be made for technical convenience.

Assumption 2.1. $b_i(\cdot, k)$ and $\sigma_{ij}(\cdot, k)$ are Borel-measurable and bounded functions of t for $i = 0, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, l$.

Assumption 2.2. $\sigma(t, i)$ satisfies the nondegeneracy condition, i.e., there exists $\delta > 0$ such that $\Sigma(t, i) := \sigma(t, i)\sigma(t, i)' \geq \delta I$, $\forall t \in [0, T]$, $i = 1, \ldots, l$, where I denotes the n-dimensional identity matrix.

Remark 2.2. The nondegeneracy condition in Assumption 2.2 could be satisfied only if the rank of $\sigma(t, i)$ is n, which implies that the dimension of $W(\cdot)$ must be at lease equal to the number of risky assets in the financial market. However, the market completeness is unnecessary, that is, m may be strictly greater than n.

Definition 2.1. A portfolio $u(\cdot)$ is said to be admissible if it is an $\mathcal{F}_t$-adapted locally integrable process, i.e., $\int_0^T \| u(t) \|^2\, dt < \infty$, a.s. and the SDE (2.1) admits a unique strong solution $x(\cdot)$ satisfying the square-integrable condition, i.e, $E\max_{0 \leq t \leq T} x(t)^2 < \infty$. Let U denote the set of all admissible portfolios.

Remark 2.3. Due to its linear structure, the wealth process (2.1) always has an explicit solution for any locally integrable process $u(\cdot)$. By Definition 2.1, therefore, the essential difficulty is to show the integrability of $\max_{0 \leq t \leq T} x(t)^2$ when verifying the admissibility of a portfolio process.

As a mean-variance investor, the agent's objective is to find an

admissible portfolio $u(\cdot)$ such that the variance of the terminal wealth $Var(x(T))$ is minimized while the expected terminal wealth is fixed at some acceptable level z. This mean-variance portfolio selection problem can be formulated as a constrained stochastic minimization problem

$$\begin{cases} \text{minimize} & J(x_0, i_0, u(\cdot)) := Var(x(T)) \\ \text{subject to} & Ex(T) = z, \quad u(\cdot) \in U. \end{cases} \tag{2.2}$$

For any given z, an optimal portfolio of (2.2) is called an efficient portfolio and the pair $(\min Var(x(T)), z)$ is said to be an efficient point. As the agent revises his expectation of the terminal wealth level z within R, a set of efficient points would be drawn as the efficient frontier.

3. Feasibility

We start with the feasibility of the problem (2.2), that is, there exists at least one admissible portfolio in U such that the expectation of the agent's terminal wealth could reach the pre-specified level z.

Lemma 3.1. *Consider the following system of linear ODEs*

$$\begin{cases} \dot{\varphi}(t, i) = a(t, i)\varphi(t, i) - \sum_{j=1}^{l} q_{ij}\varphi(t, j), \\ \varphi(T, i) = 1, \quad i \in \mathcal{M}. \end{cases} \tag{3.1}$$

If $a(t, i)$ is a bounded function, then (3.1) admits a unique strictly positive solution. Moreover, there exists $\beta > 0$ such that $\varphi(t, i) \geq \beta$ for $t \in [0, T]$, $i \in \mathcal{M}$.

Proof. See Appendix A.

Theorem 3.1. *Problem (2.2) is feasible for every $z \in R$ if and only if*

$$E \int_0^T \| B(t, \alpha(t)) \|^2 \, dt > 0. \tag{3.2}$$

Proof. We introduce a system of linear ODEs

$$\begin{cases} \dot{\phi}(t, i) = -b_0(t, i)\phi(t, i) - \sum_{j=1}^{l} q_{ij}\phi(t, j), \\ \phi(T, i) = 1, \quad i \in \mathcal{M}. \end{cases}$$

According to Lemma 3.1, this system admits a unique strictly positive solution. Given any $u(\cdot) \in U$, we apply the Itô formula to $\phi(t, \alpha(t))x(t)$ and have

$$d\phi(t, \alpha(t))x(t) = \phi(t, \alpha(t))B(t, \alpha(t))'u(t)dt + \{\cdots\}dW(t) + dM(t),$$

where $M(\cdot)$ is a zero mean local martingale with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. For simplicity, in the rest of the paper, $M(\cdot)$ is referred to as a suitable zero mean local martingale. Here, we ignore the specific form of the integrand with respect to "$W(\cdot)$", as shown in the bracket, since it does not affect the final result. Let $\{\tau_n\}_{n \geq 1}$ be a localizing sequence and then

$$E\phi(T \wedge \tau_n, \alpha(T \wedge \tau_n))x(T \wedge \tau_n)$$
$$= E \int_0^{T \wedge \tau_n} \phi(t, \alpha(t))B(t, \alpha(t))'u(t)dt + \phi(0, i_0)x_0. \tag{3.3}$$

We first prove the "if" part by constructing an admissible portfolio satisfying the constraint $Ex(T) = z$. Consider $u(t) := k\phi^{-1}(t, \alpha(t))B(t, \alpha(t))$, where k is some constant to be determined by z. From Lemma 3.1, $u(\cdot)$ is a bounded process and thus an admissible portfolio. Hence, applying the dominated convergence theorem to (3.3) gives

$$Ex(T) = kE \int_0^T \| B(t, \alpha(t)) \|^2 \, dt + \phi(0, i_0)x_0.$$

If the condition (3.2) holds, there exists a unique k such that $Ex(T) = z$.

Conversely, we prove the "only if" part by contradiction. If the

condition (3.2) does not hold, (3.3) is reduced to

$$Ex(T) = \phi(0, i_0)x_0.$$

Apparently, the expected terminal wealth is independent of any admissible portfolio chosen by the agent, and the problem (2.2) is not feasible if $z \neq \phi(0, i_0)x_0$. This leads to a contradiction with the proposition that "Problem (2.2) is feasible for every $z \in R$". □

Under the condition (3.2), the mean-variance problem (2.2) satisfies the following duality equation by Theorem 2.5.3 in Shi, 1990,

$$\min_{u(\cdot) \in U, Ex(T) = z} J(x_0, i_0, u(\cdot)) = \min_{u(\cdot) \in U} \{E[x(T) - z]^2 - \lambda^*[Ex(T) - z]\},$$

where λ^* is called a Lagrange multiplier, and solves the dual problem

$$\max_{\lambda \in R} \min_{u(\cdot) \in U} \{E[x(T) - z]^2 - \lambda[Ex(T) - z]\}.$$

For convenience, we adopt $2\lambda^*$ instead of λ^* and rewrite the above duality relation as

$$\min_{u(\cdot) \in U, Ex(T) = z} J(x_0, i_0, u(\cdot)) = \max_{\lambda \in R} \min_{u(\cdot) \in U} \{E[x(T) - z]^2 - 2\lambda[Ex(T) - z]\}$$
$$= \max_{\lambda \in R} \min_{u(\cdot) \in U} \{E[x(T) - z - \lambda]^2 - \lambda^2\}$$
$$= \max_{d \in R} \min_{u(\cdot) \in U} \{E[x(T) - d]^2 - (d - z)^2\}. \tag{3.4}$$

The last equality is obtained by changing variable "$d = z + \lambda$".

4. Solution to the unconstrained problem

In this section, we proceed to investigate a linear-quadratic stochastic minimization problem without any constraints as below

$$\begin{cases} \text{minimize} & J^d(x_0, i_0, u(\cdot)) := E[x(T) - d]^2 \\ \text{subject to} & u(\cdot) \in U. \end{cases} \tag{4.1}$$

We need to solve the problem (4.1) for every $d \in R$. To do this, we firstly introduce three systems of linear ODEs.

$$\begin{cases} \dot{f}(t, i) = [\rho(t, i) + 2\beta(t, i) + \gamma(t, i)]f(t, i) - \sum_{j=1}^{l} q_{ij}f(t, j), \\ f(T, i) = 1, \end{cases} \tag{4.2}$$

$$\begin{cases} \dot{g}(t, i) = [\rho(t, i) + \beta(t, i)]g(t, i) - \sum_{j=1}^{l} q_{ij}g(t, j), \\ g(T, i) = 1, \end{cases} \tag{4.3}$$

$$\begin{cases} \dot{h}(t, i) = \rho(t, i)\frac{g^2(t,i)}{f(t,i)} - \sum_{j=1}^{l} q_{ij}h(t, j), \\ h(T, i) = 1, \end{cases} \tag{4.4}$$

where $\rho(t, i) = B(t, i)'\Sigma(t, i)^{-1}B(t, i)$, $\beta(t, i) = B(t, i)'\Sigma(t, i)^{-1}\sigma(t, i)\sigma_0(t, i) - b_0(t, i)$ and $\gamma(t, i) = \sigma_0(t, i)'\sigma(t, i)'\Sigma(t, i)^{-1}\sigma(t, i)\sigma_0(t, i) - \| \sigma_0(t, i) \|^2$, for $t \in [0, T]$, $i = 1, \ldots, l$. For the system (4.4) to be well defined, it has to be shown that $f(t, i) \neq 0$. In fact, we can obtain a even stronger result in the next lemma.

Lemma 4.1. *Under Assumption 2.1 and 2.2, the systems of ODEs (4.2), (4.3) and (4.4) admit unique bounded solutions. In particular, $f(t, i) > 0$, $g(t, i) > 0$ and $h(t, i) < 1$ for any $t \in [0, T)$, $i \in \mathcal{M}$.*

Proof. See Appendix B.

Theorem 4.1. *The feedback control u^* as below is an optimal portfolio of the problem (4.1),*

$$u^*(t, x, i) = -\Sigma(t, i)^{-1}\left\{ x[\sigma(t, i)\sigma_0(t, i) + B(t, i)] - d\frac{g(t, i)}{f(t, i)}B(t, i) \right\}. \tag{4.5}$$

Furthermore, by adopting u^*, we have

$$\min_{u(\cdot) \in U} J^d(x_0, i_0, u(\cdot)) = J^d(x_0, i_0, u^*) = f(0, i_0)x_0^2 - 2g(0, i_0)x_0 d + h(0, i_0)d^2$$

Proof. If we adopt the feedback strategy u^* in the agent's wealth process (2.1), the associated $x^*(\cdot)$ uniquely solves a linear SDE with bounded coefficients. By Theorem 6.3 in Yong and Zhou (1999), we obtain the integrability condition of $E\max_{0 \le t \le T} x^*(t)^2 < \infty$, which implies the admissibility of u^*.

Next, we prove the optimality of u^* by "completion of square". Given any $u(\cdot) \in U$, applying the Itô formula to $f(t, \alpha(t))x(t)^2$, $g(t, \alpha(t))x(t)$ and $h(t, \alpha(t))$ respectively yields

$$df(t, \alpha(t))x(t)^2 = x(t)^2 \left\{ \dot{f}(t, \alpha(t)) + \sum_{j=1}^{l} q_{\alpha(t)j} f(t, j) \right.$$
$$\left. + [2b_0(t, \alpha(t)) + \| \sigma_0(t, \alpha(t)) \|^2] f(t, \alpha(t)) \right\} dt$$
$$+ f(t, \alpha(t)) \{ u(t)' \Sigma(t, \alpha(t)) u(t)$$
$$+ 2x(t)[B(t, \alpha(t)) + \sigma(t, \alpha(t)) \sigma_0(t, \alpha(t))]' u(t) \} dt$$
$$+ \{ \cdots \} dW(t) + dM(t), \quad dg(t, \alpha(t))x(t)$$
$$= x(t) \left[\dot{g}(t, \alpha(t)) + \sum_{j=1}^{l} q_{\alpha(t)j} g(t, j) + b_0(t, \alpha(t)) g(t, \alpha(t)) \right] dt$$
$$+ g(t, \alpha(t)) B(t, \alpha(t)) u(t)' dt + \{ \cdots \} dW(t)$$
$$+ dM(t), \quad dh(t, \alpha(t)) = \left[\dot{h}(t, \alpha(t)) + \sum_{j=1}^{l} q_{\alpha(t)j} h(t, j) \right] dt$$
$$+ dM(t).$$

Let $y(t) = f(t, \alpha(t))x(t)^2 - 2g(t, \alpha(t))x(t)d + h(t, \alpha(t))d^2$, and then

$$dy(t) = f(t, \alpha(t)) \{ u(t)' \Sigma(t, \alpha(t)) u(t)$$
$$+ 2x(t)[B(t, \alpha(t)) + \sigma(t, \alpha(t)) \sigma_0(t, \alpha(t))]' u(t) \} dt$$
$$- 2dg(t, \alpha(t)) B(t, \alpha(t))' u(t) dt + x(t)^2 \left\{ \dot{f}(t, \alpha(t)) \right.$$
$$+ \sum_{j=1}^{l} q_{\alpha(t)j} f(t, j) + [2b_0(t, \alpha(t)) + \| \sigma_0(t, \alpha(t)) \|^2] f(t, \alpha(t)) \bigg\} dt$$
$$- 2dx(t) \left[\dot{g}(t, \alpha(t)) + \sum_{j=1}^{l} q_{\alpha(t)j} g(t, j) + b_0(t, \alpha(t)) g(t, \alpha(t)) \right] dt$$
$$+ d^2 \left[\dot{h}(t, \alpha(t)) + \sum_{j=1}^{l} q_{\alpha(t)j} h(t, j) \right] dt + \{ \cdots \} dW(t) + dM(t).$$

By the "completion of square" technique, we obtain

$$y(T) - y(0) \ge \int_0^T \{ \cdots \} dW(t) + M(t).$$

Let $\{ \tau_n \}_{n \ge 1}$ be a localizing sequence, and thus

$$Ey(T \wedge \tau_n) \ge y(0) = f(0, i_0)x_0^2 - 2g(0, i_0)x_0 d + h(0, i_0)d^2. \quad (4.6)$$

Since $E\max_{0 \le t \le T} x(t)^2 < \infty$, applying the dominated convergence theorem to (4.6) yields

$$J^d(x_0, i_0, u(\cdot)) \ge f(0, i_0)x_0^2 - 2g(0, i_0)x_0 d + h(0, i_0)d^2.$$

In particular, if we adopt u^*, "$\ge$" becomes "$=$" in (4.6), and it can be found that

$$J^d(x_0, i_0, u^*) = f(0, i_0)x_0^2 - 2g(0, i_0)x_0 d + h(0, i_0)d^2. \quad \square$$

Corollary 4.1. *If f, g and h are the unique solutions of* (4.2), (4.3) *and*

(4.4) *respectively, then $f(0, i_0)h(0, i_0) - g(0, i_0)^2 \ge 0$.*

Proof. By Theorem 4.1, we obtain

$$f(0, i_0)x_0^2 - 2g(0, i_0)x_0 d + h(0, i_0)d^2 = \min_{u(\cdot) \in U} J^d(x_0, i_0, u(\cdot)) \ge 0.$$

This inequality holds for any $x_0, d \in R$. Let $x_0 = \frac{g(0, i_0)}{f(0, i_0)}$ and $d=1$. We find that

$$f(0, i_0)x_0^2 - 2g(0, i_0)x_0 d + h(0, i_0)d^2 = h(0, i_0) - \frac{g(0, i_0)^2}{f(0, i_0)} \ge 0,$$

which leads to $f(0, i_0)h(0, i_0) - g(0, i_0)^2 \ge 0. \square$

5. Efficient portfolio and efficient frontier

This section obtains the efficient portfolio and efficient frontier of the problem (2.2) by using the results derived in the preceding section.

Theorem 5.1 (*efficient portfolio and efficient frontier***).** *Suppose that the condition* (3.2) *holds. Then, the mean-variance problem* (2.2) *admits an efficient portfolio,*

$$u^*(t, x, i) = -\Sigma(t, i)^{-1} \left\{ x[\sigma(t, i)\sigma_0(t, i) + B(t, i)] - (\lambda^* + z)\frac{g(t, i)}{f(t, i)} B(t, i) \right\},$$
$$(5.1)$$

where the Lagrange multiplier $\lambda^* = \frac{z - g(0, i_0)h^{-1}(0, i_0)x_0}{h^{-1}(0, i_0) - 1}$. Moreover, the agent's efficient frontier is

$$\min Var(x(T)) = \frac{[z - g(0, i_0)h^{-1}(0, i_0)x_0]^2}{h^{-1}(0, i_0) - 1} + x_0^2[f(0, i_0) - h^{-1}(0, i_0)$$
$$g^2(0, i_0)]. \quad (5.2)$$

Proof. According to the dual relation (3.4) and Theorem 4.1, the following equation holds

$$\min_{u(\cdot) \in U, Ex(T)=z} J(x_0, i_0, u(\cdot)) = \max_{d \in R} F(d), \quad (5.3)$$

where $F(d) = f(0, i_0)x_0^2 - z^2 + [h(0, i_0) - 1]d^2 - 2[g(0, i_0)x_0 - z]d$. By Lemma 4.1, $F(\cdot)$, as a quadratic function, can be uniquely maximized at $d^* = \frac{z - g(0, i_0)x_0}{1 - h(0, i_0)}$, and its maximum is

$$\frac{[z - g(0, i_0)h^{-1}(0, i_0)x_0]^2}{h^{-1}(0, i_0) - 1} + x_0^2[f(0, i_0) - h^{-1}(0, i_0)g^2(0, i_0)].$$

From (3.4), the Lagrange multiplier $\lambda^* = d^* - z = \frac{z - g(0, i_0)h^{-1}(0, i_0)x_0}{h^{-1}(0, i_0) - 1}$. Finally, the efficient portfolio (5.1) is derived from (4.5) by setting $d = \lambda^* + z. \square$

Remark 5.1. When $n=1$, i.e., two risky assets are considered, the vectors in 5.1 become scalar. u^* is an affine function of the wealth amount x with the slope of $-\left(\frac{\sigma_0}{\sigma} + \frac{b_1 - b_0}{\sigma^2} \right)$, and the intercept of $\frac{(b_1 - b_0)g}{\sigma^2 f}(\lambda^* + z)$. Suppose that the appreciation rate of asset 1 is greater than that of asset 0, and, reasonably, we may also assume that $\sigma_1 > \sigma_0 > 0$. Therefore, the investor's position for asset 1 would move in the opposite direction with his wealth level x.

Theorem 5.2 (*global minimum variance***).** *The efficient frontier of the problem* (2.2) *admits a global minimum variance*

$$Var_{\min} = x_0^2[f(0, i_0) - h^{-1}(0, i_0)g^2(0, i_0)] \ge 0,$$

with the corresponding expected terminal wealth $z_{\min} = g(0, i_0)h^{-1}(0, i_0)x_0$. Furthermore, the efficient feedback portfolio that achieves the global minimum variance is

$$u^*_{\min}(t, x, i) = -\Sigma(t, i)^{-1}\left\{x[\sigma(t, i)\sigma_0(t, i) + B(t, i)] - z_{\min}\frac{g(t, i)}{f(t, i)}B(t, i)\right\}. \tag{5.4}$$

Proof. The global minimum variance and its corresponding expected terminal wealth are obtained immediately from the efficient frontier (5.2). When $z = z_{\min}$, the Lagrange multiplier $\lambda^* = 0$, and thus the efficient feedback portfolio (5.4) follows from (5.1).□.

Theorem 5.3 (*mutual fund theorem*). *Let* $u_0(\cdot)$ *be an efficient feedback portfolio associated with* $z_0 > z_{\min}$. *Then a feedback portfolio* $u(\cdot)$ *is efficient if and only if there exists* $\gamma \geq 0$ *such that*

$$u(\cdot) = \gamma u_0(\cdot) + (1 - \gamma)u_{\min}(\cdot). \tag{5.5}$$

Proof. We rewrite the efficient portfolio (5.1) as below

$$u^*(t, x, i) = -\Sigma(t, i)^{-1}\left\{x[\sigma(t, i)\sigma_0(t, i) + B(t, i)]\right.$$
$$\left. - \frac{z - g(0, i_0)x_0}{1 - h(0, i_0)}\frac{g(t, i)}{f(t, i)}B(t, i)\right\}. \tag{5.6}$$

If $u(\cdot)$ is a combination of $u_0(\cdot)$ and $u_{\min}(\cdot)$ given by (5.5), then we could obtain by simple algebra

$$u(t, x, i) = -\Sigma(t, i)^{-1}\left\{x[\sigma(t, i)\sigma_0(t, i) + B(t, i)]\right.$$
$$\left. - \frac{\gamma z_0 + (1 - \gamma)z_{\min} - g(0, i_0)x_0}{1 - h(0, i_0)}\frac{g(t, i)}{f(t, i)}B(t, i)\right\}.$$

Clearly, $u(\cdot)$ is an efficient feedback portfolio associated with the expected terminal wealth $\gamma z_0 + (1 - \gamma)z_{\min}$.

Conversely, suppose that $u(\cdot)$ is an efficient portfolio corresponding to $z \geq z_{\min}$. Let $\gamma = \frac{z - z_{\min}}{z_0 - z_{\min}}$, and then $z = \gamma z_0 + (1 - \gamma)z_{\min}$. (5.5) could be easily verified by (5.6).□.

Remark 5.2. The results in Theorem 5.1 are consistent with Zhou and Yin (2003). Compared to (5.2) in Zhou and Yin (2003), our efficient portfolio (5.1) is complicated by another stochastic term "$\sigma(t, i)\sigma_0(t, i)$" that would disappear under their model setting. In addition, we represent the efficient frontier in terms of three systems of linear ODEs, and avoid the calculation of the parameter θ defined in (4.9) in Zhou and Yin (2003).

Remark 5.3. As an extension, this paper comes to similar conclusions with Yao et al. (2014), including a hyperbola type of efficient frontier, the existence of nonnegative global minimum variance and mutual fund theorem. The difference is that three systems of ODEs need to be addressed as a consequence of introducing Markov-modulated parameters. Moreover, instead of the dynamic programming, we adopt the maximum principle in the stochastic control theory because the model in our formulation is essentially an LQ stochastic control problem.

6. Numerical examples

In this section, several examples are offered to illustrate how the efficient frontier (5.2) is shaped by the relevant parameters. We focus on a financial market with two risky assets driven by a standard two dimensional Brownian motion. Market regimes switch between two states, denoted as regime 1 (bearish) and regime 2 (bullish), representing bad and good economy respectively. We demonstrate the impact on the efficient frontier of the following parameters: time horizon T, initial market regime i_0 and transition intensity q_{12}. The relevant parameters would be valued as following: $b_0(1) = 0.05$, $b_0(2) = 0.1$, $\sigma_0(1) = (0.12, 0.15)'$, $\sigma_0(2) = (0.06, 0.1)'$, $B(1) = 0.2$, $B(2) = 0.4$, $\sigma(1) = (0.15, 0.3)'$, $\sigma(2) = (0.2, 0.4)'$, $q_{12} = q_{21} = 0.5$.

We rewrite the efficient frontier by dividing x_0^2 on both sides of (5.2) to obtain

$$\min Var\left(\frac{x(T)}{x_0}\right) = \frac{[\frac{z}{x_0} - g(0, i_0)h^{-1}(0, i_0)]^2}{h^{-1}(0, i_0) - 1} + [f(0, i_0) - h^{-1}(0, i_0)g^2(0, i_0)]$$

.

The proportion $\frac{z}{x_0}$ could be regarded as the expected investment return at the terminal time. The efficient frontier would be depicted as a relation between the mean and standard deviation of the investment return.

Example 6.1. We first study how the efficient frontier would shift as the time horizon T varies. As illustrated in Fig. 1, the efficient frontier would move upward regardless of the initial market regime if the investor extends the time horizon from 0.5 to 1.5. Both the minimum expected return $z_{\min}$ and global minimum variance are rising with the former at a larger scale than the latter. In addition, the steepness of the efficient frontier is improved, which implies that the investor would achieve a pre-specified expected return with much less uncertainty. More specifically, as seen from (a) and (b), the efficient frontier is raised to a greater extent with the bullish entry mode, compared to the bearish one.

Example 6.2. The numerical results are presented in Fig. 2 on the impact of the initial market regime on the efficient frontier at two time horizons. Intuitively, investors would anticipate a higher efficient frontier when they enter a bullish financial market, as reflected in Fig. 2. Moreover, the minimum expected return is improved whereas the global minimum variance is mitigated as the initial market mode is switched from "bearish" to "bullish". By comparing (a), (b) and (c), the size of the shift does not have a noticeable difference among different time horizons.

Example 6.3. The final example demonstrates how and to what extent the efficient frontier might be affected by the transition intensity q_{12} representing the likelihood of the market switching from bearish to bullish. The time horizon T is taken as 1, and the transition intensity q_{21} is 0.5. As shown in Fig. 3, the efficient frontier increases as q_{12} changes from 0.2 to 0.8, which is reasonable since a greater value of q_{12} would yield a higher expected return for the investor's portfolio. The distinction between (a) and (b) reveals that the magnitude of the shift greatly depends on the timing when investors enter the market. If the investor chooses to enter a bullish market, the transition intensity q_{12} exerts little influence on the efficient frontier.

7. Conclusion

This paper studies a continuous time mean-variance portfolio selection problem when the financial market consists of only risky assets whose price processes are modelled by Markov-modulated geometric Brownian motions. By introducing the Lagrange multiplier, the efficient frontier and efficient portfolio are expressed in closed form via three systems of ordinary differential equations. The global minimum variance is obtained, and the mutual fund theorem is proved by the fact that the efficient feedback portfolio is an affine function of the expected wealth level at the terminal time T. A few extensions can be made in our future research. The asset liability management problem could be investigated by taking into account the investor's endogenous or exogenous liability process. Additionally, to reflect limitations in the real market, constraints, such as prohibitions of short-selling the risky assets, may be imposed.
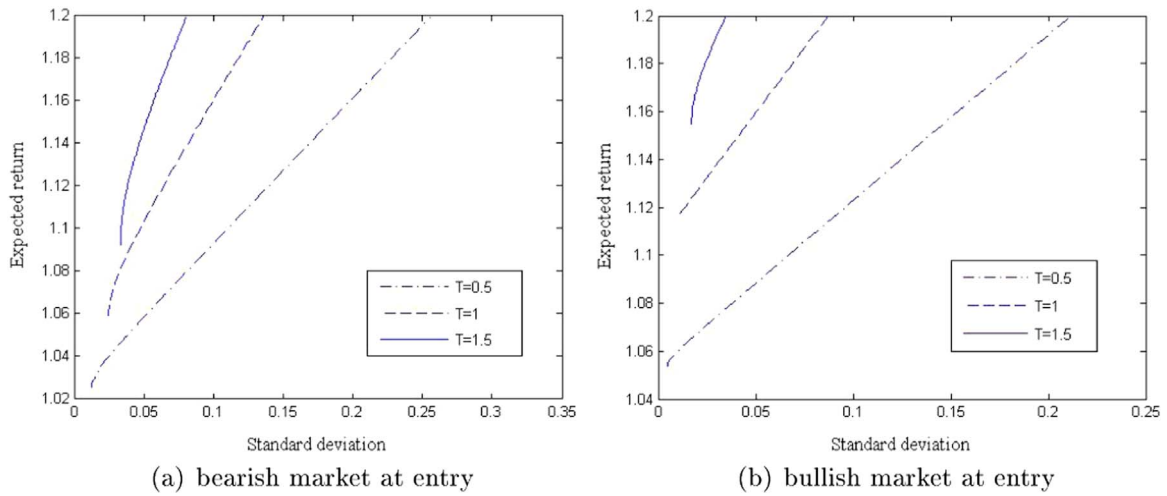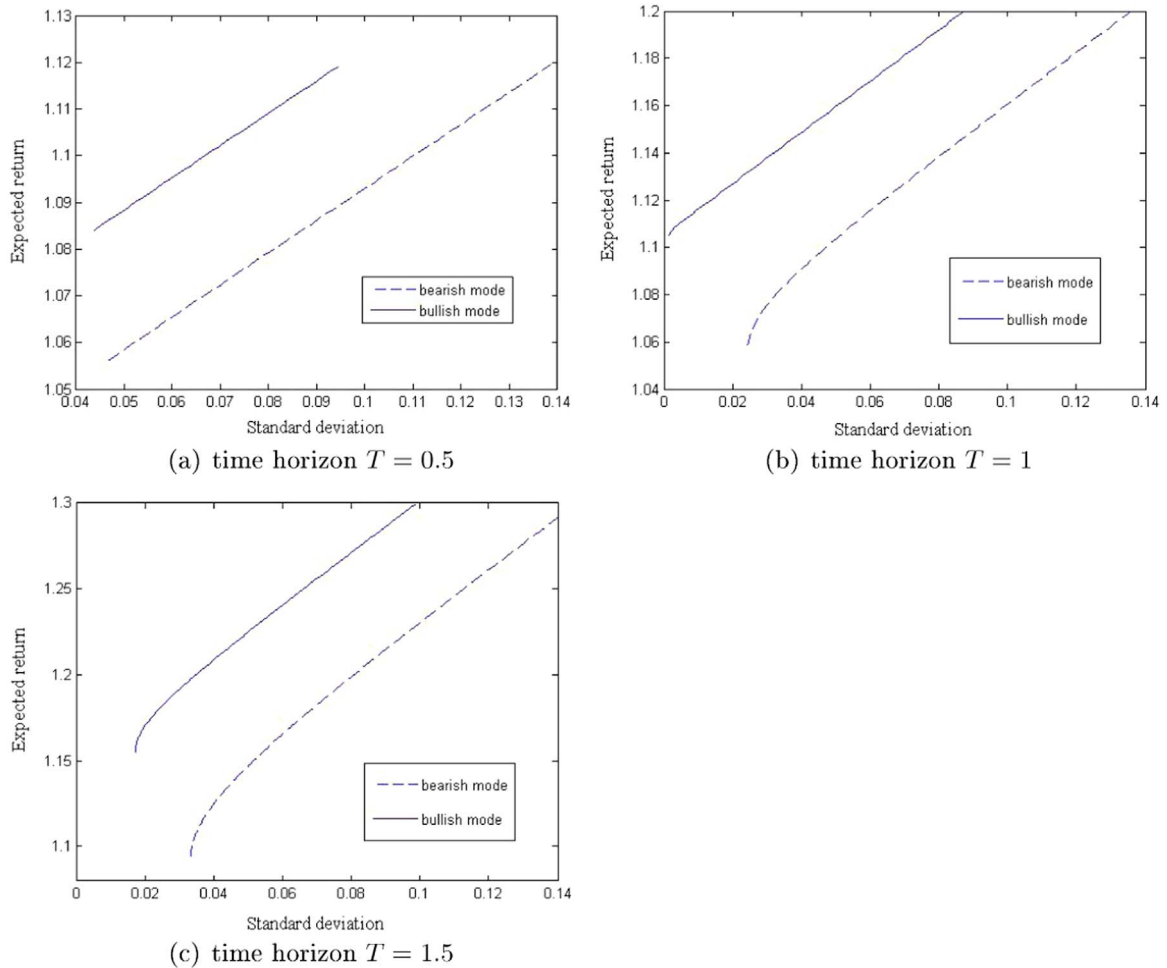
Fig. 1. The impact of the time horizon *T*.

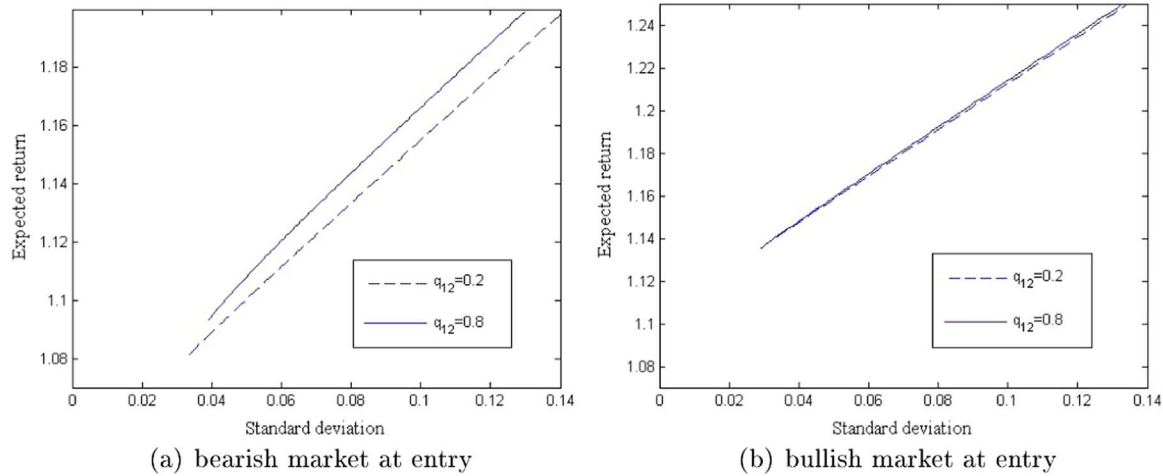Fig. 2. The impact of the initial market regime.

Fig. 3. The impact of the transition intensity q_{12}.

Appendix A

Proof of Lemma 3.1. The existence and uniqueness of solutions are evident since (3.1) is a system of linear ODEs with uniformly bounded coefficients. Next, we show that its solution is strictly positive. In fact, we rewrite (3.1) as

$$\varphi(t, i) = e^{-\int_t^T a(s,i)-q_{ii}ds} + \int_t^T e^{-\int_t^s a(u,i)-q_{ii}du} \sum_{j \neq i} q_{ij}\varphi(s, j)ds,$$

where $t \in [0, T]$, $i \in \mathcal{M}$. Since φ is equal to 1 and left continuous at time T, there exists some $\delta > 0$ such that $\varphi(t, i) > 0$ for any $t \in [T - \delta, T]$ and $i \in \mathcal{M}$. We define the set of roots for $\varphi(t) = 0$ as below

$$A := \{t \in [0, T - \delta] : \exists i \in \mathcal{M} \quad s. t. \quad \varphi(t, i) = 0\}.$$

The positivity of φ could be verified by contradiction. Suppose that φ is nonpositive at some $t \in [0, T]$ and $i \in \mathcal{M}$. Then, A must be a bounded nonempty set, and $\varphi(t) > 0$ for $t > t^* := \sup A$. Furthermore, there exists $i^* \in \mathcal{M}$ such that $\varphi(t^*, i^*) = 0$. On the other hand, we have

$$\varphi(t^*, i^*) = e^{-\int_{t^*}^T a(s,i^*)-q_{i^*i^*}ds} + \int_{t^*}^T e^{-\int_{t^*}^s a(u,i^*)-q_{i^*i^*}du} \sum_{j \neq i^*} q_{i^*j}\varphi(s, j)ds > 0.$$

This is contradictory to $\varphi(t^*, i^*) = 0$. Since φ is a continuous function of t on the closed set $[0, T]$, its minimum must be positive. Then, β could be taken as $\min_{[0,T],\mathcal{M}} \varphi(t, i)$. □

Appendix B

Proof of Lemma 4.1. The existence and uniqueness of solutions are clear due to the bounded coefficients in (4.2), (4.3) and (4.4). By taking $a(t, i) = \rho(t, i) + 2\beta(t, i) + \gamma(t, i)$ or $\rho(t, i) + \beta(t, i)$ in Lemma 3.1, we conclude that both f and g are strictly positive.

The remaining task is to show that $h < 1$. We consider $\tilde{h} := 1 - h$, and it is equivalent to prove that $\tilde{h}$ is strictly positive. Note that $\tilde{h}$ uniquely solves the following system of ODEs

$$\begin{cases} \dot{\tilde{h}}(t, i) = -\rho(t, i)\frac{g^2(t,i)}{f(t,i)} - \sum_{j=1}^l q_{ij}\tilde{h}(t, j), \\ \tilde{h}(T, i) = 0, \quad i \in \mathcal{M}. \end{cases}$$

We rewrite these ODEs in the integral form

$$\tilde{h}(t, i) = \int_t^T e^{\int_t^s q_{ii}du} \left[\rho(s, i)\frac{g^2(s, i)}{f(s, i)} + \sum_{j \neq i} q_{ij}\tilde{h}(s, j) \right] ds. \tag{B.1}$$

Since $\tilde{h}(T) = 0$, $\tilde{h}$ is left continuous at T, and $\rho(t, i)\frac{g^2(t,i)}{f(t,i)} > 0$, there exists $\delta > 0$ such that $\rho(t, i)\frac{g^2(t,i)}{f(t,i)} + \sum_{j \neq i} q_{ij}\tilde{h}(t, j) > 0$ for $t \in [T - \delta, T]$, $i \in \mathcal{M}$. From (B.1), we conclude that $\tilde{h}(T - \delta) > 0$. By treating $T - \delta$ as the terminal time, we can apply the same technique as in Lemma 3.1 to prove the positivity of $\tilde{h}$. □

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