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ASYMMETRIC BENDING OF CIRCULAR SANDWICH PLATE INCLUDING TRANSVERSE SHEAR IN FACINGS

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An approximate formulation is presented for the bending of an isotropic circular sandwich plate under transverse load, taking into account shear forces in the core and facings. The equations are derived by using the principle of minimum potential energy and are solved in Fourier Bessel series for the case when the outer edge is clamped.

INTRODUCTION

In the past few decades, the analysis of sandwich structures has been the topic of intensive investigation. To achieve lightness with strength, sandwich structures are being utilized in the construction of missiles and space crafts. A review of the work upto 1965, was done by Habip (1965), Plantemma (1966). and Allen (1968). Kao (1965, 1969) has derived the system of equations for asymmetric bending of a circular sandwich plate considering facings as membranes for linearly varying and eccentric loads. Stickney and Abdulhadi (1968), have presented small deflection theory for the analysis of orthotropic circular sandwich plates by extremizing the complementary strain energy. In all the above investigations the circular sandwich plates are analyzed by considering facings as membranes, in which transverse shear of the facings is being neglected.

In the present investigation, the asymmetric bending of a loaded isotropic circular sandwich plate of constant thickness is considered. The transverse shear is taken into account both for core and facings. The equations of equilibrium are derived by using the principal of minimum potential energy and are solved in terms of Fourier Bessel series for a load distributed over a sector either uniformly or linearly varying and a point loading.

FORMULATION OF THE PROBLEM

We consider a circular sandwich plate of thickness $2h$ and radius a. The thickness of the core and each of the two facings are taken $2h_1$ and h_2, so that $h = h_1 + h_2$

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(Fig. 1). The plate is referred to cylindrical coordinates r, θ and z by taking the axis of the plate as the line $r = 0$, the middle plane as $z = 0$ and z axis in the downward direction. The two interfaces and bottom and top surfaces are taken $z = \pm h_1, \pm h$ respectively. The two facings are taken of the same material, but different from that of the core. Therefore, the various quantities for the core, the upper facing, and the lower facing will be distinguished by the subscripts 1, 2 and 3 respectively.

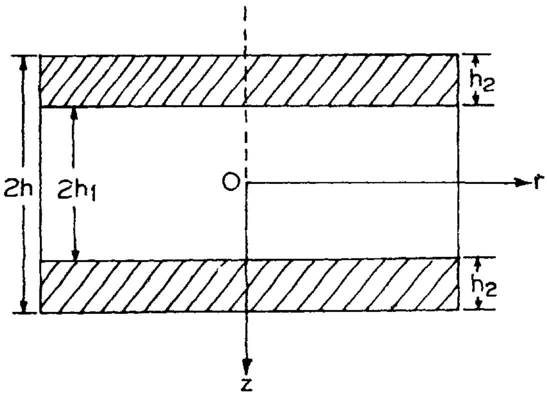

FIG. 1. Cross-section of plate.

Since we are considering asymmetric bending of the circular plate, the displacement in θ direction is taken into account. Therefore, the displacements in the core and facings are approximated as

$$u_1(r, \theta, z) = z\psi_1(r, \theta); \; u_2(r, \theta, z) = -h_1\psi_1(r, \theta) + (z + h_1)\psi_2(r, \theta)$$

$$u_3(r, \theta, z) = h_1\psi_1(r, \theta) + (z - h_1)\psi_2(r, \theta); \; v_1(r, \theta, z) = z\varphi_1(r, \theta) \quad \ldots(1)$$

$$v_2(r, \theta, z) = -h_1\varphi_1(r, \theta) + (z + h_1)\varphi_2(r, \theta); \; v_3(r, \theta, z) = h_1\varphi_1(r, \theta) + (z - h_1)\varphi_2(r, \theta);$$

$$w_i(r, \theta, z) = w(r, \theta)$$

where $i = 1, 2$ and 3.

In the above expressions u_i v_i and w_i are the displacements in the directions r, θ and z respectively, w the transverse deflection of the plate is assumed to be constant for each cross-section (r = constant), ψ_1, ψ_2 and φ_1, φ_2 are rotations in r-z and θ-z plane due to bending of the normal to the middle plane of the core and facings respectively.

The non-zero strain components are found to be

$$
\left.
\begin{aligned}
&\epsilon_{r1} = z\psi_{1,r};\ \epsilon_{r2} = -h_1(\psi_{1,r} - \psi_{2,r}) + z\psi_{2,r};\\
&\epsilon_{r3} = h_1(\psi_{1,r} - \psi_{2,r}) + z\psi_{2,r};\\
&\epsilon_{\theta 1} = \frac{z}{r}\left(\varphi_{1,\theta} + \psi_1\right);\ \epsilon_{\theta 2} = -\frac{h_1}{r}\left(\psi_1 - \psi_2 + \varphi_{1,\theta} - \varphi_{2,\theta}\right)\\
&\qquad + \frac{z}{r}\left(\psi_2 + \varphi_{2,\theta}\right);\\
&\epsilon_{\theta 3} = \frac{h_1}{r}\left(\psi_1 - \psi_2 + \varphi_{1,\theta} - \varphi_{2,\theta}\right) + \frac{z}{r}\left(\psi_2 + \varphi_{2,\theta}\right);\\
&\epsilon_{r\theta 1} = z\left(\frac{1}{r}\psi_{1,\theta} + \varphi_{1,r} - \frac{1}{r}\varphi_1\right);\\
&\epsilon_{r\theta 2} = -\frac{h_1}{r}\left\{\psi_{1,\theta} - \psi_{2,\theta} + r(\varphi_{1,r} - \varphi_{2,r}) - (\varphi_1 - \varphi_2)\right\}\\
&\qquad + \frac{z}{r}(\psi_{2,\theta} + r\varphi_{2,r} - \varphi_2);\\
&\epsilon_{r\theta 3} = \frac{h_1}{r}\left\{\psi_{1,\theta} - \psi_{2,\theta} + r(\varphi_{1,r} - \varphi_{2,r}) - (\varphi_1 - \varphi_2)\right\}\\
&\qquad + \frac{z}{r}(\psi_{2,\theta} + r\varphi_{2,r} - \varphi_2);\\
&\epsilon_{rz1} = \psi_1 + w_{,r};\ \epsilon_{rz2} = \epsilon_{rz3} = \psi_2 + w_{,r};\\
&\epsilon_{\theta z1} = \varphi_1 + \frac{1}{r}w_{,r};\ \epsilon_{\theta z2} = \epsilon_{\theta z3} = \varphi_2 + \frac{1}{r}w_{,\theta};
\end{aligned}
\right\} \qquad \ldots(2)
$$

where a comma (,) followed by a suffix denotes differentiation with respect to that variable.

The stress strain relations are

$$
\begin{aligned}
&\sigma_{ri} = \lambda_i(\epsilon_{ri} + \nu_i\epsilon_{\theta i});\ \sigma_{\theta i} = \lambda_i(\epsilon_{\theta i} + \nu_i\epsilon_{ri});\ \sigma_{rzi} = \mu_1\epsilon_{rzi};\\
&\mu_i = E_i/[2(1+\nu_i)];\ \lambda_i = E_i/(1-\nu_i^2);
\end{aligned} \qquad \ldots(3)
$$

where E_i and ν_i are the Young's modulii and Poisson's ratios respectively.

Equations of Equilibrium

The strain energy of an isotropic sandwich plate in a polar coordinate system, neglecting the direct transverse strains ϵ_{zi} may be written as

$$
V = \sum_{i=1}^{3}\int_0^a\int_0^{2\pi}\int_{x_i}^{y_i}(\sigma_{ri}\epsilon_{ri} + \sigma_{\theta i}\epsilon_{\theta i} + \sigma_{rzi}\epsilon_{rzi} + \sigma_{\theta zi}\epsilon_{\theta zi} + \sigma_{r\theta i}\epsilon_{r\theta i})\,rdzd\theta dr \qquad \ldots(4)
$$

where the limits of integration, x_i to y_i $(i = 1, 2, 3)$, stand for $-h_1$ to h_1, $-h$ to $-h_1$, h_1 to h respectively. Substituting the expressions for the strain components (2) into (4) and then performing the first integration, the strain energy of the plate may be expressed in the form

$$V = \int_0^a \int_0^{2\pi} [rP_1\psi_{1,r} + P_2\psi_{1,\theta} + P_3\psi_1 + rP_4\psi_{2,r} + P_5\psi_{2,\theta} + P_6\psi_2 + P_7\varphi_{1,\theta} + rP_8\varphi_{1,r}$$
$$+ P_8\varphi_1 + P_9\varphi_{2,\theta} + rP_5\varphi_{2,r} + P_{10}\varphi_2 + rQ_r w_{,r} + Q_\theta w_{,\theta}]\, d\theta dr\,; \qquad \ldots(5)$$

where

$$P_1 = M_{r1} - N_1\,;\ P_2 = M_{r\theta 1} - N_3\,;\ P_3 = M_{\theta 1} + rQ_{r1} - N_2\,;$$
$$P_4 = M_{r2} + M_{r3} + N_1\,;\ P_5 = M_{r\theta 2} + M_{r\theta 3} + N_3\,;\ P_6 = M_{\theta 2} + M_{\theta 3}$$
$$+ r(Q_{r2} + Q_{r3}) + N_2\,;\ P_7 = M_{\theta 1} - N_2\,;\ P_8 = rQ_{\theta 1} + N_3 - M_{r\theta 1}\,;$$
$$P_9 = M_{\theta 2} + M_{\theta 3} + N_2\,;\ P_{10} = r(Q_{\theta 2} + Q_{\theta 3}) - 2M_{r\theta 2} - N_3\,;\ Q_r = Q_{r1}$$
$$+ Q_{r2} + Q_{r3}\,;\ Q_\theta = Q_{\theta 1} + Q_{\theta 2} + Q_{\theta 3}\,;\ N_1 = h_1(N_{r2} - N_{r3})\,;$$
$$N_2 = h_1(N_{\theta 2} - N_{\theta 3})\,;\ N_3 = h_1(N_{r\theta 2} - N_{r\theta 3}).$$

The stress-resultants are given by

$$(N_{ri}, N_{\theta i}, M_{ri}, M_{\theta i}, M_{r\theta i}, Q_{ri}, Q_{\theta i}) = \int_{x_i}^{y_i} (\sigma_{ri}, \sigma_{\theta i}, z\sigma_{ri}, z\sigma_{\theta i}, z\sigma_{r\theta i},$$
$$k_s\, \sigma_{rzi}, k_s\, \sigma_{\theta zi})\, dz \qquad (6)$$

where limits of integration x_i, y_i are same as for eqn. (4). Since the bending of facings and core is considered separately, we have taken the averaging shear constant $k_s = 1$ in our analysis.

The change in potential energy of the system due to applied surface load $p(r, \theta)$ is

$$U = -\int_0^a \int_0^{2\pi} p(r, \theta)\, wr\, d\theta\, dr. \qquad \ldots(7)$$

For a system that is in equilibrium, the first variation of the total potential energy $(T = U + V)$ vanishes for any arbitrary set of variations of the dependent variables, w, ψ_1, ψ_2, φ_1 and φ_2 compatible with the prescribed boundary conditions, i.e.,

$$\delta T \equiv \delta U + \delta V = 0. \qquad \ldots(8)$$

Carrying out the first variation and integrating by parts those integrals containing derivatives of dependent variables, w, ψ_1, ψ_2, φ_1 and φ_2, we get

$$\int_0^a \int_0^{2\pi} \Big[\frac{1}{r} (P_1 + r P_{1,r} + P_{2,\theta} - P_3)\, \delta\psi_1 + \frac{1}{r} (P_4 + r P_{4,r} + P_{5,\theta} - P_6)\, \delta\psi_2$$

(*equation continued on p.* 1541.)

$$+ \frac{1}{r} (P_2 + rP_{2,r} + P_{7,\theta} - P_8) \delta\varphi_1$$

$$+ \frac{1}{r} (P_5 + rP_{5,r} + P_{9,\theta} - P_{10}) \delta\varphi_2$$

$$+ \frac{1}{r} \left\{ Q_r + rQ_{r,r} + Q_{\theta,\theta} + p(r, \theta)\right\} \delta w \Big] r \, d\theta \, dr$$

$$- \int_{\theta}^{2\pi} [rP_1\delta\psi_1 + rP_4 \delta\psi_2 + rP_2\delta\varphi_1 + rP_5\delta\varphi_2 + rQ_r\delta w]_0^a \, d\theta$$

$$- \int_0^a [P_2\delta\psi_1 + P_5\delta\psi_2 + P_7\delta\varphi_1 + P_9\delta\varphi_2 + Q_\theta\delta w]_0^{2\pi} dr = 0. \quad \ldots(9)$$

In the above equations the vanishing of the coefficients corresponding to variation $\delta\psi_1$, $\delta\psi_2$, $\delta\varphi_1$, $\delta\varphi_2$ and δw under double integration gives the five equations of equilibrium and the vanishing of the integrand in the single integrals gives the boundary conditions.

With the help of (2), (3) and (6), the equations of equilibrium obtained in terms of displacements are rendered dimensionless, to give

$$\frac{2H_1}{3} (H_1G_1 + 3H_2G_2G_3) \nabla_1^2\overline{\psi}_1 + H_2^2G_2G_3 \nabla_1^2 \overline{\psi}_2 + \frac{2}{3} H_1 \{H_1 (\nu_1G_1 + 1)$$

$$+ 3H_2G_3 (\nu_2G_2 + 1\} \frac{1}{R} \overline{\varphi}_{1,R\theta}$$

$$+ H_2^2G_3 (\nu_2G_2 + 1) \frac{1}{R} \overline{\varphi}_{2,R\theta} + \frac{2}{3} H_1 (H_1 + 3H_2G_3)$$

$$\times \frac{1}{R^2} \overline{\psi}_{1,\theta\theta} + H_2^2G_3 \frac{1}{R^2} \overline{\psi}_{2,\theta\theta} - \frac{2}{3} H_1 \{H_1 (G_1 + 1)$$

$$+ 3H_2G_3 (G_2 + 1)\} \frac{1}{R^2} \overline{\varphi}_{1,\theta} - H_2^2 G_3 (G_2 + 1)$$

$$\times \frac{1}{R} \overline{\varphi}_{2,\theta} - 2 (\overline{\psi}_1 + \overline{w}_{,R}) = 0 \quad \ldots(10)$$

$$H_1H_2G_2\nabla_1^2\overline{\psi}_1 + \frac{2}{3} H_2^2G_2\nabla_1^2 \overline{\psi}_2 + \frac{2}{3} H_2^2 \frac{1}{R^2} \overline{\psi}_{2,\theta\theta} + H_1H_2 \frac{1}{R^2} \overline{\psi}_{1,\theta\theta}$$

$$+ H_1H_2 (\nu_2G_2 + 1) \frac{1}{R} \overline{\varphi}_{1,R\theta} + \frac{2}{3} H_2^2 (\nu_2G_2 + 1)$$

$$\times \frac{1}{R} \overline{\varphi}_{2,R\theta} - H_1H_2 (G_2 + 1) \frac{1}{R^2} \overline{\varphi}_{1,\theta} - \frac{2}{3} H_2^2 (G_2 + 1)$$

$$\times \frac{1}{R^2} \overline{\varphi}_{2,\theta} - 2(\overline{\psi}_2 + \overline{w}_{,r}) = 0 \, ; \quad \ldots(11)$$

$$\frac{2}{3}H_1(H_1+3H_2G_3)\nabla_1^2\bar{\varphi}_1+H_2^2G_3\nabla_1^2\bar{\varphi}_2+\frac{2}{3}H_1(H_1G_1+3H_2G_2G_3)$$
$$\times\frac{1}{R^2}\bar{\varphi}_{1,\theta\theta}+H_2^2G_2G_3\frac{1}{R^2}\bar{\varphi}_{2,\theta\theta}+\frac{2}{3}H_1\{H_1(\nu_1G_1+1)$$
$$+3H_2G_3(\nu_2G_2+1)\}\frac{1}{R}\bar{\psi}_{1,R\theta}+H_2^2G_3(\nu_2G_2+1)$$
$$\times\frac{1}{R}\bar{\psi}_{2,R\theta}+\frac{2}{3}H_1\{H_1(G_1+1)+3H_2G_3(G_2+1)\}$$
$$\times\frac{1}{R^2}\bar{\psi}_{1,\theta}+H_2^2G_3(G_2+1)\frac{1}{R^2}\bar{\psi}_{2,\theta}-2(\bar{\varphi}_1+\frac{1}{R}\bar{w}_{,\theta})=0 \quad \ldots(12)$$

$$H_1H_2\nabla_1\bar{\varphi}_1+\frac{2}{3}H_2^2\nabla_1^2\bar{\varphi}_2+H_1H_2G_2\frac{1}{R^2}\bar{\varphi}_{1,}\;{}_{,}+\frac{2}{3}H_2^2G_2\frac{1}{R^2}\bar{\varphi}_{2,\theta}$$
$$+H_1H_2(\nu_2G_2+1)\frac{1}{R}\bar{\psi}_{1,R\theta}+\frac{2}{3}H_2^2(\nu_2G_2+1)\frac{1}{R}\bar{\psi}_{2,R\theta}$$
$$+H_1H_2(G_2+1)\frac{1}{R^2}\bar{\psi}_{1,\theta}+\frac{2}{3}H_2^2(G_2+1)\frac{1}{R^2}\bar{\psi}_{2,\theta}$$
$$-2(\bar{\varphi}_2+\frac{1}{R}\bar{w}_{,\theta})=0 \quad \ldots(13)$$

and

$$(H_1+H_2G_3)\left(\bar{w}_{,RR}+\frac{1}{R}\bar{w}_{,R}+\frac{1}{R^2}\bar{w}_{,\theta\theta}\right)+H_1\left(\bar{\psi}_{1,R}+\frac{1}{R}\bar{\psi}_1\right.$$
$$\left.+\frac{1}{R}\bar{\varphi}_{1,\theta}\right)+H_2G_3\left(\bar{\psi}_{2,R}+\frac{1}{R}\psi_2+\frac{1}{R}\varphi_{2,\theta}\right)$$
$$+\frac{p(R,\theta)}{2P}=0 \quad \ldots(14)$$

$$\left.\begin{aligned}
&\text{where } \nabla_1^2\equiv\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}-\frac{1}{R^2};\\
&R=r/a;\ H_1=h_1/a;\ H_2=h_2/a;\\
&G_1=\lambda_1/\mu_1;\ G_2=\lambda_2/\mu_2;\ G_3=\mu_2/\mu_1;\\
&\bar{\psi}_1=\mu_1\psi_1/P;\ \bar{\psi}_2=\mu_1\psi_2/P;\ \bar{w}=\mu_1 w/aP;\\
&\bar{\varphi}_1=\mu_1\varphi_1/P;\ \bar{\varphi}_2=\mu_1\varphi_2/P;
\end{aligned}\right\} \quad \ldots(15)$$

and P is the average load per unit area

The boundary conditions found are such that one element of each following pairs should vanish

(P_1, ψ_1) ; (P_5, ψ_2) ; (P_2, φ_1) ; (P_5, φ_2) ; (Q_r, w).

It is found convenient to introduce five auxiliary variables as follows :

$$\left.\begin{aligned}
\bar{\psi}_1 &= \sum_{m=0}^{\infty} \left(S_{1m,R} + \frac{m}{R} T_{1m} \right) \cos m\theta \\
\bar{\psi}_2 &= \sum_{m=0}^{\infty} \left(S_{2m,R} + \frac{m}{R} T_{2m} \right) \cos m\theta \\
\bar{\varphi}_1 &= -\sum_{m=0}^{\infty} \left(\frac{m}{R} S_{1m} + T_{1m,R} \right) \sin m\theta \\
\bar{\varphi}_2 &= -\sum_{m=0}^{\infty} \left(\frac{m}{R} S_{2m} + T_{2m,R} \right) \sin m\theta \\
\bar{w} &= \sum_{m=0}^{\infty} W_m \cos m\theta
\end{aligned}\right\} \qquad \ldots(16)$$

where S_{1m}, S_{2m}, T_{1m}, T_{2m} and W_m are functions of R alone.

We further take the load term as

$$p(R,\theta) = \sum_{m=0}^{\infty} p_m(R) \cos m\theta \,, \qquad \ldots(17)$$

Now operating eqn. (10) by $\left(\frac{\partial}{\partial R} + \frac{1}{R} \right)$ and eqn. (12) by $\frac{1}{R} \frac{\partial}{\partial \theta}$, adding and then introducing auxiliary variables from eqn. (16), we get

$$a_1 (\nabla^2 - 2) \nabla^2 S_{1m} + a_2 \nabla^4 S_{2m} - 2\nabla^2 W_m = 0. \qquad \ldots(18)$$

Similarly from eqns. (11) and (13), we get

$$a_3 \nabla^4 S_{1m} + a_4 (\nabla^2 - 2) \nabla^2 S_{2m} - 2\nabla^2 W_m = 0. \qquad \ldots(19)$$

Introducing the auxiliary variables in eqn. (14), we get

$$H_1 \nabla^2 S_{1m} + a_5 \nabla^2 S_{2m} + a_6 \nabla^2 W_m = -\bar{p}_m(R)/2. \qquad \ldots(20)$$

Operating eqn. (10) by $\frac{1}{R} \frac{\partial}{\partial \theta}$ and eqn. (13) by $\left(\frac{\partial}{\partial R} + \frac{1}{R} \right)$, subtracting latter from former and then introducing auxiliary variables from eqn. (16), we get

$$a_7 (\nabla^2 - 2) T_{1m} + a_8 \nabla^2 T_{2m} = 0. \qquad \ldots(21)$$

Similarly from eqns. (11) and (13), we get

$$a_9 \nabla^4 T_{1m} + a_{10} (\nabla^2 - 2) T_{2m} = 0 \qquad ...(22)$$

where

$$\nabla^2 = \frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} - \frac{m^2}{R^2}$$

$$a_1 = 2 (H_1^2 G_1 + 3 H_1 H_2 G_2 G_3)/3 \; ; \; a_2 = H_2^2 G_2 G_3 \; ; \; a_3 = H_1 H_2 G_2 ; \; a_4 = 2 H_2^2 G_2/3 ;$$

$$a_5 = H_2 G_2 G_3 \; ; \; a_6 = H_1 + a_5 \; ; \; a_7 = 2 (H_1^2 + 3 H_2 H_1 G_3)/3 \; ; \; a_8 = H_2^2 G_3 \; ;$$

$$a_9 = H_1 H_2 \; ; \; a_{10} = 2 H_2^2/3 : \bar{p}_m(R) = p_m(R)/p.$$

Solution of Equations

The solutions of the eqns. (18) to (20) consists of two parts, the complementary function and the particular integral. For complementary function, the eqns. (18) to (20) are made homogeneous by putting their right hand side equal to zero and then an equation in a single dependent variable is obtained by eliminating the other two, to give

$$(b_1 \nabla^2 + b_2) \nabla^4 F_m = 0 \qquad ...(23)$$

where F_m stands for any of the variables S_{1m}, S_{2m} and W_m.

A solution of the equation

$$\nabla^4 F_m = 0 \qquad ...(24)$$

will be a solution of eqn. (23).

Let us assume

$$F_m = R^{\xi}. \qquad ...(25)$$

Substituting for F_m in eqn. (24), we get

$$(\xi^2 - 4 \xi + 4 - m^2)(\xi^2 - m^2) = 0.$$

This gives $\xi = \pm m, 2 \pm m$. The values with negative m will be rejected because of singularity at the origin $R = 0$. Hence R^m and R^{m+2} are solutions of eqn. (23).

A solution of

$$(b_1 \nabla^2 + b_2) F_m = 0$$

or

$$R^2 \frac{d^2 F_m}{dR^2} + R \frac{dF_m}{dR} + \left(n_3^2 R^2 - m^2 \right) F_m = 0 \qquad ...(26)$$

will also be a solution of eqn. (23). Eqn. (26) is Bessel's equation with $n_3^2 = b_2/b_1$, whose solution will be

$$F_m = J_m (n_3 R). \qquad ...(27)$$

The other solution $Y_m(n_3R)$ is rejected because of singularity at the origin $R = 0$.

Hence the solution for S_{1m}, S_{2m} and W_m can be taken as

$$S_{1m} = (A_{1m} + A_{2m} R^2)R^m + A_{3m} J_m(n_3R) \quad ...(28)$$

$$S_{2m} = (B_{1m} + B_{2m} R^2)R^m + B_{3m} J_m(n_3R) \quad ...(29)$$

$$W_m = (C_{1m} + C_{2m} R^2)R^m + C_{3m} J_m(n_3R) \quad ...(30)$$

where A_{im}, B_{im}, C_{im} $(i = 1, 2, 3)$ are integration constants.
Equations (28) to (30) are substituted in eqns. (18) and (19) and B_{im} and C_{im} are obtained in terms of A_{im} $(i = 1, 2, 3)$ as

$$\left.\begin{aligned} B_{1m} &= 2(m+1)(a_3a_4 - a_1a_2) A_{2m} + a_1 A_{1m}/a_4 \\ C_{1m} &= 2(m+1)a_1(a_2 + a_4) A_{2m}/a_4 - a_1A_{1m} \\ B_{2m} &= -a_1 A_{2m}/a_2 \; ; \; C_{2m} = -a_1 A_{2m} \\ B_{3m} &= [\{(a_1 - a_3) n_3^2 + 2a_1\}/\{(a_4 - a_2) n_3 + 2a_4\}] A_{1m} \\ C_{3m} &= [\{a_2a_3n_3^4 - a_1a_4(n_3 + 2)\}/\{(a_4 - a_2) n_3^2/2 + a_4\}] A_{1m} \end{aligned}\right\} \quad ...(31)$$

For particular integral, we take

$$\bar{p}_m(R) = \sum_{n=0}^{\infty} D_m^n J_m(K_{mn} R) \quad ...(32)$$

and seek the solution for S_{1m}, S_{2m} and W_m in the form

$$\left.\begin{aligned} S_{1m} &= \sum_{n=0}^{\infty} A_{1m}^n J_m(K_{mn} R) \\ S_{2m} &= \sum_{n=0}^{\infty} B_{1m}^n J_m(K_{mn} R) \\ W_m &= \sum_{n=0}^{\infty} C_m^n J_m(K_{mn} R) \end{aligned}\right. \quad ...(33)$$

where K_{mn} $(n = 0, 1, 2, ...)$ are roots of $J_m(K_{mn}) = 0$. ...(34a)

D_m^n being the coefficients in the load function are known and hence substituting eqns. (32) and (33) into eqns. (18) to (20) and solving for the constants A_{1m}^n, B_{1m}^n and C_m^n in terms of D_m^n, we get

$$A_{1m}^{n} = \left(b_3 K_{mn}^2 + 2 \right) D_m^n \Big/ D$$

$$B_{1m}^{n} = \left(b_4 K_{mn}^2 + 2 \right) D_m^n \Big/ D \qquad \ldots(34b)$$

$$C_m^n = \left(b_5 K_{mn}^4 - b_6 K_{mn}^2 - 2 \right) D_m^n \Big/ D$$

where $D = K_{mn}^6 (K_{mn}^2 b_1 - b_2)$; $b_1 = a_6 (a_1 a_4 - a_2 a_3)$,

$b_2 = 2 (a_6 b_6 - H_1 b_3 - a_5 b_4)$; $b_3 = a_4 - a_2$; $b_4 = a_1 - a_3$;

$b_5 = (a_2 a_3 - a_1 a_4)/2$; $b_6 = a_1 + a_4$.

To obtain the solutions for T_{1m} and T_{2m}, we eliminate one of them from the eqns. (21) and (22), to give

$$\{\nabla^4 + b_7 (\nabla^2 - 1)\} T_{im} = 0 \qquad \ldots(34c)$$

where $i = 1, 2$ and $b_7 = - 4 a_7 a_{10}/(a_7 a_{10} - a_8 a_9)$.

Equation (34) has two solutions based upon the solution of the Bessel's equation

$$\nabla^2 T_{im} = - n^2 T_{im} . \qquad \ldots(35)$$

Therefore, we can take the solution of eqn. (34b) as

$$\left.\begin{aligned} T_{1m} &= A_{4m} J_m (Rn_1) + A_{5m} J_m (Rn_2) \\ T_{2m} &= B_{4m} J_m (Rn_1) + B_{5m} J_m (Rn_2) \end{aligned}\right\} \qquad \ldots(36)$$

where $\pm n_1$ and $\pm n_2$ are the roots of the equation

$$n_i^4 + b_7 \left(- n_i^2 - 1 \right) = 0. \qquad \ldots(37)$$

The solution $Y_m (Rn_1)$ and $Y_m (R_{n_2})$ are rejected due to singularity at the origin $R = 0$.

The solution (36) substituted in eqn. (21) give

$$\left.\begin{aligned} B_{4m} &= - a_7 \left(2 + n_1^2 \right) A_{4m} \Big/ \left(a_8 n_1^2 \right) \\ B_{5m} &= - a_7 \left(2 + n_2^2 \right) A_{5m} \Big/ \left(a_8 n_2^2 \right) . \end{aligned}\right\} \qquad \ldots(38)$$

Thus the solutions for $\overline{\psi}_1$, $\overline{\psi}_2$, $\overline{\varphi}_1$, $\overline{\varphi}_2$ and $\overline{w}$ are given by

$$\overline{\psi}_1 = \sum_{m=0}^{\infty} \Big[\Big\{ m A_{1m} + (m + 2) A_{2m} R^2 \Big\} R^{m-1} - n_3 A_{3m} J_m' (n_3 R)$$

$$- \sum_{n=0}^{\infty} K_{mn} A_{1m}^n J_m' (K_{mn} R) + m \{A_{4m} J_m (Rn_1)$$

$$+ A_{5m} J_m (Rn_2)\}/R \Big] \cos m\theta$$

(equation continued on p. 1547)

$$\bar{\psi}_2 = \sum_{m=0}^{\infty} \Big[\Big\{ mB_{1m} + (m+2) B_{2m}R^2 \Big\} R^{m-1} - n_3B_{3m}J_m'(n_3R)$$

$$- \sum_{n=0}^{\infty} K_{mn} B_{1m}^n J_m'(K_{mn}R) + m \Big\{ B_{4m}J_m(Rn_1)$$

$$+ B_{5m}J_m(R_{n_2}) \Big\} \Big/ R \Big] \cos m\theta$$

$$\bar{\varphi}_1 = - \sum_{m=0}^{\infty} [m (A_{1m} + A_{2m}R^2)R^{m-1} + A_{3m}J_m ((n_3R)$$

$$+ \sum_{n=0}^{\infty} A_{1m}^n J_m (K_{mn}R) - \{n_1A_{4m}J_m'(Rn_1)$$

$$+ n_2 A_{5m}J_m'(Rn_2)\}] \sin m\theta$$

$$\bar{\varphi}_2 = - \sum_{m=0}^{\infty} [m (B_{1m} + B_{2m}R^2)R^{m-1} + B_{3m}J_m(n_3R)$$

$$+ \sum_{n=0}^{\infty} B_{1m}^n J_m (K_{mn}R) - \{n_1B_{4m}J_m'(Rn_1)$$

$$+ n_2 B_{5m}J_m'(Rn_2)\}] \sin m\theta$$

$$\bar{w} = \sum_{m=0}^{\infty} [(C_{1m} + C_{2m}R^2) R^m + C_{3m}J_m(n_3R)$$

$$+ \sum_{n=0}^{\infty} C_m^n J_m(K_{mn}R)] \cos m\theta. \qquad \ldots(39)$$

The above solutions involve only 5 arbitrary constants, A_{im} $(i = 1, 2, 3, 4, 5)$ since B_{im} and C_{im} are givn by eqns. (31), (33) and (38). These constants can be evaluated by the five boundary conditions at the edge $R = 1$.

Boundary Conditions

For a clamped edge, the boundary conditions are

$$\psi_1 = \psi_2 = \varphi_1 = \varphi_2 = w = 0 \text{ at } R = 1. \qquad \ldots(40)$$

These give

$$mA_{1m} + (m+2) A_{2m} - n_3A_{3m}J_m'(n_3) - \sum_{n=0}^{\infty} K_{mn} A_{1m}^n J_m'(K_{mn})$$

(*equation continued on p.* 1548)

$$+ m\{A_{4m}J_m(n_1) + A_{5m}J_m(n_2)\} = 0$$

$$mB_{1m} + (m+2)B_{2m} - n_3B_{3m}J_m'(n_3) - \sum_{n=0}^{\infty} K_{mn} B_{1m}^n J_m'(K_{mn})$$

$$+ m\{B_{4m}J_m(n_1) + B_{5m}J_m(n_2)\} = 0$$

$$m(A_{1m} + A_{2m}) + A_{3m}J_m(n_3) - n_1A_{4m}J_m'(n_1) - n_2A_{5m}J_m(n_2) = 0$$

$$m(B_{1m} + B_{2m}) + B_{3m}J_m(n_3) - n_1B_{3m}J_m'(n_1) - n_2B_{5m}J_m(n_2) = 0$$

$$C_{1m} + C_{2m} + C_{3m}J_m(n_3) = 0. \qquad \ldots(41)$$

Evaluation of the Coefficients D_m^n

Multiplying by $R\,J_m(K_{mn}R)$ in eqn. (32) and integrating from 0 to 1, we get

$$\int_0^1 R\,\bar{p}_m(R)J_m(K_{mn}R)\,dR = D_m^n \int_0^1 R\,J_m^2(K_{mn}R)\,dR$$

or

$$\int_0^1 R\,\bar{p}_m(R)J_m(K_{mn}R)\,dR = \tfrac{1}{2}\,D_m^n \left\{ J_m'(K_{mn}) \right\}^2$$

(since other integrals vanish due to orthogonality conditions).
Therefore

$$D_m^n = \frac{2}{\{J_m'(K_{mn})\}^2} \int_0^1 R\,\bar{p}_m(R)J_m(K_{mn}R)\,dR\,. \qquad \ldots(42)$$

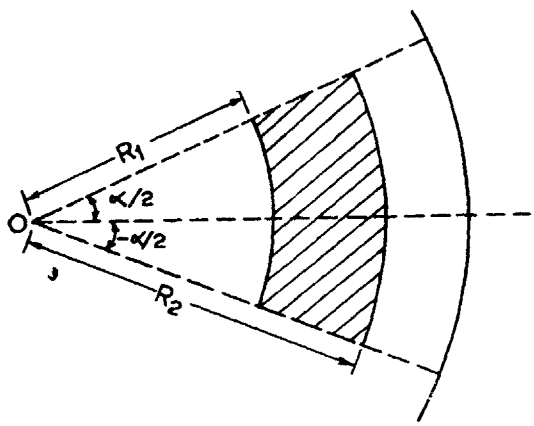

FIG. 2. Loaded sector of plate.

(i) *Uniformly distributed load over a sector*—If we consider a load uniformly distributed over a sector bounded by radii vectors $\theta = -\alpha/2$ to $\theta = \alpha/2$ and the circular areas $R = R_1$ to $R = R_2$ $(R_1 < R_2)$, (Fig. 2), then

$$\text{Load area} = \alpha\left(R_2^2 - R_1^2\right)\Big/2 = \alpha\, b\, \bar{R}$$

where $b = R_2 - R_1$ and $\bar{R} = (R_1 + R_2)/2$.

Since the total load is kept constant, therefore

$$\begin{aligned} p(R,\theta) &= \pi P/(\alpha \bar{R} b) \qquad (R_1 \leqslant R \leqslant R_2\ ;\ -\alpha/2 \leqslant \theta \leqslant \alpha/2) \\ &= 0 \qquad \text{elsewhere.} \end{aligned} \qquad ...(43)$$

Therefore

$$p_m(R) = \frac{2}{\pi}\int_0^{\pi} p(R,\theta)\cos m\theta\, d\theta = \frac{2}{\pi}\left[\int_0^{\alpha/2} p(R,\theta)\cos m\theta\, d\theta\right.$$

$$\left. + \int_{\alpha/2}^{\pi} p(R,\theta)\cos m\theta\, d\theta\right] = \frac{2}{\pi}\int_0^{\alpha/2} \frac{\pi P}{\alpha \bar{R} b}\cos m\theta\, d\theta$$

or

$$\bar{p}_m(R) = p_m(R)/P = \frac{2}{m\alpha\bar{R}b}\sin(m\alpha/2)\ R_1 \leqslant R \leqslant R_2\,. \qquad ...(44)$$

Substituting in eqn. (42) for $\bar{p}_m(R)$, we get

$$D_m^n = \frac{2}{\{J_m'(K_{mn})\}^2}\int_0^1 R\bar{p}_m(R) J_m(K_{mn}R)\, dR$$

$$= \frac{2}{\{J_m'(K_{mn})\}^2}\left[\int_0^{R_1} R\,\bar{p}_m(R) J_m(K_{mn}R)\, dR + \int_{R_1}^{R_2} R\,\bar{p}_m(R) J_m(K_{mn}R)\, dR\right.$$

$$\left. + \int_{R_2}^{1} R\,\bar{p}_m(R) J_m(K_{mn}R)\, dR\right]$$

or

$$D_m^n = \frac{4\ \mathrm{Sin}\,(m\,\alpha/2)}{m\alpha\bar{R}b\,\{J_m'(K_{mn})\}^2}\int_{R_1}^{R_2} R J_m(K_{mn}R)\, dR\,. \qquad ...(45)$$

(The first and last integrals vanish, because $\bar{p}_m = 0$ for their ranges of integration.)

(ii) *Linearly varying load over a sector*—If we consider such a load over a sector as in (i), then

$$p(R, \theta) = p_0 (1 - \gamma R) \qquad (R \leq R \leq R_2 \; ; \; -\alpha/2 \leq \theta \leq \alpha/2)$$
$$= 0 \qquad \text{elsewhere.} \qquad ...(46)$$

Therefore

$$\bar{p}_m(R) = 2p_0 (1 - \gamma R) \sin (m \alpha/2)/(\pi m P). \qquad ...(47)$$

Since the total load is kept constant, therefore

$$\int_{R_1}^{R_2} \int_{-\alpha/2}^{\alpha/2} p_0 (1 - \gamma R) R \, d\theta \, dR = \pi P$$

or

$$p_0/P = 6\pi/\Big[\alpha b \Big\{ 6\bar{R} - 2\gamma \Big(R_1^2 + R_2^2 + R_1 R_2 \Big)\Big\}\Big] . \qquad ...(48)$$

Substituting in eqn. (42) for $\bar{p}_m(R)$, we get

$$D_m^n = \frac{24 \sin (m \alpha/2)}{m\alpha b \{6\bar{R} - 2\gamma (R_1^2 + R_2^2 + R_1 R_2)\}} \int_{R_1}^{R_2} (1 - \gamma R) R J_m(K_{mn}R) \, dR. \qquad ...(49)$$

[The other integrals vanish as in (i)]

(iii) *Point loading*—If we make $b \to 0$ and $\alpha \to 0$, then the load gets concentrated at a point on r-axis at a distance d (say) from the centre of the plate, then D_m^n from (45) is given by

$$D_m^n = \frac{2J_m (K_{mn}d)}{\{J_m'(K_{mn})\}^2} . \qquad ...(50)$$

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