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Eigenvalues of rank-one updated matrices with some applications[☆]

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Abstract

We prove a spectral perturbation theorem for rank-one updated matrices of special structure. Two applications of the result are given to illustrate the usefulness of the theorem. One is for the spectrum of the Google matrix and the other is for the algebraic simplicity of the maximal eigenvalue of a positive matrix.

Keywords: Rank-one update; Determinant; Spectrum

1. Introduction

In this paper we give a simple proof of a spectral perturbation theorem for rank-one perturbed matrices of special structure, using a well known determinant identity. This work was motivated by the recent spectral analysis of the so-called Google matrix in the computation of the PageRank for the Google Web search engine [2,3,7,8,10]. The Google matrix is a positive matrix obtained by a special rank-one perturbation of a stochastic matrix which represents the hyperlink structure of the webpages. In our more general result, the unperturbed matrix is arbitrary, but the perturbation satisfies some natural condition.

Because we apply a classic determinant equality to our spectral analysis, we are able to find the explicit expression of the characteristic polynomial of the rank-one perturbed matrix. All the eigenvalues of the matrix are immediately available. Then, as a consequence, the eigenvalues of the Google matrix can be obtained easily. It would be interesting to note that our general result may also be applied to derive other useful results, for instance, the algebraic simplicity of the maximal eigenvalue of a positive matrix.

The main idea behind our proof is from the following simple relation between the determinants of a matrix and its rank-one perturbation.

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Lemma 1.1. *If A is an invertible $n \times n$ matrix, and u and v are two n-dimensional column vectors, then*

$$\det(A + uv^{\mathrm{T}}) = (1 + v^{\mathrm{T}} A^{-1} u) \det A. \tag{1}$$

Proof. We may assume $A = I$, the $n \times n$ identity matrix, since then (1) follows from $A + uv^{\mathrm{T}} = A(I + A^{-1}uv^{\mathrm{T}})$ in the general case. In this special case, the result comes from the equality

$$\begin{bmatrix} I & 0 \\ v^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} I + uv^{\mathrm{T}} & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ -v^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} I & u \\ 0 & 1 + v^{\mathrm{T}}u \end{bmatrix}. \quad \square$$

In the next section we present the main result, and we give two applications in Section 3.

2. Spectral perturbation of rank-one updated matrices

Let A be an $n \times n$ matrix. The eigenvalues of A are all the complex zeros of the *characteristic polynomial* $p_A(\lambda) \equiv \det(\lambda I - A)$ of A. Let $\sigma(A) \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the set of the eigenvalues of A, counting algebraic multiplicity. Our purpose is to find the eigenvalues of a special rank-one updated matrix of A and their multiplicity. The following is our main theorem.

Theorem 2.1. *Let u and v be two n-dimensional column vectors such that u is an eigenvector of A associated with eigenvalue λ_1. Then, the eigenvalues of*

$$A + uv^{\mathrm{T}}$$

are

$$\{\lambda_1 + v^{\mathrm{T}}u, \lambda_2, \ldots, \lambda_n\},$$

counting algebraic multiplicity.

Proof. Let $\lambda \notin \sigma(A)$ be any complex number. Then, by applying Lemma 1.1 to the equality

$$\lambda I - (A + uv^{\mathrm{T}}) = (\lambda I - A) - uv^{\mathrm{T}},$$

we have

$$\det[\lambda I - (A + uv^{\mathrm{T}})] = [1 - v^{\mathrm{T}}(\lambda I - A)^{-1}u] \det(\lambda I - A). \tag{2}$$

The condition $Au = \lambda_1 u$ implies that

$$(\lambda I - A)^{-1}u = \frac{1}{\lambda - \lambda_1} u, \tag{3}$$

so (2) becomes

$$\begin{aligned} \det[\lambda I - (A + uv^{\mathrm{T}})] &= \left(1 - \frac{v^{\mathrm{T}}u}{\lambda - \lambda_1}\right) \det(\lambda I - A) \\ &= \frac{[\lambda - (\lambda_1 + v^{\mathrm{T}}u)](\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)}{\lambda - \lambda_1} \\ &= [\lambda - (\lambda_1 + v^{\mathrm{T}}u)](\lambda - \lambda_2) \cdots (\lambda - \lambda_n). \end{aligned} \tag{4}$$

Since the above equality is true for all $\lambda \notin \sigma(A)$, the theorem is proved. $\square$

Remark 2.1. By Theorem 2.1, the characteristic polynomial of $A + uv^{\mathrm{T}}$ is

$$p_{A+uv^{\mathrm{T}}}(\lambda) = [\lambda - (\lambda_1 + v^{\mathrm{T}}u)](\lambda - \lambda_2) \cdots (\lambda - \lambda_n). \tag{5}$$

Remark 2.2. Since A and A^{T} have the same eigenvalues counting algebraic multiplicity, the conclusion of Theorem 2.1 also holds for $A + uv^{\mathrm{T}}$, where v is a *left* eigenvector of A corresponding to eigenvalue λ_1.

3. Applications of the theorem

A direct consequence of Theorem 2.1 and Remark 2.2 is the following

Proposition 3.1. *Let A be an $n \times n$ matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, counting algebraic multiplicity, let u and v be n-dimensional column vectors such that either u is an eigenvector of A or v is a left eigenvector of A, associated with eigenvalue λ_1, and let $\alpha \in [0, 1]$. Then the eigenvalues of the matrix*

$$\alpha A + (1 - \alpha) u v^{\mathrm{T}}$$

are $\alpha\lambda_1 + (1 - \alpha) v^{\mathrm{T}} u, \alpha\lambda_2, \ldots, \alpha\lambda_n$, counting algebraic multiplicity.

Proposition 3.1 can be used to find the eigenvalues of the Google matrix in the Google Web search engine. Let S be an $n \times n$ column-stochastic matrix, i.e. a nonnegative matrix that satisfies $e^{\mathrm{T}} S = e^{\mathrm{T}}$, where the n-dimensional vector $e^{\mathrm{T}} = (1, 1, \ldots, 1)$. The *Google matrix G* is defined by

$$G = \alpha S + (1 - \alpha) u e^{\mathrm{T}},$$

where $0 < \alpha < 1$, and u is an n-dimensional positive vector normalized by $u^{\mathrm{T}} e = 1$ (i.e. u is a probability vector). It is obvious that G is also column-stochastic. The eigenvector of G corresponding to the maximal eigenvalue 1 is called the *PageRank*, the computation of which is a major talk of Google. Because of the huge size of the Google matrix, the only practical method for computing the PageRank is the power method [6,8] whose convergence rate depends on the second largest eigenvalue of G in magnitude.

The spectrum of the Google matrix G is given below, which is a direct consequence of Proposition 3.1. See [5,7,8] for related works.

Corollary 3.1. *Let S be a column-stochastic matrix with eigenvalues $1, \lambda_2, \ldots, \lambda_n$, counting algebraic multiplicity, let u be an n-dimensional probability vector, and let $\alpha \in (0, 1)$. Then the eigenvalues of the Google matrix $G = \alpha S + (1 - \alpha) u e^{\mathrm{T}}$ are $1, \alpha\lambda_2, \alpha\lambda_3, \ldots, \alpha\lambda_n$, counting algebraic multiplicity.*

Proof. Let $v = e$ and $A = S$ in Proposition 3.1. Then all the conditions there are satisfied and $\alpha\lambda_1 + (1 - \alpha) v^{\mathrm{T}} u = \alpha \cdot 1 + (1 - \alpha) \cdot 1 = 1$. Therefore, Corollary 3.1 follows. □

Our next application of Theorem 2.1 is for a new, short, and more direct proof of the following result supplementing the Perron theorem for positive matrices; other proofs in textbooks and monographs known to the authors seem quite long and complicated, using various techniques such as a game theory argument (Theorem 1.4.4(v) of [1]), Schur's triangulation theorem (Theorem 8.2.10 of [4]), and a derivative approach (Theorem 4.3 in [9]). A real matrix is *positive* if all of its entries are positive. The *spectral radius* $r(A)$ of a matrix A is the maximal magnitude of its eigenvalues.

Theorem 3.1. *Let A be an $n \times n$ positive matrix. Then $r(A)$ is an eigenvalue of A of algebraic multiplicity* 1*; that is, $r(A)$ is a simple zero of the characteristic polynomial of A.*

Proof. From Perron's theorem [1,4,9], $r \equiv r(A)$ is an eigenvalue of A with geometric multiplicity 1,

$$\lim_{k \to \infty} (r^{-1} A)^k = x y^{\mathrm{T}},$$

where x and y are the positive (right) eigenvector and left eigenvector of A respectively corresponding to eigenvalue r such that $y^{\mathrm{T}} x = 1$, and r is not an eigenvalue of the matrix $A - r x y^{\mathrm{T}}$. Let

$$r, \lambda_2, \ldots, \lambda_n$$

be the eigenvalues of A, counting algebraic multiplicity. Then the condition of Theorem 2.1 is satisfied with $u = -r x$, and $v = y$. Thus, the eigenvalues of $A - r x y^{\mathrm{T}}$ are, counting algebraic multiplicity,

$$\mu, \lambda_2, \ldots, \lambda_n,$$

where $\mu = r + y^{\mathrm{T}}(-r x) = r - r = 0$. Therefore, $r \neq \lambda_i$ for all $i = 2, 3, \ldots, n$. In other words, $r(A)$ is an algebraically simple eigenvalue of A. □

Remark 3.1. The conclusion of Theorem 3.1 and its above proof are still true if A is a nonnegative irreducible matrix.

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