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Unified response probability distribution analysis of two hybrid uncertain acoustic fields

Baizhan Xia, Dejie Yu *, Xu Han, Chao Jiang

State Key Laboratory of Advanced Design and Manufacturing for Vehicle Body, Hunan University, Changsha, Hunan 410082, People's Republic of China

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Abstract

For the response analysis of uncertain acoustic fields with limited information, two hybrid uncertain models have been developed. One is the hybrid probability and interval model in which the probability and interval variables exist simultaneously. The other one is the hybrid interval probability model in which some distribution parameters of probability variables are expressed as interval variables. For the unified response probability distribution analysis of acoustic fields under two hybrid uncertain models, an inverse mapping hybrid perturbation method (IMHPM) is proposed. In IMHPM, the responses of two hybrid uncertain acoustic fields are converted to invertible functions of probability variables based on the Taylor series expansion. According to the inverse mapping relationships between responses and probability variables, the formal expressions of the response probability distributions are yielded on the basis of the change-of-variable technique. The variational ranges of the response probability distributions are estimated by the interval perturbation technique. The effectiveness and efficiency of the proposed method for the unified response probability distribution analysis of two hybrid uncertain acoustic fields are investigated by a numerical example.

Keywords: Matrix perturbation method; Change-of-variable technique; Acoustic field; Hybrid probability and interval model; Hybrid interval probability model

1. Introduction

In the last several decades, the numerical method for the response analysis of the acoustic field with deterministic parameters has achieved great attention [1]. However, due to the influences of model inaccuracies, physical imperfections, multiphase characteristics of materials and unpredictability of environment, uncertainties existing in acoustic fields are unavoidable. The responses of uncertain acoustic fields are very

* Corresponding author. Tel.: +86 731 88821915; fax: +86 731 88823946.
E-mail address: djyu@hnu.edu.cn (D. Yu).

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sensitive to these uncertain parameters. The traditional mathematical model used to treat uncertain parameters is the probability model. In the probability model, probability distributions of uncertain parameters are unambiguously defined on the basis of available information. For the response analysis of acoustic fields with probability variables, a lot of probabilistic approaches have been developed. James and Dowling proposed a field shifting approximation technique to evaluate the probability distribution of the computed field amplitude under known environmental uncertainties [2]. Finette et al. introduced the spectral stochastic method to predict the ocean acoustic propagation in an uncertain waveguide environment [3,4]. The field shifting approximation technique and the spectral stochastic method are approximate methods. Recently, the change-of-variable technique is introduced to calculate the analytic solution of the probability distribution of the array beam power of the acoustic field in the uncertain ocean environment [5]. The application precondition of the change-of-variable technique is that the response of the acoustic field with probability variables has to be an invertible function. For complex acoustic fields, this application precondition may be not satisfied. Based on the stochastic perturbation method and the change-of-variable technique, a change-of-variable stochastic perturbation method has been proposed for the response probability distribution analysis of complex random acoustic fields [6] and complex random structures [7]. From the overall perspective, probability methods have achieved significant successes in the response analysis of acoustic fields with probability variables. The main advantage of these probability methods is that the response probability distributions of acoustic fields can be obtained. The common disadvantage of these probability methods is that the precise probability distributions of uncertain parameters have to be available. Unfortunately, in most cases, the information to construct the precise probability distributions of uncertain parameters is limited.

For the response analysis of the acoustic field with limited information, the interval model can be employed. In the interval model, the uncertain parameters are expressed as interval variables whose variational ranges are well-defined. The interval model was firstly applied to the response analysis of uncertain structures. If the responses of structures with interval variables can be defined as non-convex problems, the vertex method can be employed to calculate their exact response intervals [8,9]. The vertex method has two inherent disadvantages. The first one is that its computational cost increases exponentially with the increase of the number of interval variables. The second one is that the extreme values which are not at the vertexes cannot be considered. With the matrix perturbation technique and the interval extension theory, the interval perturbation method has been applied to the static response analysis, the dynamic response analysis and the eigenvalue analysis of structures with interval variables [10–15]. Recently, the interval perturbation method was developed to evaluate the response variational ranges of acoustic fields with interval variables [16–18]. The main shortcoming of the interval model is that only the response variational ranges can be yielded. The response probability distributions in variational ranges are missing.

To keep the advantages of the probability model and the interval model simultaneously, two hybrid uncertain models have been developed. The first one is the hybrid probability and interval model. In the hybrid probability and interval model, the uncertain parameters with sufficient information to construct the corresponding probability distributions are treated as probability variables, while the uncertain parameters with limited information are treated as interval variables. Significant successes have been achieved in the hybrid probability and interval model. The stochastic seismic analysis of buildings with interval mass distributions was done by Cacciola et al. [19]. The probabilistic characteristics of the stationary stochastic response (mean-value vector, power spectral density function and covariance matrix) of structures with interval parameters under random excitation were investigated by Muscolino and Sofi [20–22]. The uncertainty propagation of uncertain systems with both probability and interval variables was investigated by Zaman et al. [23]. The reliability of uncertain structures with both probability and interval variables was investigated by Qiu and Wang [24]. Recently, a random interval perturbation method for the response analysis of uncertain structures with both probability and interval parameters was proposed by Gao et al. [25,26]. Based on the work of Gao et al., two numerical methods named as the hybrid perturbation Monte-Carlo method and the hybrid perturbation vertex method have been developed by Xia et al. [27].

The second type of the hybrid uncertain model is the hybrid interval probability model. In the hybrid interval probability model, the uncertain parameter is treated as the probability variable whose distribution parameters with limited information can only be given variation intervals but not precise values. Some significant successes have also been achieved in the hybrid interval probability model. Based on the classical reliability theory and the interval analysis technique, the failure probabilistic intervals of structures with interval probability variables were evaluated by Qiu et al. [28]. By combining the Monte-Carlo simulation process into the interval analysis, the variational ranges of the system reliabilities of structures with interval probability variables have been investigated by Zhang et al. [29,30] and Hurtado [31]. To obtain the variational ranges of expectations and variances of responses of acoustic fields and structural–acoustic systems with interval probability variables, an interval random perturbation method has been developed by Xia et al. [32,33].

Though significant developments have been achieved in two hybrid uncertain models, some important issues remain unsolved. First, the unified numerical method which can be used for the response analysis of acoustic fields under two hybrid uncertain models is still unreported. The common element of two hybrid uncertain models is that probability variables and interval variables exist simultaneously. Based on this common element, we have reasons to believe that a unified numerical method can be developed for the response analysis of acoustic fields under two hybrid uncertain models. Second, in the earlier researches done by Xia et al. [27,32], only the variational ranges of expectations and variances of responses of acoustic fields under two hybrid uncertain models were obtained, while the response probability distributions of acoustic fields under two hybrid uncertain models are still not investigated. However, the response probability distribution plays an important role in the engineering practice. For example, if the response probability distribution is known, the relative likelihood of response at any value can be obtained. Furthermore, based on the response probability distribution, the cumulative distribution and the confidence interval of response can be yielded. Thus, it is desired to develop a new hybrid uncertain method for the unified response probability distribution analysis of acoustic fields under two hybrid uncertain models.

In this paper, a new numerical method named as the inverse mapping hybrid perturbation method (IMHPM) is proposed for the response probability distribution analysis of acoustic fields under two hybrid uncertain models. The main computing process of IMHPM is divided into two steps. The first step is the probability analysis in which the uncertainties of probability variables are considered, while the uncertainties of interval variables are temporarily neglected. The second step is the interval analysis in which the uncertainties of interval variables are considered. In the probability analysis, the responses of two hybrid uncertain acoustic fields are converted to the invertible functions of probability variables based on the Taylor series expansion. According to the inverse mapping relationships between responses and probability variables, the formal expressions of the response probability distributions are yielded by using the change-of-variable technique. In the interval analysis, the variational ranges of the response probability distributions are estimated by the interval perturbation technique. The effectiveness and efficiency of IMHPM for the unified response probability distribution analysis of two hybrid uncertain acoustic fields will be investigated by a numerical example.

2. Two hybrid uncertain acoustic fields

2.1. Hybrid uncertain acoustic field with both probability and interval variables

An acoustic field Ω is encircled by the boundary Γ is shown in Fig. 1. The boundary Γ is divided to three parts: the pressure boundary Γ_D, the normal velocity boundary Γ_N and the normal impedance boundary Γ_R.

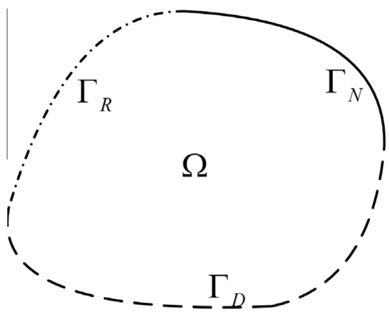

Fig. 1. An acoustic field model.

The steady-state acoustic pressure p in the acoustic field governed by the Helmholtz equation can be expressed as

$$\Delta p + (\omega/c)^2 p = 0 \tag{1}$$

where Δ is the Laplace operator; ω is the angular frequency; c is the sound speed.

The pressure boundary condition Γ_D, the normal velocity boundary condition Γ_N and the normal impedance boundary condition Γ_R can be expressed as

$$\begin{aligned} &p = p_D \text{ on } \Gamma_D \\ &\mathbf{n} \cdot \nabla p = -\mathrm{j}\rho\omega v_n \text{ on } \Gamma_N \\ &\mathbf{n} \cdot \nabla p = -\mathrm{j}\rho A_n p \text{ on } \Gamma_R \end{aligned} \tag{2}$$

where p_D is the sound pressure on the pressure boundary Γ_D; $\mathrm{j} = \sqrt{-1}$ is an imaginary unit; $\mathbf{n}$ is the exterior unitnormal vector; ρ is the density of the fluid in the acoustic field; v_n is the normal velocity applied on the normal velocity boundary Γ_N; A_n is the admittance coefficient that models the structural damping on the normal impedance boundary Γ_R.

Based on Eqs. (1) and (2), the finite element equation of the acoustic field can be expressed as

$$(\mathbf{K} - \omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C})\mathbf{p} = \mathbf{F} \tag{3}$$

where $\mathbf{K}$, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{F}$ stand for the acoustic system stiffness matrix, mass matrix, damping matrix and load vector, respectively. They can be expressed as

$$\begin{aligned} &\mathbf{K} = \sum_{i=1}^{N_{cell}} \int_{\Omega_i} (\nabla\mathbf{N})^{\mathrm{T}}(\nabla\mathbf{N})\mathrm{d}\Omega, \\ &\mathbf{M} = \sum_{i=1}^{N_{cell}} \frac{1}{c^2} \int_{\Omega_i} \mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Omega, \\ &\mathbf{C} = \sum_{i=1}^{N_{cell}} \rho A_n \int_{\Gamma_R^i} \mathbf{N}^{\mathrm{T}}\mathbf{N}\mathrm{d}\Gamma, \\ &\mathbf{F} = -\mathrm{j}\sum_{i=1}^{N_{cell}} \omega\rho \int_{\Gamma_N^i} \mathbf{N}^{\mathrm{T}} v_n \mathrm{d}\Gamma \end{aligned} \tag{4}$$

where the summation represents an assembly process of the system matrices and vectors; N_{cell} is the total number of elements; Ω_i is the ith element in the acoustic field; $\Gamma_R^i = \Gamma_R \cap \Omega_i$ is the normal impedance boundary Γ_R which is related with element Ω_i; $\Gamma_N^i = \Gamma_N \cap \Omega_i$ is the normal velocity boundary Γ_N which is related with the element Ω_i; $\mathbf{N}$ is the Lagrange shape function of the isoparametric element.

Assuming the acoustic dynamic stiffness matrix is $\mathbf{Z}$, which can be expressed as

$$\mathbf{Z} = \mathbf{K} - \omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} \tag{5}$$

then the finite element equation of the acoustic field can be rewritten as

$$\mathbf{Zp} = \mathbf{F} \tag{6}$$

Due to the influences of model inaccuracies, physical imperfections, multiphase characteristics of materials and unpredictability of environment, uncertainties existing in the acoustic field are unavoidable. The uncertain parameters, whose probability distributions are well determined, can be modeled as probability variables. The uncertain parameters, whose variational ranges are well defined but the probability distributions within variational ranges are lost, can be modeled as interval variables. The probability vector composed of probability variables is marked as $\mathbf{a} = \{a_1, a_2, \ldots, a_r, \ldots\}^{\mathrm{T}}$. $r = 1, 2, \ldots, l_1$. l_1 is the number of probability variables. The interval vector composed of interval variables is marked as $\mathbf{b} = \{b_1, b_2, \ldots, b_t, \ldots\}^{\mathrm{T}}$. $t = 1, 2, \ldots, l_2$. l_2 is the number of interval variables. The detailed definitions of the probability vector and the interval vector can be referred to Ref. [27].

The finite element equation of the acoustic field with both probability and interval variables can be rewritten as

$$\mathbf{Z}(\mathbf{a},\mathbf{b})\mathbf{p} = \mathbf{F}(\mathbf{a},\mathbf{b}) \tag{7}$$

where $\mathbf{Z}(\mathbf{a},\mathbf{b})$ is the hybrid probability and interval acoustic dynamic stiffness matrix; $\mathbf{F}(\mathbf{a},\mathbf{b})$ is the hybrid probability and interval load vector. The sound pressure vector $\mathbf{p}$ is also a hybrid probability and interval vector.

2.2. Hybrid uncertain acoustic field with interval probability variables

Assume that the uncertain parameters of acoustic field with limited information can be treated as the independent interval probability variables. The interval probability vector composed of interval probability variables is marked as $\mathbf{a}(\mathbf{b})$

$$\begin{aligned} &\mathbf{a}(\mathbf{b}) = \{a_1(\mathbf{b}), a_2(\mathbf{b}), \ldots, a_r(\mathbf{b}), \ldots\}^{\mathrm{T}}, \quad r = 1,2,\ldots,l_1 \\ &\mathbf{b} = \{b_1, b_2, \ldots, b_t, \ldots\}^{\mathrm{T}}, \quad t = 1,2,\ldots,l_2 \end{aligned} \tag{8}$$

where l_1 is the number of interval probability variables. $a_r(\mathbf{b})$ is the rth interval probability parameter. $\mathbf{b}$ is the interval vector associated with the interval probability vector $\mathbf{a}(\mathbf{b})$. The detailed definitions of the interval probability vector can be referred to Ref. [32].

As is defined above, the probability vector $\mathbf{a}$ and the interval vector $\mathbf{b}$ simultaneously exist in two hybrid uncertain models. However, in the hybrid probability and interval model, $\mathbf{a}$ and $\mathbf{b}$ are parallel. $\mathbf{a}$ is used to treat the uncertain parameters with sufficient information, while $\mathbf{b}$ is used to treat the uncertain parameters with limited information. As $\mathbf{a}$ and $\mathbf{b}$ in the hybrid probability and interval model are independent of each other, they can be marked as $(\mathbf{a},\mathbf{b})$. In the hybrid interval probability model, $\mathbf{b}$ is attached to $\mathbf{a}$. $\mathbf{a}$ is used to treat the uncertain parameters with limited information. $\mathbf{b}$ is used to treat the uncertain distribution parameters of $\mathbf{a}$. As $\mathbf{b}$ in the hybrid interval probability model is dependent of $\mathbf{a}$, they should be marked as $(\mathbf{a}(\mathbf{b}))$.

The finite element equation of the acoustic field with interval probability variables can be rewritten as

$$\mathbf{Z}(\mathbf{a}(\mathbf{b}))\mathbf{p} = \mathbf{F}(\mathbf{a}(\mathbf{b})) \tag{9}$$

where $\mathbf{Z}(\mathbf{a}(\mathbf{b}))$ is the interval probability acoustic dynamic stiffness matrix; $\mathbf{F}(\mathbf{a}(\mathbf{b}))$ is the interval probability load vector. The sound pressure vector $\mathbf{p}$ is also an interval probability vector.

3. IMHPM for two hybrid uncertain acoustic fields

3.1. Probability analysis of two hybrid uncertain acoustic fields

If the interval vector $\mathbf{b}$ is assumed as a constant vector, the finite element equations of two hybrid uncertain acoustic fields (shown in Eqs. (7) and (9)) are simplified to a probability finite element equation

$$\mathbf{Z}(\mathbf{a})\mathbf{p} = \mathbf{F}(\mathbf{a}) \tag{10}$$

The Taylor series expansion of the interval probabilistic response vector can be expressed as

$$\mathbf{p} = \mathbf{p}|_{\mathbf{a}=E(\mathbf{a})} + \sum_{r=1}^{l_1} \frac{\partial \mathbf{p}}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} (a_r - E(a_r)) + \frac{1}{2!}\sum_{r=1}^{l_1}\sum_{\gamma=1}^{l_1} \frac{\partial^2 \mathbf{p}^R}{\partial a_r \partial a_\gamma}\bigg|_{\mathbf{a}^R=E(\mathbf{a}^R)} (a_r - E(a_r))(a_\gamma - E(a_\gamma)) + \cdots \tag{11}$$

where $\mathbf{p}|_{\mathbf{a}=E(\mathbf{a})}$ is the value of $\mathbf{p}$ when the probability vector $\mathbf{a}$ has its expectation $E(\mathbf{a})$. $E(\mathbf{a}) = \{E(a_1), E(a_1), \ldots, E(a_r), \ldots\}$. $E(a_r)$ is the expectation of the probability variable a_r.

For most engineering problems, the improvement in accuracy obtained by using the higher order perturbation terms is rather small when compared with the increase of the computational complexity and the computational burden. Neglecting the higher order perturbation terms, Eq. (11) can be rewritten as

$$\begin{aligned}\mathbf{p} &= \mathbf{p}|_{\mathbf{a}=E(\mathbf{a})} + \sum_{r=1}^{l_1} \frac{\partial \mathbf{p}}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} (a_r - E(a_r)) = \mathbf{p}|_{\mathbf{a}=E(\mathbf{a})} - \sum_{r=1}^{l_1} \frac{\partial \mathbf{p}}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} E(a_r) + \sum_{r=1}^{l_1} \frac{\partial \mathbf{p}}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} a_r \\ &= \mathbf{p}_0 + \sum_{r=1}^{l_1} \partial \mathbf{p}'_r a_r \end{aligned} \tag{12}$$

where

$$\begin{aligned}\mathbf{p}_0 &= \mathbf{p}|_{\mathbf{a}=E(\mathbf{a})} - \sum_{r=1}^{l_1} \frac{\partial \mathbf{p}}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} E(a_r) = (\mathbf{Z}(E(\mathbf{a})))^{-1}\mathbf{F}(E(\mathbf{a})) \\ &\quad - \sum_{r=1}^{l_1} (\mathbf{Z}(E(\mathbf{a})))^{-1} \left(\frac{\partial \mathbf{F}(\mathbf{a})}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} - \frac{\partial \mathbf{Z}(\mathbf{a})}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} (\mathbf{Z}(E(\mathbf{a})))^{-1}\mathbf{F}(E(\mathbf{a})) \right) E(a_r) \\ \partial \mathbf{p}'_r &= \sum_{r=1}^{l_1} (\mathbf{Z}(E(\mathbf{a})))^{-1} \left(\frac{\partial \mathbf{F}(\mathbf{a})}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} - \frac{\partial \mathbf{Z}(\mathbf{a})}{\partial a_r}\bigg|_{\mathbf{a}=E(\mathbf{a})} (\mathbf{Z}(E(\mathbf{a})))^{-1}\mathbf{F}(E(\mathbf{a}^R)) \right)\end{aligned} \tag{13}$$

$\mathbf{p}$, $\mathbf{p}_0$ and $\mathbf{p}'_r$ are n-vectors. n is the degree of freedom of the hybrid uncertain acoustic fields. The kth element p_k in $\mathbf{p}$ can be expressed as

$$p_k = p_{0,k} + \sum_{r=1}^{l_1} p'_{r,k} a_r, \quad k = 1, 2, \ldots, n \tag{14}$$

where $p_{0,k}$ is the kth element in $\mathbf{p}_0$; $p'_{r,k}$ is the kth element in $\mathbf{p}'_r$.

The Monte Carlo method can be considered as a potential method to estimate the probability distribution of p_k. However, the probability distribution yielded by the Monte Carlo method is discrete. Compared with the discrete version of the probability distribution, the formal expression of the probability distribution of response plays a more important role in the design and optimization of acoustic field. For examples, if the formal expression of the probability distribution of response is yielded, the likelihood of response at any considered values can be obtained directly and the influence of each random variable on the probability density function of random response can be analyzed intuitively.

To obtain the formal expression of the probability distribution of p_k, the change-of-variable technique [5] will be introduced. In the probability theory, if the mapping relationships between dependent variables and independent variables are invertible and the probability distributions of independent variables are defined unambiguously, the probability distributions of dependent variables can be yielded by the change-of-variable technique. As is shown in Eq. (14), p_k is an invertible function whose inverse function can be obtained by simple mathematical operations. Thus, the change-of-variable technique may be an effective approach to calculate the formal expression of the probability distribution of p_k.

According to the number of random variables, three cases (the acoustic field with one random variable, the acoustic field with two random variables and the acoustic field with more than two random variables) will be discussed in detail.

Case 1: The acoustic field with one random variable.

For the acoustic field with one random variable, Eq. (14) can be rewritten as

$$p_k = p_{0,k} + p'_{1,k} a_1, \quad k = 1, 2, \ldots, n \tag{15}$$

The probability distribution function of p_k can be expressed as

$$f_{p_k}(p_k) = \frac{1}{|p'_{1,k}|} f_{a_1}\left(\frac{p_k - p_{0,k}}{p'_{1,k}}\right) \tag{16}$$

where $f_{p_k}(\cdot)$ is the probability distribution function of p_k. $f_{a_1}(\cdot)$ is the probability distribution function of a_1.

Case 2: The acoustic field with two random variables.

For the acoustic field with two random variables, Eq. (14) can be rewritten as

$$p_k = p_{0,k} + p'_{1,k}a_1 + p'_{2,k}a_2, \quad k = 1, 2, \ldots, n \tag{17}$$

The probability distribution function of p_k can be expressed as

$$f_{p_k}(p_k) = \frac{1}{|p'_{2,k}|}\int_{-\infty}^{\infty} f_{a_1}(a_1) f_{a_2^R}\left(\frac{p_k - p_{0,k} - p'_{1,k}a_1}{p'_{2,k}}\right)\mathrm{d}a_1 \tag{18}$$

where $f_{a_2}(\cdot)$ is the probability distribution function of a_2 .

Case 3: The acoustic field with more than two random variables.

Assume that $x_{1,k} = p'_{1,k}a_1 + p'_{2,k}a_2$. The probability distribution function of $x_{1,k}$ can be expressed as

$$f_{x_{1,k}}(x_{1,k}) = \frac{1}{|p'_{2,k}|}\int_{-\infty}^{\infty} f_{a_1}(a_1) f_{a_2}\left(\frac{x_{1,k} - p'_{1,k}a_1}{p'_{2,k}}\right)\mathrm{d}a_1 \tag{19}$$

where $f_{x_{1,k}}(\cdot)$ is the probability distribution function of $x_{1,k}$.

Next, assume that $x_{2,k} = x_{1,k} + p'_{3,k}a_3$. The probability distribution function of $x_{2,k}$ can be expressed as

$$f_{x_{2,k}}(x_{2,k}) = \int_{-\infty}^{\infty} f_{a_3}(a_3) f_{x_{1,k}}(x_{2,k} - p'_{3,k}a_3)\mathrm{d}a_3 \tag{20}$$

where $f_{x_{2,k}}(\cdot)$ is the probability distribution function of $x_{2,k}$; $f_{a_3}(\cdot)$ is the probability distribution function of a_3 .

With this analogy, the probability distribution function of $x_{l_1-2,k} = x_{l_1-3,k} + p'_{l_1-1,k}a_{l_1-1}$ can be expressed as

$$f_{x_{l_1-2,k}}(x_{l_1-2,k}) = \int_{-\infty}^{\infty} f_{a_{l_1-1}}(a_{l_1-1}) f_{x_{l_1-3,k}}(x_{l_1-2,k} - p'_{l_1-1,k}a_{l_1-1})\mathrm{d}a_{l_1-1} \tag{21}$$

where $f_{x_{l_1-3,k}}(\cdot)$ is the probability distribution function of $x_{l_1-3,k}$; $f_{x_{l_1-2,k}}(\cdot)$ is the probability distribution function of $x_{l_1-2,k}$; $f_{a_{l_1-1}}(\cdot)$ is the probability distribution function of a_{l_1-1}.

When the probability distribution function of $x_{l_1-2,k}$ is obtained, the unknown probability distribution function of p_k can be expressed as

$$f_{p_k}(p_k) = \int_{-\infty}^{\infty} f_{a_{l_1}}(a_{l_1}) f_{x_{l_1-2,k}}(p_k - p_{0,k} - p'_{l_1,k}a_{l_1})\mathrm{d}a_{l_1} \tag{22}$$

where $f_{a^R_{l_1}}(\cdot)$ is the probability distribution function of $a^R_{l_1}$.

From the derivative procedure mention above, we can obtain that the formal expression of the probability distribution of p_k can be yielded by the change-of-variable technique effectively. For two hybrid uncertain acoustic fields, interval variables actually exist in the acoustic field. Thus, the probability distribution function of p_k (namely, $f_{p_k}(p_k)$) is a function of the interval vector $\mathbf{b}$. The uncertainty of $f_{p_k}(p_k)$ rising from the interval vector $\mathbf{b}$ can be expressed as the variational range of $f_{p_k}(p_k)$.

3.2. Interval analysis of two hybrid uncertain acoustic fields

$f_{p_k}(p_k)$ is a function of the interval vector $\mathbf{b}$. To investigate the variational range of $f_{p_k}(p_k)$, the interval perturbation method will be introduced.

The first-order Taylor expansions of $f_{p_k}(p_k)$ at the mean values of interval variables b_t ($t = 1, 2, \ldots, l_2$) can be expressed as

$$f_{p_k}(p_k) = f^m_{p_k}(p_k) + \sum_{t=1}^{l_2} \left.\frac{\partial f_{p_k}(p_k)}{\partial b_t}\right|_{\mathbf{b}=\mathbf{b}^m} \Delta b_t e^I \tag{23}$$

where $f^m_{p_k}(p_k)$ is the value of $f_{p_k}(p_k)$ when the interval variables b_t ($t = 1, 2, \ldots, l_2$) has its mean values. $\mathbf{b}^m = \{b^m_1, b^m_2, \ldots, b^m_t, \ldots, b^m_{l_2}\}$ is the mean value of the interval vector $\mathbf{b}$. b^m_t is the mean value of b_t. Δb_t is the deviation radius of b_t. $e^I = [-1, 1]$.

Due to the complexity of the formal expression of the probability distribution of p_k, the first derivative of $f_{p_k}(p_k)$ related with the interval variable b_t may not be obtained directly. Thus, the central difference method which can be easily implemented in the engineering practice will be introduced to approximate the first derivative of $f_{p_k}(p_k)$:

$$\left.\frac{\partial f_{p_k}(p_k)}{\partial b_t}\right|_{\mathbf{b}=\mathbf{b}^m} = \frac{f_{p_k}(p_k(\bar{\mathbf{b}}_t)) - f_{p_k}(p_k(\underline{\mathbf{b}}_t))}{2\Delta b_t} \tag{24}$$

where

$$\begin{aligned} \bar{\mathbf{b}}_t &= \{b_1^m, b_2^m, \ldots, \bar{b}_t, \ldots, b_{l_2}^m\} \\ \underline{\mathbf{b}}_t &= \{b_1^m, b_2^m, \ldots, \underline{b}_t, \ldots, b_{l_2}^m\} \end{aligned} \tag{25}$$

where $\underline{b}_t$ and $\bar{b}_t$ are the lower and upper bounds of b_t.

Substituting Eq. (24) into Eq. (23), one gets

$$f_{p_k}(p_k) = f_{p_k}^m(p_k) + \sum_{t=1}^{l_2} \frac{f_{p_k}(p_k(\bar{\mathbf{b}}_t)) - f_{p_k}(p_k(\underline{\mathbf{b}}_t))}{2} e^I = f_{p_k}^m(p_k) + \Delta f_{p_k}(p_k) e^I \tag{26}$$

where $\Delta f_{p_k}(p_k)$ is the deviation radius of $f_{p_k}(p_k)$. $\Delta f_{p_k}(p_k)$ can be expressed as

$$\Delta f_{p_k}(p_k) = \sum_{t=1}^{l_2} \left| \frac{f_{p_k}(p_k(\bar{\mathbf{b}}_t)) - f_{p_k}(p_k(\underline{\mathbf{b}}_t))}{2} \right| \tag{27}$$

where $|\cdot|$ denotes the absolute value.

Based on the interval algorithm, the lower and upper bounds of $f_{p_k}(p_k)$ can be expressed as

$$\begin{aligned} f_{p_k}(p_k)_{\text{lower}} &= f_{p_k}^m(p_k) - \Delta f_{p_k}(p_k) \\ f_{p_k}(p_k)_{\text{upper}} &= f_{p_k}^m(p_k) + \Delta f_{p_k}(p_k) \end{aligned} \tag{28}$$

It should be noted that the cumulative probability of the lower bound of $f_{p_k}(p_k)$ in the definition domain is less than 1, while the cumulative probability of the upper bound of $f_{p_k}(p_k)$ in the definition domain is larger than 1. Thus, the lower and upper bounds of $f_{p_k}(p_k)$ are not a standardized probability distribution. The lower and upper bounds of $f_{p_k}(p_k)$ are used to determine an interval in which all potential response probability distributions of two hybrid uncertain acoustic fields will be included.

The main steps of IMHPM to predict the variational ranges of the response probability distributions of two hybrid uncertain acoustic fields can be summed as:

Step 1: Calculation of $\mathbf{p}_0$ and $\mathbf{p}'_r$ shown in Eq. (13).
Step 2: Calculation of the probability distribution function of p_k (namely $f_{p_k}(p_k)$) according to Eqs. (16) and (18)–(22).
Step 3: Calculation of deviation radius of $f_{p_k}(p_k)$ (namely $\Delta f_{p_k}(p_k)$) according to Eq. (27).
Step 4: Calculation of the lower and upper bounds of $f_{p_k}(p_k)$ (namely $f_{p_k}(p_k)_{\text{lower}}$ and $f_{p_k}(p_k)_{\text{upper}}$) according to Eq. (28).

4. Numerical examples

U-shaped tube is widely used in air-conditioning condenser cooling equipments. The prediction of the sound pressure response of the U-shaped tube plays an important in role in the noise vibration harshness (NVH) optimization of air-conditioning condenser cooling equipments. Fig. 2 depicts a U-shaped tube model. A discontinuous normal velocity $v_n = v_{n0}e^{\mathrm{j}2\pi\lambda t}$ is imposed on the left top side of the tube cavity. v_{n0} and λ are the amplitude and the frequency of the discontinuous normal velocity v_n. The remaining sides are perfectly rigid. This U-shaped tube is divided into the finite element model. The size of hexahedron element is $0.0125\ \mathrm{m} \times 0.0125\ \mathrm{m} \times 0.0125\ \mathrm{m}$.

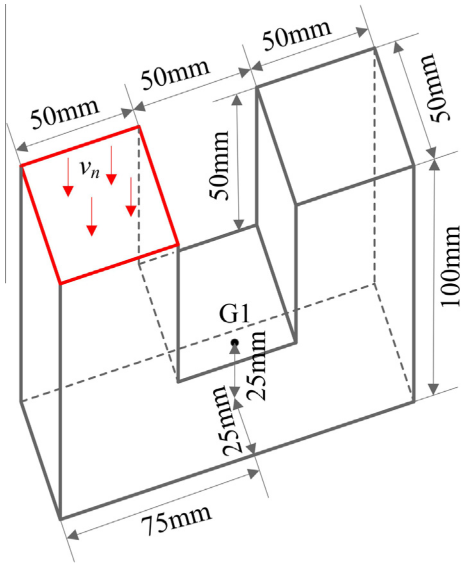

Fig. 2. A U-shaped tube model.

This U-shaped tube model is surrounded by air. Considering the unpredictability of the environment temperature, the air density and the sound speed are considered as interval variables. The variational range of the air density ρ is $[1.204\ \mathrm{kg/m^3}, 1.225\ \mathrm{kg/m^3}]$. The variational range of the sound speed c is $[340.5\ \mathrm{m/s}, 343.4\ \mathrm{m/s}]$. The amplitude v_{n0} of the discontinuous normal velocity v_n is assumed as the probability variable which follows the Gumbel distribution. The Gumbel distribution is a particular case of the generalized extreme value distribution which can be used to model the distribution of the maximum (or the minimum) of the number of samples of various distributions. Distribution parameters of the amplitude v_{n0} are $\mu_{v_{n0}} = 0.15\ \mathrm{m/s}$ and $\beta_{v_{n0}} = 0.05\ \mathrm{m/s}$. The frequency λ of the discontinuous normal velocity v_n is assumed as the probability variable which follows the Gaussian distribution. Four types of distribution parameters of the frequency λ are presented in Table 1.

As the interval variables (ρ and c) and the probability variables (v_{n0} and λ) exist simultaneously, this U-shaped tube model belongs to the first type of hybrid uncertain acoustic field. Simulations of this U-shaped tube model are carried out by Matlab 7.0 on a 3.39 GHz Intel(R) Core(TM) i7-4770.

The lower and upper bounds of the response probability distribution of node G1 (marked in Fig. 2) yielded by IMHPM are plotted in Figs. 3–6. The types of distribution parameters of the frequency λ in Figs. 3–6 are Type I, Type II, Type III and Type IV, respectively. The Monte Carlo method will be introduced to calculate the potential response probability distribution of node G1. In the implementation of the Monte Carlo method, the sample number of interval variables is 100 and the sample number of random variables is 100,000. For each combination of interval variables, a potential response probability distribution can be obtained by the Monte Carlo method. Thus, there are 100 potential response probability distributions for each considered type of distribution parameters of the frequency λ. These potential response probability distributions are also plotted in Figs. 3–6. In Figs. 3–6 and the following context, the results obtained by IMHPM and the Monte Carlo method are marked as IMHPM and MCM, respectively.

Table 1
Four types of distribution parameters of the frequency λ.

Type	Expectation (μ_λ) (Hz)	Standard variance (σ_λ) (Hz)
I	100	5
II	200	10
III	300	15
IV	450	22.5

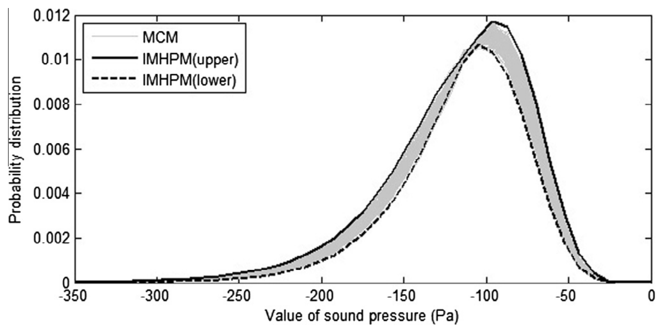

Fig. 3. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type I of distribution parameters of the frequency λ.

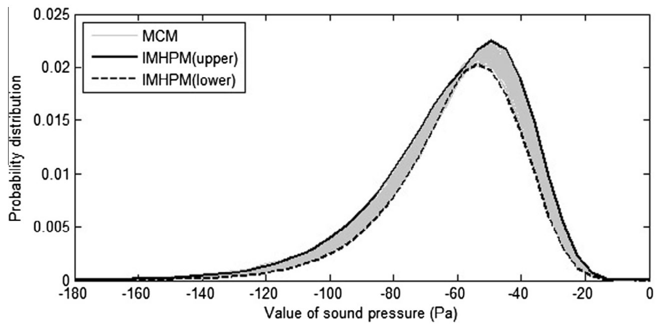

Fig. 4. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type II of distribution parameters of the frequency λ.

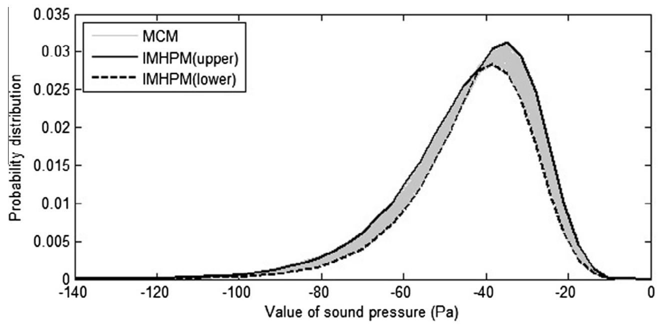

Fig. 5. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type III of distribution parameters of the frequency λ.

Figs. 3–6 show that the potential response probability distributions yielded by the Monte Carlo method are included in the interval constrained by the lower and upper bounds of the response probability distributions obtained by IMHPM. Thus, the lower and upper bounds of probability distribution yielded by IMHPM can be used to determine an interval in which the potential response probability distributions of the hybrid uncertain acoustic field with both probability and interval variables are included.

The effectiveness of IMHPM for the response probability distribution analysis of the second type of the hybrid uncertain acoustic field will be investigated. The air density ρ, the sound speed c and the amplitude v_{n0} of discontinuous normal velocity v_n are assumed as the interval probability variables. The distributions of these interval probability variables are listed in Table 2.

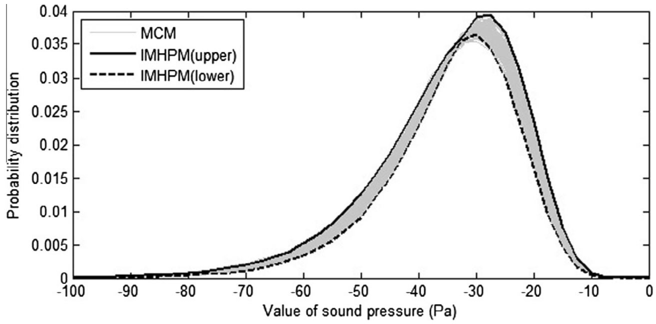

Fig. 6. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type IV of distribution parameters of the frequency λ.

Table 2
Distributions of interval probability variables for the U-shaped tube.

Interval random variable	Distribution type	Probability distribution parameters	
c (m/s)	Gaussian distribution	$\mu_c = [340.5, 343.4]$	$\sigma_c = [4.9, 5.1]$
ρ (kg/m^3)	Gaussian distribution	$\mu_\rho = [1.204,\ 1.225]$	$\sigma_\rho = [0.0198, 0.0202]$
v_{n0} (m/s)	Gumbel distribution	$\mu_{v_{n0}} = [0.14, 0.16]$	$\beta_{v_{n0}} = [0.045, 0.055]$

Table 3
Four types of distribution parameters of the frequency λ.

Type	Expectation (μ_λ) (Hz)	Standard variance (σ_λ)
I	100	[4.95 Hz, 5.05 Hz]
II	200	[9.9 Hz, 10.1 Hz]
III	300	[14.85 Hz, 15.15 Hz]
IV	450	[22.275 Hz, 22.725 Hz]

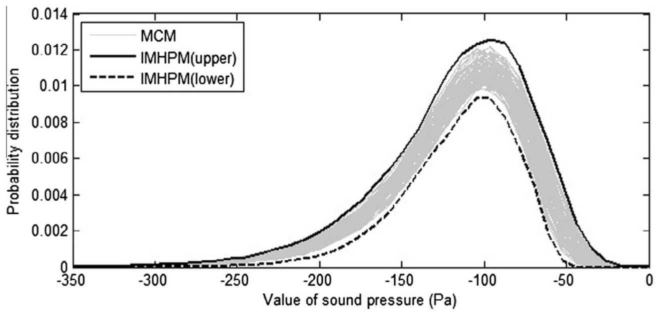

Fig. 7. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type I of distribution parameters of the frequency λ.

The frequency λ of the discontinuous normal velocity v_n is also assumed as the interval Gaussian random variable. Four types of distribution parameters of the frequency λ are presented in Table 3.

The lower and upper bounds of the response probability distribution of node G1 yielded by IMHPM are plotted in Figs. 7–10. The potential response probability distribution of node G1 yielded by the Monte Carlo method are also plotted in Figs. 7–10. The types of distribution parameters of the frequency λ in Figs. 7–10 are Type I, Type II, Type III and Type IV, respectively. In the implementation of the Monte Carlo method, the sample number of interval variables is 100 and the sample number of random variables is 100,000. Figs. 7–10

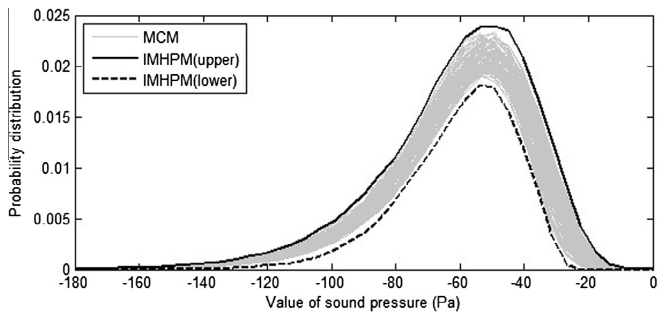

Fig. 8. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type II of distribution parameters of the frequency λ.

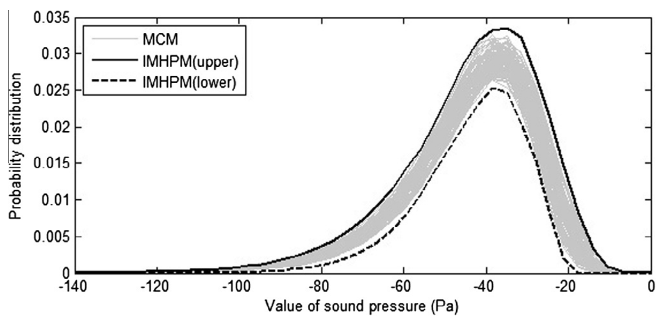

Fig. 9. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type III of distribution parameters of the frequency λ.

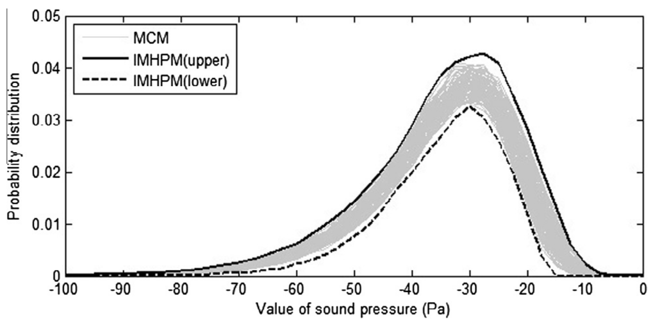

Fig. 10. The lower and upper bounds of the response probability distribution of node G1 calculated by IMHPM and the potential response probability distribution of node G1 yielded by the Monte Carlo method for Type IV of distribution parameters of the frequency λ.

show that the potential response probability distributions yielded by the Monte Carlo method are included in the interval constrained by the lower and upper bounds of the response probability distributions obtained by IMHPM. Thus, the lower and upper bounds of the response probability distribution yielded by IMHPM can be used to determine an interval in which the potential response probability distributions of the hybrid uncertain acoustic field with interval probability variables are included.

Computational efficiency is an important factor to evaluate the performances of numerical methods. The computational time of the Monte Carlo method (with 10^7 samples) and IMHPM for the response probability distribution analysis of node G1 for Type I of distribution parameters of the frequency λ is listed in Table 4. As is shown in Table 4, the computational cost of IMHPM for the response probability distribution analysis of two hybrid uncertain acoustic fields is much less than that of the Monte Carlo method. In other words, the

Table 4
The computational cost of the Monte Carlo method and IMHPM for the response probability distribution analysis of two hybrid uncertain acoustic fields.

	MCM (s)	IMHPM (s)
The first type	123,343	0.5
The second type	123,468	70.8

computational efficiency of IMHPM for the response probability distribution analysis of two hybrid acoustic fields is much higher than that of the Monte Carlo method. The computational cost of the Monte Carlo method for the response probability distribution analysis of two hybrid uncertain acoustic fields is approximately the same. The reason is that the number of samples used by the Monte Carlo method for two hybrid uncertain acoustic fields is the same. The computational cost of IMHPM for the hybrid interval probability acoustic field is larger than that of IMHPM for the hybrid probability and interval acoustic field. The reason will be discussed in the following.

For the hybrid probability and interval acoustic field, the numbers of both probability and interval variables are 2. For the hybrid interval probability acoustic field, the numbers of probability and interval variables are 4 and 7, respectively. Based on Eqs. (17) and (18), the probability of the sound pressure p_k of the hybrid probability and interval acoustic field can be expressed as

$$f_{p_k}(p_k) = \frac{1}{|p'_{v_{n0},k}|} \int_{-\infty}^{\infty} f_\lambda(\lambda) f_{v_{n0}}\left(\frac{p_k - p_{0,k} - p'_{\lambda,k}\lambda}{p'_{v_{n0},k}}\right) \mathrm{d}\lambda \tag{29}$$

where $f_\lambda(\cdot)$ is the probability distribution function of the frequency λ; $f_{v_{n0}}(\cdot)$ is the probability distribution function of the amplitude v_{n0}.

Based on Eqs. (19)–(22), the probability of sound pressure p_k of the hybrid interval probability acoustic field can be expressed as

$$f_{p_k}(p_k) = \frac{1}{|p'_{v_{n0},k}|} \int_{-\infty}^{\infty} f_\rho(\rho) \int_{-\infty}^{\infty} f_c(c) \int_{-\infty}^{\infty} f_\lambda(\lambda) f_{v_{n0}}\left(\frac{p_k - p_{0,k} - p'_{\rho,k}\rho - p'_{c,k}c - p'_{\lambda,k}\lambda}{p'_{v_{n0},k}}\right) \mathrm{d}\lambda \mathrm{d}c \mathrm{d}\rho \tag{30}$$

where $f_\rho(\cdot)$ is the probability distribution function of the density ρ; $f_c(\cdot)$ is the probability distribution function of the sound speed c.

It is obvious that the computational complexity and cost of Eq. (30) are much larger than those of Eq. (29). Furthermore, we can obtain from Eqs. (27) and (28) that when evaluating the lower and upper bounds of the response probability distribution, $f_{p_k}(p_k)$, which should be calculated, is related with the number of interval variables. When the number of interval variables is n, the number of $f_{p_k}(p_k)$ which should be calculated is $2n+1$. Thus, for the hybrid probability and interval acoustic field with 2 interval variables, the number of $f_{p_k}(p_k)$ which should be calculated is $2 \times 2 + 1 = 5$. For the hybrid interval probability acoustic field with 7 interval variables, the number of $f_{p_k}(p_k)$ which should be calculated is $2 \times 7 + 1 = 15$. The larger the number of $f_{p_k}(p_k)$ should be calculated, the higher the computational cost would be. Based on the discussions above, we can obtain that the computational cost of IMHPM for the hybrid interval probability acoustic field is larger than that of IMHPM for the hybrid probability and interval acoustic field.

5. Conclusion

For the response probability distribution analysis of the acoustic field with limited information, two types of hybrid uncertain models have been introduced. The first type of hybrid uncertain model is the hybrid probabilistic and interval model. In the hybrid probabilistic and interval model, the uncertain parameters with and without sufficient information to construct the corresponding probability distributions are treated as probabilistic variables and interval variables, respectively. The second type of hybrid uncertain model is the interval probabilistic model. In the interval probabilistic model, the uncertain parameters are treated as probabilistic variables whose distribution parameters may be treated as interval variables. The common element of two hybrid uncertain models is that probability variables and interval variables exist simultaneously. Based on this common element, an inverse mapping hybrid perturbation method (IMHPM) is proposed for the unified response probability distribution analysis of two hybrid uncertain acoustic fields. In the proposed method, the responses of two hybrid uncertain acoustic fields are converted to invertible functions of probability variables based on the stochastic perturbation analysis. And then, the change-of-variable technique is introduced to calculate the response probability distributions. Last, the variational ranges of the response probability distributions are estimated by the interval perturbation technique. Numerical results on a U-shaped tube verify that the lower and upper bounds of the response probability distribution yielded by IMHPM can be used to determine an interval in which the potential response probability distributions of the acoustic field under either of two hybrid uncertain models are included. Furthermore, the numerical results also show that the computational efficiency of IMHPM is very high, when compared with the Monte-Carlo method. Hence, IMHPM can be considered as an effective and efficient engineering method to quantify the effects of parametric uncertainties on the sound pressure response of two hybrid uncertain acoustic fields.

IMHPM, which is proposed in this paper for evaluating the variational ranges of the response probability distributions of hybrid uncertain acoustic fields, can be extended to the response probability distribution analysis of the hybrid uncertain static structures, the hybrid uncertain dynamic structures, the hybrid uncertain structural–acoustic systems, the hybrid uncertain heat transfer problems, etc. As the finite element method is limited to the low-frequency response analysis of acoustic field, the proposed IMHPM is also limited to the low-frequency response analysis of two hybrid uncertain acoustic fields. The mid- and high-frequency response analysis of hybrid uncertain acoustic fields is a problem which is worth to research further.

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