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A posteriori error estimates for the stabilization of low-order mixed finite elements for the Stokes problem☆

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Abstract

This paper studies a posteriori error estimates for the stabilization of low-order mixed finite elements for the Stokes problem. An interesting property of stabilized P_1/P_0 and P_1/P_1 finite element methods is proposed and used to construct new reliable and efficient error estimators. Moreover, an average technique is applied to improve the error estimators. Numerical results verify the theoretical results and show the improvement of such an average technique.

Keywords: Stabilized finite element method; A posteriori error estimates; Incompressible Stokes equations; Adaptive method

1. Introduction

It is well known that the low-order mixed finite element P_1/P_0 (linear velocity, constant pressure) and P_1/P_1 (linear velocity and pressure) pairs do not satisfy the inf–sup condition (see, e.g., [1]). Since the low-order pairs remain a popular practical choice in mixed finite element approximation of incompressible models, several stabilized finite element methods have been developed in last two decades (see, e.g., [2–8]). The stabilized methods aim to relax the continuity equation so as to allow application of unstable pairs by adding extra stabilization terms. Bochev and his co-workers [4] pointed out that the unstable pairs satisfy the weaker form of the discrete inf–sup condition, and terms

$$-\left(\sum_{e\in\mathcal{E}_h} h_e\|[p_h]\|_e^2\right)^{\frac{1}{2}} \quad \text{and} \quad -\sum_{T\in\tau_h} h_T\|\nabla p_h\|_T \tag{1}$$

reflect the inf–sup 'deficiency' of unstable P_1/P_0 and P_1/P_1 pairs, respectively. Stabilized methods introduce stabilization terms to counterbalance these key terms. In this paper, we propose an interesting property of the stabilized methods, that is, these key terms in (1) can be bounded by true errors. The observation provides useful arguments in a posteriori error estimates for the stabilization of low-order mixed finite element elements for the Stokes problem.

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It is of practical importance to devise reliable and efficient a posteriori error estimators in order to control approximation error and perform adaptive mesh refinement (see, e.g., [9,10]). Many works have been devoted to construct a posteriori error estimators for the Stokes problem (see, e.g., [11–13]). For stabilized P_1/P_0 finite element methods, residual-based a posteriori error estimators have been studied by Kay and Silvester [14] and J. Wang, Y. Wang, and Ye [15] for the penalizing jump method and by Zheng, Hou and Shi [16] for the projection method; recovery-based a posteriori error estimators have been proposed by Song, Hou and Cai [17] for both stabilized methods. Moreover, a posteriori error estimates for low-order nonconforming finite element methods have also been developed e.g. in [18–20]. Although nonconforming finite element methods usually result in a much larger discrete system than the conforming methods, they easily fulfill the inf–sup condition and have advantage on parallel computers (see, e.g., [21]). In this paper, we use the terms reflecting the inf–sup 'deficiency' to construct error estimators for stabilized P_1/P_0 and P_1/P_1 finite element methods. We prove their reliability and efficiency without any regularity assumption of true solution. The notable difference of our estimates from existing ones lies in using the property of stabilized methods.

Moreover, we use an average technique to improve the error estimators. The average technique is a post-processing method that reconstructs numerical approximations to achieve better results. Carstensen [22] studied the averaging error estimators and made some remarks on the history and future of averaging techniques in a posteriori finite element error analysis. Differing from his idea, we use the average technique to post process estimators rather than to construct error estimators. This approach opens up a possibility for improving error estimators for low-order finite elements. That is, some given classical error estimators can be improved by such post-processing. Numerical results verify the improvements.

The paper is organized as follows. Section 2 reviews stabilized low-order mixed finite element methods. The interesting property of stabilized methods is proposed in Section 3. Section 4 presents the error estimators based on the property of stabilized methods and establishes a posteriori error estimates to show their reliability and efficiency. Section 5 uses an average technique to improve the error estimators. Numerical results are reported in Section 6. They verify the theoretical results and show the practical effectivity of error estimators.

2. Stabilized low-order mixed finite element methods

As a simple model problem exhibits the main features for our arguments, we consider the Stokes problem with homogeneous Dirichlet boundary conditions in a two-dimensional polygonal domain Ω. For given $\boldsymbol{f} \in L^2(\Omega)^2$, the problem seeks the velocity $\boldsymbol{u}$ and the pressure p satisfying

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Throughout the paper, we employ the standard notations $H^l(\Omega)$ and $\|\cdot\|_l l \geq 0$ for the Sobolev space and associated standard norm, respectively. In general, $\|\cdot\|_D$ and $(\cdot,\cdot)_D$ denote the L^2 norm and inner product associated with any domain D. When $D = \Omega$, we drop the index D from the norm and inner product designation.

The standard weak form of the Stokes problem (2) reads: find $(\boldsymbol{u}, p) \in \boldsymbol{X} \times M$ such that

$$\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = \boldsymbol{f}(\boldsymbol{v}) \quad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{X} \times M, \tag{3}$$

where the bilinear and linear forms are defined by

$$\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \operatorname{div} \boldsymbol{v}) + (q, \operatorname{div} \boldsymbol{u}) \quad \text{and} \quad \boldsymbol{f}(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}),$$

and

$$\boldsymbol{X} := H_0^1(\Omega)^2 \quad \text{and} \quad M := L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

Furthermore, the bilinear form $\mathcal{L}$ satisfies the inf–sup condition: there exists a positive constant β such that

$$\inf_{(\boldsymbol{u},p)\in \boldsymbol{X}\times M} \sup_{(\boldsymbol{v},q)\in \boldsymbol{X}\times M} \frac{\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q))}{|||(\boldsymbol{v}, q)|||\; |||(\boldsymbol{u}, p)|||} \geq \beta, \tag{4}$$

where the energy norm $\|\cdot\|$ is defined by $\|(\boldsymbol{v}, q)\| = (\|\boldsymbol{v}\|_1^2 + \|q\|^2)^{\frac{1}{2}}$. This ensures the unique solvability of (3); see [1].

We consider the mixed finite element method for (3). Let $\tau_h = \{T\}$ be a family of triangulation of the domain Ω with the mesh parameter $h = \max_{T\in\tau_h} \operatorname{diam}(T)$. Denote by $\mathcal{E}_h$ the set of all edges in τ_h lying inside Ω. For any piecewise constant q, let

$$[q]_e := q|_{T_e^+} - q|_{T_e^-}$$

denote its jump on the edge $e \in \mathcal{E}_h$, where T_e^+ and T_e^- are two elements sharing the common edge e. For any piecewise constant tensor $\boldsymbol{\sigma}$, let

$$[\boldsymbol{\sigma}\cdot\boldsymbol{n}_e] := \boldsymbol{\sigma}|_{T_e^+}\cdot\boldsymbol{n}_e - \boldsymbol{\sigma}|_{T_e^-}\cdot\boldsymbol{n}_e$$

denote the jump of the normal component of $\boldsymbol{\sigma}$ on the edge e. Denote by h_e the length of a given edge $e \in \mathcal{E}_h$. The edge norm

$$\|u\|_{\mathcal{E}_h} := \left(\sum_{e\in\mathcal{E}_h} h_e\|u\|_e^2\right)^{\frac{1}{2}}$$

will prove useful in what follows.

Let us denote by

$$R_{1,h} := \{v_h \in C^0(\Omega) : v_h|_T \in P_1(T) \quad \forall T \in \tau_h\}$$

the piecewise linear finite element space, where $P_1(T)$ is a space of linear polynomials on element T. Moreover, we introduce the piecewise constant finite element space

$$R_{0,h} := \{q_h \in L^2(\Omega) :\ q_h|_T \in P_0(T) \quad \forall T \in \tau_h\},$$

where $P_0(T)$ is a constant polynomial space on element T.

We recall the classical trace inequality for finite element functions. There exists $c > 0$ independent of the mesh size such that

$$\|q_h\|_{\partial T} \le c\, h_T^{-\frac{1}{2}}\|q_h\|_T \quad \forall q_h \in P_1(T). \tag{5}$$

For simplicity, symbols $c,\ c_1,\ c_2$ here and in the rest may represent different quantities at different occurrences, but they are always independent of the mesh size.

In this paper, we consider the lowest order conforming pair

$$\boldsymbol{X}_h = \boldsymbol{X} \cap R_{1,h}^2 \quad \text{and} \quad M_h = M \cap R_{0,h}, \tag{6}$$

and the lowest equal order C^0 pair

$$\boldsymbol{X}_h = \boldsymbol{X} \cap R_{1,h}^2 \quad \text{and} \quad M_h = M \cap R_{0,h}. \tag{7}$$

It is well known that the velocity–pressure pair $(\boldsymbol{X}_h, M_h)$ does not satisfy the discrete inf–sup condition. However, the unstable velocity–pressure pairs satisfy the following weaker form of the discrete inf–sup condition; See [4].

Let $\boldsymbol{X}_h$ and M_h be the spaces defined in (6). Then, there exist positive constants c_1 and c_2 independent of the mesh size such that

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h} \frac{\int_\Omega p_h\nabla\cdot\boldsymbol{v}_h\,\mathrm{d}x}{\|\boldsymbol{v}_h\|_1} \ge c_1\|p_h\| - c_2\|[p_h]_e\|_{\mathcal{E}_h} \quad \forall p_h \in M_h.$$

Let $\boldsymbol{X}_h$ and M_h be the spaces defined in (7). Then, there exist positive constants c_1 and c_2 independent of the mesh size such that

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{X}_h} \frac{\int_\Omega p_h\nabla\cdot\boldsymbol{v}_h\,\mathrm{d}x}{\|\boldsymbol{v}_h\|_1} \ge c_1\|p_h\| - c_2\sum_{T\in\tau_h} h_T\|\nabla p_h\|_T \quad \forall p_h \in M_h.$$

The terms

$$-\,\|[p_h]_e\|_{\mathcal{E}_h} \quad \text{and} \quad -\sum_{T\in\tau_h} h_T\|\nabla p_h\|_T \tag{8}$$

reflect the inf–sup 'deficiency' of the unstable pairs (6) and (7), respectively. In some sense, stabilized methods aim to introduce additional terms to counterbalance terms in (8).

The stabilized low-order mixed finite element methods can be regularized as following formulation.

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times M_h$ which satisfy

$$\begin{aligned} &(\nabla\boldsymbol{u}_h, \nabla\boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h,\\ &(\nabla\cdot\boldsymbol{u}_h, q_h) + S(p_h, q_h) = 0 \qquad\qquad \forall q_h \in M_h. \end{aligned} \tag{9}$$

The stabilization term $S(p_h, q_h)$ is added to the continuity equation to help offset the inf–sup 'deficiency' of the unstable pairs.

The stabilized formulation (9) can be rewritten as

$$\mathcal{L}((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) + S(p_h, q_h) = \boldsymbol{f}(\boldsymbol{v}_h) \quad \forall\, (\boldsymbol{v}_h, q) \in \boldsymbol{X}_h \times M_h. \tag{10}$$

Some typical choices of the stabilization term are defined as follows.

2.1. *Direct stabilized methods*

The idea here is to directly use the terms in (8) to offset the inf–sup 'deficiency' of the unstable pair; see [7,8,14].

For P_1/P_0 approximation, Hughes and Franca [7] stabilized the approximation by penalizing jumps in pressure across internal interelement edges. The stabilization term is defined by

$$S(p_h, q_h) = \beta_0 \sum_{e\in\mathcal{E}_h} h_e([p_h], [q_h])_e \quad \forall q_h \in M_h. \tag{11}$$

The stabilization parameter β_0 must be chosen carefully, for example, the incompressibility will be destroyed if β_0 is too large.

For P_1/P_1 approximation, Brezzi and Pitkaranta [8] directly used the second term in (8) to obtain the stabilized method. The stabilization term is defined by

$$S(p_h, q_h) = c_0 \sum_{T\in\tau_h} h_T^2(\nabla p_h, \nabla q_h) \quad \forall q_h \in M_h. \tag{12}$$

The stabilization parameter c_0 also needs be chosen carefully.

2.2. *Projection stabilized method*

The method uses the terms that characterize the LBB 'deficiency' of the unstable spaces to stabilize the approximation, see [4].

The stabilization term in (10) is given by

$$S(p_h, q_h) = (p_h - \Pi p_h, q_h - \Pi q_h) \quad \forall q_h \in M_h. \tag{13}$$

The operator Π is defined as follows:

$$\Pi = \begin{cases} \Pi_1, & \text{if } M_h \text{ is defined by (6)},\\ \Pi_0, & \text{if } M_h \text{ is defined by (7)}, \end{cases}$$

and

$$\Pi_0 : L^2(\Omega) \to R_{0,h} \quad \text{and} \quad \Pi_1 : L^2(\Omega) \to R_{1,h},$$

which satisfy the following assumptions:

$$\|q_h - \Pi q_h\| \le c\, h\, \|q_h\|_1 \quad \forall\, q_h \in R_{1,h}, \tag{14}$$
$$\|p - \Pi p_h\| \le c\, \|p - p_h\|, \tag{15}$$

where p is the true pressure solution of (3) and p_h is the finite element approximation solution of (10).

In particular,

$$c\, \|[q_h]_e\|_{\mathcal{E}_h} \le \|q_h - \Pi_1 q_h\| \le C\, \|[q_h]_e\|_{\mathcal{E}_h} \quad \forall\, q_h \in R_{0,h}. \tag{16}$$

Since the definition of operator Π_1 in [4] is in the same fashion of the average operator $A(\cdot)$ in (26), using Lemma 5.1 gives the proof of (16).

3. Property of stabilized low-order mixed finite element methods

In this section, we present two theorems to show the property of stabilized methods. The property is the key terms

$$\|[p_h]_e\|_{\mathcal{E}_h} \quad \text{and} \quad \sum_{T\in\tau_h} h_T \|\nabla p_h\|_T$$

in (8) can be bounded by true errors in the energy norm.

For P_1/P_0 approximation, we have the following theorem.

Theorem 3.1. *Let $\boldsymbol{X}_h$ and M_h be the spaces defined in (6), $(\boldsymbol{u}, p)$ be the solution of problem (3) and $(\boldsymbol{u}_h, p_h)$ be the finite element approximation solution of problem (10) with stabilization term (11) or (13). There exists a positive constant c independent of the mesh size such that*

$$\|[p_h]_e\|_{\mathcal{E}_h} \le c\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|.$$

Proof. Refer to Lemmas 3.4 and 3.5 in [17] for details. □

For P_1/P_1 approximation, we first analyze the error between true solutions and approximation solutions of stabilized methods in L^2 norm.

Lemma 3.1. *Let $\boldsymbol{X}_h$ and M_h be the spaces defined in (7), $(\boldsymbol{u}, p)$ be the solution of problem (3) and $(\boldsymbol{u}_h, p_h)$ be the finite element approximation solution of problem (10) with stabilization term (12) or (13). There exists a positive constant c independent of the mesh size such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| + \|p - p_h\|_{-1} \le ch\, (\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|p - p_h\|).$$

Proof. We use the Aubin–Nitsche duality argument to estimate.

For $(\varphi, \phi) \in L^2(\Omega)^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$, consider the dual Stokes problem: seek $(\Phi, \Psi) \in (H^2(\Omega)^2 \cap H_0^1(\Omega)^2) \times (H^1(\Omega) \cap L_0^2(\Omega))$ such that

$$\mathcal{L}((\boldsymbol{v}, q), (\Phi, \Psi)) = (\boldsymbol{v}, \varphi) + (q, \phi) \quad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{X} \times M. \tag{17}$$

This problem admits a unique solution (Φ, Ψ) satisfying

$$\|\Phi\|_2 + \|\Psi\|_1 \le c\, (\|\varphi\| + \|\phi\|_1). \tag{18}$$

Moreover, there exist $\Phi_h \in \boldsymbol{X}_h$ and $\Psi_h \in M_h$ such that

$$\|\Phi - \Phi_h\|_1 \le c\, h\, \|\Phi\|_2 \quad \text{and} \quad \|\Psi - \Psi_h\| \le c\, h\, \|\Psi\|_1. \tag{19}$$

Taking $(\boldsymbol{v}, q) = (\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)$ in (17) yields

$$\mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\Phi, \Psi)) = (\boldsymbol{u} - \boldsymbol{u}_h, \varphi) + (p - p_h, \phi). \tag{20}$$

In addition, subtracting (10) from (3) gives

$$\mathcal{L}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\boldsymbol{v}_h, q_h)) - S(p_h, q_h) = 0 \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{X}_h \times M_h.$$

Taking $(\boldsymbol{v}_h, q_h) = (\Phi_h, \Psi_h)$ yields

$$\mathcal{L}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\Phi_h, \Psi_h)) - S(p_h, \Psi_h) = 0,$$

which together with (20) gives

$$(\varphi, \boldsymbol{u}-\boldsymbol{u}_h) + (\phi, p-p_h) = \mathcal{L}((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h),(\Phi-\Phi_h, \Psi-\Psi_h)) + S(p_h, \Psi_h). \tag{21}$$

Firstly, we estimate the term $S(p_h, \Psi_h)$.

If the stabilization term is defined by (13), then using the property of Π_0 in (14) and (15) gives

$$S(p_h, \Psi_h) = ((I-\Pi_0)p_h, (I-\Pi_0)\Psi_h) \le c\, \|p-p_h\| h \|\Psi_h\|_1 \le c\, h\|p-p_h\|\, \|\Psi\|_1,$$

where we use

$$\|\Psi_h\|_1 \le \|\Psi-\Psi_h\|_1 + \|\Psi\|_1 \le c\, h^{-1}\|\Psi-\Psi_h\|_0 + \|\Psi\|_1 \le c\, \|\Psi\|_1.$$

If the stabilization term is defined by (12), then using the inverse inequality and the property of Π_0 in (15) gives

$$\begin{aligned} S(p_h, \Psi_h) = c_0 \sum_{T\in\tau_h} h_T^2(\nabla p_h, \nabla\Psi_h) &\le c\, h^2\, \|p_h - \Pi_0 p_h\|_1 \|\Psi_h\|_1 \\ &\le c\, h\, \|p_h - \Pi_0 p_h\|\, \|\Psi\|_1 \\ &\le c\, h\, \|p-p_h\|\, \|\Psi\|_1. \end{aligned}$$

Therefore, the inequality

$$S(p_h, \Psi_h) \le c\, h\, \|p-p_h\|\, \|\Psi\|_1 \tag{22}$$

holds for all the stabilized methods in above subsection.

Combining (22) with (21) and using the continuity of $\mathcal{L}(\cdot,\cdot)$, (18) and (19) gives

$$\begin{aligned} (\varphi, \boldsymbol{u}-\boldsymbol{u}_h) + (\phi, p-p_h) &\le c|||(\boldsymbol{u}-\boldsymbol{u}_h),(p-p_h)|||\,|||(\Phi-\Phi_h),(\Psi-\Psi_h)||| + c\, h\, \|p-p_h\|\, \|\Psi\|_1 \\ &\le c\, h\, (\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|)(\|\Phi\|_2 + \|\Psi\|_1) \\ &\le c\, h\, (\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|)(\|\varphi\|_0 + \|\phi\|_1). \end{aligned}$$

Therefore

$$\|\boldsymbol{u}-\boldsymbol{u}_h\| + \|p-p_h\|_{-1} \le c\, h\, (\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 + \|p-p_h\|).$$

This ends the proof. □

Theorem 3.2. *Let $\boldsymbol{X}_h$ and M_h be the spaces defined in (7), $(\boldsymbol{u}, p)$ be the solution of problem (3) and $(\boldsymbol{u}_h, p_h)$ be the finite element approximation solution of problem (10) with stabilization term (12) or (13). There exists a positive constant c independent of the mesh size such that*

$$\sum_{T\in\tau_h} h_T\|\nabla p_h\|_T \le c|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)|||.$$

Proof. Using the second equation of (9) and letting $q_h = p_h$ give

$$(\nabla\cdot\boldsymbol{u}_h, p_h) + S(p_h, p_h) = 0.$$

For $p_h \in L_0^2(\Omega)\cap P_1(\Omega)$, there exists $p_h^0 \in P_0(\Omega)$ such that

$$\|p_h - p_h^0\| \le ch\|p_h\|_1. \tag{23}$$

We use the fact that $\nabla \cdot \boldsymbol{u} = 0$, the integration by parts, the Cauchy–Schwarz inequality, (23) and trace inequality (5) to get

$$\begin{aligned} S(p_h, p_h) &= -(\nabla \cdot \boldsymbol{u}_h, p_h) = (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), p_h) = -(\boldsymbol{u} - \boldsymbol{u}_h, \nabla p_h) \\ &= -\sum_{T \in \tau_h} (\boldsymbol{u} - \boldsymbol{u}_h, \nabla (p_h - p_h^0))_T \\ &= (\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h), (p_h - p_h^0)) + \sum_{e \in \mathcal{E}_h} ((\boldsymbol{u} - \boldsymbol{u}_h)\boldsymbol{n}_e, [p_h - p_h^0]_e)_e \\ &\le c \sum_{T \in \tau_h} h_T \|\nabla p_h\| \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|(\boldsymbol{u} - \boldsymbol{u}_h)\boldsymbol{n}_e\|_e^2 \right)^{\frac{1}{2}} \|[p_h - p_h^0]_e\|_{\mathcal{E}_h} \\ &\le c \left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|(\boldsymbol{u} - \boldsymbol{u}_h)\boldsymbol{n}_e\|_e^2 \right)^{\frac{1}{2}} \right) \sum_{T \in \tau_h} h_T \|\nabla p_h\|_T. \end{aligned} \tag{24}$$

On another hand, we estimate the stabilization term $S(p_h, p_h)$ in two cases.

If the stabilization term is defined by (12), then

$$S(p_h, p_h) \ge c \left(\sum_{T \in \tau_h} h_T \|\nabla p_h\|_T \right)^2.$$

If the stabilization term is defined by (13), then using the inverse inequality gives

$$S(p_h, p_h) = \|p_h - \Pi_0 p_h\|^2 \ge c \left(\sum_{T \in \tau_h} h_T \|\nabla p_h\|_T \right)^2.$$

Combining above expressions with (24), using the trace inequality and Lemma 3.1 gives

$$\begin{aligned} \sum_{T \in \tau_h} h_T \|\nabla p_h\|_T &\le c \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|(\boldsymbol{u} - \boldsymbol{u}_h)\boldsymbol{n}_e\|_e^2 \right)^{\frac{1}{2}} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \\ &\le c\, h^{-\frac{1}{2}} \|\boldsymbol{u} - \boldsymbol{u}_h\|^{\frac{1}{2}} \|\boldsymbol{u} - \boldsymbol{u}_h\|_1^{\frac{1}{2}} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \\ &\le c \|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|. \end{aligned}$$

This completes the proof. □

4. A posteriori error estimates

Based on the property of stabilized methods, we present the following error estimator η_S for the stabilization of low-order mixed finite elements for the Stokes equations. It can be constructed locally by

Case 1: if $\boldsymbol{X}_h$ and M_h are spaces defined in (6):

$$\eta_{0,T}^2 := \frac{1}{2} \sum_{e \in \partial T \cap \Omega} h_e \|[\nabla \boldsymbol{u}_h \cdot \boldsymbol{n}_e]\|_e^2 + \frac{1}{2} \sum_{e \in \partial T \cap \Omega} h_e \|[p_h]_e\|_e^2 + \|\nabla \cdot \boldsymbol{u}_h\|_T^2.$$

Case 2: if $\boldsymbol{X}_h$ and M_h are spaces defined in (7):

$$\eta_{1,T}^2 := \frac{1}{2} \sum_{e \in \partial T \cap \Omega} h_e \|[\nabla \boldsymbol{u}_h \cdot \boldsymbol{n}_e]\|_e^2 + h_T^2 \|\nabla p_h\|_T^2 + \|\nabla \cdot \boldsymbol{u}_h\|_T^2.$$

The global estimators can be defined as follows:

$$\eta_S^2 := \begin{cases} \eta_0^2 := \sum_{T \in \tau_h} \eta_{0,T}^2 = \|[\nabla \boldsymbol{u}_h \cdot \boldsymbol{n}_e]\|_{\mathcal{E}_h}^2 + \|[p_h]_e\|_{\mathcal{E}_h}^2 + \|\nabla \cdot \boldsymbol{u}_h\|^2, & \text{Case 1}, \\ \eta_1^2 := \sum_{T \in \tau_h} \eta_{1,T}^2 = \|[\nabla \boldsymbol{u}_h \cdot \boldsymbol{n}_e]\|_{\mathcal{E}_h}^2 + \sum_{T \in \tau_h} h_T^2 \|\nabla p_h\|_T^2 + \|\nabla \cdot \boldsymbol{u}_h\|^2, & \text{Case 2}. \end{cases}$$

4.1. Reliability

Firstly, we introduce the weighted Clément-type interpolation operator I_h defined in [23]. It is often used for establishing the reliability bound of a posteriori error estimators; see [23–25]. It has following properties.

Lemma 4.1. *There is a constant c, which depends only on the shape parameter such that*

$$(\boldsymbol{f}, \boldsymbol{v} - I_h \boldsymbol{v}) \leq c\ H_{\boldsymbol{f}} \|\boldsymbol{v}\|_1 \quad \forall \boldsymbol{v} \in \boldsymbol{X},$$

and

$$\left(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|\boldsymbol{v} - I_h \boldsymbol{v}_h\|_e^2\right)^{\frac{1}{2}} \leq c\ \|\boldsymbol{v}\|_1 \quad \forall \boldsymbol{v} \in \boldsymbol{X}.$$

Here $H_{\boldsymbol{f}} := \left(\sum_{z\in\mathcal{N}\setminus\mathcal{N}_h} |\omega_z|\ \|\boldsymbol{f}\|^2_{\omega_z} + \sum_{z\in\mathcal{N}_h} |\omega_z|\ \|\boldsymbol{f} - \fint_{\omega_z} \boldsymbol{f}\, dx\|^2_{\omega_z}\right)^{\frac{1}{2}}$ *and* $\fint_{\omega_z} \boldsymbol{f}\, dx$ *denotes the average of* $\boldsymbol{f}$ *over* ω_z. $\mathcal{N}$ *and* $\mathcal{N}_h$ *denote the vertices in* τ_h *and those lying inside* Ω*, respectively.* ω_z *denotes the union of all triangles that share the same vertex z.*

Proof. Refer to the Lemmas 3.1, 6.1 and 6.2 in [23]. □

Remark 1. The second term in $H_{\boldsymbol{f}}$ is a higher-order term for $\boldsymbol{f} \in L^2(\Omega)^2$ and so is the first term for $\boldsymbol{f} \in L^p(\Omega)^2$ with $p > 2$; (see [23,24]).

Theorem 4.1 (*Reliability*)**.** *Let* $(\boldsymbol{u}, p)$ *be the solution of problem* (3) *and* $(\boldsymbol{u}_h, p_h)$ *be the finite element approximation solution of problem* (10) *with stabilization term* (11)*,* (12) *or* (13)*. There exists a positive constant c independent of the mesh size such that*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| \leq c\ (\eta_S + H_{\boldsymbol{f}}).$$

Proof. Subtracting (10) from (3) gives

$$\mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}_h, q_h)) - S(p_h, q_h) = 0 \quad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{X}_h \times M_h.$$

Taking $q_h = 0$ and $\boldsymbol{v}_h = I_h \boldsymbol{v}$ yields

$$\mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (I_h \boldsymbol{v}, 0)) = 0 \quad \forall\ \boldsymbol{v} \in \boldsymbol{X}. \tag{25}$$

For any $(\boldsymbol{v}, q) \in \boldsymbol{X} \times M$, (25), integration by parts, Lemma 4.1 and the Cauchy–Schwarz inequality imply in the standard way

$$\begin{aligned}
\mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}, q)) &= \mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v} - I_h \boldsymbol{v}, q)) \\
&= (\boldsymbol{f}, \boldsymbol{v} - I_h \boldsymbol{v}) + \sum_{T\in\tau_h} (\Delta \boldsymbol{u}_h - \nabla p_h, \boldsymbol{v} - I_h \boldsymbol{v})_T + \sum_{e\in\mathcal{E}_h} ([(\nabla \boldsymbol{u}_h - p_h \boldsymbol{I}) \cdot \boldsymbol{n}_e], \boldsymbol{v} - I_h \boldsymbol{v})_e - (q, \nabla \cdot \boldsymbol{u}_h) \\
&\leq c \left(H_{\boldsymbol{f}} + \sum_{T\in\tau_h} h_T \|\nabla p_h\|_T + \|[\nabla \boldsymbol{u}_h \cdot \boldsymbol{n}_e]\|_{\mathcal{E}_h} + \|[p_h]\|_{\mathcal{E}_h} + \|\nabla \cdot \boldsymbol{u}_h\| \right) |||(\boldsymbol{v}, q)|||.
\end{aligned}$$

According to definition of $\boldsymbol{X}_h$ and M_h, it is easy to check

$$\mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}, q)) \leq c \left(H_{\boldsymbol{f}} + \eta_S\right)\ |||(\boldsymbol{v}, q)|||,$$

together with the inf–sup condition (4) yields

$$\begin{aligned}
|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)||| &\leq \beta^{-1} \sup_{(\boldsymbol{v},q)\in\boldsymbol{X}\times M} \frac{\mathcal{L}((\boldsymbol{u} - \boldsymbol{u}_h, p - p_h), (\boldsymbol{v}, q))}{|||(\boldsymbol{v}, q)|||} \\
&\leq c\ (\eta_S + H_{\boldsymbol{f}}).
\end{aligned}$$

It completes the proof of the reliability. □

4.2. Efficiency

In this subsection, we use the properties of general stabilized methods to show the estimator η_S is efficient.

Lemma 4.2. *Let $(\boldsymbol{u}, p)$ be the solution of problem (3) and $(\boldsymbol{u}_h, p_h)$ be the finite element approximation solution of problem (10) with stabilization term (11), (12) or (13). There exists a positive constant c independent of the mesh size such that*

$$h_e^{\frac{1}{2}} \|[\boldsymbol{\sigma}_h \cdot \boldsymbol{n}_e]\|_e \le c\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|_{\omega_e} + c h_e \inf_{\boldsymbol{f}_h \in \boldsymbol{X}_h} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_e},$$

where $\boldsymbol{\sigma}_h := \nabla \boldsymbol{u}_h - p_h \boldsymbol{I}$ and ω_e denotes the union of the two triangles sharing the edge e.

Proof. The proof is straightforward extensions of the works of Verfürth [11,10] for the Stokes equations. Reader can refer to [17] for details. □

Theorem 4.2 (*Efficiency*). *Let $(\boldsymbol{u}, p)$ be the solution of problem (3) and $(\boldsymbol{u}_h, p_h)$ be the finite element approximation solution of problem (10) with stabilization term (11), (12) or (13). There exists a positive constant c independent of the mesh size such that*

$$\eta_S \le c\, \|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\| + c \sum_{e \in \mathcal{E}_h} h_e \inf_{\boldsymbol{f}_h \in \boldsymbol{X}_h} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_e}.$$

Proof. If $\boldsymbol{X}_h$ and M_h are spaces defined in (6), then using Lemma 4.2, the property of the stabilized method in Theorem 3.1 and the fact that $\nabla \cdot \boldsymbol{u} = 0$ give

$$\begin{aligned}\eta_0^2 &\le c \sum_{e \in \mathcal{E}_h} h_e (\|[\boldsymbol{\sigma}_h \cdot \boldsymbol{n}_e]\|_e^2 + 2\|[p_h]_e\|_e^2) + \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\|^2 \\ &\le c\, \|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|^2 + c \left(\sum_{e \in \mathcal{E}_h} \inf_{\boldsymbol{f}_h \in \boldsymbol{X}_h} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_e} \right)^2.\end{aligned}$$

If $\boldsymbol{X}_h$ and M_h are spaces defined in (7), then using Lemma 4.2, the property of the stabilized method in Theorem 3.2 and the fact that $\nabla \cdot \boldsymbol{u} = 0$ give

$$\begin{aligned}\eta_1^2 &\le c \sum_{e \in \mathcal{E}_h} h_e \|[\boldsymbol{\sigma}_h \cdot \boldsymbol{n}_e]\|_e^2 + \sum_{T \in \tau_h} h_T^2 \|\nabla p_h\|_T^2 + \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{u}_h)\|^2 \\ &\le c\, \|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\|^2 + c \left(\sum_{e \in \mathcal{E}_h} \inf_{\boldsymbol{f}_h \in \boldsymbol{X}_h} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_e} \right)^2.\end{aligned}$$

Combination of above expressions completes the proof of the efficiency bound. □

5. Postprocess with the average technique

The average techniques are post-processing methods that reconstruct numerical approximations from finite element solution to achieve better results. The procedure is to approximate a quality q_h by some globally continuous piecewise polynomials of higher degree $A(q_h)$.

A classical example [22] for piecewise constant function q_h reads as follows. Let the nodal value of $A(q_h)$ at any node $z \in \mathcal{N}$ be the area-weighted average of q_h over ω_z. ω_z is the union of all triangles that share the same vertex z. It can be written as

$$A(q_h)(z) = \fint_{\omega_z} q_h \, \mathrm{d}x := \int_{\omega_z} q_h \, \mathrm{d}x \Big/ \int_{\omega_z} 1 \, \mathrm{d}x.$$

Then define $A(q_h)$ by interpolation with nodal basis function ϕ_z associated with z,

$$A(q_h) = \sum_{z \in \mathcal{N}} A(q_h)(z) \phi_z. \tag{26}$$

This average technique has the following property.

Lemma 5.1. *Let q_h be piecewise constant finite element function and $A(q_h)$ is defined by (26). Then, there exist two positive constants c_1 and c_2 independent of the mesh size such that*

$$c_1 \, \|[q_h]_e\|_{\mathcal{E}_h} \le \|q_h - A(q_h)\| \le c_2 \, \|[q_h]_e\|_{\mathcal{E}_h}.$$

Proof. Refer to Lemma 2.3 in [4] and Theorem 5 in [22]. □

We use such an average technique to improve the piecewise constant finite element functions in the estimator η_S. The improved estimator is defined by

$$\eta_{SA}^2 := \begin{cases} \eta_{0,A}^2 := \|\nabla \boldsymbol{u}_h - A(\nabla \boldsymbol{u}_h)\|^2 + \|p_h - A(p_h)\|^2 + \|\nabla \cdot \boldsymbol{u}_h\|^2, & \text{Case 1,} \\ \eta_{1,A}^2 := \|\nabla \boldsymbol{u}_h - A(\nabla \boldsymbol{u}_h)\|^2 + \sum_{T \in \tau_h} h_T^2 \|\nabla p_h\|_T^2 + \|\nabla \cdot \boldsymbol{u}_h\|^2, & \text{Case 2.} \end{cases}$$

Here, for vector $\boldsymbol{v} = (v_1, v_2)$, we denote $A(\boldsymbol{v}) = (A(v_1), A(v_2))$.

Theorem 5.1. *Let $(\boldsymbol{u}, p)$ be the solution of problem (3) and $(\boldsymbol{u}_h, p_h)$ be the finite element approximation solution of problem (10) with stabilization term (11), (12) or (13). There exist positive constants c_1 and c_2 independent of the mesh size such that*

$$|\|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)|\| \le c_1 \, (\eta_{SA} + H_{\boldsymbol{f}}),$$
$$\eta_{SA} \le c_2 \|(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)\| + c_2 \sum_{e \in \mathcal{E}_h} h_e \inf_{\boldsymbol{f}_h \in \boldsymbol{X}_h} \|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_e}.$$

Proof. The combination of Lemma 5.1, Theorems 4.1 and 4.2 completes the proof. □

6. Numerical results

In this section, we aim to show that both the estimators η_S and η_{SA} work well for stabilized P_1/P_0 and P_1/P_1 finite element methods. Meanwhile, we present the practical effectivity of the average technique and show the improved estimator η_{SA} is more exact than the estimator η_S.

We consider two numerical tests to illustrate the practical effectivity of estimators η_S and η_{SA}. One is a flow problem with a smooth solution. Another models a flow problem in a cracked domain with a singular solution.

Moreover, to show the wide application of these estimators, we consider all the stabilized methods mentioned in Section 2. One is the penalizing jump stabilized method, namely, the stabilized method (10) with stabilization term (11). Another is the pressure gradient stabilized method, referring to the stabilized method (10) with stabilization term (12). The third one is the projection stabilized method, namely, the stabilized method (10) with stabilization term (13).

The experiments are implemented by the public software Freefem++ [26]. The adaptive strategy is carried out as follows. Given a user-specified tolerance η^* and an initial mesh τ^0. Refine the mesh by using the mesh refinement strategy in Freefem++ until the global error estimator η (i.e., either η_S or η_{SA}) satisfies $\eta \le \eta^*$. See [17] for details of the mesh refinement strategy.

For the sake of convenience, we introduce the following notions.

- DOF^j := number of elements for the triangulation τ_h^j;
- $\boldsymbol{e}_r^j := \|(\boldsymbol{u} - \boldsymbol{u}_h^j, p - p_h^j)\| / \|(\boldsymbol{u}, p)\|$ denotes the relative error in the energy norm.
- $\eta_{SA,r}^j := \eta_{SA}^j / \|(\boldsymbol{u}, p)\|$ denotes the relative value of global estimator η_{SA} on τ_h^j.
- $\eta_{S,r}^j := \eta_S^j / \|(\boldsymbol{u}, p)\|$ denotes the relative value of global estimator η_S on τ_h^j.
- Order := $\frac{2 \log(\boldsymbol{e}_r^{j+1} / \boldsymbol{e}_r^j)}{\log(\text{DOF}^j / \text{DOF}^{j+1})}$ denotes the convergence rate of the error.
- $E_S^j := \eta_{S,r}^j / \boldsymbol{e}_r^j$ denotes effectivity index for the global estimator η_S^j on τ_h^j.
- $E_{SA}^j := \eta_{SA,r}^j / \boldsymbol{e}_r^j$ denotes effectivity index for the global estimator η_{SA}^j on τ_h^j.

Table 1
Results for adaptive refinements via penalizing jump stabilized P_1/P_0 method with η_S.

j	DOF^j	$\boldsymbol{e}_r^j$	Order	$\eta_{S,r}^j$	E_S^j
0	208	0.2670	–	0.8526	3.1928
1	368	0.2067	0.8973	0.6556	3.1716
2	625	0.1407	1.4505	0.4714	3.3486
3	1111	0.1106	0.8363	0.3695	3.3392
4	2054	0.0758	1.2296	0.2601	3.4294

Table 2
Results for adaptive refinements via projection stabilized P_1/P_0 method with η_S.

j	DOF^j	$\boldsymbol{e}_r^j$	Order	$\eta_{S,r}^j$	E_S^j
0	208	0.3072	–	0.9146	2.9771
1	346	0.2423	0.9322	0.7176	2.9612
2	569	0.1628	1.5987	0.5175	3.1784
3	1008	0.1299	0.7890	0.4088	3.1464
4	1693	0.0919	1.3339	0.3014	3.2782

Table 3
Results for adaptive refinements via pressure gradient stabilized P_1/P_1 method with η_S.

j	DOF^j	$\boldsymbol{e}_r^j$	Order	$\eta_{S,r}^j$	E_S^j
0	208	0.2379	–	0.8183	3.4391
1	384	0.1774	0.9573	0.6120	3.4496
2	633	0.1238	1.4390	0.4427	3.5752
3	1200	0.0938	0.8658	0.3403	3.6251
4	2094	0.0665	1.2351	0.2431	3.6518

6.1. A smooth problem

The first example is a flow problem with a smooth solution, given by

$$u_1 = 2\pi \sin^2(\pi x)\sin(\pi y)\cos(\pi y),$$
$$u_2 = -2\pi \sin(\pi x)\cos(\pi x)\sin^2(\pi y),$$
$$p = \cos(\pi x)\cos(\pi y),$$

where domain $\Omega = (0, 1) \times (0, 1)$.

Firstly, we solve the problem via different stabilized low-order FE methods and use the adaptive strategy with estimator η_S.

Tables 1 and 2 report the results obtained by the penalizing jump stabilized and projection stabilized P_1/P_0 methods, respectively. It is clear that the refinements get a good approximate solution as $h \to 0$ and optimal convergence order (about 1.0). In addition, the effectivity index E_S^j of estimator η_S keeps stable.

Tables 3 and 4 report the results obtained by pressure gradient stabilized and projection stabilized P_1/P_1 methods, respectively. They show the convergence rate of energy norm keeps order 1.0 and the effectivity index E_S^j is around 3.5 and keeps stable. The results are consistent with those in Tables 1 and 2.

From all the results in Tables 1–4, we can see the estimator η_S works well for general stabilized methods of low-order mixed finite elements for the Stokes equations.

In order to show improvements of the average technique, we solve the problem again by using the adaptive strategy with improved estimator η_{SA}. Table 5 reports the results associated with η_{SA} on the successive adaptive refined

Table 4
Results for adaptive refinements via projection stabilized P_1/P_1 method with η_S.

j	DOF^j	$\boldsymbol{e}_r^j$	Order	$\eta_{S,r}^j$	E_S^j
0	208	0.2483	–	0.8487	3.4170
1	374	0.1844	1.0139	0.6349	3.4416
2	634	0.1249	1.4769	0.4474	3.5813
3	1225	0.0954	0.8179	0.3447	3.6122
4	2150	0.0675	1.2296	0.2462	3.6469

Table 5
Results for adaptive refinements via stabilized methods with η_{SA} ($j = 2, 3, 4$).

	Stabilized methods	DOF^j	$\boldsymbol{e}_r^j$	Order	$\eta_{SA,r}^j$	$E_{S,A}^j$
P_1/P_0	Penalizing jump	548	0.1440	1.1551	0.1475	1.0247
		950	0.1082	1.0379	0.1095	1.0118
		1639	0.0811	1.0560	0.0817	1.0077
	Projection	528	0.1550	1.2144	0.1575	1.0157
		923	0.1171	1.0041	0.1172	1.0013
		1569	0.0872	1.1122	0.0874	1.0028
P_1/P_1	Pressure gradient	597	0.1284	1.1924	0.1325	1.0317
		1068	0.0953	1.0233	0.0970	1.0174
		1923	0.0703	1.0369	0.0710	1.0107
	Projection	586	0.1299	1.2854	0.1328	1.0227
		1040	0.0968	1.0255	0.0976	1.0082
		1862	0.0722	1.0025	0.0726	1.0045

meshes. Comparing the results with those in Tables 1–4, it is clear that η_{SA} can work as well as η_S in terms of $\boldsymbol{e}_r^j$ and Order. Moreover, the fact that $E_{S,A}^j$ keeps almost 1.0 shows that estimator η_{SA} stays much close to the true error. In this sense, η_{SA} is more exact than η_S.

6.2. A singular problem

In the second example, we consider Ω to be a disk of radius 1 with a crack joining the center to the boundary as presented in [11] and the exact solution $\boldsymbol{u} = (u_1, u_2)$ and p are given as follows:

$$u_1 = 1.5r^{1/2}(\cos(0.5\theta) - \cos(1.5\theta)),$$
$$u_2 = 1.5r^{1/2}(3\sin(0.5\theta) - \sin(1.5\theta)),$$
$$p = -6r^{-1/2}\cos(0.5\theta),$$

where (r, θ) is the polar representation of a point in the disk. This problem is singular at the end of the crack, i.e., at the center of the disk. $\boldsymbol{f}$ is determined by (2) and $\boldsymbol{u}$ is enforced with appropriate inhomogeneous boundary conditions.

We solve this problem via the projection stabilized method and use the adaptive strategy with both estimators η_S and η_{SA}.

In Table 6, we compare the results of uniform refinements with those of adaptive refinements. The comparison shows that the refinements based on estimators η_S and η_{SA} perform similarly, and both get much better approximation than the uniform refinements. Moreover, the convergence rate for adaptive refinements is higher than that for the uniform refinements.

Next, we will show the improved estimator η_{SA} is more exact than η_S.

Figs. 1 and 2 show the adaptive meshes and comparisons of true error with estimators η_S and η_{SA} via the projection stabilized P_1/P_0 method. From the refined mesh in Fig. 1, we can see both estimators η_S and η_{SA} capture the singularity at the origin and produce very similar meshes. From the comparisons of true error with η_S and η_{SA} in Fig. 2, it is clear that η_{SA} stays more close to the true error than η_S.

Table 6
Comparison of the uniform and adaptive refinements via projection stabilized method.

	Strategy	DOF	e_r	Order
P_1/P_0	Uniform	6412	0.1759	0.5213
	Adaptive with η_S	2033	0.0899	1.0405
	Adaptive with η_{SA}	2097	0.0845	0.9092
P_1/P_1	Uniform	6412	0.2066	0.5742
	Adaptive with η_S	2234	0.0824	1.0463
	Adaptive with η_{SA}	2213	0.0863	1.0585

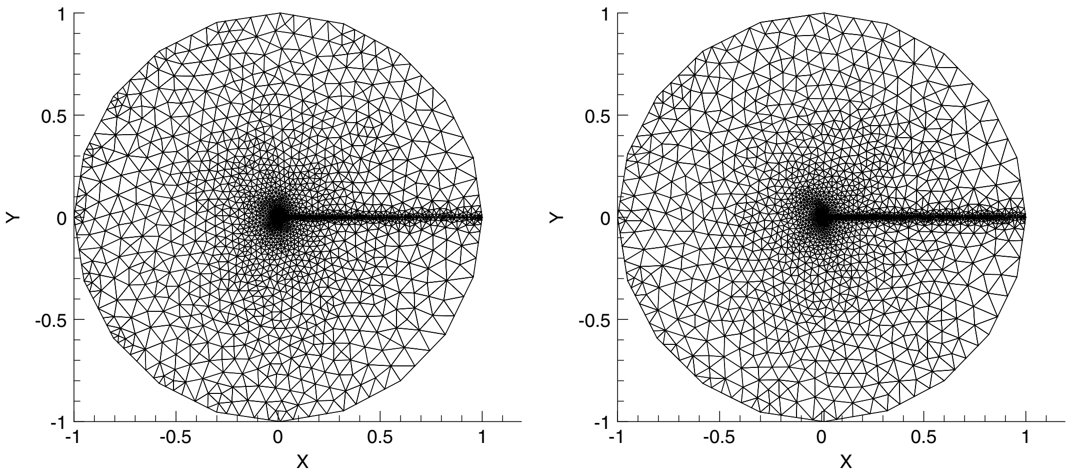

Fig. 1. Refined meshes with η_S (left) and η_{SA} (right) via projection stabilized P_1/P_0 method.

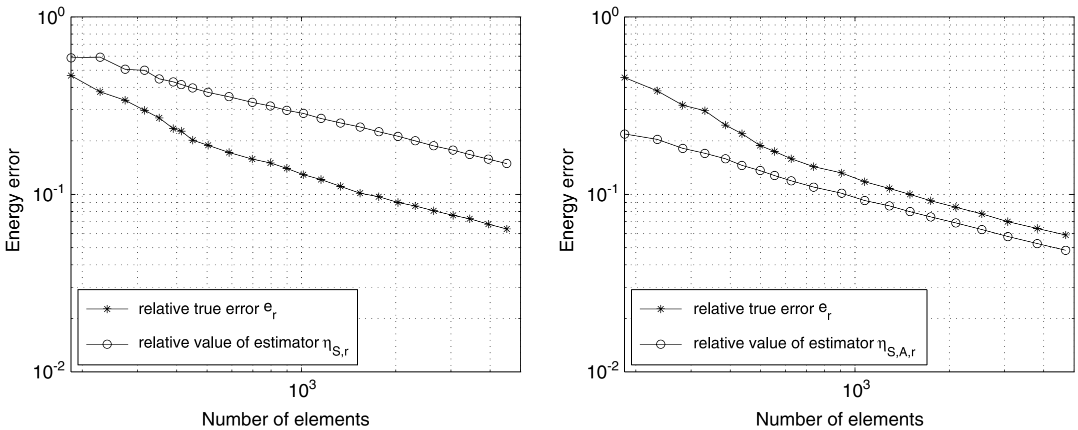

Fig. 2. Comparison of true error with estimators η_S (left) and η_{SA} (right) via projection stabilized P_1/P_0 method.

Similarly, Figs. 3 and 4 show the adaptive meshes and comparisons of true error with estimator η_S and η_{SA} via the projection stabilized P_1/P_1 method. Fig. 3 shows mesh refinements based on both η_S and η_{SA} occur at the origin. Fig. 4 shows that the estimator η_{SA} is more exact than η_S. These observations are consistent with the results in Figs. 1 and 2.

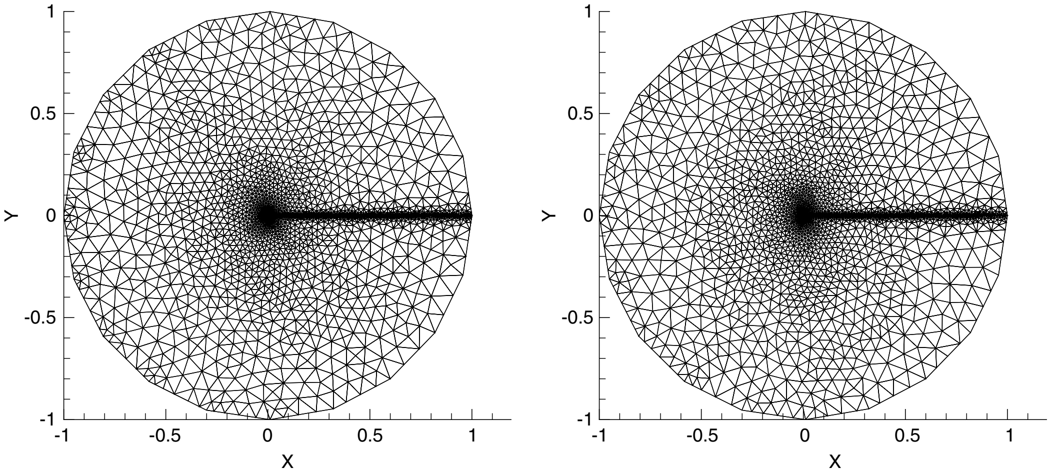

Fig. 3. Refined meshes with η_S (left) and η_{SA} (right) via projection stabilized P_1/P_1 method.

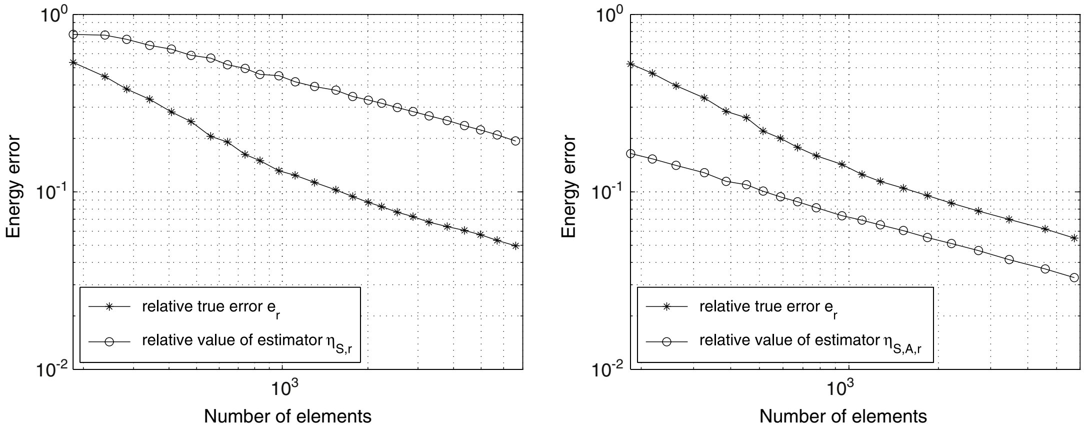

Fig. 4. Comparison of true error with estimators η_S (left) and η_{SA} (right) via projection stabilized P_1/P_1 method.

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