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A priori and posteriori error analysis for time-dependent Maxwell's equations

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Dedicated to Mary F. Wheeler on the occasion of her 75th birthday

Abstract

We consider time-dependent Maxwell's equations discretized by variable time steps in time domain and edge elements in spatial domain. First, the stability and optimal a priori error estimate are proved for both semi and fully discrete schemes. Then a posteriori error analysis is carried out for both schemes.

MSC: 78M10; 65N30; 35L15

Keywords: Maxwell's equations; Backward Euler scheme; Posteriori error analysis; Edge elements

1. Introduction

Wave propagation phenomena happen in a broad range of applications. Examples include sound waves, light waves and water waves, which arise in acoustics, electromagnetics, and fluid dynamics, respectively. The wave propagation problem is often described by the second-order hyperbolic equation, also called the wave equation. Upon considering time-harmonic (steady-state) waves, the wave equation reduces to the Helmholtz equation. Over the last four decades there has been considerable interest in developing various finite element methods (FEMs) for solving the wave equation (e.g., [1–6], and references therein). To solve the wave equation more efficiently, adaptive FEMs are often used. Adaptive FEMs are often based on a posteriori error estimates, i.e., some computable quantities that estimate the FEM solution error in a suitable norm.

Over the last three decades many a posteriori error estimates have been developed for time-independent problems such as elliptic equations and Helmholtz equation (e.g., [7–13], and references therein). As for time-dependent prob-

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lems, considerable results on a posteriori error estimates for parabolic problems have also been obtained (e.g., [14–17], and references therein). However, according to Bernardi and Süli back in 2005 [18, p. 199]: "the a posteriori error analysis of finite element approximations to second-order hyperbolic problems is in a less complete state". Indeed, compared to results on elliptic, parabolic, and first order hyperbolic problems, "hyperbolic problems of second order have been much less studied" as Picasso mentioned in 2010 [19, p. 2213]. Though some a posteriori error estimates have been obtained over the years (e.g., [20–22,18,19]), "the design and implementation of adaptive algorithms for the wave equation based on rigorous a posteriori error estimators is a largely unexplored subject" as Georgoulis, Lakkis and Makridakis remarked in their 2013 paper [23, p. 1262].

Time-dependent Maxwell's equations are used in simulating electromagnetic wave propagation problems, and can be reduced to the second-order wave equation under special circumstances. Since the first a posteriori error estimate obtained for time-harmonic Maxwell's equations by Monk in 1998 [24], some excellent estimates have been proved (e.g., [25–29], [30, Ch. 6] and references therein). However, to the best of our knowledge, there seems no publication on a posteriori error estimates for time-dependent Maxwell's equations [31,32], except one paper by Zheng, Chen and Wang [33] for time-dependent eddy current problems and one very recent work by Creusé, Nicaise and Tittarelli [34] for the $A-\varphi$ magnetodynamic problem. Note that both problems of [33,34] are described by parabolic type equations, and many a posteriori error estimates techniques developed for parabolic equations can be applied. At the end of their 2005 paper [18], Bernardi and Süli mentioned that similar estimates for time-dependent Maxwell system can be shown using their techniques developed for the second-order wave equation, no such a study has ever been published so far. In this paper, we initiate this task by developing some a posteriori error estimates for time-dependent Maxwell's equations.

The outline of the paper is as follows. In Section 2, we present the model problem and a stability result. Following the technique introduced by Bernardi and Süli [18], Section 3 is devoted to the description of a backward Euler scheme with variable time steps, and a fully discrete edge element method. Stability and optimal a priori error estimate are proved for both the semi and fully discrete schemes. Section 4 is devoted to the a posteriori error analysis of both schemes. We conclude the paper in Section 5.

2. The governing equations and notation

Before we present the governing equations, let us introduce some notation. We assume that Ω is a bounded Lipschitz polyhedral domain in $\mathcal{R}^3$ with connected boundary $\partial\Omega$. For any domain $\omega \subset R^3$, we let $L^2(\omega)$ be the Hilbert space, equipped with inner product $(\cdot,\cdot)_{0,\omega}$ and norm $\|\cdot\|_{L^2(\omega)}$. When $\omega = \Omega$, we simply write $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$. We let $H^s(\omega)$ be the standard Sobolev space of order $s \geq 0$ equipped with norm $\|\cdot\|_{s,\omega}$ and semi-norm $|\cdot|_{s,\omega}$. For vector functions, we simply denote $\boldsymbol{L}^2(\omega) = (L^2(\omega))^3$ and $\boldsymbol{H}^s(\omega) = (H^s(\omega))^3$. To deal with Maxwell's equations, we need the spaces

$$H^s(\text{curl};\,\Omega) = \left\{\boldsymbol{u} \in H^s(\Omega);\ \nabla\times\boldsymbol{u} \in H^s(\Omega)\right\}$$

and

$$H_0^s(\text{curl};\,\Omega) = \left\{\boldsymbol{u} \in H^s(\text{curl};\,\Omega);\ \hat{\boldsymbol{n}}\times\boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega\right\},$$

where $\hat{\boldsymbol{n}}$ is the unit outward normal to $\partial\Omega$. When $s = 1$, we simply write $H^1(\text{curl};\,\Omega) = H(\text{curl};\,\Omega)$ and $H_0^1(\text{curl};\,\Omega) = H_0(\text{curl};\,\Omega)$. For time $T > 0$ and any separable Banach space X, we consider the space $L^1(0,T;X)$ of integrable functions on $(0,T)$ with values in X. We also need the space $C^s(0,T;X)$ of continuously differentiable functions on $[0,T]$ up to the order s with values in X.

To model electromagnetic wave propagation, we usually have to solve the famous Maxwell's equations:

$$\epsilon\frac{\partial\boldsymbol{E}}{\partial t} - \nabla\times\boldsymbol{M} = -\boldsymbol{J}_s, \quad \text{in } \Omega\times(0,T), \tag{1}$$

$$\mu\frac{\partial\boldsymbol{M}}{\partial t} + \nabla\times\boldsymbol{E} = 0, \quad \text{in } \Omega\times(0,T), \tag{2}$$

where ϵ and μ denote the permittivity and permeability, respectively, $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{M}(\boldsymbol{x},t)$ are the electric and magnetic fields, respectively, and $\boldsymbol{J}_s(\boldsymbol{x},t)$ is a given source. To make the problem well-posed, we further assume that the system

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(1)–(2) is subject to the perfectly conducting boundary condition:

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{3}$$

and the initial conditions:

$$\boldsymbol{E}(\mathbf{x}, 0) = \boldsymbol{E}_0(\mathbf{x}), \qquad \boldsymbol{M}(\mathbf{x}, 0) = \boldsymbol{M}_0(\mathbf{x}), \tag{4}$$

where $\boldsymbol{E}_0$ and $\boldsymbol{M}_0$ are some given functions.

Differentiating (1) with respect to t and replacing $\boldsymbol{M}$ by using (2), we obtain

$$\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) = -\frac{\partial \boldsymbol{J}_s}{\partial t} := \boldsymbol{f}. \tag{5}$$

To make the problem (5) complete, we assume that (5) is supplemented with the boundary condition (3) and initial conditions:

$$\boldsymbol{E}(\mathbf{x}, 0) = \boldsymbol{E}_0(\mathbf{x}), \qquad \boldsymbol{E}_t(\mathbf{x}, 0) = \boldsymbol{E}_1(\mathbf{x}), \tag{6}$$

here and below we denote the derivative $\boldsymbol{E}_{t^k} = \frac{\partial^k \boldsymbol{E}}{\partial t^k}$ for $k = 1, 2, 3$. For the second derivative, sometimes we write $\boldsymbol{E}_{t^2} = \boldsymbol{E}_{tt} = \partial_{tt}\boldsymbol{E}$.

A popular way to deal with the second order hyperbolic wave equation is to introduce a new variable and reduce the original problem into a system of differential equations with one becoming parabolic type equation, which can be easily dealt with. Without confusion, let us introduce the new variable $\boldsymbol{H} = \epsilon \boldsymbol{E}_t$ and rewrite (5) in the system form:

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} + \begin{pmatrix} 0 & -\epsilon^{-1} \\ \nabla \times (\mu^{-1} \nabla \times) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{f} \end{pmatrix}. \tag{7}$$

To obtain more general results below, we consider the right hand side of (7) to be more general, i.e., the problem becomes as

$$\frac{\partial}{\partial t}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} + \begin{pmatrix} 0 & -\epsilon^{-1} \\ \nabla \times (\mu^{-1} \nabla \times) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{f} \end{pmatrix} \tag{8}$$

supplemented with boundary condition (3) and initial conditions (4).

Theorem 2.1. *For any $\boldsymbol{f} \in L^1(0, T; \boldsymbol{L}^2(\Omega)), \boldsymbol{g} \in L^1(0, T; H_0(curl; \Omega))$, and $t \in [0, T]$, the solution $(\boldsymbol{E}, \boldsymbol{H})$ of (8) satisfies the following stability*

$$(\|\epsilon^{-\frac{1}{2}}\boldsymbol{H}\|^2 + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{E}\|^2)^{1/2}(t) \le (\|\epsilon^{-\frac{1}{2}}\boldsymbol{H}_0\|^2 + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{E}_0\|^2)^{1/2} + \int_0^t (\|\epsilon^{-\frac{1}{2}}\boldsymbol{f}\| + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{g}\|)(s)ds.$$

Proof. Taking the inner product of (8) with vector $W = (\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}), \epsilon^{-1}\boldsymbol{H})'$, here and below the prime $'$ denotes the transpose, we have

$$(\boldsymbol{E}_t, \nabla \times (\mu^{-1}\nabla \times \boldsymbol{E})) + (\boldsymbol{H}_t, \epsilon^{-1}\boldsymbol{H}) = (\boldsymbol{g}, \nabla \times (\mu^{-1}\nabla \times \boldsymbol{E})) + (\boldsymbol{f}, \epsilon^{-1}\boldsymbol{H}),$$

which, after integration by parts and using (3), leads to

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[(\mu^{-1}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{E}) + (\epsilon^{-1}\boldsymbol{H}, \boldsymbol{H})] &= (\nabla \times \boldsymbol{g}, \mu^{-1}\nabla \times \boldsymbol{E}) + (\boldsymbol{f}, \epsilon^{-1}\boldsymbol{H}) \\ &\le (\|\epsilon^{-\frac{1}{2}}\boldsymbol{f}\|^2 + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{g}\|^2)^{1/2}(\|\epsilon^{-\frac{1}{2}}\boldsymbol{H}\|^2 + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{E}\|^2)^{1/2}, \end{aligned} \tag{9}$$

from which we obtain

$$\frac{d}{dt}(\|\epsilon^{-\frac{1}{2}}\boldsymbol{H}\|^2 + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{E}\|^2)^{1/2} \le (\|\epsilon^{-\frac{1}{2}}\boldsymbol{f}\|^2 + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{g}\|^2)^{1/2} \le \|\epsilon^{-\frac{1}{2}}\boldsymbol{f}\| + \|\mu^{-\frac{1}{2}}\nabla \times \boldsymbol{g}\|,$$

which completes the proof by integrating both sides from 0 to t. □

To simplify the notation and reduce the technicality, below we just assume that $\epsilon = \mu = 1$, since similar results can be obtained for variable coefficients ϵ and μ under the assumption that they are bounded below and above in Ω.

3. The a priori error analysis

3.1. A semi-discrete scheme

First, let us consider a time discretization scheme for (8). To construct the scheme, we partition the time interval $[0, T]$ into non-uniform subintervals $[t_{n-1}, t_n]$, $1 \le n \le N$, such that $0 = t_0 < t_1 < \cdots < t_N = T$. Furthermore, we denote the length $\tau_n = t_{n+1} - t_n$ and $\tau = \max_{0\le n\le N-1} \tau_n$.

We construct a backward Euler scheme for (8): For any $0 \le n \le N-1$,

$$\frac{U^{n+1} - U^n}{\tau_n} + \begin{pmatrix} 0 & -1 \\ \nabla\times\nabla\times & 0 \end{pmatrix} U^{n+1} = \begin{pmatrix} \boldsymbol{g}^{n+1} \\ \boldsymbol{f}^{n+1} \end{pmatrix}, \tag{10}$$

supplemented with boundary condition

$$\hat{\boldsymbol{n}} \times \boldsymbol{E}^n = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{11}$$

and initial conditions

$$U^0 = (\boldsymbol{E}^0, \boldsymbol{H}^0)' = (\boldsymbol{E}_0(\boldsymbol{x}), \boldsymbol{H}_0(\boldsymbol{x}))'. \tag{12}$$

In the above, we denote $\boldsymbol{g}^{n+1} = \boldsymbol{g}(t_{n+1}), \boldsymbol{f}^{n+1} = \boldsymbol{f}(t_{n+1})$ and $U^n = (\boldsymbol{E}^n, \boldsymbol{H}^n)'$.

Theorem 3.1. *For $0 \le n \le N-1$, the solution $(\boldsymbol{E}^{n+1}, \boldsymbol{H}^{n+1})$ satisfies the following estimate*

$$(\|\nabla\times\boldsymbol{E}^{n+1}\|^2 + \|\boldsymbol{H}^{n+1}\|^2)^{1/2} \le (\|\nabla\times\boldsymbol{E}^0\|^2 + \|\boldsymbol{H}^0\|^2)^{1/2} + \sum_{m=0}^{n} \tau_m (\|\nabla\times\boldsymbol{g}^{m+1}\| + \|\boldsymbol{f}^{m+1}\|).$$

Proof. Taking the inner product of (10) with $W = (\nabla\times\nabla\times\boldsymbol{E}^{n+1}, \boldsymbol{H}^{n+1})'$ and using integration by parts, we have

$$\begin{aligned} &(\nabla\times(\boldsymbol{E}^{n+1} - \boldsymbol{E}^n), \nabla\times\boldsymbol{E}^{n+1}) + (\boldsymbol{H}^{n+1} - \boldsymbol{H}^n, \boldsymbol{H}^{n+1}) \\ &\quad = \tau_n(\nabla\times\boldsymbol{g}^{n+1}, \nabla\times\boldsymbol{E}^{n+1}) + \tau_n(\boldsymbol{f}^{n+1}, \boldsymbol{H}^{n+1}) \\ &\quad \le \tau_n(\|\nabla\times\boldsymbol{g}^{n+1}\|^2 + \|\boldsymbol{f}^{n+1}\|^2)^{1/2}(\|\nabla\times\boldsymbol{E}^{n+1}\|^2 + \|\boldsymbol{H}^{n+1}\|^2)^{1/2}. \end{aligned} \tag{13}$$

Denote $S^{n+1} = (\|\nabla\times\boldsymbol{E}^{n+1}\|^2 + \|\boldsymbol{H}^{n+1}\|^2)^{1/2}$. From (13), we have

$$\begin{aligned} (S^{n+1})^2 &\le (\nabla\times\boldsymbol{E}^n, \nabla\times\boldsymbol{E}^{n+1}) + (\boldsymbol{H}^n, \boldsymbol{H}^{n+1}) + \tau_n(\|\nabla\times\boldsymbol{g}^{n+1}\|^2 + \|\boldsymbol{f}^{n+1}\|^2)^{1/2} S^{n+1} \\ &\le S^n \cdot S^{n+1} + \tau_n(\|\nabla\times\boldsymbol{g}^{n+1}\| + \|\boldsymbol{f}^{n+1}\|) S^{n+1}, \end{aligned}$$

which leads to

$$S^{n+1} \le S^n + \tau_n(\|\nabla\times\boldsymbol{g}^{n+1}\| + \|\boldsymbol{f}^{n+1}\|),$$

which concludes the proof by summing up both sides from 0 to n. □

Below let us consider the error estimate for the scheme (10). Denote the errors

$$e_E^n = \boldsymbol{E}(t_n) - \boldsymbol{E}^n, \qquad e_H^n = \boldsymbol{H}(t_n) - \boldsymbol{H}^n, \qquad V^n = \begin{pmatrix} e_E^n \\ e_H^n \end{pmatrix}.$$

Theorem 3.2. *For $0 \le n \le N-1$, the solution $(\boldsymbol{E}^{n+1}, \boldsymbol{H}^{n+1})$ of (10) and the solution $(\boldsymbol{E}, \boldsymbol{H})$ of (8) at $t = t_{n+1}$ satisfy the following error estimate*

$$(\|\nabla\times(\boldsymbol{E}(t_{n+1}) - \boldsymbol{E}^{n+1})\|^2 + \|\boldsymbol{H}(t_{n+1}) - \boldsymbol{H}^{n+1}\|^2)^{1/2} \le \tau \int_0^{t_{n+1}} (\|\partial_{tt}\nabla\times\boldsymbol{E}\| + \|\partial_{tt}\boldsymbol{H}\|) ds. \tag{14}$$

Proof. Subtracting (10) from (8) evaluated at t_{n+1}, we obtain the error equation

$$\frac{V^{n+1}-V^n}{\tau_n}+\begin{pmatrix}0 & -1\\ \nabla\times\nabla\times & 0\end{pmatrix}V^{n+1}=\begin{pmatrix}\gamma_E^{n+1}\\ \gamma_H^{n+1}\end{pmatrix}, \tag{15}$$

where we denote

$$\gamma_E^{n+1}=\frac{\boldsymbol{E}(t_{n+1})-\boldsymbol{E}(t_n)}{\tau_n}-\partial_t\boldsymbol{E}(t_{n+1}),\qquad \gamma_H^{n+1}=\frac{\boldsymbol{H}(t_{n+1})-\boldsymbol{H}(t_n)}{\tau_n}-\partial_t\boldsymbol{H}(t_{n+1}).$$

Applying Theorem 3.1 to the error equation (15) and using the fact that $e_E^0=e_H^0=0$, we obtain

$$(\|\nabla\times e_E^{n+1}\|^2+\|e_H^{n+1}\|^2)^{1/2}\le\sum_{m=0}^{n}\tau_m(\|\nabla\times\gamma_E^{m+1}\|+\|\gamma_H^{m+1}\|). \tag{16}$$

Note that

$$\begin{aligned}\|\gamma_H^{m+1}\|&=\left\|\frac{\boldsymbol{H}(t_{m+1})-\boldsymbol{H}(t_m)}{\tau_m}-\partial_t\boldsymbol{H}(t_{m+1})\right\|\\ &=\left\|-\frac{1}{\tau_m}\int_{t_m}^{t_{m+1}}(s-t_m)\partial_{tt}\boldsymbol{H}(s)ds\right\|\le\int_{t_m}^{t_{m+1}}\|\partial_{tt}\boldsymbol{H}\|ds.\end{aligned} \tag{17}$$

Similarly, we have

$$\|\nabla\times\gamma_E^{m+1}\|\le\int_{t_m}^{t_{m+1}}\|\partial_{tt}\nabla\times\boldsymbol{E}\|ds. \tag{18}$$

Substitution of (17) and (18) into (16) concludes the proof. □

3.2. A fully-discrete scheme

To design a finite element method for solving (5), we partition Ω by a family of regular cubic or tetrahedral meshes T_h with maximum mesh size h. An arbitrary order curl conforming finite element space $\boldsymbol{V}_h$ can be defined as follows (cf. [35]): For any $k\ge 1$, on tetrahedral elements,

$$\boldsymbol{V}_h=\{\boldsymbol{v}_h\in H(\text{curl};\Omega):\ \boldsymbol{v}_h|_K\in(p_{k-1})^3\oplus S_k,\ \forall K\in T_h\},$$

where the subspace $S_k=\{\vec{p}\in(\tilde{p}_k)^3:\ \boldsymbol{x}\cdot\vec{p}=0\}$; while on cubic elements,

$$\boldsymbol{V}_h=\{\boldsymbol{v}_h\in H(\text{curl};\Omega):\ \boldsymbol{v}_h|_K\in Q_{k-1,k,k}\times Q_{k,k-1,k}\times Q_{k,k,k-1},\ \forall K\in T_h\}.$$

Here $\tilde{p}_k$ denotes the space of homogeneous polynomials of degree k, and $Q_{i,j,k}$ denotes the space of polynomials whose degrees are less than or equal to i, j, k in variables x, y, z, respectively. To accommodate the boundary condition (3), we define a subspace of $\boldsymbol{V}_h$:

$$\boldsymbol{V}_h^0=\{\boldsymbol{v}_h\in\boldsymbol{V}_h:\hat{\boldsymbol{n}}\times\boldsymbol{v}_h=\boldsymbol{0}\}. \tag{19}$$

We need to define the $H_0(\text{curl};\Omega)$ orthogonal projection operator $\Pi_h\boldsymbol{u}:H_0(\text{curl};\Omega)\to V_h^0$, which satisfies [35, p. 171]

$$(\nabla\times(\Pi_h-I)\boldsymbol{u},\nabla\times\phi_h)+((\Pi_h-I)\boldsymbol{u},\phi_h)=0,\quad\forall\phi_h\in V_h^0. \tag{20}$$

Moreover, Π_h satisfies the interpolation error estimate [35]: If $\boldsymbol{u}\in\boldsymbol{H}^s(\Omega)$ and $\nabla\times\boldsymbol{u}\in\boldsymbol{H}^s(\Omega)$ for $\frac{1}{2}+\delta\le s\le k$ for $\delta>0$ then

$$\|\boldsymbol{u}-\Pi_h\boldsymbol{u}\|+\|\nabla\times(\boldsymbol{u}-\Pi_h\boldsymbol{u})\|\le Ch^s(\|\boldsymbol{u}\|_{\boldsymbol{H}^s(\Omega)}+\|\nabla\times\boldsymbol{u}\|_{\boldsymbol{H}^s(\Omega)}). \tag{21}$$

We consider the following fully-discrete scheme: Given appropriate approximations $\boldsymbol{E}_h^0$ and $\boldsymbol{E}_h^1$ in V_h^0, for $1 \le n \le N-1$ find $\boldsymbol{E}_h^{n+1} \in V_h^0$ such that

$$\left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n} - \frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}}, \phi_h\right) + \tau_n(\nabla\times\boldsymbol{E}_h^{n+1}, \nabla\times\phi_h) = \tau_n(\boldsymbol{f}^{n+1}, \phi_h), \quad \forall\, \phi_h \in V_h^0. \tag{22}$$

Theorem 3.3. *For any* $1 \le n \le N-1$, *the solution* $\boldsymbol{E}_h^{n+1}$ *of* (22) *satisfies the following stability:*

$$\left(\left\|\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n}\right\|^2 + \|\nabla\times\boldsymbol{E}_h^{n+1}\|^2\right)^{1/2} \le \left(\left\|\frac{\boldsymbol{E}_h^{1}-\boldsymbol{E}_h^0}{\tau_0}\right\|^2 + \|\nabla\times\boldsymbol{E}_h^{1}\|^2\right)^{1/2} + \sum_{m=1}^{n}\tau_m\|\boldsymbol{f}^{m+1}\|.$$

Proof. Denote $S^{n+1} = (\|\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n}\|^2 + \|\nabla\times\boldsymbol{E}_h^{n+1}\|^2)^{1/2}$. Choosing $\phi_h = \frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n}$ in (22) and following the proof of Theorem 3.1, we easily obtain

$$S^{n+1} \le S^n + \tau_n\|\boldsymbol{f}^{n+1}\|,$$

summing up which from $n = 1$ to n concludes the proof. □

To prove the error estimate for scheme (22), we need the following result.

Lemma 3.1. *The following estimates hold for any* $0 \le n \le N-1$:

$$\text{(i)}\ \left|\frac{u(t_{n+1})-u(t_n)}{\tau_n} - \frac{u(t_n)-u(t_{n-1})}{\tau_n}\right| \le \int_{t_{n-1}}^{t_{n+1}} |u_{tt}(s)|ds,\ \forall\, u_{tt} \in L^1(t_{n-1}, t_{n+1}), \tag{23}$$

$$\text{(ii)}\ \left|\left(u_t(t_{n+1}) - \frac{u(t_{n+1})-u(t_n)}{\tau_n}\right) - \left(u_t(t_n) - \frac{u(t_n)-u(t_{n-1})}{\tau_n}\right)\right|$$
$$\le \tau_n\int_{t_{n-1}}^{t_{n+1}} |u_{t^3}(s)|ds, \quad \forall\, u_{t^3} \in L^1(t_{n-1}, t_{n+1}). \tag{24}$$

Proof. (i) Using identity

$$u_t(t_{n+1}) - \frac{u(t_{n+1})-u(t_n)}{\tau_n} = \frac{1}{\tau_n}\int_{t_n}^{t_{n+1}} (s-t_n)u_{tt}(s)ds, \tag{25}$$

we obtain

$$\begin{aligned}
&\frac{u(t_{n+1})-u(t_n)}{\tau_n} - \frac{u(t_n)-u(t_{n-1})}{\tau_{n-1}}\\
&= u_t(t_{n+1}) - u_t(t_n) + \frac{1}{\tau_n}\int_{t_n}^{t_{n+1}}(t_n-s)u_{tt}(s)ds - \frac{1}{\tau_{n-1}}\int_{t_{n-1}}^{t_n}(t_{n-1}-s)u_{tt}(s)ds\\
&= \int_{t_n}^{t_{n+1}} u_{tt}(s)ds + \frac{1}{\tau_n}\int_{t_n}^{t_{n+1}}(t_n-s)u_{tt}(s)ds + \frac{1}{\tau_{n-1}}\int_{t_{n-1}}^{t_n}(s-t_{n-1})u_{tt}(s)ds\\
&= \frac{1}{\tau_n}\int_{t_n}^{t_{n+1}}(t_{n+1}-s)u_{tt}(s)ds + \frac{1}{\tau_{n-1}}\int_{t_{n-1}}^{t_n}(s-t_{n-1})u_{tt}(s)ds\\
&\le \int_{t_n}^{t_{n+1}}|u_{tt}(s)|ds + \int_{t_{n-1}}^{t_n}|u_{tt}(s)|ds,
\end{aligned}$$

which completes the proof.

(ii) From (25), we have

$$\left(u_t(t_{n+1}) - \frac{u(t_{n+1}) - u(t_n)}{\tau_n}\right) - \left(u_t(t_n) - \frac{u(t_n) - u(t_{n-1})}{\tau_{n-1}}\right)$$
$$= \int_{t_n}^{t_{n+1}} \left(\frac{s - t_n}{\tau_n}\right) u_{tt}(s)ds - \int_{t_{n-1}}^{t_n} \left(\frac{\hat{s} - t_{n-1}}{\tau_{n-1}}\right) u_{tt}(\hat{s})d\hat{s}. \tag{26}$$

Introducing the transformation $s - t_n = \frac{\tau_n}{\tau_{n-1}}(\hat{s} - t_{n-1})$ for the second integral, we can reduce the right hand side of (26) to

$$\int_{t_n}^{t_{n+1}} \left(\frac{s - t_n}{\tau_n}\right) u_{tt}(s)ds - \int_{t_n}^{t_{n+1}} \left(\frac{s - t_n}{\tau_n}\right) u_{tt}\left(t_{n-1} + \frac{\tau_{n-1}}{\tau_n}(s - t_n)\right) \cdot \frac{\tau_{n-1}}{\tau_n} ds$$
$$= \int_{t_n}^{t_{n+1}} \frac{s - t_n}{\tau_n} \left(\int_{t_{n-1}+\frac{\tau_{n-1}}{\tau_n}(s-t_n)}^{s} u_{t^3}(\mu)d\mu\right) ds$$
$$\leq \tau_n \int_{t_{n-1}}^{t_{n+1}} |u_{t^3}(s)|ds, \tag{27}$$

which, along with (26), concludes the proof. □

Theorem 3.4. *Let $k \geq 1$ be the order of the finite element space $\boldsymbol{V}_h$. Then under the assumptions*

$$\boldsymbol{f}_t, \boldsymbol{E}_{t^3} \in L^1(0, T; \boldsymbol{L}^2(\Omega)), \qquad \boldsymbol{E}_{t^2} \in L^1(0, T; H^k(curl; \Omega)), \qquad \boldsymbol{E} \in C^1(0, T; H^k(curl; \Omega)), \tag{28}$$

and

$$\left(\left\|\frac{(\boldsymbol{E}_h^1 - \Pi_h \boldsymbol{E}(t_1)) - (\boldsymbol{E}_h^0 - \Pi_h \boldsymbol{E}(t_0))}{\tau_0}\right\|^2 + \|\nabla \times (\boldsymbol{E}_h^1 - \Pi_h \boldsymbol{E}(t_1))\|^2\right)^{1/2} \leq C(\tau + h^k), \tag{29}$$

the following error estimate

$$\left(\left\|\frac{e_h^{n+1} - e_h^n}{\tau_n}\right\|^2 + \|\nabla \times e_h^{n+1}\|^2\right)^{1/2} \leq C(\tau + h^k),$$

holds true, where $e_h^n = \boldsymbol{E}_h^n - \boldsymbol{E}(t_n)$ denotes the error between the exact solution $\boldsymbol{E}$ of (5) at t_n and the solution $\boldsymbol{E}_h^n$ of scheme (22).

Proof. Integrating the continuous equation (5) from t_n to t_{n+1}, then multiplying the resultant by a test function $\phi \in H_0(\text{curl}; \Omega)$, we obtain

$$(\boldsymbol{E}_t(t_{n+1}) - \boldsymbol{E}_t(t_n), \phi) + \left(\int_{t_n}^{t_{n+1}} \nabla \times \boldsymbol{E}(s)ds, \nabla \times \phi\right) = \left(\int_{t_n}^{t_{n+1}} \boldsymbol{f}(s)ds, \phi\right). \tag{30}$$

From (22) and (30), we obtain the error equation: For any $\phi_h \in V_h^0$,

$$\left(\frac{e_h^{n+1} - e_h^n}{\tau_n} - \frac{e_h^n - e_h^{n-1}}{\tau_{n-1}}, \phi_h\right) + \tau_n(\nabla \times e_h^{n+1}, \nabla \times \phi_h)$$
$$= \tau_n(\boldsymbol{f}^{n+1}, \phi_h) - \left(\frac{\boldsymbol{E}(t_{n+1}) - \boldsymbol{E}(t_n)}{\tau_n} - \frac{\boldsymbol{E}(t_n) - \boldsymbol{E}(t_{n-1})}{\tau_{n-1}}, \phi_h\right) - \tau_n(\nabla \times \boldsymbol{E}(t_{n+1}), \nabla \times \phi_h)$$
$$= \left(\tau_n \boldsymbol{f}^{n+1} - \int_{t_n}^{t_{n+1}} \boldsymbol{f}(s)ds, \phi_h\right) + \left(\int_{t_n}^{t_{n+1}} \nabla \times \boldsymbol{E}(s)ds - \tau_n \nabla \times \boldsymbol{E}(t_{n+1}), \nabla \times \phi_h\right)$$
$$+ \left(\left(\boldsymbol{E}_t(t_{n+1}) - \frac{\boldsymbol{E}(t_{n+1}) - \boldsymbol{E}(t_n)}{\tau_n}\right) - \left(\boldsymbol{E}_t(t_n) - \frac{\boldsymbol{E}(t_n) - \boldsymbol{E}(t_{n-1})}{\tau_{n-1}}\right), \phi_h\right). \tag{31}$$

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Let us decompose the error e_h^n as

$$e_h^n = \rho_h^n + \xi_h^n = \left(\boldsymbol{E}_h^n - \Pi_h \boldsymbol{E}(t_n)\right) + \left(\Pi_h \boldsymbol{E}(t_n) - \boldsymbol{E}(t_n)\right).$$

Furthermore, we denote $S^{n+1} = (\|\frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\|^2 + \|\nabla\times\rho_h^{n+1}\|^2)^{1/2}$.

Choosing $\phi_h = \frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}$ in (31) and using the following identity

$$\begin{aligned}
&\left(\frac{\rho_h^{n+1}-\rho_h^n}{\tau_n} - \frac{\rho_h^n-\rho_h^{n-1}}{\tau_{n-1}}, \phi_h\right) + \tau_n(\nabla\times\rho_h^{n+1}, \nabla\times\phi_h)\\
&\quad = \left(\frac{e_h^{n+1}-e_h^n}{\tau_n} - \frac{e_h^n-e_h^{n-1}}{\tau_{n-1}}, \phi_h\right) + \tau_n(\nabla\times e_h^{n+1}, \nabla\times\phi_h)\\
&\qquad - \left(\frac{\xi_h^{n+1}-\xi_h^n}{\tau_n} - \frac{\xi_h^n-\xi_h^{n-1}}{\tau_{n-1}}, \phi_h\right) - \tau_n(\nabla\times\xi_h^{n+1}, \nabla\times\phi_h),
\end{aligned}$$

we have

$$\begin{aligned}
(S^{n+1})^2 &= \left(\frac{\rho_h^n-\rho_h^{n-1}}{\tau_{n-1}}, \frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right) + (\nabla\times\rho_h^{n+1}, \nabla\times\rho_h^n)\\
&\quad + \left(\tau_n \boldsymbol{f}^{n+1} - \int_{t_n}^{t_{n+1}} \boldsymbol{f}(s)ds, \frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right)\\
&\quad + \left(\left(\boldsymbol{E}_t(t_{n+1}) - \frac{\boldsymbol{E}(t_{n+1})-\boldsymbol{E}(t_n)}{\tau_n}\right) - \left(\boldsymbol{E}_t(t_n) - \frac{\boldsymbol{E}(t_n)-\boldsymbol{E}(t_{n-1})}{\tau_{n-1}}\right), \frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right)\\
&\quad + \left(\int_{t_n}^{t_{n+1}} \nabla\times\boldsymbol{E}(s)ds - \tau_n\nabla\times\boldsymbol{E}(t_{n+1}), \nabla\times\frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right)\\
&\quad - \left((\Pi_h - I)\left(\frac{\boldsymbol{E}(t_{n+1})-\boldsymbol{E}(t_n)}{\tau_n} - \frac{\boldsymbol{E}(t_n)-\boldsymbol{E}(t_{n-1})}{\tau_{n-1}}\right), \frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right)\\
&\quad + \tau_n\left((\Pi_h - I)\boldsymbol{E}(t_{n+1}), \frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right) = \sum_{i=1}^{7} Err_i,
\end{aligned} \tag{32}$$

where we used the projection property of Π_h and the fact that

$$\xi_h^{n+1} - \xi_h^n = (\Pi_h - I)(\boldsymbol{E}(t_{n+1}) - \boldsymbol{E}(t_n)).$$

Below we need to estimate each term Err_i. It is easy to see that

$$Err_1 + Err_2 \le S^{n+1}\cdot S^n.$$

Note that

$$\begin{aligned}
\left|\tau_n \boldsymbol{f}^{n+1} - \int_{t_n}^{t_{n+1}} \boldsymbol{f}(s)ds\right| &= \left|\int_{t_n}^{t_{n+1}} (\boldsymbol{f}(t_{n+1}) - \boldsymbol{f}(s))ds\right|\\
&= \left|\int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}} \boldsymbol{f}_t(\mu)d\mu ds\right| \le \tau_n \int_{t_n}^{t_{n+1}} |\boldsymbol{f}_t(s)|ds,
\end{aligned} \tag{33}$$

from which we have

$$Err_3 \le \tau_n \int_{t_n}^{t_{n+1}} \|\boldsymbol{f}_t(s)\|ds \cdot \left\|\frac{\rho_h^{n+1}-\rho_h^n}{\tau_n}\right\|.$$

By Lemma 3.1, we obtain

$$Err_4 \leq \tau_n \int_{t_n}^{t_{n+1}} \|\boldsymbol{E}_{t^3}(s)\| ds \cdot \left\| \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right\|.$$

Using integration by parts and (33) with $\boldsymbol{f} = \nabla \times \nabla \times \boldsymbol{E}$, we have

$$\begin{aligned} Err_5 &= \left(\int_{t_n}^{t_{n+1}} \nabla \times \nabla \times \boldsymbol{E}(s) ds - \tau_n \nabla \times \nabla \times \boldsymbol{E}(t_{n+1}), \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right) \\ &\leq \tau_n \int_{t_n}^{t_{n+1}} \|\partial_t \nabla \times \nabla \times \boldsymbol{E}(s)\| ds \cdot \left\| \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right\| \\ &\leq \tau_n \int_{t_n}^{t_{n+1}} (\|\boldsymbol{E}_{t^3}\| + \|\boldsymbol{f}_t\|) ds \cdot \left\| \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right\|, \end{aligned}$$

where in the last step we used Eq. (5).

By Lemma 3.1 and the interpolation error estimate (21), we have

$$\begin{aligned} Err_6 &\leq \int_{t_{n-1}}^{t_{n+1}} \|(\Pi_h - I)\boldsymbol{E}_{t^2}(s)\| ds \cdot \left\| \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right\| \\ &\leq Ch^k \int_{t_{n-1}}^{t_{n+1}} \|\boldsymbol{E}_{t^2}(s)\|_{H^k(\mathrm{curl};\Omega)} ds \cdot \left\| \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right\|. \end{aligned}$$

Finally, by the interpolation error estimate (21), we have

$$Err_7 \leq \tau_n \cdot Ch^k \|\boldsymbol{E}_{t^2}(t_{n+1})\|_{H^k(\mathrm{curl};\Omega)} \cdot \left\| \frac{\rho_h^{n+1} - \rho_h^n}{\tau_n} \right\|.$$

Substituting the above estimates of Err_i into (32), we have

$$\begin{aligned} S^{n+1} &\leq S^n + \tau_n \int_{t_n}^{t_{n+1}} (\|\boldsymbol{f}_t(s)\| + \|\boldsymbol{E}_{t^3}(s)\|) ds + \tau_n \int_{t_{n-1}}^{t_{n+1}} (\|\boldsymbol{f}_t(s)\| + \|\boldsymbol{E}_{t^3}(s)\|) ds \\ &\quad + Ch^k \int_{t_{n-1}}^{t_{n+1}} \|\boldsymbol{E}_{t^2}(s)\|_{H^k(\mathrm{curl};\Omega)} ds + \tau_n \cdot Ch^k \|\boldsymbol{E}(t_{n+1})\|_{H^k(\mathrm{curl};\Omega)}. \end{aligned} \tag{34}$$

Summing up (34) from $n = 1$ to n, we obtain

$$\begin{aligned} S^{n+1} &\leq S^1 + \tau \int_0^{t_{n+1}} (\|\boldsymbol{f}_t(s)\| + \|\boldsymbol{E}_{t^3}(s)\|) ds \\ &\quad + Ch^k \left(\int_0^{t_{n+1}} \|\boldsymbol{E}_{t^2}(s)\|_{H^k(\mathrm{curl};\Omega)} ds + t_{n+1} \cdot \|\boldsymbol{E}(t_{n+1})\|_{H^k(\mathrm{curl};\Omega)} \right), \end{aligned} \tag{35}$$

which, along with the assumption (29), triangle inequality and the interpolation error estimate (21), concludes the proof. □

4. The a posteriori error analysis

In this section, we develop some a posteriori error estimates for the problem (5).

Denote the linear interpolation $\boldsymbol{E}_\tau$ on each interval $[t_n, t_{n+1}]$ for the time discretization solution of (5), i.e., the solution of (10) with $\boldsymbol{g}^{n+1} = 0$:

$$\boldsymbol{E}_\tau(\cdot, t) = \frac{t - t_n}{\tau_n} \boldsymbol{E}^{n+1} + \frac{t_{n+1} - t}{\tau_n} \boldsymbol{E}^n, \quad t \in [t_n, t_{n+1}]. \tag{36}$$

An interesting property $\partial_t \boldsymbol{E}_\tau = \frac{\boldsymbol{E}^{n+1} - \boldsymbol{E}^n}{\tau_n}$ will be used later.

Let us consider the error equation

$$\partial_t \begin{pmatrix} \boldsymbol{E}-\boldsymbol{E}_\tau \\ \boldsymbol{H}-\boldsymbol{H}_\tau \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ \nabla\times\nabla\times & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{E}-\boldsymbol{E}_\tau \\ \boldsymbol{H}-\boldsymbol{H}_\tau \end{pmatrix} = \begin{pmatrix} R_E \\ R_H \end{pmatrix}. \tag{37}$$

The right hand side residuals of (37) can be obtained as follows:

$$R_E = \partial_t(\boldsymbol{E}-\boldsymbol{E}_\tau) - (\boldsymbol{H}-\boldsymbol{H}_\tau) = \boldsymbol{H}_\tau - \partial_t \boldsymbol{E}_\tau = \boldsymbol{H}_\tau - \boldsymbol{H}^{n+1},$$

and

$$\begin{aligned} R_H &= \partial_t(\boldsymbol{H}-\boldsymbol{H}_\tau) + \nabla\times\nabla\times(\boldsymbol{E}-\boldsymbol{E}_\tau) = \boldsymbol{E}_{tt} - \partial_t\boldsymbol{H}_\tau + \nabla\times\nabla\times\boldsymbol{E} - \nabla\times\nabla\times\boldsymbol{E}_\tau \\ &= \boldsymbol{f} - \frac{\boldsymbol{H}^{n+1}-\boldsymbol{H}^n}{\tau_n} - \nabla\times\nabla\times\boldsymbol{E}_\tau = \boldsymbol{f} - \boldsymbol{f}^{n+1} + \nabla\times\nabla\times(\boldsymbol{E}^{n+1}-\boldsymbol{E}_\tau). \end{aligned}$$

We can now prove our first a posteriori error estimates.

Theorem 4.1. *For any* $0 \le n \le N-1$, *we have*

$$\begin{aligned} &\|\nabla\times(\boldsymbol{E}(t_{n+1})-\boldsymbol{E}^{n+1})\| + \|\boldsymbol{H}(t_{n+1})-\boldsymbol{H}^{n+1}\| \\ &\quad \le \sum_{m=0}^{n}\left[\frac{\tau_m}{\sqrt{2}}(\|\nabla\times(\boldsymbol{H}^{m+1}-\boldsymbol{H}^m)\| + 2\|\nabla\times\nabla\times(\boldsymbol{E}^{m+1}-\boldsymbol{E}^m)\|) + 2\int_{t_m}^{t_{m+1}}\|\boldsymbol{f}-\boldsymbol{f}^{m+1}\|ds\right]. \end{aligned}$$

Proof. Denote $S(t) = (\|\nabla\times(\boldsymbol{E}-\boldsymbol{E}_\tau)\|^2 + \|\boldsymbol{H}-\boldsymbol{H}_\tau\|^2)^{1/2}$. Taking the inner product of (37) with $\begin{pmatrix} \nabla\times\nabla\times(\boldsymbol{E}-\boldsymbol{E}_\tau) \\ \boldsymbol{H}-\boldsymbol{H}_\tau \end{pmatrix}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{dS^2}{dt} &= (\nabla\times(\boldsymbol{H}_\tau-\boldsymbol{H}^{n+1}), \nabla\times(\boldsymbol{E}-\boldsymbol{E}_\tau)) + (\boldsymbol{f}-\boldsymbol{f}^{n+1}, \boldsymbol{H}-\boldsymbol{H}_\tau) \\ &\quad + (\nabla\times\nabla\times(\boldsymbol{E}^{n+1}-\boldsymbol{E}_\tau), \boldsymbol{H}-\boldsymbol{H}_\tau) \\ &\le (\|\nabla\times(\boldsymbol{H}_\tau-\boldsymbol{H}^{n+1})\|^2 + 2\|\boldsymbol{f}-\boldsymbol{f}^{n+1}\|^2 + 2\|\nabla\times\nabla\times(\boldsymbol{E}^{n+1}-\boldsymbol{E}_\tau)\|^2)^{1/2}\cdot S, \end{aligned}$$

integrating which with respect to t from t_n to t_{n+1} leads to

$$S(t_{n+1}) - S(t_n) \le \int_{t_n}^{t_{n+1}}\left(\|\nabla\times(\boldsymbol{H}_\tau-\boldsymbol{H}^{n+1})\|^2 + 2\|\boldsymbol{f}-\boldsymbol{f}^{n+1}\|^2 + 2\|\nabla\times\nabla\times(\boldsymbol{E}^{n+1}-\boldsymbol{E}_\tau)\|^2\right)^{1/2} ds. \tag{38}$$

Summing up (38) from $n=0$ to n, and using the fact that $S(0)=0$ and

$$\int_{t_m}^{t_{m+1}}\|\nabla\times(\boldsymbol{H}_\tau-\boldsymbol{H}^{m+1})\|(s)ds = \|\nabla\times(\boldsymbol{H}^{m+1}-\boldsymbol{H}^m)\|\int_{t_m}^{t_{m+1}}\frac{t_{m+1}-s}{\tau_m}ds = \frac{\tau_m}{2}\|\nabla\times(\boldsymbol{H}^{m+1}-\boldsymbol{H}^m)\|,$$

we have

$$\begin{aligned} &\frac{1}{\sqrt{2}}(\|\nabla\times(\boldsymbol{E}(t_{n+1})-\boldsymbol{E}^{n+1})\| + \|\boldsymbol{H}(t_{n+1})-\boldsymbol{H}^{n+1}\|) \le S(t_{n+1}) \\ &\quad \le \sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\left(\|\nabla\times(\boldsymbol{H}_\tau-\boldsymbol{H}^{m+1})\| + \sqrt{2}\|\boldsymbol{f}-\boldsymbol{f}^{m+1}\| + \sqrt{2}\|\nabla\times\nabla\times(\boldsymbol{E}^{m+1}-\boldsymbol{E}_\tau)\|\right)(s)ds \\ &\quad = \sum_{m=0}^{n}\left[\frac{\tau_m}{2}(\|\nabla\times(\boldsymbol{H}^{m+1}-\boldsymbol{H}^m)\| + \sqrt{2}\|\nabla\times\nabla\times(\boldsymbol{E}^{m+1}-\boldsymbol{E}^m)\|) + \sqrt{2}\int_{t_m}^{t_{m+1}}\|\boldsymbol{f}-\boldsymbol{f}^{m+1}\|ds\right], \end{aligned}$$

which completes the proof. □

Now let us consider a posteriori error estimates for the fully discrete scheme (22). Let $e_h^{n+1} = \boldsymbol{E}_h^{n+1} - \boldsymbol{E}^{n+1}$ be the error between the solution $\boldsymbol{E}^{n+1}$ of (10) and the solution $\boldsymbol{E}_h^{n+1}$ of (22).

From (10) and (22), we obtain the error equation

$$
\begin{aligned}
&\left(\frac{e_h^{n+1}-e_h^n}{\tau_n}-\frac{e_h^n-e_h^{n-1}}{\tau_{n-1}},\phi\right)+\tau_n(\nabla\times e_h^{n+1},\nabla\times\phi)\\
&=\left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n}-\frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}},\phi\right)+\tau_n(\nabla\times\boldsymbol{E}_h^{n+1},\nabla\times\phi)\\
&\quad-\left(\frac{\boldsymbol{E}^{n+1}-\boldsymbol{E}^n}{\tau_n}-\frac{\boldsymbol{E}^n-\boldsymbol{E}^{n-1}}{\tau_{n-1}},\phi\right)-\tau_n(\nabla\times\boldsymbol{E}^{n+1},\nabla\times\phi)\\
&=\left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n}-\frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}},\phi\right)+\tau_n(\nabla\times\boldsymbol{E}_h^{n+1},\nabla\times\phi)-\tau_n(\boldsymbol{f}^{n+1},\phi)\equiv r(\phi),
\end{aligned}
\tag{39}
$$

which holds true for any $\phi\in H_0(\text{curl};\Omega)$.

Before we prove the a posteriori error estimates for the fully discrete scheme, some auxiliary tools are needed.

Lemma 4.1 *([36, Lemma 2.1]).* *The space $H_0(curl;\Omega)$ has the following orthogonal decomposition*

$$H_0(curl;\Omega)=H_0^0(curl;\Omega)\oplus H_0^{\perp}(curl;\Omega),$$

where $H_0^0(curl;\Omega)\equiv\{\boldsymbol{v}\in H_0(curl;\Omega):\ \nabla\times\boldsymbol{v}=\boldsymbol{0}\}$ and $H_0^{\perp}(curl;\Omega)\equiv\{\boldsymbol{v}\in H_0(curl;\Omega):\ (\boldsymbol{v},\boldsymbol{v}^0)=0,\ \boldsymbol{v}^0\in H_0^0(curl;\Omega)\}$.

The spaces $H_0^0(\text{curl};\Omega)$ and $H_0^{\perp}(\text{curl};\Omega)$ have the following characteristics.

Lemma 4.2 *([36, Lemma 2.2]).* *If the domain Ω is simply connected with connected boundary, then we have $H_0^0(curl;\Omega)=\nabla H_0^1(\Omega)$. On the other hand, for any $\boldsymbol{v}\in H_0^{\perp}(curl;\Omega)$ we have $\|\boldsymbol{v}\|\le C\|\nabla\times\boldsymbol{v}\|$, where the constant C only depends on Ω.*

Below we shall need the following regular decomposition for space $H_0(\text{curl};\Omega)$.

Lemma 4.3 *([36, Lemma 2.3]).* *If the domain Ω is a bounded Lipschitz domain, then for any $\boldsymbol{v}\in H_0(curl;\Omega)$, there exists some $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)\cap H_0(curl;\Omega)$ and $\phi\in H_0^1(\Omega)$ such that $\boldsymbol{v}=\boldsymbol{w}+\nabla\phi$ with the estimate $\|\boldsymbol{w}\|_1+\|\phi\|_1\le C\|\boldsymbol{v}\|_{H(curl;\Omega)}$.*

Moreover, we shall use the following approximation property.

Lemma 4.4 *([36, Lemma 2.4]).* *Denote S_h for the continuous piecewise linear finite element subspace of $H_0^1(\Omega)$ on T_h, the operators $I_h: H_0^1(\Omega)\to S_h$ from [37] and $\Pi_h:\boldsymbol{H}^1(\Omega)\cap H_0(curl;\Omega)\to V_h$ from [38]. Then for any $\phi\in H_0^1(\Omega)$ and any $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)\cap H_0(curl;\Omega)$, we have*

$$
\begin{aligned}
&\|\phi-I_h\phi\|_{0,K}\le Ch_K|\phi|_{1,D_K}, &&\forall K\in T_h,\\
&\|\phi-I_h\phi\|_{0,F}\le Ch_F^{1/2}|\phi|_{1,D_F}, &&\forall F\in F_h,\\
&\|\boldsymbol{w}-\Pi_h\boldsymbol{w}\|_{0,K}\le Ch_K|\boldsymbol{w}|_{1,D_K}, &&\forall K\in T_h,\\
&\|\boldsymbol{w}-\Pi_h\boldsymbol{w}\|_{0,F}\le Ch_F^{1/2}|\boldsymbol{w}|_{1,D_F}, &&\forall F\in F_h,
\end{aligned}
$$

where the constant C depends on the shape regularity of the mesh T_h, and D_K (resp. D_F) denotes the union of elements in T_h with non-empty intersection with K (resp. F).

Denote the error

$$S^{n+1}=\left(\left\|\frac{e_h^{n+1}-e_h^n}{\tau_n}\right\|^2+\|\nabla\times e_h^{n+1}\|^2\right)^{1/2}, \tag{40}$$

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the differential equation residuals on each element K of T_h:

$$R_K(\boldsymbol{E}_h) = \frac{1}{\tau_n}\left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n} - \frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}}\right) + \nabla\times\nabla\times\boldsymbol{E}_h^{n+1} - \boldsymbol{f}^{n+1},$$

$$D_K(\boldsymbol{E}_h) = \nabla\cdot\left(\frac{1}{\tau_n}\left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n} - \frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}}\right) - \boldsymbol{f}^{n+1}\right),$$

and the normal jump

$$J_F(\boldsymbol{E}_h) = \left[\left(\frac{1}{\tau_n}\left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n} - \frac{\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}}\right) - \boldsymbol{f}^{n+1}\right)\cdot\hat{\boldsymbol{n}}\right]_F.$$

Theorem 4.2. *Given proper approximations $\boldsymbol{E}_h^0$ and $\boldsymbol{E}_h^1$, for any $1 \le n \le N-1$, we have*

$$S^{n+1} \le \left(\left\|\frac{(\boldsymbol{E}_h^1-\boldsymbol{E}^1)-(\boldsymbol{E}_h^0-\boldsymbol{E}^0)}{\tau_0}\right\|^2 + \|\nabla\times(\boldsymbol{E}_h^1-\boldsymbol{E}^1)\|^2\right)^{1/2}$$
$$+ C\sum_{m=1}^{n}\left(\sum_{K\in T_h} h_K(\|R_K(\boldsymbol{E}_h)\|_{0,K} + \|D_K(\boldsymbol{E}_h)\|_{0,K})\right.$$
$$\left. + \sum_{F\in F_h} h_F^{1/2}(\|[\hat{\boldsymbol{n}}\times curl\boldsymbol{E}_h^{m+1}]_F\|_{0,F} + \|J_F(\boldsymbol{E}_h)\|_{0,F})\right).$$

Proof. Choosing $\phi = \frac{e_h^{n+1}-e_h^n}{\tau_n}$ in (39), and using the Cauchy–Schwarz inequality, we have

$$(S^{n+1})^2 \le S^n\cdot S^{n+1} + r(\phi). \tag{41}$$

Below we shall estimate $r(\phi)$. Since $\phi = \frac{e_h^{n+1}-e_h^n}{\tau_n} \in H_0(\text{curl};\Omega)$, by Lemma 4.1 we can decompose ϕ as

$$\phi = \phi^\perp + \phi^0,$$

where $\phi^\perp \in H_0^\perp(\text{curl};\Omega)$ and $\phi^0 \in H_0^0(\text{curl};\Omega)$. By Lemma 4.2, ϕ^0 can be written as $\phi^0 = \nabla\xi$ for some $\xi \in H_0^1(\Omega)$. Furthermore, by Lemma 4.3, $\phi^\perp$ can be decomposed as

$$\phi^\perp = \varphi + \nabla\psi,$$

where $\varphi \in \boldsymbol{H}^1(\Omega)\cap H_0(\text{curl};\Omega)$ and $\psi \in H_0^1(\Omega)$.

With the above decompositions, we see that

$$r(\phi) = r(\varphi) + r(\nabla\psi) + r(\nabla\xi). \tag{42}$$

Using integration by parts and Lemma 4.4, we have

$$\begin{aligned} r(\varphi) &= r(\varphi - \Pi_h\varphi)\\ &= \tau_n\sum_{K\in T_h}(R_K(\boldsymbol{E}_h), \varphi-\Pi_h\varphi)_K + \tau_n\sum_{F\in F_h}([\hat{\boldsymbol{n}}\times \text{curl}\boldsymbol{E}_h^{n+1}]_F, \varphi-\Pi_h\varphi)_F\\ &\le \tau_n\left(\sum_{K\in T_h}\|R_K(\boldsymbol{E}_h)\|_{0,K}\cdot h_K + \sum_{F\in F_h} h_F^{1/2}\|[\hat{\boldsymbol{n}}\times\text{curl}\boldsymbol{E}_h^{n+1}]_F\|_{0,F}\right)|\varphi|_1. \end{aligned} \tag{43}$$

By Lemmas 4.1–4.3, and definition (40), we obtain

$$|\varphi|_1 \le C\|\phi^\perp\|_{H(\text{curl};\Omega)} \le C\|\nabla\times\phi^\perp\| = C\|\nabla\times\phi\| \le \frac{C}{\tau_n}(S^{n+1}+S^n). \tag{44}$$

Substituting (44) into (43), we have

$$r(\varphi) \le C\left(\sum_{K\in T_h} h_K\|R_K(\boldsymbol{E}_h)\|_{0,K} + \sum_{F\in F_h} h_F^{1/2}\|[\hat{\boldsymbol{n}}\times \mathrm{curl}\boldsymbol{E}_h^{n+1}]_F\|_{0,F}\right)(S^{n+1}+S^n). \tag{45}$$

Similarly, using integration by parts and Lemma 4.4, we can obtain

$$\begin{aligned} r(\nabla\psi) &= r(\nabla(\psi - I_h\psi)) \\ &= \left(\frac{\boldsymbol{E}_h^{n+1}-\boldsymbol{E}_h^n}{\tau_n} - \frac{\boldsymbol{E}_h^{n}-\boldsymbol{E}_h^{n-1}}{\tau_{n-1}} - \tau_n \boldsymbol{f}^{n+1}, \nabla(\psi - I_h\psi)\right) \\ &= -\tau_n\sum_{K\in T_h}(D_K(\boldsymbol{E}_h), \psi - I_h\psi)_K + \tau_n\sum_{F\in F_h}(J_F(\boldsymbol{E}_h), \psi - I_h\psi)_F \\ &\le \tau_n\left(\sum_{K\in T_h}\|D_K(\boldsymbol{E}_h)\|_{0,K}\cdot h_K + \sum_{F\in F_h} h_F^{1/2}\|J_F(\boldsymbol{E}_h)\|_{0,F}\right)|\psi|_1 \\ &\le \left(\sum_{K\in T_h} h_K\|D_K(\boldsymbol{E}_h)\|_{0,K} + \sum_{F\in F_h} h_F^{1/2}\|J_F(\boldsymbol{E}_h)\|_{0,F}\right)(S^{n+1}+S^n), \end{aligned} \tag{46}$$

where in the last step we used the estimate

$$|\psi|_1 \le C\|\phi^\perp\|_{H(\mathrm{curl};\Omega)} \le C\|\nabla\times\phi^\perp\| = C\|\nabla\times\phi\| \le \frac{C}{\tau_n}(S^{n+1}+S^n).$$

By the same technique, we can obtain

$$\begin{aligned} r(\nabla\xi) &= r(\nabla(\xi - I_h\xi)) \\ &\le \tau_n\left(\sum_{K\in T_h}\|D_K(\boldsymbol{E}_h)\|_{0,K}\cdot h_K + \sum_{F\in F_h} h_F^{1/2}\|J_F(\boldsymbol{E}_h)\|_{0,F}\right)|\xi|_1 \\ &\le \tau_n\left(\sum_{K\in T_h} h_K\|D_K(\boldsymbol{E}_h)\|_{0,K} + \sum_{F\in F_h} h_F^{1/2}\|J_F(\boldsymbol{E}_h)\|_{0,F}\right)S^{n+1}, \end{aligned} \tag{47}$$

where in the last step we used the following estimate

$$|\xi|_1 = \|\phi^0\| \le \|\phi\| \le S^{n+1}. \tag{48}$$

Substituting the estimates (42)–(47) into (41), we have

$$\begin{aligned} (S^{n+1})^2 \le S^n\cdot S^{n+1} + C\Bigg[&\sum_{K\in T_h} h_K(\|R_K(\boldsymbol{E}_h)\|_{0,K} + \|D_K(\boldsymbol{E}_h)\|_{0,K}) \\ &+ \sum_{F\in F_h} h_F^{1/2}(\|[\hat{\boldsymbol{n}}\times \mathrm{curl}\boldsymbol{E}_h^{n+1}]_F\|_{0,F} + \|J_F(\boldsymbol{E}_h)\|_{0,F})\Bigg](S^{n+1}+S^n). \end{aligned} \tag{49}$$

Adding $(S^n)^2$ to both sides of (49), we obtain

$$\begin{aligned} \frac{1}{2}(S^{n+1}+S^n)^2 \le (S^{n+1})^2 + (S^n)^2 \le S^n(S^{n+1}+S^n) + C\Bigg[&\sum_{K\in T_h} h_K(\|R_K(\boldsymbol{E}_h)\|_{0,K} + \|D_K(\boldsymbol{E}_h)\|_{0,K}) \\ &+ \sum_{F\in F_h} h_F^{1/2}(\|[\hat{\boldsymbol{n}}\times \mathrm{curl}\boldsymbol{E}_h^{n+1}]_F\|_{0,F} + \|J_F(\boldsymbol{E}_h)\|_{0,F})\Bigg](S^{n+1}+S^n), \end{aligned} \tag{50}$$

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which can be reduced to

$$S^{n+1} \le S^n + 2C\left(\sum_{K\in T_h} h_K(\|R_K(\boldsymbol{E}_h)\|_{0,K} + \|D_K(\boldsymbol{E}_h)\|_{0,K})\right.$$
$$\left. + \sum_{F\in F_h} h_F^{1/2}(\|[\hat{\boldsymbol{n}} \times \mathrm{curl}\boldsymbol{E}_h^{n+1}]_F\|_{0,F} + \|J_F(\boldsymbol{E}_h)\|_{0,F})\right). \tag{51}$$

Summing up both sides of (51) with respect to n concludes the proof. □

5. Conclusions

In this paper, we initiate the study of a posteriori error estimates for time-dependent Maxwell's equations. Estimators are obtained for both semi and fully discrete schemes. Though our proofs are given for $\epsilon = \mu = 1$, we believe that similar results can be directly proved for variable parameters ϵ and μ if they are bounded below and above in Ω. Further derivation of other error estimators for more general cases (including other type boundary conditions) will be explored. The numerical implementation of the a posteriori error estimators for time-dependent Maxwell's equations will be considered in the future, since our past works on wave propagation simulation in metamaterials [30] showed that the adaptive finite element method seems necessary for 3D time-domain cloaking simulation [39].

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