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Constructing B-spline representation of quadratic Sibson–Thomson splines ☆

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ABSTRACT

In this paper, we show how to construct a normalized B-spline basis for a special $\mathcal{C}^1$ continuous splines of degree 2, defined on Sibson–Thomson refinement. The basis functions have a local support, they are nonnegative, and they form a partition of unity. The dilatation equation can be found by applying the dyadic subdivision scheme directly to the Sibson–Thomson spline basis functions. As an application, a quasi-interpolation method, based on this Sibson–Thomson B-spline representation, is described which can be used for the efficient visualization of gridded surface data.

1. Introduction

Sibson–Thomson (ST) splines are functions in the space $S_2^1(\Delta_2^*)$ of $\mathcal{C}^1$ continuous piecewise quadratic functions on a ST refinement (uniform criss-cross triangulation). Such a refinement Δ_2^* can be obtained from a rectangular mesh by dividing each rectangular patch into four rectangles by arbitrary parallels to its vertical and horizontal edges. Then, each subrectangle is subdivided into four triangles by its diagonals. The structure of these splines has been analyzed in Beatson and Ziegler (1985), Lorente-Pardo et al. (2000), Sibson and Thomson (1981), Sibson (1980) and they can be computed in Bernstein–Bézier form (Farin, 1986; Lorente-Pardo et al., 2000). In Dubuc and Merrien (1999), given f and ∇f at the vertices of a rectangular mesh, Dubuc and Merrien have studied an algorithm HR^1 building an interpolating $\mathcal{C}^1$ function. As an example, they show that the ST element can be obtained by HR^1. The connection between the refinable Hermite interpolant and ST Hermite interpolation on subrectangles was described in Han et al. (2003).

Sorokina and Zeilfelder (2004) have presented a method to construct quasi-interpolation operators based on quadratic $\mathcal{C}^1$-splines on uniform type-2 triangulation. The Bernstein–Bézier coefficients of the piecewise quadratic polynomials are directly determined by appropriate combinations of the data values. Their motivation, it seems difficult – if not impossible – to construct local and stable bases for splines $\mathcal{C}^1$-splines on uniform type-2 triangulation. However, representing complex surfaces requires the use of a large number of Bézier triangles. Preserving a certain degree of continuity between all patches results in a large set of non-trivial relations between their Bézier ordinates. These relations are hard to implement

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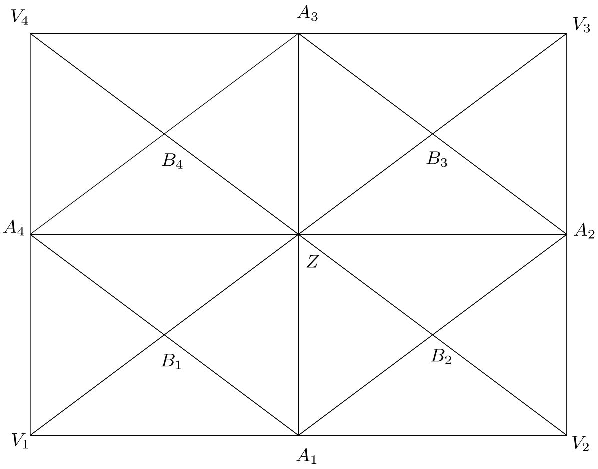

Fig. 1. ST subdivision of a rectangle.

and they hinder the designer to make predictable local changes (Vanraes, 2004). On the other hand, the operator described in Sorokina and Zeilfelder (2004) was first developed in Chui and He (1986) (see also Dagnino and Lamberti, 2005; Sablonnière, 2003) by using a completely different approach based on the so-called box-splines, which are locally supported and not linearly independent splines. A well-known example where linear independence does not hold is the $\mathcal{C}^1$-quadratic Zwart–Powell element defined on a four-directional (criss-cross) partition of the plane (Wang, 2001).

The aim of this paper is to show that ST splines could be compactly represented in a normalized B-spline basis. The basis functions have a local support, they are nonnegative, and they form a partition of unity. This representation has an intuitive geometric interpretation involving tangent control triangles. There exist many sets of basis functions with a similar construction based on Powell–Sabin partitions with different degrees (see for example Dierckx, 1997; Lamnii et al., 2014; Speleers, 2010a, 2013b). Similar B-spline representation for bivariate reduced Clough–Tocher splines has been considered in Speleers (2010b). Recently, a simplex spline basis for the $\mathcal{C}^1$-quadratics on the Powell–Sabin-12 with all the usual properties of the univariate B-spline basis was discovered (Cohen et al., 2013).

In addition, we discuss dyadic subdivision scheme for the ST spline surfaces. The goal is to calculate the B-spline representation of a surface on a dyadic refinement of the given triangulation. More precisely, we show that the basis functions are translated and dilated of one vectors of scaling function. Hence, the dilatation equation can be found by applying the subdivision scheme directly to the ST spline basis functions. Moreover, we use some results on blossoming to establish general Marsden identities representing polynomials of at most degree 2 in terms of ST B-splines of $\mathcal{C}^1$-smoothness. As a consequence, we employ the bivariate polarization formulas to construct different families of differential and discrete quasi-interpolants (q.i.s.) reproducing bivariate polynomials of degree 2 and have an optimal approximation order.

The paper is organized as follows. Section 2 recalls the definition of the ST spline space. Section 3 describes the construction of a normalized B-spline basis, which is based on determining a set of triangles that contain a specific set of points, and the Marsden's identity for the $\mathcal{C}^1$-continuous ST splines. In Section 4 we discuss dyadic subdivision scheme for the ST spline surfaces. In Section 5, we present an approach to construct the q.i.s. based on this representation. The approach is illustrated by the construction of Hermite interpolant and particular q.i.s. based on function evaluations. Finally, in order to illustrate our results, we give in Section 6 some numerical examples.

2. ST splines

Let $h > 0$, $n, m \geq 2$ and $\{(x_i, y_j) = (ih, jh),\ 0 \leq i \leq n,\ 0 \leq j \leq m\}$ be the set of $(n+1) \times (m+1)$ points in the rectangular domain $R := [0, nh] \times [0, mh]$. Then the collection of rectangles $R_{i,j} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where $i = 0, \ldots, n-1$, $j = 0, \ldots, m-1$, forms a partition of R. The so-called ST refinement Δ_2^* of R is defined by dividing each rectangle $R_{i,j}$ into four rectangles by arbitrary parallels to its vertical and horizontal edges. Then, each subrectangle is subdivided into four triangles by its diagonals: thus, ST partition does define a uniform criss-cross triangulation consisting of 16 triangles (see Fig. 1).

The ST space $S_2^1(\Delta_2^*)$ is the space of piecewise quadratic $\mathcal{C}^1$ continuous functions on a ST refinement Δ_2^*, with additional smoothness around some edges. More precisely, for each function $s \in S_2^1(\Delta_2^*)$, $\partial s/\partial x$ is linear along the edges $x = x_i$ and $x = x_{i+1}$ of the subrectangle $R_{i,j}$ and the derivative $\partial s/\partial y$ is linear along the edges $y = y_j$ and $y = y_{j+1}$. It is well known (cf. Sibson and Thomson, 1981) that the dimension of $S_2^1(\Delta_2^*)$ is $3(n+1)(m+1)$. Furthermore, any element of $s \in S_2^1(\Delta_2^*)$ is uniquely specified by its value and its gradient at $V_{i,j} := (x_i, y_j)^T$, for $i = 0, \ldots, n$ and $j = 0, \ldots, m$, and can be locally constructed on each triangle of Δ_2^* once these values and gradients are given, see Farin (1986), Sibson and Thomson (1981). Hence, for any given set of $(f_{i,j}, f_{x,i,j}, f_{y,i,j})$-values, a spline $s \in S_2^1(\Delta_2^*)$ can be defined by means of the following Hermite interpolation problem:

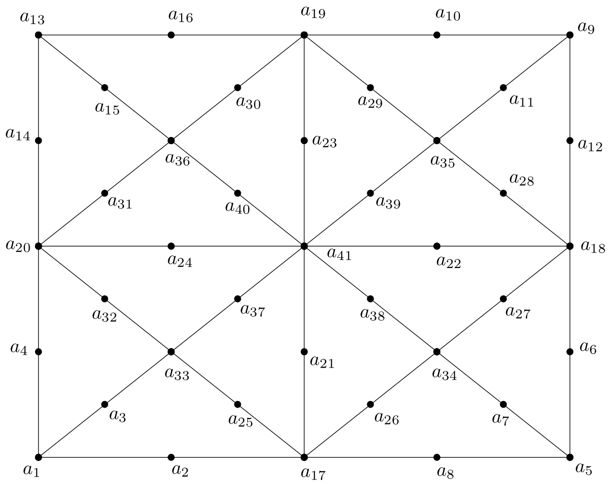

Fig. 2. B-coefficients of s.

$$s(V_{i,j}) = f_{i,j}, \quad \frac{\partial s}{\partial x}(V_{i,j}) = f_{x,i,j}, \quad \frac{\partial s}{\partial y}(V_{i,j}) = f_{y,i,j}, \tag{1}$$

for $i = 0, \dots, n$ and $j = 0, \dots, m$.

Let $\mathbb{P}_d$ denote the linear space of algebraic polynomials of degree less than or equal to d. We use the piecewise Bernstein–Bézier representation (B-form) of the splines, i.e., for each spline $s \in S_2^1(\Delta_2^*)$, the polynomial piece $p = s_{|T} \in \mathbb{P}_2$ on a triangle $T \in \Delta_2^*$ is given by

$$p(x, y) = \sum_{i+j+k=2} b_{i,j,k} \mathfrak{B}^d_{i,j,k}(\lambda_1, \lambda_2, \lambda_3).$$

Here $\mathfrak{B}^d_{i,j,k} = \frac{2}{i!j!k!}\lambda_1^i \lambda_2^j \lambda_3^k$ are the six quadratic Bernstein polynomials associated with T, and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ are the barycentric coordinates of a point $(x, y)^T \in \mathbb{R}^2$ with respect to the triangle T.

We now show how to compute the Bernstein–Bézier representation of the ST spline satisfying (1). In order to simplify the construction, we develop the ST element on a reference square $S(V_1, V_2, V_3, V_4)$ of vertices V_i, $i = 1, \dots, 4$ (see Fig. 1). Setting

$$A_1 = \frac{1}{2}(V_1 + V_2), \;\; A_2 = \frac{1}{2}(V_2 + V_3), \;\; A_3 = \frac{1}{2}(V_3 + V_4), \;\; A_4 = \frac{1}{2}(V_4 + V_1),$$
$$B_1 = \frac{1}{2}(Z + V_1), \;\; B_2 = \frac{1}{2}(Z + V_2), \;\; B_3 = \frac{1}{2}(Z + V_3), \;\; B_4 = \frac{1}{2}(Z + V_4),$$
$$Z = \frac{1}{2}(V_1 + V_3).$$

Suppose that the ST-spline s is defined by means of interpolation problem (1). On each micro-triangle in the square $S(V_1, V_2, V_3, V_4)$, the spline s is a quadratic polynomial that can be represented in its Bernstein–Bézier formulation. The corresponding Bézier ordinates are schematically represented in Fig. 2.

In view of the C^1-smoothness at vertex V_1, the Bézier ordinates in the neighborhood of V_1 are found as

$$a_1 = s(V_1), \tag{2}$$
$$a_2 = a_1 + \frac{1}{2}(A_1 - V_1).\nabla s(V_1), \tag{3}$$
$$a_3 = a_1 + \frac{1}{2}(B_1 - V_1).\nabla s(V_1). \tag{4}$$

Similarly, the coefficients $a_5, \dots, a_{16}$ can be computed from the C^1 smoothness condition at V_i, $i = 2, 3, 4$. By the C^1 smoothness at A_1 we obtain

$$a_{17} = \frac{1}{2}(a_2 + a_8). \tag{5}$$

The coefficients a_{18}, a_{19}, a_{20} can be computed in the same way.

Using straightforward computations, we obtain

$$a_{21} = -\frac{a_1}{2} - \frac{a_5}{2} + a_3 + a_7, \tag{6}$$

with similar formulas for a_{22}, a_{23} and a_{24}.

We also have

$$a_{25} = \frac{1}{2}(a_2 + a_{21}). \tag{7}$$

In the same way, $a_{27}, \dots, a_{32}$ can be determined.

Similarly, we have

$$a_{33} = \frac{1}{2}(a_{32} + a_{25}), \tag{8}$$

$$a_{37} = \frac{1}{2}(a_{24} + a_{21}). \tag{9}$$

Finally, we get

$$a_{41} = \frac{1}{2}(a_{21} + a_{23}). \tag{10}$$

By using a similar argument, we can determine the expression of $a_{34}, a_{35}, a_{36}, a_{38}, a_{39}, a_{40}$.

3. A normalized B-spline representation for ST-splines

In this section, a basis of $S_2^1(\Delta_2^*)$ can be constructed in a similar way as for the basis described in Dierckx (1997) for the quadratic Powell–Sabin spline space. More precisely, we are looking a B-spline representation for ST-splines

$$s(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3} c_{i,j,k} B_{i,j,k}(x, y), \quad (x, y)^T \in R, \tag{11}$$

where the basis functions satisfy

$$B_{i,j,k}(x, y) \geq 0, \qquad 1 = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3} B_{i,j,k}(x, y), \tag{12}$$

and have local support.

To construct the basis functions $B_{i,j,k}$, we use a geometric method suggested by Dierckx (1997) and some results presented in Speleers (2013b).

Let $M_{i,j}$ be the subset of R consisting of the points belonging to the union of all the rectangles $R_{i_1,j_1} = [x_{i_1}, x_{i_1+1}] \times [y_{j_1}, y_{j_1+1}]$ containing the vertex $V_{i,j}$. From the Bézier–Bernstein representation, it is immediate to see that if we would like ST B-splines to have minimal support then the support for any $B_{i,j,k}$ is contained in $M_{i,j}$.

For each vertex $V_{i,j}$ the three functions $B_{i,j,k}(x, y)$, $k = 1, 2, 3$, can be locally constructed over the $M_{i,j}$ once their values and gradients at each vertex are given. Due to the structure of the support we have $B_{i,j,k}$ is zero with its first derivatives at any vertex except for $V_{i,j}$. Moreover, we denote:

$$\beta_{i,j,k}^{ab} := \frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{i,j,k}(V_{i,j}) \text{ for } 0 \leq a + b \leq 1. \tag{13}$$

For each vertex $V_{i,j}$ we associate an arbitrary triangle $t_{i,j}(Q_{i,j,1}, Q_{i,j,2}, Q_{i,j,3})$ with vertices $Q_{i,j,k} = (X_{i,j,k}, Y_{i,j,k})^T$, $k = 1, 2, 3$. From such a ST triangle one can uniquely determine the values $\{\beta_{i,j,k}^{ab},\ 0 \leq a + b \leq 1\}$ of the three B-splines $B_{i,j,k}$ at the vertex $V_{i,j}$ as

$$\begin{pmatrix} \beta_{i,j,1}^{00} & \beta_{i,j,2}^{00} & \beta_{i,j,3}^{00} \\ \beta_{i,j,1}^{10} & \beta_{i,j,2}^{10} & \beta_{i,j,3}^{10} \\ \beta_{i,j,1}^{01} & \beta_{i,j,2}^{01} & \beta_{i,j,3}^{01} \end{pmatrix} \begin{pmatrix} X_{i,j,1} & Y_{i,j,1} & 1 \\ X_{i,j,2} & Y_{i,j,2} & 1 \\ X_{i,j,3} & Y_{i,j,3} & 1 \end{pmatrix} = \begin{pmatrix} x_i & y_j & 1 \\ x_i & y_j & 1 \\ x_i & y_j & 1 \end{pmatrix}$$

More precisely, from Speleers (2013b) we have

$$\beta_{i,j,1}^{ab} = \frac{\partial^{a+b}}{\partial x^a \partial y^b} \mathfrak{B}_{100}^1(V_{i,j}), \beta_{i,j,2}^{ab} = \frac{\partial^{a+b}}{\partial x^a \partial y^b} \mathfrak{B}_{010}^1(V_{i,j}), \beta_{i,j,3}^{ab} = \frac{\partial^{a+b}}{\partial x^a \partial y^b} \mathfrak{B}_{001}^1(V_{i,j}), \tag{14}$$

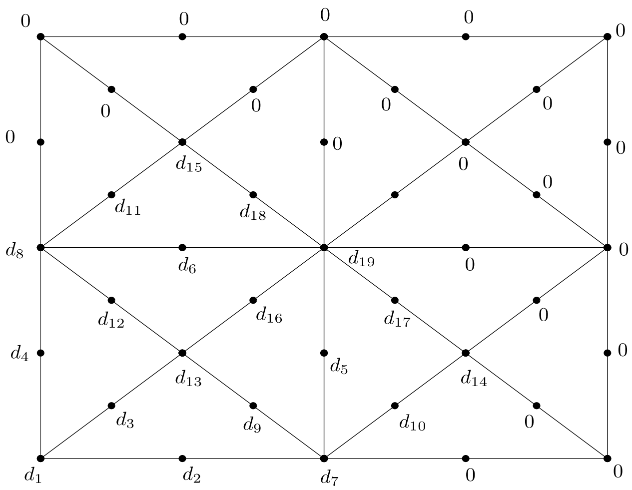

Fig. 3. Schematic representation of the Bézier ordinates of the B-spline.

for all $0 \le a + b \le 1$, where $\mathfrak{B}^1_{i_1 i_2 i_3}$ are the Bernstein basis polynomials of degree 1 with respect to a triangle $t_{i,j}$. Consequently, the B-splines $B_{i,j,k}(x, y)$ constructed using the set of triangles $t_{i,j}$ form a partition of unity and the following expansions hold

$$x = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=1}^{3} X_{i,j,k} B_{i,j,k}(x, y), \quad y = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=1}^{3} Y_{i,j,k} B_{i,j,k}(x, y).$$

The positivity of a normalized ST-basis depends on finding a set of ST-triangles that must contain a number of specified points. Let

$$\mathcal{Z} := \{Z_{i,j} = (V_{i,j} + V_{i+1,j+1})/2,\ 0 \le i \le n-1,\ 0 \le j \le m-1\}$$

the set of the centers of all squares $R_{i,j}$. We can then give a geometric approach to satisfy the nonnegativity conditions.

Theorem 1. *The B-splines $B_{i,j,k}(x, y)$ constructed using a set of triangles $t_{i,j}$ are nonnegative on $R_{i,j}$ if each triangle $t_{i,j}$ contains its corresponding vertex $V_{i,j}$ and the points*

$$\frac{3}{4}V_{i,j} + \frac{1}{4}V_{i+1,j},\ \frac{1}{2}(V_{i,j} + Z_{i,j}),\ \frac{3}{4}V_{i,j} + \frac{1}{4}V_{i,j+1}.$$

Proof. We use a similar line of argument as in the Powell–Sabin splines (see Section 4 of Dierckx, 1997) and reduced cubic Clough–Tocher splines (see Section 3.2 of Speleers, 2010b). We consider again the macro-square $\mathcal{S}(V_1, V_2, V_3, V_4)$ depicted in Fig. 1 such that the corresponding B-splines with respect to a vertex V_1 are noted by $B_{1,k}$, $k = 1, 2, 3$. The Bernstein–Bézier representation of the B-spline $B_{1,k}$ is schematically represented in Fig. 3. In order to derive the conditions for the nonnegativity of $B_{1,k}$ on the square $\mathcal{S}(V_1, V_2, V_3, V_4)$, it is sufficient to request

$$d_i \ge 0, \quad i = 1, \ldots, 6. \tag{15}$$

Put

$$P_1 = V_1,\ P_2 = \frac{1}{2}\big(V_1 + A_1\big),\ P_3 = B_1,\ P_4 = \frac{1}{2}\big(V_1 + A_4\big).$$

Without loss of generality we assume $k = 1$. We recall from (13) and (14) that

$$\frac{\partial^{a+b}}{\partial x^a \partial y^b} B_{1,1}(V_1) = \beta^{ab}_{1,1} \text{ for } 0 \le a + b \le 1, \tag{16}$$

where

$$\beta^{ab}_{1,1} = \frac{\partial^{a+b}}{\partial x^a \partial y^b} \mathfrak{B}^1_{100}(V_1),$$

with $\mathfrak{B}^1_{100}$ is the Bernstein polynomial of degree 1 with respect to a triangle $t_1(Q_1, Q_2, Q_3)$.

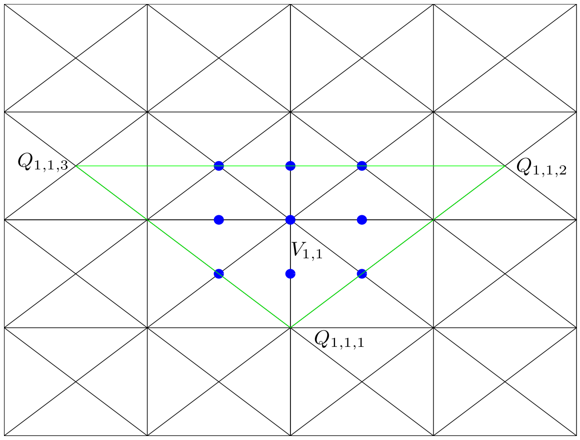

Fig. 4. The split of a domain triangle.

Let us denote $f(V) := \mathfrak{B}^1_{100}(V)$, then

$$f(V) = f(V_1) + (V - V_1).\nabla f(V_1). \tag{17}$$

As presented in above section, we have

$$d_1 = f(V_1) = f(P_1),$$
$$d_2 = d_1 + \frac{1}{2}(A_1 - V_1).\nabla f(V_1) = f\left(\frac{A_1 + V_1}{2}\right) = f(P_2),$$
$$d_3 = f\left(\frac{P_1 + P_3}{2}\right), \quad d_4 = f(P_4),$$
$$d_6 = d_5 = -\frac{1}{2}d_1 + d_3 = \frac{1}{2}f(V_1) + \frac{1}{2}(B_1 - V_1).\nabla f(V_1) = \frac{1}{2}f(P_3).$$

Each Bernstein polynomial is nonnegative on its domain triangle. It follows that $f(V) \geq 0$ for all $V \in t_1$. It follows that the nonnegativity conditions (15) for B-spline $B_{1,1}$ on rectangle $\mathcal{S}(V_1, V_2, V_3, V_4)$ are satisfied when the triangle t_1 contains the points P_1, P_2, P_3 and P_4. □

Summarizing, the ST B-splines associated to each vertex $V_{i,j}$ are uniquely associated to the triple of points $Q_{i,j,k}$, $k = 1, 2, 3$, forming the ST triangle. ST triangles are very useful to geometrically identify and describe ST B-splines and their properties. ST triangles are not uniquely defined (see Fig. 4). One possibility for their construction is to calculate a triangle of minimal area subjected to the constraints of Theorem 1. As Powell–Sabin B-splines (Dierckx, 1997), we can propose an optimization strategy to select triangles with minimal area ensuring positivity of the corresponding ST B-splines.

In the following result, we give Hermite interpolation rules for $\mathcal{C}^1$ quadratic ST-splines in the normalized B-spline representation which is completely identical to what is known for Powell–Sabin splines, see Section 4.1 of Speleers (2013a) and reduced Clough–Tocher splines, see Eq. (4.2) given in Speleers (2010b).

Proposition 2. *Let $s \in \mathcal{S}^1_2(\Delta^*_2)$, then*

$$s(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3}(s(V_{i,j}) + (Q_{i,j,k} - V_{i,j}).\nabla s(V_{i,j}))B_{i,j,k}(x, y), \ (x, y)^T \in R.$$

We shall study a ST B-spline representation of ST-splines of class $\mathcal{C}^1$ or all quadratic polynomials in terms of their blossoms. The blossom of a polynomial $p \in \mathbb{P}_d$ (introduced in Ramshaw, 1989) is the unique multivariate polynomial $\mathcal{B}[p_d]$ that satisfies the following three conditions:

- $\mathcal{B}[p_d]$ is symmetric,

  $$\mathcal{B}[p_d](z_1, \ldots, z_d) = \mathcal{B}[p_d](z_{\pi(1)}, \ldots, z_{\pi(d)}),$$

  for any permutation π of the integers $1, \ldots, d$.

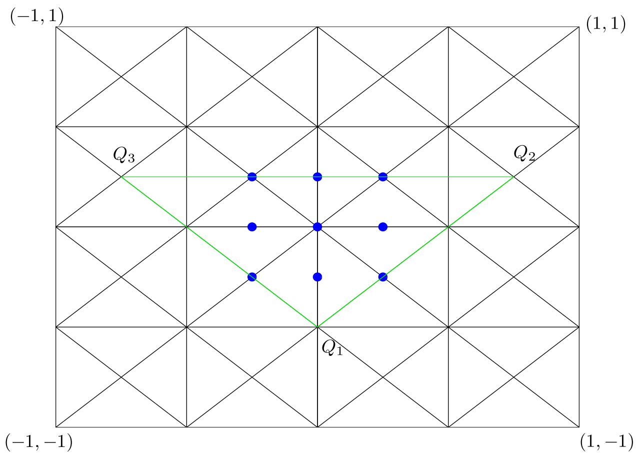

Fig. 5. ST-points and ST-triangle containing the ST-points.

- $\mathcal{B}[p_d]$ is multiaffine,

$$\mathcal{B}[p_d](z_1, (\alpha\widehat{z} + \beta\widetilde{z}), z_3, \ldots, z_d) = \alpha\mathcal{B}[p_d](z_1, \widehat{z}, z_3, \ldots, z_d) + \beta\mathcal{B}[p_d](z_1, \widetilde{z}, z_3, \ldots, z_d),$$

  where $\alpha + \beta = 1$.
- $\mathcal{B}[p_d]$ is diagonal, $p_d(z) = \mathcal{B}[p_d](\underbrace{z, \ldots, z}_{d})$, for all $z \in \mathbb{R}^2$.

With blossoming we have a simple but powerful tool for determining the B-spline coefficients of ST-splines. Put $\widetilde{Q}_{i,j,k} = -V_{i,j} + 2Q_{i,j,k}$, $i = 0, \ldots, n$, $j = 0, \ldots, m$, and $k = 1, 2, 3$. We can also express the coefficients of ST splines using blossoming in a similar way as in Theorem 7 of Sbibih et al. (2009) (see also Theorem 7 in Lamnii et al., 2014).

Proposition 3. *For any $s \in \mathcal{S}_2^1(\Delta_2^*)$ we have*

$$s(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{j=1}^{3}\mathcal{B}[s_{i,j}]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right)B_{i,j,k}(x, y),$$

where $s_{i,j}$ is the restriction of s to one of the triangles of Δ_2^* having $V_{i,j}$ as vertex.

Corollary 1. *For any polynomial $p \in \mathbb{P}_2$, we have*

$$p(x, y) = \sum_{i=0}^{n}\sum_{i=0}^{m}\sum_{k=1}^{3}\mathcal{B}[p]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right)B_{i,j,k}(x, y), \ \forall (x, y)^T \in R.$$

4. Dyadic subdivision scheme

Dubuc and Merrien (1999) show that the quadratic finite element of ST can be constructed by Hermite dyadic interpolation. Lounsberg et al. (1997) proved that with every subdivision scheme, a sequence of nested linear spaces, with the corresponding basis functions, the scaling functions can be associated. It is then possible to associate the ST B-splines and HR^1 presented in Dubuc and Merrien (1999).

4.1. Uniform ST B-splines

In order to simplify the construction the uniform ST B-splines, we give explicitly the 3 B-spline functions $B_k(x, y)$, $k = 1, 2, 3$, on a reference square $[-1, 1] \times [-1, 1]$, where the center of their support is at $(0, 0)^T$. The three ST B-splines associated to a vertex $(0, 0)^T$ are uniquely associated to the triple of points $Q_1 = (0, -\frac{1}{2})^T$, $Q_2 = (\frac{3}{4}, \frac{1}{4})^T$ and $Q_3 = (-\frac{3}{4}, \frac{1}{4})^T$, see Fig. 5.

Any point $V = (x, y)^T$ in the plane of the triangle can be uniquely expressed in terms of the barycentric coordinates $\lambda_j(x, y)$, $j = 1, 2, 3$ with respect to $t(Q_1, Q_2, Q_3)$.

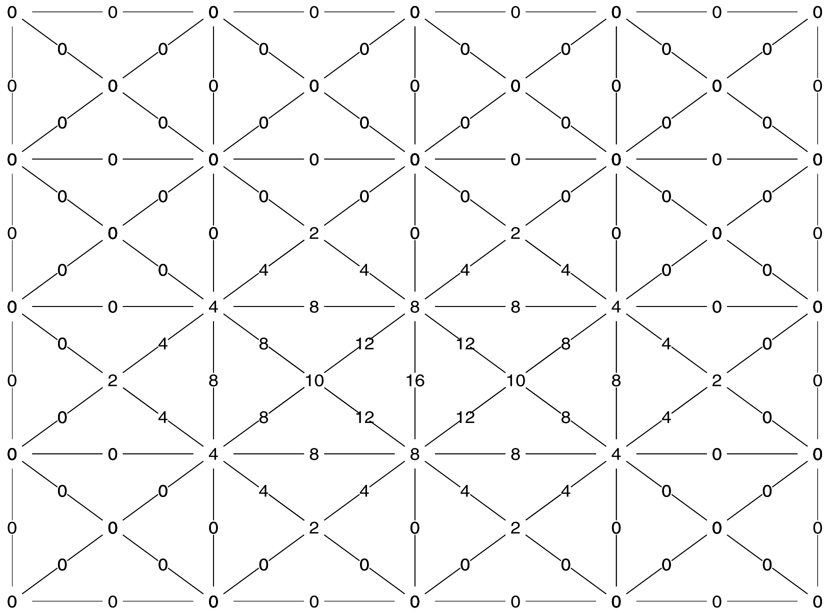

Fig. 6. The Bézier coefficients of B-spline B_1 divided by 24.

Since,

$$x = \frac{3}{4}\lambda_2(x, y) - \frac{3}{4}\lambda_3(x, y),$$
$$y = -\frac{1}{2}\lambda_1(x, y) + \frac{1}{4}\lambda_2(x, y) + \frac{1}{4}\lambda_3(x, y),$$
$$1 = \lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y),$$

it follows that

$$\lambda_1(x, y) = \frac{1}{3} - \frac{4y}{3}, \tag{18}$$
$$\lambda_2(x, y) = \frac{2x}{3} + \frac{2y}{3} + \frac{1}{3}, \tag{19}$$
$$\lambda_3(x, y) = -\frac{2x}{3} + \frac{2y}{3} + \frac{1}{3}. \tag{20}$$

From (13), the three B-splines $B_k(x, y)$, $k = 1, 2, 3$ are determined by

$$(\beta^{00}_{0,1}, \beta^{10}_{0,1}, \beta^{01}_{0,1}) = (\frac{1}{3}, 0, -\frac{4}{3}), \quad (\beta^{00}_{0,2}, \beta^{10}_{0,2}, \beta^{01}_{0,2}) = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$$
$$(\beta^{00}_{0,3}, \beta^{10}_{0,3}, \beta^{01}_{0,3}) = (\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}).$$

Hence, Figs. 6, 7 and 8 show respectively the Bézier representation of B-splines $B_k(x, y)$, $k = 1, 2, 3$. From the previous, it follows that for a given uniform ST-triangulation, the basis functions are translations of three functions B_k:

$$\mathbf{B}(u) = \begin{bmatrix} B_1(u) \\ B_2(u) \\ B_3(u) \end{bmatrix}$$

4.2. Refinement of B-spline functions

Noted by ϕ_j, $j = 1, 2, 3$ the classical Hermite basis of ST element. Let $\Phi = (\phi_1, \phi_2, \phi_3)^T$. From Dubuc and Merrien (1999), we have the following refinement equation:

$$\Phi = \sum_{\alpha \in \mathbb{Z}^2} A(\alpha)\Phi(2. - \alpha), \tag{21}$$

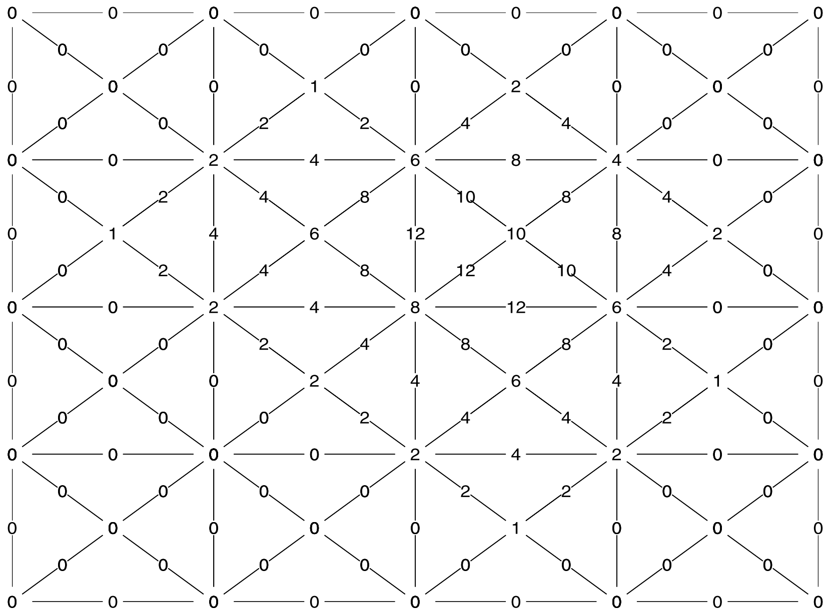

Fig. 7. The Bézier coefficients of B-spline B_2 divided by 24.

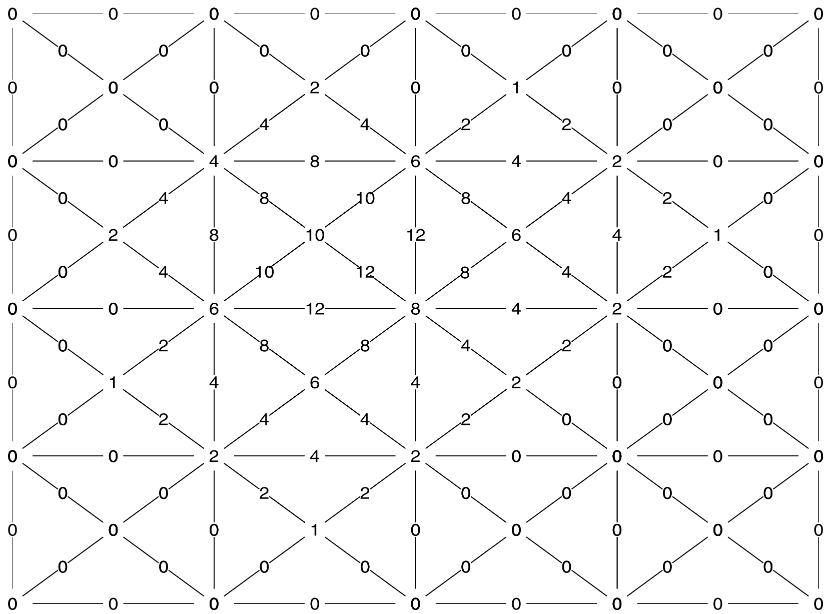

Fig. 8. The Bézier coefficients of B-spline B_3 divided by 24.

where

$$A(0,0)=\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&\frac{1}{2}\end{bmatrix},\quad A(\eta_1,0)=\begin{bmatrix}\frac{1}{2}&-\eta_1&0\\\frac{\eta_1}{8}&-\frac{1}{4}&0\\0&0&\frac{1}{4}\end{bmatrix},$$

$$A(0,\eta_2)=\begin{bmatrix}\frac{1}{2}&0&-\eta_2\\0&\frac{1}{4}&0\\\frac{\eta_2}{8}&0&-\frac{1}{4}\end{bmatrix},\quad A(\eta_1,\eta_2)=\begin{bmatrix}\frac{1}{4}&-\frac{\eta_1}{2}&-\frac{\eta_2}{2}\\\frac{\eta_1}{16}&-\frac{1}{8}&-\frac{\eta_1\eta_2}{8}\\\frac{\eta_2}{16}&-\frac{\eta_1\eta_2}{8}&-\frac{1}{8}\end{bmatrix},$$

with $\eta_1,\ \eta_2\ \in\{\pm 1\}$ and $\operatorname{supp} A=\{(0,0),\pm(1,0),\pm(0,1),\pm(1,1)\}$.

Let us denote

$$M=\begin{pmatrix}\beta_{0,1}^{00} & \beta_{0,2}^{00} & \beta_{0,3}^{00}\\ \beta_{0,1}^{10} & \beta_{0,2}^{10} & \beta_{0,3}^{10}\\ \beta_{0,1}^{01} & \beta_{0,2}^{01} & \beta_{0,3}^{01}\end{pmatrix}.$$

It is easy to see that

$$\mathbf{B}(u)=M.\Phi(u). \tag{22}$$

The dilatation equation, which expresses the multi-scaling function in terms of translations and dilations of itself, can be found by applying the subdivision scheme directly to the uniform ST-spline basis functions:

$$\mathbf{B}=\sum_{\alpha\in\mathbb{Z}^2} MA(\alpha)M^{-1}\mathbf{B}(2\cdot-\alpha). \tag{23}$$

Here, we are primarily interested in the case where

$$Q_1=(0,-\frac{1}{2}),\ Q_2=(\frac{3}{4},\frac{1}{4})\text{ and }Q_3=(-\frac{3}{4},\frac{1}{4}).$$

Then, we have

$$M=\begin{bmatrix}\frac{1}{3} & 0 & -\frac{4}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3}\end{bmatrix}\quad\text{and}\quad M^{-1}=\begin{bmatrix}1 & 1 & 1\\ 0 & \frac{3}{4} & -\frac{3}{4}\\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\end{bmatrix}.$$

By (23), we have

$$\mathbf{B}=\sum_{\alpha\in\mathbb{Z}^2}\overline{A}(\alpha)\mathbf{B}(2.-\alpha),$$

where

$$\overline{A}(0,0)=\begin{bmatrix}\frac{2}{3} & \frac{1}{6} & \frac{1}{6}\\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3}\end{bmatrix},\ \overline{A}(\eta_1,0)=\begin{bmatrix}\frac{1}{3} & \frac{1}{12}(1-3\eta_1) & \frac{1}{12}(3\eta_1+1)\\ \frac{1}{12}(\eta_1+1) & \frac{1}{12}(1-2\eta_1) & \frac{1}{3}(\eta_1+1)\\ \frac{1}{12}(1-\eta_1) & \frac{1}{3}(1-\eta_1) & \frac{1}{12}(2\eta_1+1)\end{bmatrix},$$

$$\overline{A}(0,\eta_2)=\begin{bmatrix}0 & \frac{1}{4}(1-\eta_2) & \frac{1}{4}(1-\eta_2)\\ \frac{1}{4}(\eta_2+1) & \frac{1}{4} & 0\\ \frac{1}{4}(\eta_2+1) & 0 & \frac{1}{4}\end{bmatrix},$$

$$\overline{A}(\eta_1,\eta_2)=\begin{bmatrix}0 & \frac{1}{8}(\eta_1-1)(\eta_2-1) & -\frac{1}{8}(\eta_1+1)(\eta_2-1)\\ \frac{1}{24}(\eta_1+3)(\eta_2+1) & -\frac{1}{12}\eta_1(\eta_2+1) & \frac{1}{24}(\eta_1(\eta_2+4)+3)\\ -\frac{1}{24}(\eta_1-3)(\eta_2+1) & \frac{1}{24}(3-\eta_1(\eta_2+4)) & \frac{1}{12}\eta_1(\eta_2+1)\end{bmatrix},$$

and $\operatorname{supp}\overline{A}=\{(0,0),\pm(1,0),\pm(0,1),\pm(1,1)\}$.

5. Local quadratic $\mathcal{C}^1$ spline quasi-interpolants

We can now define a linear operator $\mathcal{Q}$ mapping $\mathcal{C}(R)$ onto the space $S_2^1(\Delta_2^*)$. More precisely, for each $f\in\mathcal{C}(R)$ we define

$$\mathcal{Q}f=\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3}\lambda_{i,j,k}(f)B_{i,j,k} \tag{24}$$

where $\lambda_{i,j,k}$ are suitable linear functionals. First of all we note that, if

$$f(V_{i,j}),\ \nabla f(V_{i,j}),\ i=0,\ldots,n,\ j=0,\ldots,m, \tag{25}$$

are given, setting

$$\lambda_{i,j,k}(f):=f(V_{i,j})+(Q_{i,j,k}-V_{i,j}).\nabla f(V_{i,j}), \tag{26}$$

then, from Proposition 2, expression (24) provides the unique element in $S_2^1(\Delta_2^*)$ which interpolates the data (25). So, the scheme (24) with coefficients given by (26) is a quasi-interpolating (actually Hermite interpolating) scheme in $S_2^1(\Delta_2^*)$ which obviously reproduces $S_2^1(\Delta_2^*)$.

5.1. A general method for constructing quasi-interpolants based ST splines

In this subsection we introduce general methods that will be applied in Subsections 5.2 and 5.3 for the construction of bivariate quadratic q.i.s. More precisely, we introduce general methods for constructing q.i.s. of the form

$$Q^{[r]}f = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3} \lambda_{i,j,k}^{[r]} f B_{i,j,k}(x,y) \tag{27}$$

and satisfying

$$Q^{[r]}f = f, \quad \text{for all } f \in \mathbb{P}_r, \tag{28}$$

where $r = 0, 1, 2$ and $\lambda_{i,j,k}^{[r]}$, $i = 0, \ldots, n$, $j = 0, \ldots, m$, $k = 1, 2, 3$, are suitable linear functionals. The following results are essentially taken from Sbibih et al. (2009).

Theorem 4. *Let $r = 0, 1, 2$. For each $0 \le i \le n$, $0 \le j \le m$ and $k = 1, 2, 3$, let $I_{i,j,k}^{[r]} f$ be the unique polynomial in $\mathbb{P}_r|_{M_{i,j}}$ that interpolates or approximates some scattered data values and derivatives of f such that for all $p \in \mathbb{P}_r$, we have*

$$I_{i,j,k}^{[r]} p(x,y) = p(x,y), \quad \text{for all } (x,y)^T \in M_{i,j}.$$

Then, the q.i. $Q^{[r]}$ of the form (27) with $\lambda_{i,j,k}^{[r]}$ defined by

$$\lambda_{i,j,k}^{[r]} = \mathcal{B}\left[I_{i,j,k}^{[r]} f\right](V_{i,j}, \widetilde{Q}_{i,j,k})$$

satisfies (28).

Proposition 5. *For each $0 \le i \le n$, $0 \le j \le m$ and $k = 1, 2, 3$, suppose that there exists a subtriangle $\tau_{i,j,k}$ of Δ_2^*, with $V_{i,j}$ as vertex, that contains the data sites which determine the polynomial $I_{i,j,k}^{[2]} f$. Then the q.i. $Q^{[2]} f$ of the form (27) reproduces $\mathcal{S}_2^1(\Delta_2^*)$, i.e.,*

$$Q^{[2]} f = f, \quad \text{for all } f \in \mathcal{S}_2^1(\Delta_2^*).$$

5.2. A quasi-interpolant based on the Taylor polynomial

Let f be a function of class $\mathcal{C}^3$ on R. For each $0 \le i \le n$, $0 \le j \le m$, and $k = 1, 2, 3$, we let $I_{i,j,k}^{[r]} f$ be the Taylor polynomial of degree $r \le 2$ at the point $Z_{i,j}^{(k)} = \left(x_{i,j}^{(k)}, y_{i,j}^{(k)}\right)^T$,

$$I_{i,j,k}^{[r]} f(x,y) := \sum_{0 \le l_1 + l_2 \le r} \frac{1}{l_1! l_2!} \frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial y^{l_2}} f\left(x_{i,j}^{(k)}, y_{i,j}^{(k)}\right) \left(x - x_{i,j}^{(k)}\right)^{l_1} \left(y - y_{i,j}^{(k)}\right)^{l_2}.$$

Consider the q.i.

$$Q^{[r]} f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3} \mathcal{B}[I_{i,j,k}^{[r]} f](V_{i,j}, \widetilde{Q}_{i,j,k}) B_{i,j,k}(x,y). \tag{29}$$

Hence, from Theorem 4, $Q^{[r]} f$ reproduces all polynomials of degree r.

We now give some examples with different choices of $Z_{i,j}^{(k)}$. The Schoenberg–Marsden type scheme

$$Q^{[1]} f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=1}^{3} f(Q_{i,j,k})\, B_{i,j,k}(x,y),$$

can be obtained by choosing $Z_{i,j}^{(k)} = Q_{i,j,k}$ in (29), with $r = 1$. So that,

$$Q^{[1]} p = p, \quad \forall\, p \in \mathbb{P}_1.$$

This Schoenberg operator has excellent shape preserving properties. For instance, it maps convex functions on convex functions.

For $r=2$, if we choose $Z_{i,j}^{(k)} = V_{i,j}$, $0 \le i \le n$, $0 \le j \le m$ and $k = 1,2,3$. Then, we have

$$\mathcal{B}[I_{i,j,k}^{[2]} f]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) = f(V_{i,j}) + (Q_{i,j,k} - V_{i,j}).\nabla f(V_{i,j}). \tag{30}$$

Hence, from Proposition 2 we deduce that the scheme (29) with $Z_{i,j}^{(k)} = V_{i,j}$, is the Hermite interpolating scheme which obviously reproduces the whole spline space $\mathcal{S}_2^1(\Delta_2^*)$.

Now, by choosing $Z_{i,j}^{(k)} = Q_{i,j,k}$ in (29), for $0 \le i \le n$, $0 \le j \le m$, $k = 1,2,3$, we get

$$\mathcal{B}[I_{i,j,k}^{[2]} f]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) = f(Q_{i,j,k}) - \frac{1}{2}(Q_{i,j,k} - V_{i,j})^T \nabla^2 f(Q_{i,j,k})(Q_{i,j,k} - V_{i,k}).$$

Finally, as $\nabla^2 p(Q_{i,j,k}) = \nabla^2 p(V_{i,j})$ for all $p \in \mathbb{P}_2$, we deduce that the q.i. (29) with coefficients

$$f(Q_{i,j,k}) - \frac{1}{2}(Q_{i,j,k} - V_{i,j})^T \nabla^2 f(V_{i,j})(Q_{i,j,k} - V_{i,k}) \tag{31}$$

reproduces $\mathbb{P}_2$. We note that the formula (31) has also an equivalent for Powell–Sabin splines, see Theorem 2 in Manni and Sablonnière (2007).

5.3. Quasi-interpolants based on point evaluators

Frequently in practice one has to approximate given values at scattered data points where no derivative information is provided. In this case, we propose some q.i.s. based on point evaluators. More precisely, we look for q.i.s. of the form (27) with

$$\lambda_{i,j,k}^{[2]}(f) = \sum_{l=1}^{6} q_{i,j,k}^{(l)} f\left(Z_{i,j}^{(k,l)}\right), \quad q_{i,j,k}^{(l)} \in \mathbb{R}. \tag{32}$$

More precisely, let $Z_{i,j}^{(k,l)}$, $l = 1, \ldots, 6$, be six points satisfying a specific geometric configuration, for example the GC condition (see Chui and He, 1986). Then there exists a Lagrange basis $L_{i,j}^{(k,l)}(x,y)$, such that $L_{i,j}^{(k,l)}\left(Z_{i,j}^{(k,m)}\right) = \delta_{l,m}$, $l,m = 1, \ldots, 6$, and the polynomial of degree 2

$$I_{i,j,k}^{[2]} f(x,y) = \sum_{l=1}^{6} f\left(Z_{i,j}^{(k,l)}\right) L_{i,j}^{(k,l)}(x,y),$$

interpolates f at the points $Z_{i,j}^{(k,l)}$, $l = 1, \ldots, 6$.

Consequently, from Theorem 4, we have the following result.

Proposition 6. *Let $Q^{[2]} f$ be any q.i. of the form (27) such that*

$$\lambda_{i,j,k}^{[2]}(f) = \sum_{l=1}^{6} \mathcal{B}\left[L_{i,j}^{(k,l)}\right]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) f\left(Z_{i,j}^{(k,l)}\right). \tag{33}$$

Then, $Q^{[2]} p = p$, for all $p \in \mathbb{P}_2$.

In order to minimize the number of needed values of f, it is convenient to select the points $Z_{i,j}^{(k,l)}$, $l = 1, \ldots, 6$ as follows.

Proposition 7. *For each $0 \le i \le n$, $0 \le j \le m$ and $k = 1,2,3$, if the points $Z_{i,j}^{(k,l)}$, $l = 1,2,3$ are collinear with $V_{i,j}$ and $\widetilde{Q}_{i,j,k}$. Then, we have*

$$\mathcal{B}\left[L_{i,j}^{(k,l)}\right]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) = 0, \quad \textit{for } l = 4,5,6. \tag{34}$$

Proof. For each $0 \le i \le m$, $0 \le j \le n$ and $k = 1,2,3$ we assume that $Z_{i,j}^{(k,l)}$, $l = 1, \ldots, 6$, satisfy the GC condition. The GC condition implies that

$$L_{i,j}^{(k,l)}(x,y) = \frac{R_{i,j,k}^{(l,1)}(x,y) R_{i,j,k}^{(l,2)}(x,y)}{R_{i,j,k}^{(l,1)}(Z_{i,j}^{(k,l)}) R_{i,j,k}^{(l,2)}(Z_{i,j}^{(k,l)})}, \tag{35}$$

where $R^{(l,1)}_{i,j,k}$, $R^{(l,2)}_{i,j,k}$ are two lines containing $Z^{(k,m)}_{i,j}$, $m=1,\dots,6$, and $m \neq l$. Hence, if $Z^{(k,l)}_{i,j}$, $l=1,2,3$ are collinear with $V_{i,j}$ and $\widetilde{Q}_{i,j,k}$, we deduce that for $l=4,5,6$, $R^{(l,1)}_{i,j,k}$ or $R^{(l,2)}_{i,j,k}$ is the line $\left(V_{i,j}\widetilde{Q}_{i,j,k}\right)$. Consequently, it is easy to verify that (34) is satisfied. □

We will work out a specific case study. Given a polynomial $p \in \mathbb{P}_2$. Let Z_{ijk}, $i+j+k=2$, be 6 points satisfying a specific geometric configuration, for example the GC condition (see Chui and He, 1986). The geometric condition for bivariate interpolation is equivalent to the existence of a Lagrange formulas whose terms are products of linear factors. Then there exists a Lagrange basis p_{ijk} such that

$$p_{ijk}(V) = \prod_{\mu=0}^{i-1} \frac{a_\mu(V)}{a_\mu(Z_{ijk})} \prod_{\nu=0}^{j-1} \frac{b_\nu(V)}{b_\nu(Z_{ijk})} \prod_{\kappa=0}^{k-1} \frac{c_\kappa(V)}{c_\kappa(Z_{ijk})}$$

where a_μ, b_ν and c_κ are linear polynomials such that

$$\begin{aligned}
a_\mu &: \text{ the line passing through the points } Z_{\mu jk} \text{ with } \mu+j+k=2,\\
b_\nu &: \text{ the line passing through the points } Z_{i\nu k} \text{ with } i+\nu+k=2,\\
c_\kappa &: \text{ the line passing through the points } Z_{ij\kappa} \text{ with } i+j+\kappa=2.
\end{aligned}$$

Then,

$$p = \sum_{i+j+k=2} p(Z_{ijk}) p_{ijk}.$$

We can then find the polar form $\mathcal{B}[p]$ by substituting the elementary symmetric functions into the Lagrange interpolant. Consequently, we get

$$\mathcal{B}[p] = \sum_{i+j+k=2} p(Z_{ijk}) \mathcal{B}[p_{ijk}].$$

An interesting question is how to construct sets of points satisfying the GC condition. Busch (1990) has been proved that the set of 6 points in the plane satisfying the GC condition must contain 3 collinear points. Some important examples have been given in Chui and He (1986), such as natural lattices and principal lattices. First example, consider the triangle $\mathcal{T}$ with vertices A_1, A_2, A_3. Let

$$Z_{ijk} = \frac{iA_1 + jA_2 + kA_3}{2}, \quad i+j+k=2.$$

Then, these points satisfy the geometric characterization, see Chui and He (1986). Let $\lambda^l = (\lambda^l_1, \lambda^l_2, \lambda^l_3)$ be the barycentric coordinates of the points P_l, $l=1,2$, with respect to the triangle $\mathcal{T}$. Using straightforward computations (see Speleers, 2014), we obtain

$$\begin{aligned}
\mathcal{B}[p_{ijk}](P_1, P_2) &= \mathcal{B}[p_{ijk}](\lambda^1, \lambda^2)\\
&= \frac{1}{i!j!k!} \frac{1}{2} \sum_{\pi \in S_2} \prod_{\mu=0}^{i-1} \left(2\lambda_1^{\pi(\mu+1)} - \mu\right)\\
&\quad \times \prod_{\nu=0}^{j-1} \left(2\lambda_2^{\pi(i+\nu+1)} - \nu\right) \prod_{\kappa=0}^{k-1} \left(2\lambda_2^{\pi(i+j+\kappa+1)} - \kappa\right),
\end{aligned}$$

where S_2 is the group of all permutations on the 2 first natural numbers.

We then arrive at the following theorems.

Theorem 8. *Let $Q^{[2]}f$ be any q.i. of the form (27) with $\lambda^{[2]}_{i,j,k}(f)$ defined according to (32). For $i=0,\dots,n$, $j=0,\dots,m$ and $k=1,2,3$, let*

$$Z^{(k,1)}_{i,j} = V_{i,j}, \; Z^{(k,2)}_{i,j} = \zeta^{(k,2)}_{i,j} V_{i,j} + (1-\zeta^{(k,2)}_{i,j})\widetilde{Q}_{i,j,k}, \; Z^{(k,3)}_{i,j} = \widetilde{Q}_{i,j,k} \quad \textit{with } \zeta^{(k,2)}_{i,j} \neq 0, 1,$$

and

$$q_{i,j}^{(k,1)} = \frac{1}{2}\left(1 - \frac{1}{\zeta_{i,j}^{(k,2)}}\right), \; q_{i,j}^{(k,2)} = \frac{1}{2}\frac{1}{\zeta_{i,j}^{(k,2)}\left(1-\zeta_{i,j}^{(k,2)}\right)}, \; q_{i,j}^{(k,3)} = -\frac{1}{2}\frac{\zeta_{i,j}^{(k,2)}}{\left(1-\zeta_{i,j}^{(k,2)}\right)},$$
$$q_{i,j}^{(k,4)} = q_{i,j}^{(k,5)} = q_{i,j}^{(k,6)} = 0,$$

the q.i. $Q^{[2]}f$ satisfies $Q^{[2]}p = p, \; \forall\, p \in \mathbb{P}_2$.

Theorem 9. *Let $Q^{[2]}f$ be any q.i. of the form (27) such that $\lambda_{i,j,k}^{[2]}(f)$ are defined according to (32). For $i = 0, \ldots, n$, $j = 0, \ldots, m$ and $k = 1, 2, 3$, let*

$$Z_{i,j}^{(k,1)} = \left(1 - \frac{1}{\zeta_{i,j}^{(k)}}\right) V_{i,j} + \frac{1}{\zeta_{i,j}^{(k)}} Q_{i,j,k}, \; Z_{i,j}^{(k,2)} = -\frac{\zeta_{i,j}^{(k)}}{1-\zeta_{i,j}^{(k)}} V_{i,j} + \frac{1}{1-\zeta_{i,j}^{(k)}} Q_{i,j,k},$$

with $\zeta_{i,j}^{(k)} \neq 0, \frac{1}{2}, 1$, and

$$q_{i,j}^{(k,1)} = \frac{\left(\zeta_{i,j}^{(k)}\right)^2}{2\zeta_{i,j}^{(k)} - 1}, \; q_{i,j}^{(k,2)} = \frac{\left(1-\zeta_{i,j}^{(k)}\right)^2}{2\zeta_{i,j}^{(k)} - 1}, \; q_{i,j}^{(k,3)} = q_{i,j}^{(k,4)} = q_{i,j}^{(k,5)} = q_{i,j}^{(k,6)} = 0,$$

then, the q.i. $Q^{[2]}f$ satisfies $Q^{[2]}p = p, \; \forall\, p \in \mathbb{P}_2$.

For the second example, we consider

$$Z_{i,j}^{(k,1)} = V_{i,j} = (ih, jh)^T, \; Z_{i,j}^{(k,2)} = \left(ih, jh - \frac{h}{2}\right)^T, \; Z_{i,j}^{(k,3)} = \left(ih, jh + \frac{h}{2}\right)^T,$$
$$Z_{i,j}^{(k,4)} = \left(ih - \frac{h}{2}, jh\right)^T, \; Z_{i,j}^{(k,5)} = \left(ih + \frac{h}{2}, jh - \frac{h}{2}\right)^T, \; Z_{i,j}^{(k,6)} = \left(ih + \frac{h}{2}, jh + \frac{h}{2}\right)^T.$$

Then, it is well known that $\left\{Z_{i,j}^{(k,l)}\right\}_{l=1}^{6}$ satisfies node configuration A (see Chui and Lai, 1987), hence it admits unique Lagrange interpolation. Using straightforward computations, we have an interesting spline q.i.

Theorem 10. *Let $Q^{[2]}f$ be any q.i. of the form (27) such that $\lambda_{i,j,k}(f)$ are defined according to (32). For $i = 0, \ldots, n$, $j = 0, \ldots, m$ and $k = 1, 2, 3$, let*

$$\begin{aligned}
\lambda_{i,j,1}^{[2]}(f) &= f(ih, jh) + \frac{1}{2}\left(f\left(ih, jh - \frac{h}{2}\right) - f\left(ih, jh + \frac{h}{2}\right)\right) \\
\lambda_{i,j,2}^{[2]}(f) &= \frac{1}{4}\left(-f\left(ih, jh - \frac{h}{2}\right) - 3f\left(ih - \frac{h}{2}, jh\right) + 4f(ih, jh)\right. \\
&\quad \left. + 3f\left(ih + \frac{h}{2}, jh\right) + f\left(ih, jh + \frac{h}{2}\right)\right) \\
\lambda_{i,j,3}^{[2]}(f) &= \frac{1}{4}\left(-f\left(ih, jh - \frac{h}{2}\right) + 3f\left(ih - \frac{h}{2}, jh\right) + 4f(ih, jh)\right. \\
&\quad \left. - 3f\left(ih + \frac{h}{2}, jh\right) + f\left(ih, jh + \frac{h}{2}\right)\right).
\end{aligned} \tag{36}$$

The q.i. $Q^{[2]}f$ satisfies $Q^{[2]}p = p, \; \forall\, p \in \mathbb{P}_2$.

5.4. Bounding the norm of quasi-interpolants

For $f \in \mathcal{C}(R)$ and any compact subset $G \subseteq R$, we let $\|f\|_D := \sup\{f(u), \; u \in G\}$ be the uniform norm. By standard arguments (see, e.g., Lai and Schumaker, 2007) we can establish an optimal order error bound of $\|f - Q^{[2]}f\|_R$, where f is a function in the classical space $\mathcal{C}^3(R)$ and $Q^{[2]}$ is a q.i. defined by (27) which satisfies (28). In the next theorem, for any $f \in \mathcal{C}^3(R)$, we let

$$\|D^3 f\|_G := \sup\{\|D_x^a D_y^b f\|_G, \; a + b = 3\}$$

where $G \subseteq R$ and $D_x f$ and $D_y f$ denote the first derivatives of a function f in x and y directions, respectively.

Proposition 11. *There exists a constant C_2, depending only on the smallest angle in Δ_2^* such that for every $f \in \mathcal{C}^3(R)$,*

$$\|D_x^a D_y^b (f - Q^{[2]} f)\|_R \le C_2 h^{3-a-b} \|D^3 f\|_R. \tag{37}$$

We now derive error bounds for the discrete q.i.s. $Q^{[2]}$ defined in the previous section where the interpolation points verify the GC condition. Denoting by $\|Q^{[2]}\|_R$ the corresponding induced norm, then if $Q^{[2]}$ reproduces quadratic polynomials, we have

$$\|f - Q^{[2]} f\|_R \le (1 + \|Q^{[2]}\|_R) \inf_{p \in \mathbb{P}_2} \|f - p\|_R.$$

From properties (12) and (27), we have

$$\|Q^{[2]}\|_R \le \max_{i=0,\ldots,n} \max_{j=0,\ldots,m} \max_{k=1,2,3} \left|\lambda_{i,j,k}^{[2]}\right|. \tag{38}$$

Using (33), we obtain

$$\begin{aligned} \|Q^{[2]}\|_R &\le \max_{i=0,\ldots,n} \max_{j=0,\ldots,m} \max_{k=1,2,3} \left| \sum_{l=1}^{6} \mathcal{B}[L_{i,j,k}^{(l)}]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) f\left(Z_{i,j}^{(k,l)}\right) \right|, \\ &\le 6\|f\|_R \max_{i=0,\ldots,n} \max_{j=0,\ldots,n} \max_{k=1,2,3} \max_{l=1,\ldots,6} \left| \mathcal{B}[L_{i,j,k}^{(l)}]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) \right|. \end{aligned}$$

Using some elementary manipulations (see also Theorem 8 in Speleers, 2014), we can show that

$$\begin{aligned} \left| \mathcal{B}[L_{i,j,k}^{(l)}]\left(V_{i,j}, \widetilde{Q}_{i,j,k}\right) \right| \le & \left(\max_{\{\mathcal{A}_{i,j,k}^{(p,q,r)}>0,\ p,q,r=1,\ldots,6\}} \frac{1}{\mathcal{A}_{i,j,k}^{(p,q,r)}} \right)^2 \\ & \left(\max_{p=1,\ldots,6} \|V_{i,j} - Z_{i,j}^{(k,p)}\|_2 \right)^2 \left(\max_{p=1,\ldots,6} \|\widetilde{Q}_{i,j,k} - Z_{i,j}^{(k,p)}\|_2 \right)^2, \end{aligned}$$

where $\mathcal{A}_{i,j,k}^{(p,q,r)} = area(Z_{i,j}^{(k,p)}, Z_{i,j}^{(k,q)}, Z_{i,j}^{(k,r)})$.

We now give an error bound for $f - Q^{[2]} f$, where $Q^{[2]} f$ is the q.i. presented in Theorem 10.

Theorem 12. *Let $Q^{[2]}$ be any q.i. of the form (27) with $\lambda_{i,j,k}^{[2]}$ defined according to (36). Then, for any $f \in \mathcal{C}^3(R)$,*

$$\|f - Q^{[2]} f\|_R \le 18 h^3 \|D^3 f\|_R. \tag{39}$$

Proof. Since the proof is similar to the one of Theorem 6 given in Sorokina and Zeilfelder (2004), we can be brief. Let us fix T a split of Δ_2^*, then there exists a $p \in \mathbb{P}_2$ such that

$$\|f - p\|_T \le \|f - p\|_{\Omega_T} \le \frac{9}{2} h^3 \|D^3 f\|_{\Omega_T}, \tag{40}$$

where Ω_T is the union of the triangles in the $star(T)$ (for the definition of $star(T)$, see Lai and Schumaker, 2007). Using the linearity of $Q^{[2]}$ and the fact that $Q^{[2]}$ reproduces polynomials of degree 2, we can write

$$\|f - Q^{[2]} f\|_T \le \|f - p\|_T + \|Q^{[2]}(f - p)\|_T. \tag{41}$$

It suffices to estimate the second quantity. By applying (27) and (38), it follows that

$$\|Q^{[2]}(f - p)\|_T \le 3\|f - p\|_{\Omega_T}. \tag{42}$$

Finally, we take the maximum over all $T \in \Delta_2^*$ to get (39). □

6. Numerical results

In order to illustrate the approximation properties and the visual quality of our splines, in this section we present some numerical results. We approximate the smooth bivariate test function of Franke type

$$\begin{aligned} f(x, y) = & \frac{3}{4} \exp\left(-\frac{(9x-2)^2 + (9y-2)^2}{4} \right) + \frac{3}{4} \exp\left(-\frac{(9x+1)^2}{49} - \frac{(9y+1)}{10} \right) \\ & \frac{1}{2} \exp\left(-\frac{(9x-7)^2 + (9y-3)^2}{4} \right) - \frac{1}{5} \exp\left(-(9x-4)^2 - (9y-7)^2 \right), \end{aligned}$$

on the domain $R = [0, 1] \times [0, 1]$.

Table 1
Error behavior of different q.i.s.

h	Hermite interp.	q.i. TH. 8	q.i. TH. 9	q.i. TH. 10
1/8	1.1089×10^{-2}	5.4427×10^{-2}	1.2617×10^{-1}	2.0352×10^{-2}
1/16	1.2872×10^{-3}	1.32746×10^{-2}	3.0541×10^{-2}	2.5762×10^{-3}
1/32	7.7198×10^{-5}	1.3437×10^{-3}	3.6389×10^{-3}	1.3656×10^{-4}

To measure the accuracy of a q.i. $Q^{[2]}f$ we have computed the maximum error on a 50×50 uniform grid on R:

$$\max_{r,s=1,\ldots,50} |f(x_r, y_s) - Q^{[2]}f(x_r, y_s)|. \tag{43}$$

The numerical results are given in Table 1. The first column indicates the refinement level. In the remaining columns, we have the values of the tabulated absolute error (43) for different q.i.s. In column 2 we have considered the Hermite interpolant see (30). Column 3 shows the results for the q.i. presented in Theorem 8 with $\zeta_{i,j}^{(k,2)} = \frac{1}{2}$. Column 4 refers to the q.i. presented in Theorem 9 with $\zeta_{i,j}^{(k)} = \frac{1}{4}$. Finally, column 5 shows the results for the q.i. presented in Theorem 10.

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