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Reduced curvature formulae for surfaces, offset surfaces, curves on a surface and surface intersections ☆

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A B S T R A C T

We introduce the concept of reduced curvature formulae for 3-D space entities (surfaces, curves). A reduced formula entails only derivatives of the functions involved in the entity's representation and admits no further algebraic simplifications. Although not always the most compact, reduced curvature formulae entail only basic arithmetic operators and are more efficient computationally compared to alternative unreduced formulae. Reduced formulae are presented for the normal, mean and Gaussian curvatures of a surface and the curvature of curves on a surface, where each surface or curve on a surface may be defined parametrically or implicitly. Reduced formulae are also presented for the curvature of surface intersection curves, where each of the intersecting surfaces may be a given surface or an offset of a given surface and each given surface may be defined parametrically or implicitly. Known formulae are cited, without derivation, to form a collection, in one place, of new and of known results scattered in the literature. Each curve curvature formula is presented together with a formula for the respective binormal vector, from which formulae for the Frenet frame and torsion of the curve can be derived.

1. Introduction

Formulae for computing various curvature measures are important in Geometric Modeling, CNC machining and other applications. Texts on classical differential geometry (Knoblauch, 1913; Struik, 1950; Lipschutz, 1969; Spivak, 1975; Do Carmo, 1976; Kreyszig, 1991) focus on the Gaussian and mean curvatures of surfaces, which determine local surface shape. They generally limit their discussion on the normal curvature to the classic formula expressing the normal curvature of a parametric surface as the ratio of the second to first fundamental forms of the surface, which is at the root of the classical theory on surface curvature. Regarding the curvature of curves, they provide only a general formula for parametric space curves, on the tacit assumption that specific formulae for curves on a surface or surface intersection curves can somehow be derived. Both Ye and Maekawa (1999) and Goldman (2005) have noted the scarcity of English literature on the differential geometry of intersection curves. A notable exception is Willmore (1959) who describes procedures (but gives no closed formulae) for computing the curvature, torsion and Frenet frame vectors of intersection curves of two implicit surfaces.

In recent years, interest in the differential geometry of curves on a surface and surface intersection curves has revived, motivated by research in Geometric Modeling (Faux and Pratt, 1981; Hartmann, 1996; Ye and Maekawa, 1999; Goldman, 2005) and the need for advanced CNC controls (Papaioannou and Patrikoussakis, 2011). The curvature of a plane

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or space curve is a function of the first and second derivatives of the curve's position vector. Similarly, the normal curvature of a surface and, by extension, its Gaussian and mean curvatures, are functions of the first and second partial derivatives of the surface's position vector. These derivatives can, in turn, be expressed in terms of first and second derivatives, or partial derivatives of the functions involved in the actual curve or surface representation. We shall call a curvature formula entailing only representation function derivatives *reduced*, if it does not evidently admit further algebraic simplifications. We allow, however, the use of placeholders for reduced expressions in reduced curvature formulae, for the sake of compactness.

The task of generating a reduced formula for the curvature of the intersection curve of a parametric surface $\mathbf{r}(u,v) = [x(u,v)\ \ y(u,v)\ \ z(u,v)]^{\mathrm{T}}$ with an implicit surface $f(x,y,z)=0$ will serve to illustrate the issues involved. An established fact in differential geometry is the dependence of the intersection curvature k of two surfaces $\mathrm{S_a}$, $\mathrm{S_b}$ on their local normal curvatures k_{an}, k_{bn}, which reduces the task of computing k to computing k_{an}, k_{bn} and combining them, by means of an equation expressing this dependence, to produce k. The following expression of k^2 in terms of k_{an}, k_{bn} is given in Lipschutz (1969)

$$k^2 = \frac{k_{an}^2 + k_{bn}^2 - 2k_{an}k_{bn}\cos\varphi}{\sin^2\varphi} \tag{1}$$

where φ is the angle formed by the local normal vectors of $\mathrm{S_a}$, $\mathrm{S_b}$. Previous authors (Faux and Pratt, 1981; Ye and Maekawa, 1999) suggest substituting the values of k_{an}, k_{bn}, $\cos\varphi$, $\sin\varphi$ into Equ. (1) to compute k. Efficiency gains can, however, be obtained by introducing known formulae for k_{an}, k_{bn} into Equ. (1), or an equivalent expression, to produce reduced formulae for k, by taking advantage of possible simplifications. This is a more systematic and efficient approach. It is also less complicated than trying to compute k, without regard to its dependence on k_{an}, k_{bn}.

Thus, to reduce Equ. (1) for the above case, we introduce into it the classic expression for the normal curvature k_{rn} of a parametric surface $\mathbf{r}(u,v) = [x(u,v)\ \ y(u,v)\ \ z(u,v)]^{\mathrm{T}}$

$$\begin{aligned} &k_{rn} = \frac{c_{rn}}{|\mathbf{t}|^2|\mathbf{n_r}|}, \quad c_{rn} = L'a^2 + 2M'ab + N'b^2, \\ &L' = \mathbf{r_{uu}}\mathbf{n_r}, \qquad M' = \mathbf{r_{uv}}\mathbf{n_r}, \qquad N' = \mathbf{r_{vv}}\mathbf{n_r}, \qquad \mathbf{t} = a\mathbf{r_u} + b\mathbf{r_v} \end{aligned} \tag{2}$$

the expression for the normal curvature k_{fn} of an implicit surface $f(x,y,z)=0$

$$\begin{aligned} &k_{fn} = \frac{c_{fn}}{|\mathbf{t}|^2|\mathbf{n_f}|}, \quad c_{fn} = -[\mathbf{t}]^{\mathrm{T}}\mathrm{H_f}[\mathbf{t}] \\ &c_{fn} = -\left(f_{xx}t_x^2 + f_{yy}t_y^2 + f_{zz}t_z^2 + 2f_{xy}t_xt_y + 2f_{xz}t_xt_z + 2f_{yz}t_yt_z\right) \end{aligned} \tag{3}$$

where $\mathrm{H_f}$ is the Hessian of $f(x,y,z)$, and the following expressions for $\cos\varphi$ and $\sin\varphi$

$$\cos\varphi = \frac{\mathbf{n_r}\cdot\mathbf{n_f}}{|\mathbf{n_r}||\mathbf{n_f}|}, \qquad \sin\varphi = \frac{|\mathbf{n_r}\times\mathbf{n_f}|}{|\mathbf{n_r}||\mathbf{n_f}|} = \frac{|\mathbf{t}|}{|\mathbf{n_r}||\mathbf{n_f}|} \tag{4}$$

where $\mathbf{n_r} = \mathbf{r_u}\times\mathbf{r_v}$ and $\mathbf{n_f} = \nabla f$ are the normal vectors of the intersecting surfaces and $\mathbf{t} = \mathbf{n_r}\times\mathbf{n_f}$ is the tangent vector of the intersection curve.

There have been, however, two missing links for this reduction process to succeed. Formula (3) for the normal curvature of an implicit surface has been provided only recently by Ye and Maekawa (1999), although implicit forms of this formula can be traced back to classical works (Spivak, 1975; Do Carmo, 1976). And unlike this formula, in which $\mathbf{t}$ is represented by its Cartesian coordinates, formula (2) for the normal curvature of a parametric surface entails the coordinates a, b of $\mathbf{t}$ in the basis $\mathbf{r_u}$, $\mathbf{r_v}$ of the surface's tangent plane. Reduced formulae for a, b when $\mathbf{t}$ is a tangent vector of the intersection curve of a parametric surface by another surface have not been known, but we provide this link in Section 3, Equ. (20b), which here assume the form

$$a = -\mathbf{r_v}\cdot\mathbf{n_f}, \qquad b = \mathbf{r_u}\cdot\mathbf{n_f} \tag{5}$$

Introducing Equ. (2)–(4) into Equ. (1) and simplifying, we obtain

$$k = \frac{|c_{rn}\mathbf{n_f} - c_{fn}\mathbf{n_r}|}{|\mathbf{t}|^3} \tag{6a}$$

where L', M', N', c_{rn} (Equ. (2)), c_{fn} (Equ. (3)), a, b (Equ. (5)) and $\mathbf{n_r}$, $\mathbf{n_f}$, $\mathbf{t}$ are placeholders for reduced derivative expressions of the representation functions $x(u,v)$, $y(u,v)$, $z(u,v)$, $f(x,y,z)$ and formula (6a) is also reduced as evidenced by its explicit form

$$k = \frac{((c_{rn}f_x - c_{fn}n_{rx})^2 + (c_{rn}f_y - c_{fn}n_{ry})^2 + (c_{rn}f_z - c_{fn}n_{rz})^2)^{1/2}}{(t_x^2 + t_y^2 + t_z^2)^{3/2}} \tag{6b}$$

The formulae presented in this paper fall into three categories. Known formulae, which are cited without derivation. Derived reduced formulae for curvature measures for which alternative unreduced formulae or procedures have been presented by other authors. Completely new formulae for the curvature of curves on implicit surfaces and the normal and intersection curvatures of offset surfaces.

The rest of the paper is organized as follows: Following a brief review of classical differential geometry (Section 2), we derive in Section 3 reduced formulae for the tangential coordinates a, b of the surface tangent vector $\mathbf{t}$ for use with Equ. (2) and in Section 4 reduced curvature formulae for curves on both parametric and implicit surfaces. In Section 5, from the Faux and Pratt (1981) expression of the $k\mathbf{B}$ vector, we derive reduced formulae for the curvature of intersection curves, for the remaining two representation modes (implicit/implicit and parametric/parametric) of the intersecting surfaces. In Section 6 we present reduced formulae for the normal and intersection curvatures of offset surfaces, a subject not treated in the English literature. The paper concludes with some final remarks (Section 7) and two appendices, the first of which compares the efficiency of proposed reduced to existing unreduced formulae and the second rederives curvature formulae presented earlier, by reformulating the problem as a surface intersection problem and applying proposed formulae for this problem.

A word on notation: We use capital bold letters to distinguish unit vectors from other vectors. The principal normal vector of a curve and the unit normal vector of a surface are denoted by $\mathbf{N_C}$ and $\mathbf{N}$, respectively. Otherwise, indexes indicate the representation of curves/surfaces and their differential quantities (r for parametric, f, g for implicit, h for explicit representations) and for partial derivatives of implicit functions (f, g) of position vectors ($\mathbf{r}$) and of their coordinates (x, y, z) the associated parameters. Dots signify derivatives w.r.t. arc length and primes derivatives w.r.t. any other variable.

2. Brief review of classical differential geometry

Classical differential geometry starts from the Frenet–Serret equations of a curve

$$\begin{aligned}\dot{\mathbf{T}} &= k\mathbf{N}\\ \dot{\mathbf{N}} &= -k\mathbf{T} + \tau \boldsymbol{B}\\ \dot{\mathbf{B}} &= -\tau\mathbf{N}\end{aligned} \tag{7}$$

and derives two basic curvature formulae. The first for the *binormal curvature vector* $k\mathbf{B}$ of a parametric space curve $\mathbf{r}(t) = [x(t)\ \ y(t)\ \ z(t)]^{\mathrm{T}}$

$$k\mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}'|^3} \tag{8a}$$

from which follow expressions for the curvature k and the binormal vector $\mathbf{B}$

$$k = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \qquad \mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \tag{8b}$$

of the curve. In fact, all formulae for the curvature of curves derived in the sequel (curves on a surface and surface intersection curves) originate at expressions of the $k\mathbf{B}$ vector and this close relationship between k and $k\mathbf{B}$ implies that for any such curvature formula of the form $k = |\mathbf{v}|/|\mathbf{t}|^3$, where $|\mathbf{v}|$ is the norm of a vector expression and $\mathbf{t}$ is the curve's tangent vector, there is a respective expression $\mathbf{B} = \mathbf{v}/|\mathbf{v}|$ of the binormal vector of the curve. The curve's unit tangent vector $\mathbf{T}$, principal normal vector $\mathbf{N_C}$ and torsion τ can then be obtained from the formulae

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \qquad \mathbf{N_C} = \mathbf{B} \times \mathbf{T}, \qquad \tau = \frac{\mathbf{r}''' \cdot \mathbf{B}}{|\mathbf{r}' \times \mathbf{r}''|} \tag{9}$$

The second basic curvature formula

$$\begin{aligned}&\ddot{\mathbf{r}}_{\mathbf{C}} \cdot \mathbf{N_r} = k_{rn} = \frac{II}{I} = \frac{La^2 + 2Mab + Nb^2}{Ea^2 + 2Fab + Gb^2}, \quad \mathbf{N_r} = \frac{\mathbf{r_u} \times \mathbf{r_v}}{|\mathbf{r_u} \times \mathbf{r_v}|}\\ &E = \mathbf{r_u} \cdot \mathbf{r_u}, \qquad F = \mathbf{r_u} \cdot \mathbf{r_v}, \qquad G = \mathbf{r_v} \cdot \mathbf{r_v}, \qquad L = \mathbf{r_{uu}} \cdot \mathbf{N_r} = -\mathbf{r_u} \cdot \mathbf{N_{ru}},\\ &M = \mathbf{r_{uv}} \cdot \mathbf{N_r} = -\mathbf{r_u} \cdot \mathbf{N_{rv}}, \qquad N = \mathbf{r_{vv}} \cdot \mathbf{N_r} = -\mathbf{r_v} \cdot \mathbf{N_{rv}}\end{aligned} \tag{10}$$

gives the normal curvature k_{rn} of a parametric surface $\mathrm{S_r}$ at a point P, in terms of the first and second fundamental forms $I = Ea^2 + 2Fab + Gb^2$ and $II = La^2 + 2Mab + Nb^2$ of $\mathrm{S_r}$ at P. Both forms are associated with a tangent vector $\mathbf{t} = a\mathbf{r_u} + b\mathbf{r_v}$ of $\mathrm{S_r}$. In particular, I is a surface metric allowing lengths, areas and angles on $\mathrm{S_r}$ to be expressed in terms of its first fundamental coefficients E, F, G. Thus, the norms of $\mathbf{t}$ and of the normal vector $\mathbf{n_r} = \mathbf{r_u} \times \mathbf{r_v}$ of $\mathrm{S_r}$ at P are

$$\begin{aligned}|\mathbf{t}| &= \big((a\mathbf{r_u} + b\mathbf{r_v})(a\mathbf{r_u} + b\mathbf{r_v})\big)^{1/2} = \big(Ea^2 + 2Fab + Gb^2\big)^{1/2}\\ |\mathbf{n_r}| &= |\mathbf{r_u} \times \mathbf{r_v}| = \big(\mathbf{r_u^2 r_v^2} \sin^2\gamma\big)^{1/2} = \big(\mathbf{r_u^2 r_v^2} - \mathbf{r_u^2 r_v^2} \cos^2\gamma\big)^{1/2} = \big(EG - F^2\big)^{1/2}\end{aligned} \tag{11}$$

The normal curvature k_{rn} is the curvature of the section curve of S_r by the normal plane spanned by $\mathbf{t}$ and $\mathbf{N_r}$. It measures the curving of S_r in the tangent direction $\mathbf{t}$ and its variation as $\mathbf{t}$ rotates around $\mathbf{N_r}$ reveals the local surface geometry at P. This leads to an examination of the stationary values of k_{rn}, as the direction ratio a/b of $\mathbf{t}$ varies, with the following results: k_{rn} has either two real distinct stationary values (minimum k_{rn1}, maximum k_{rn2}, termed *principal curvatures*) or it is constant in all directions. In the latter case, the surface looks locally like a spherical cup and the point is an *umbilic*. Apart from umbilics, the principal curvatures are distinct and the associated *principal directions* are orthogonal. The local surface shape depends on the relative signs of k_{rn1}, k_{rn2}. This gives rise to two new curvature measures, the *Gaussian curvature* $K = k_{rn1}k_{rn2}$ and the *mean curvature* $H = (k_{rn1} + k_{rn2})/2$. Since $k_{rn1} \le k_{rn} \le k_{rn2}$, $K > 0$ implies that k_{rn} does not change sign and the local surface shape is cup-like. In particular, if $H^2 - K = 0$, then $k_{rn1} = k_{rn2}$ and the point is an umbilic. $K < 0$, on the other hand, implies a sign change of k_{rn} and the local surface shape is saddle-like.

The left part of formula (10) expresses the normal curvature k_{rn} of S_r at P as the projection on the surface unit normal vector $\mathbf{N_r}$ of the curvature vector $\ddot{\mathbf{r}}_\mathbf{C} = k\mathbf{N_C}$ of any curve C on S_r which passes through P and is tangent to $\mathbf{t}$. Since $\mathbf{N_r}$, $\mathbf{N_C}$ are unit vectors, this part can also be written in the form

$$k_{rn} = k\mathbf{N_C} \cdot \mathbf{N_r} = k\cos\theta \tag{12}$$

For reasons that will become apparent when we come to intersection curves, it is convenient to introduce into formula (10) the surface normal vector $\mathbf{n_r} = \mathbf{r_u} \times \mathbf{r_v}$ in place of the unit normal vector $\mathbf{N_r} = \mathbf{n_r}/|\mathbf{n_r}|$ of S_r and write

$$L = \frac{L'}{|\mathbf{n_r}|}, \qquad L' = \mathbf{r_{uu}}\mathbf{n_r}, \qquad M = \frac{M'}{|\mathbf{n_r}|}, \qquad M' = \mathbf{r_{uv}}\mathbf{n_r}, \qquad N = \frac{N'}{|\mathbf{n_r}|}, \qquad N' = \mathbf{r_{vv}}\mathbf{n_r} \tag{13}$$

Formula (10) then reduces to formula (2).

The classic formulae for the Gaussian and mean curvatures of a parametric surface are

$$\begin{aligned} K_r &= \frac{c_{rK}}{|\mathbf{n_r}|^4}, \quad c_{rK} = L'N' - M'^2, \\ H_r &= \frac{c_{rH}}{2|\mathbf{n_r}|^3}, \quad c_{rH} = EN' + GL' - 2FM' \end{aligned} \tag{14}$$

Spivak (1975, vol. 3), Belyaev et al. (1998), Turkiyyah et al. (1997), Patrikalakis and Maekawa (2002) and Osher and Fedkiw (2003) provide the following formulae for the Gaussian and mean curvatures of a surface with explicit representation $z = h(x, y)$ and implicit representation $f(x, y, z) = 0$

$$\begin{aligned} K_h &= \frac{c_{hK}}{|\mathbf{n_h}|^4}, \quad c_{hK} = h_{xx}h_{yy} - h_{xy}^2, \; |\mathbf{n_h}| = \left(h_x^2 + h_y^2 + 1\right)^{1/2}, \\ H_h &= \frac{c_{hH}}{2|\mathbf{n_h}|^3}, \quad c_{hH} = \left(1 + h_x^2\right)h_{yy} - 2h_xh_yh_{xy} + \left(1 + h_y^2\right)h_{xx} \end{aligned} \tag{15}$$

$$\begin{aligned} K_f &= \frac{c_{fK}}{|\mathbf{n_f}|^4}, \quad |\mathbf{n_f}| = \left(f_x^2 + f_y^2 + f_z^2\right)^{1/2}, \\ c_{fK} &= f_x^2\left(f_{yy}f_{zz} - f_{yz}^2\right) + f_y^2\left(f_{xx}f_{zz} - f_{xz}^2\right) + f_z^2\left(f_{xx}f_{yy} - f_{xy}^2\right) + 2f_xf_y(f_{xz}f_{yz} - f_{xy}f_{zz}) \\ &\quad + 2f_xf_z(f_{xy}f_{yz} - f_{xz}f_{yy}) + 2f_yf_z(f_{xy}f_{xz} - f_{yz}f_{xx}) \\ H_f &= \frac{c_{fH}}{2|\mathbf{n_f}|^3}, \quad c_{fH} = 2(f_xf_yf_{xy} + f_xf_zf_{xz} + f_yf_zf_{yz}) - f_x^2(f_{yy} + f_{zz}) - f_y^2(f_{xx} + f_{zz}) - f_z^2(f_{xx} + f_{yy}) \end{aligned} \tag{16}$$

It is important to note that we have cast the normal, Gaussian and mean curvatures of a surface in the generic forms

$$k_{in} = \frac{c_{in}}{|\mathbf{t}|^2|\mathbf{n_i}|}, \qquad K_i = \frac{c_{iK}}{|\mathbf{n_i}|^4}, \qquad H_i = \frac{c_{iH}}{2|\mathbf{n_i}|^3}, \quad i \in (r, h, f, g) \tag{17}$$

where index i indicates the surface representation. We shall call the numerator of each of these forms *curvature factor* of the respective curvature.

3. The classic normal curvature formula revisited

To apply formula (2) for the normal curvature of a parametric surface S_r in a tangent direction $\mathbf{t}$, one must express $\mathbf{t}$ in the form

$$\mathbf{t} = a\mathbf{r_u} + b\mathbf{r_v} \tag{18a}$$

This task is trivially simple when $\mathbf{t}$ is defined as tangent to a curve C on S_r, represented by $u = u(t)$, $v = v(t)$. Then

$$\mathbf{t} = \mathbf{r}' = \mathbf{r_u}u' + \mathbf{r_v}v' \tag{18b}$$

so that $a = u'$, $b = v'$. When C is defined on S_r by an implicit equation $f(u, v) = 0$ then, by the implicit function theorem, as long as $f_u \neq 0$, u is an explicit function $u = u(v)$ of v and C can be represented parametrically as $\mathbf{r}(v) = [x(u(v), v)\ y(u(v), v)\ z(u(v), v)]^T$. Then $v' = 1$ and

$$f' = f_u u' + f_v = 0 \rightarrow u' = -\frac{f_v}{f_u}$$
$$\mathbf{t} = \mathbf{r}' = \mathbf{r_u} u' + \mathbf{r_v} = -\frac{f_v}{f_u}\mathbf{r_u} + \mathbf{r_v} \tag{18c}$$

so that $a = -f_v/f_u$, $b = 1$ and formula (2) becomes

$$k_{rn} = \frac{c_{rn}}{|\mathbf{t}|^2|\mathbf{n_r}|}, \quad c_{rn} = L' f_v^2 - 2M' f_u f_v + N' f_u^2$$
$$|\mathbf{t}|^2 = E f_v^2 - 2F f_u f_v + G f_u^2 \tag{19}$$

regardless of which of the parameters u, v is a function of the other.

Ye and Maekawa (1999) give the following general expressions for a, b, when $\mathbf{t}$ is an arbitrary tangent vector of S_r

$$a = \frac{G(\mathbf{t}\cdot\mathbf{r_u}) - F(\mathbf{t}\cdot\mathbf{r_v})}{EG - F^2}, \qquad b = \frac{E(\mathbf{t}\cdot\mathbf{r_v}) - F(\mathbf{t}\cdot\mathbf{r_u})}{EG - F^2} \tag{20a}$$

which are obtained by taking the dot product of Equ. (18a), first with $\mathbf{r_u}$, then with $\mathbf{r_v}$ and solving the resulting system for a, b. We shall reduce these expressions for the case when $\mathbf{t}$ is tangent to the curve of intersection of S_r by another surface with local normal vector $\mathbf{n}$. Then $\mathbf{t} = (\mathbf{r_u} \times \mathbf{r_v}) \times \mathbf{n}$ and using the vector identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, we obtain

$$\mathbf{t} = (\mathbf{r_u}\cdot\mathbf{n})\mathbf{r_v} - (\mathbf{r_v}\cdot\mathbf{n})\mathbf{r_u}$$
$$\mathbf{t}\cdot\mathbf{r_u} = (\mathbf{r_u}\cdot\mathbf{n})F - (\mathbf{r_v}\cdot\mathbf{n})E$$
$$\mathbf{t}\cdot\mathbf{r_v} = (\mathbf{r_u}\cdot\mathbf{n})G - (\mathbf{r_v}\cdot\mathbf{n})F \tag{21}$$

Introduction of the last two expressions into Equ. (20a) yields

$$a = -(\mathbf{r_v}\cdot\mathbf{n}), \qquad b = \mathbf{r_u}\cdot\mathbf{n} \tag{20b}$$

4. Curvature of curves on a surface

For a curve C on a parametric surface S_r, defined parametrically by $u = u(t)$, $v = v(t)$, the classic expression (8a) of the $k\mathbf{B}$ vector yields (Hartmann, 1996)

$$k\mathbf{B} = \frac{(\mathbf{r_u}u' + \mathbf{r_v}v') \times (\mathbf{r_{uu}}u'^2 + 2\mathbf{r_{uv}}u'v' + \mathbf{r_{vv}}v'^2) + \mathbf{r_u}\times\mathbf{r_v}(u'v'' - u''v')}{|\mathbf{r_u}u' + \mathbf{r_v}v'|^3} \tag{21a}$$

If C is defined implicitly by $f(u, v) = 0$, Equ. (21a) becomes

$$k\mathbf{B} = \frac{(\mathbf{r_v}f_u - \mathbf{r_u}f_v) \times \mathbf{a} + \beta(\mathbf{r_u}\times\mathbf{r_v})}{|\mathbf{r_v}f_u - \mathbf{r_u}f_v|^3}$$
$$\mathbf{a} = \mathbf{r_{uu}}f_v^2 - 2\mathbf{r_{uv}}f_u f_v + \mathbf{r_{vv}}f_u^2, \qquad \beta = f_{uu}f_v^2 + f_{vv}f_u^2 - 2f_{uv}f_u f_v \tag{21b}$$

Hartmann suggests computing the curvature of C as $k = \sqrt{|k\mathbf{B}|^2}$. The resulting formulae when Equ. (21a) or (21b) is introduced into this expression are far from been reduced.

We shall use the general expression for the curvature k of a space curve C (Equ. (8b)) as a starting point to generate reduced curvature formulae for curves on S_r, by substituting in it reduced expressions of the derivatives $\mathbf{r}'$, $\mathbf{r}''$ of the position vector of C, in terms of derivatives of the functions involved in the representation of C. Curves on a surface may be represented parametrically, explicitly or implicitly, as the surface itself. In all cases, the domain or range of the curve (depending on the type of variables involved in the curve's definition) must be contained within the domain of definition of the surface, otherwise, numerical problems will arise. For parametric representations $\mathbf{r}(u, v) = [x(u, v)\ y(u, v)\ z(u, v)]^T$ of S_r and $u = u(t)$, $v = v(t)$ of C, $\mathbf{r}'$, $\mathbf{r}''$ are expressed in terms of representation function derivatives, using the chain rule

$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix}, \qquad \mathbf{r}'' = \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} x_{uu} & x_{uv} & x_{vv} \\ y_{uu} & y_{uv} & y_{vv} \\ z_{uu} & z_{uv} & z_{vv} \end{bmatrix}\begin{bmatrix} u'^2 \\ 2u'v' \\ v'^2 \end{bmatrix} + \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}\begin{bmatrix} u'' \\ v'' \end{bmatrix} \tag{22a}$$

and formulae (8b) become

$$k = \frac{((x_p y_{pp} - x_{pp} y_p)^2 + (y_p z_{pp} - y_{pp} z_p)^2 + (z_p x_{pp} - z_{pp} x_p)^2)^{1/2}}{(x_p^2 + y_p^2 + z_p^2)^{3/2}}$$
$$x_p = x_u u' + x_v v', \qquad y_p = y_u u' + y_v v', \qquad z_p = z_u u' + z_v v'$$
$$x_{pp} = x_{uu} u'^2 + 2x_{uv} u' v' + x_{vv} v'^2 + x_u u'' + x_v v'',$$
$$y_{pp} = y_{uu} u'^2 + 2y_{uv} u' v' + y_{vv} v'^2 + y_u u'' + y_v v'',$$
$$z_{pp} = z_{uu} u'^2 + 2z_{uv} u' v' + z_{vv} v'^2 + z_u u'' + z_v v''$$
$$\mathbf{B} = \frac{[(y_p z_{pp} - y_{pp} z_p)(z_p x_{pp} - z_{pp} x_p)(x_p y_{pp} - x_{pp} y_p)]^{\mathrm{T}}}{((x_p y_{pp} - x_{pp} y_p)^2 + (y_p z_{pp} - y_{pp} z_p)^2 + (z_p x_{pp} - z_{pp} x_p)^2)^{1/2}} \tag{23}$$

When C is defined in the parametric plane of $\mathrm{S_r}$ by an implicit equation $f(u, v) = 0$, we distinguish two cases. If this equation can be solved in the form say $u = u(v)$, C can be represented as a space curve $\mathbf{r}(v) = [x(u(v), v)\ \ y(u(v), v)\ \ z(u(v), v)]^{\mathrm{T}}$ and its curvature obtained by means of formula (8b). Otherwise, we need a special curvature formula entailing partial derivatives of $f(u, v)$. Assuming $f_u \neq 0$, we can stipulate the existence of a function $u = u(v)$ and of a representation of C as above. Then $v' = 1$, $v'' = 0$ and

$$f' = f_u u' + f_v = 0 \;\rightarrow\; u' = -\frac{f_v}{f_u}$$
$$f'_u = f_{uu} u' + f_{uv} = \frac{-f_{uu} f_v + f_{uv} f_u}{f_u}, \qquad f'_v = \frac{-f_{uv} f_v + f_{vv} f_u}{f_u}$$
$$u'' = \frac{f'_u f_v - f_u f'_v}{f_u^2} = \frac{2 f_{uv} f_u f_v - f_{uu} f_v^2 - f_{vv} f_u^2}{f_u^3} \tag{24}$$

so that Equ. (22a) are adapted as follows:

$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} -f_v/f_u \\ 1 \end{bmatrix},$$
$$\mathbf{r}'' = \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} x_{uu} & x_{uv} & x_{vv} \\ y_{uu} & y_{uv} & y_{vv} \\ z_{uu} & z_{uv} & z_{vv} \end{bmatrix} \begin{bmatrix} f_v^2/f_u^2 \\ -2f_v/f_u \\ 1 \end{bmatrix} + \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} u'' \\ 0 \end{bmatrix} \tag{22b}$$

Further

$$\begin{aligned} x'y'' - x''y' &= [x_u \quad x_v] \begin{bmatrix} -f_v/f_u \\ 1 \end{bmatrix} \left([y_{uu} \quad y_{uv} \quad y_{vv}] \begin{bmatrix} f_v^2/f_u^2 \\ -2f_v/f_u \\ 1 \end{bmatrix} + y_u u'' \right) \\ &\quad - [y_u \quad y_v] \begin{bmatrix} -f_v/f_u \\ 1 \end{bmatrix} \left([x_{uu} \quad x_{uv} \quad x_{vv}] \begin{bmatrix} f_v^2/f_u^2 \\ -2f_v/f_u \\ 1 \end{bmatrix} + x_u u'' \right) \\ &= \frac{(x_{uu} f_v^2 - 2x_{uv} f_u f_v + x_{vv} f_u^2)(y_u f_v - y_v f_u)}{f_u^3} \\ &\quad - \frac{(y_{uu} f_v^2 - 2y_{uv} f_u f_v + y_{vv} f_u^2)(x_u f_v - x_v f_u)}{f_u^3} \\ &\quad - \frac{(x_u y_v - x_v y_u)(2f_{uv} f_u f_v - f_{uu} f_v^2 - f_{vv} f_u^2)}{f_u^3} \end{aligned} \tag{25}$$

and deriving similar expressions for $y'z'' - y''z'$, $z'x'' - z''x'$, we finally obtain the formulae

$$k = \frac{((c_{xy} - c_{yx} - n_z u_{pp})^2 + (c_{yz} - c_{zy} - n_x u_{pp})^2 + (c_{zx} - c_{xz} - n_y u_{pp})^2)^{1/2}}{(c_{xl}^2 + c_{yl}^2 + c_{zl}^2)^{3/2}}$$
$$c_{xq} = x_{uu} f_v^2 - 2x_{uv} f_u f_v + x_{vv} f_u^2, \qquad c_{yl} = y_u f_v - y_v f_u, \qquad c_{xy} = c_{xq} c_{yl}$$
$$c_{yq} = y_{uu} f_v^2 - 2y_{uv} f_u f_v + y_{vv} f_u^2, \qquad c_{xl} = x_u f_v - x_v f_u, \qquad c_{yx} = c_{yq} c_{xl}$$
$$c_{zq} = z_{uu} f_v^2 - 2z_{uv} f_u f_v + z_{vv} f_u^2, \qquad c_{zl} = z_u f_v - z_v f_u, \qquad c_{yz} = c_{yq} c_{zl}$$
$$c_{zy} = c_{zq} c_{yl}, \qquad c_{zx} = c_{zq} c_{xl}, \qquad c_{xz} = c_{xq} c_{zl}, \qquad u_{pp} = 2f_{uv} f_u f_v - f_{uu} f_v^2 - f_{vv} f_u^2$$

$$n_x = y_u z_v - y_v z_u, \qquad n_y = z_u x_v - z_v x_u, \qquad n_z = x_u y_v - x_v y_u$$
$$\mathbf{B} = \frac{[(c_{yz} - c_{zy} - n_x u_{pp})(c_{zx} - c_{xz} - n_y u_{pp})(c_{xy} - c_{yx} - n_z u_{pp})]^{\mathrm{T}}}{((c_{xy} - c_{yx} - n_z u_{pp})^2 + (c_{yz} - c_{zy} - n_x u_{pp})^2 + (c_{zx} - c_{xz} - n_y u_{pp})^2)^{1/2}} \tag{26}$$

Example 1. Given a spherical surface S_r, represented as $\mathbf{r} = R[\cos u \cos v \ \ \sin u \cos v \ \ \sin v]^{\mathrm{T}}$, the curvature of the surface curve $f = u - v = 0$, $-\pi/2 \le u, v \le \pi/2$, is found by means of curvature formula (8b) to be

$$k = \frac{(3\cos^2 v + 5)^{1/2}}{R(\cos^2 v + 1)^{3/2}}$$

Verify this expression using curvature formula (26).

Partial derivatives of f: $f_u = 1$, $f_v = -1$, $f_{uu} = f_{uv} = f_{vv} = 0$, $u_{pp} = 0$.
Partial derivatives of S_r: $\mathbf{r_u} = R[-\sin u \cos v \ \ \cos u \cos v \ \ 0]^{\mathrm{T}}$, $\mathbf{r_v} = R[-\cos u \sin v \ \ -\sin u \sin v \ \ \cos v]^{\mathrm{T}}$, $\mathbf{r_{uu}} = R[-\cos u \cos v \ -\sin u \cos v \ 0]^{\mathrm{T}}$, $\mathbf{r_{uv}} = R[\sin u \sin v \ -\cos u \sin v \ 0]^{\mathrm{T}}$, $\mathbf{r_{vv}} = -R[\cos u \cos v \ \ \sin u \cos v \ \ \sin v]^{\mathrm{T}}$.

$\underline{c_{xl}, c_{yl}, c_{zl}}$: $c_{xl} = R(\sin u \cos v + \cos u \sin v)$, $c_{yl} = R(-\cos u \cos v + \sin u \sin v)$, $c_{zl} = -R\cos v$, $(c_{xl}^2 + c_{yl}^2 + c_{zl}^2)^{3/2} = R^3(\cos^2 v + 1)^{3/2}$.
$\underline{c_{xq}, c_{yq}, c_{zq}}$: $c_{xq} = 2R(\sin u \sin v - \cos u \cos v)$, $c_{yq} = -2R(\sin u \cos v + \cos u \sin v)$, $c_{zq} = -R\sin v$.
$\underline{c_{xy}, c_{yx}, c_{xz}, c_{zx}, c_{yz}, c_{zy}}$: $c_{xy} = c_{xq}c_{yl} = 2R^2(\sin u \sin v - \cos u \cos v)^2$, $c_{yx} = c_{yq}c_{xl} = -2R^2(\sin u \cos v + \cos u \sin v)^2$, $c_{xz} = c_{xq}c_{zl} = -2R^2 \cos v(\sin u \sin v - \cos u \cos v)$, $c_{zx} = c_{zq}c_{xl} = -R^2 \sin v(\sin u \cos v + \cos u \sin v)$, $c_{yz} = c_{yq}c_{zl} = 2R^2 \cos v(\sin u \cos v + \cos u \sin v)$, $c_{zy} = c_{zq}c_{yl} = -R^2 \sin v(-\cos u \cos v + \sin u \sin v)$.

Curvature formula (26): $(c_{xy} - c_{yx})^2 = 4R^4$, $(c_{yz} - c_{zy})^2 = R^4(\sin u \cos^2 v + \sin u + \cos u \sin v \cos v)^2$, $(c_{zx} - c_{xz})^2 = R^4(\sin u \sin v \cos v - \cos u \cos^2 v - \cos u)^2$

$$k = \frac{((c_{xy} - c_{yx})^2 + (c_{yz} - c_{zy})^2 + (c_{zx} - c_{xz})^2)^{1/2}}{(c_{xl}^2 + c_{yl}^2 + c_{zl}^2)^{3/2}} = \frac{(3\cos^2 v + 5)^{1/2}}{R(\cos^2 v + 1)^{3/2}}$$

The task of developing curvature formulae for curves lying on an implicit surface S_f has been an open question, according to Goldman (2005). The rest of this section is our attempt to provide an answer. When S_f is represented as $f(x, y, z) = 0$, a parametric definition $x = x(u)$, $y = y(u)$, $z = z(u)$ of C on S_f, involving all three coordinates cannot be distinguished from an ordinary parametric representation of C as a space curve, even though incidentally $x = x(u)$, $y = y(u)$, $z = z(u)$ satisfy the surface equation $f(x, y, z) = 0$ identically. Also an implicit definition $g(x, y, z) = 0$ of C as a curve on S_f cannot be distinguished from its definition as an intersection curve of the surfaces $f(x, y, z) = 0$, $g(x, y, z) = 0$. In the first case, the curvature of C is provided by curvature formula (8b), while the second case is treated in Section 5.

Distinct definitions of C on S_f result by imposing on only two of the three coordinates of S_f, say on x, y, a parametric restriction $x = x(u)$, $y = y(u)$ or an implicit restriction $g(x, y) = 0$. In the first case, a value of u fixes x, y, but z can only be obtained from the implicit equation $f(x, y, z) = 0$ if $f_z \ne 0$. The curvature formula must somehow reflect this fact. The derivatives $\mathbf{r}'$, $\mathbf{r}''$ in the general curvature formula (8b) are then obtained for the x and y components from the given functions $x = x(u)$, $y = y(u)$, while for z', z'', the curve representation $f(x(u), y(u), z) = 0$ yields

$$f' = f_x x' + f_y y' + f_z z' = 0 \rightarrow z' = \frac{z_p}{f_z}, \quad z_p = -\left(f_x x' + f_y y'\right) \tag{27}$$

and by differentiating z'

$$z'' = -\frac{f_z(f_x' x' + f_y' y' + f_x x'' + f_y y'') - (f_x x' + f_y y') f_z'}{f_z^2} \tag{28a}$$

where, by the chain rule

$$\begin{bmatrix} f_x' \\ f_y' \\ f_z' \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \tag{29}$$

and by introducing Equ. (29) into Equ. (28a), the latter becomes

$$z'' = \frac{z_{pp}}{f_z^3}, \quad z_{pp} = -\left(f_z^2\left(f_{xx}x'^2 + f_{yy}y'^2 + 2f_{xy}x'y' + f_x x'' + f_y y''\right) + 2f_z\left(f_{xz}x' + f_{yz}y'\right)z_p + f_{zz}z_p^2\right) \tag{28b}$$

Then, introduction of the above expressions for z', z'' into formulae (8b), yields

$$k=\frac{(f_z^6(x'y''-x''y')^2+(y'z_{pp}-y''f_z^2z_p)^2+(f_z^2z_px''-z_{pp}x')^2)^{1/2}}{(f_z^2(x'^2+y'^2)+z_p^2)^{3/2}}$$
$$\mathbf{B}=\frac{[(y'z_{pp}-y''f_z^2z_p)(f_z^2z_px''-z_{pp}x')f_z^3(x'y''-x''y')]^{\mathrm{T}}}{(f_z^6(x'y''-x''y')^2+(y'z_{pp}-y''f_z^2z_p)^2+(f_z^2z_px''-z_{pp}x')^2)^{1/2}} \tag{30}$$

These expressions are symmetric in x, y but non-robust since, when $f_z=0$ we have $z_p=z_{pp}=0$ and k, $\mathbf{B}$ assume the indeterminate values $0/0$, $\mathbf{0}/0$. This is expected since, as noted above, $f(x,y,z)=0$ cannot be solved for z when this condition exists.

When both $\mathrm{S_f}$ and C are represented implicitly as $f(x,y,z)=0$ and $g(x,y)=0$, respectively, assuming $g_y\neq 0$, we can stipulate the existence of a function $y=y(x)$ and represent again C on S parametrically by $x=x$, $y=y(x)$, with parameter x this time. Then $x'=1$, $x''=0$ and

$$g'=g_x+g_yy'=0\ \rightarrow\ y'=-\frac{g_x}{g_y},\qquad y''=\frac{g_y'g_x-g_yg_x'}{g_y^2}$$
$$f'=f_x+f_yy'+f_zz'=0\ \rightarrow\ z'=\frac{z_p}{g_yf_z},\quad z_p=f_yg_x-f_xg_y$$
$$z''=\frac{(f_y'g_x+f_yg_x'-f_x'g_y-f_xg_y')g_yf_z-z_p(g_y'f_z+g_yf_z')}{g_y^2f_z^2} \tag{31a}$$

Introducing the derivative expressions

$$\begin{bmatrix}g_x'\\ g_y'\end{bmatrix}=\begin{bmatrix}g_{xx}&g_{xy}\\ g_{xy}&g_{yy}\end{bmatrix}\begin{bmatrix}1\\ y'\end{bmatrix}=\frac{1}{g_yf_z}\begin{bmatrix}g_{xx}&g_{xy}\\ g_{xy}&g_{yy}\end{bmatrix}\begin{bmatrix}g_yf_z\\ -g_xf_z\end{bmatrix}$$
$$\begin{bmatrix}f_x'\\ f_y'\\ f_z'\end{bmatrix}=\begin{bmatrix}f_{xx}&f_{xy}&f_{xz}\\ f_{xy}&f_{yy}&f_{yz}\\ f_{xz}&f_{yz}&f_{zz}\end{bmatrix}\begin{bmatrix}1\\ y'\\ z'\end{bmatrix}=\frac{1}{g_yf_z}\begin{bmatrix}f_{xx}&f_{xy}&f_{xz}\\ f_{xy}&f_{yy}&f_{yz}\\ f_{xz}&f_{yz}&f_{zz}\end{bmatrix}\begin{bmatrix}g_yf_z\\ -g_xf_z\\ z_p\end{bmatrix} \tag{32}$$

we obtain

$$y''=\frac{y_{pp}}{g_y^3},\quad y_{pp}=2g_{xy}g_xg_y-g_{xx}g_y^2-g_{yy}g_x^2,\qquad z''=\frac{z_{pp}}{g_y^3f_z^3}$$
$$z_{pp}=2f_{xy}g_xg_y^2f_z^2-2f_{xz}g_y^2f_zz_p+2f_{yz}g_xg_yf_zz_p-f_{xx}g_y^3f_z^2-f_{yy}g_x^2g_yf_z^2$$
$$\quad-f_{zz}g_yz_p^2-2g_{xy}g_xg_yf_yf_z^2+g_{xx}f_yg_y^2f_z^2+g_{yy}g_x^2f_yf_z^2 \tag{31b}$$

$$y'z''-y''z'=\frac{c_{yz}}{g_y^3f_z^3}$$
$$c_{yz}=-2f_{xy}g_x^2g_yf_z^2+2f_{xz}g_xg_yf_zz_p-2f_{yz}g_x^2f_zz_p+f_{xx}g_xg_y^2f_z^2$$
$$\quad+f_{yy}g_x^3f_z^2+f_{zz}g_xz_p^2+2g_{xy}f_xg_xg_yf_z^2-g_{xx}f_xg_y^2f_z^2-g_{yy}f_xg_x^2f_z^2 \tag{33}$$

With the above expressions of y', y'', z', z'' and the values $x'=1$, $x''=0$, formulae (8b) become

$$k=\frac{((y_{pp}f_z^3)^2+(c_{yz})^2+(z_{pp})^2)^{1/2}}{((g_x^2+g_y^2)f_z^2+z_p^2)^{3/2}}$$
$$\mathbf{B}=\frac{[c_{yz}-z_{pp}\,y_{pp}f_z^3]^{\mathrm{T}}}{((y_{pp}f_z^3)^2+(c_{yz})^2+(z_{pp})^2)^{1/2}} \tag{34}$$

Curvature formula (34) encompasses the known formula for the curvature of a plane implicit curve $g(x,y)=0$

$$k=\frac{y_{pp}}{(g_x^2+g_y^2)^{3/2}}=\frac{2g_{xy}g_xg_y-g_{xx}g_y^2-g_{yy}g_x^2}{(g_x^2+g_y^2)^{3/2}} \tag{35}$$

as a special case, when $\mathrm{S_f}$ is the x–y coordinate plane, defined by $f(x,y,z)=z=0$. Then, $f_z=1$, $z_p=z_{pp}=c_{yz}=0$ and formula (34) reduces to formula (35).

The curvatures of special curves on a surface are also of interest. Geodesics have their principal normal vector $\mathbf{N_C}$ aligned with the surface unit normal vector $\mathbf{N}$. Consequently (Equ. (12)), their curvature k is equal to the local normal curvature k_n

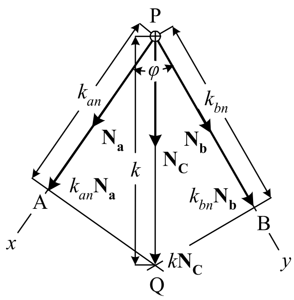

Fig. 1. Construction of an intersection curve's curvature vector $k\mathbf{N_C}$, from the normal curvature vectors $k_{an}\mathbf{N_a}$, $k_{bn}\mathbf{N_b}$ of the intersecting surfaces.

of the surface, in the direction of the geodesic. The curvatures of lines of curvature (principal curvatures) are roots of the quadratic equation

$$k^2 - 2Hk + K = 0 \tag{36}$$

Direct but unreduced formulae for the curvatures of curvature lines on implicit surfaces have been provided by Che et al. (2007). Curvature formulae for topologically important curves such as ridges and ravines are presented in Che et al. (2011).

5. Curvature of surface intersection curves

Let S_a, S_b be two intersecting surfaces, C their curve of intersection, P a point on C and $\mathbf{N_a}$, $\mathbf{N_b}$ the unit normal vectors of S_a, S_b at P. Then, the *normal curvature vectors* $k_{an}\mathbf{N_a}$, $k_{bn}\mathbf{N_b}$ of S_a, S_b determine the curvature k of their intersection curve C at P (Fig. 1). This is a consequence of Equ. (12), as follows: The plane spanned by $\mathbf{N_a}$, $\mathbf{N_b}$ is normal to C at P and contains the principal normal $\mathbf{N_C}$ of C and its curvature vector $k\mathbf{N_C}$. Since C belongs to both S_α and S_b, according to Equ. (12), the projections of its curvature vector $k\mathbf{N_C}$ on the lines Px, Py normal to S_α, S_b will be equal in magnitude to the normal curvatures k_{an}, k_{bn} of S_a, S_b, in the direction of the tangent to C at P. Conversely, if on the lines Px, Py drawn normal to S_a, S_b we take segments $PA = k_{an}$, $PB = k_{bn}$ and from their end points A, B draw normals to these lines in their plane, the intersection Q of the normals defines the curvature vector $\overrightarrow{\mathbf{PQ}} = k\mathrm{N_C}$ of C and, consequently, the curvature k of C at P. The dependence of k on k_{an}, k_{bn} and φ revealed by this construction, is expressed by Equ. (1). There are two alternative equations, which express the same relationship. The first

$$k\mathbf{B} = \frac{k_{an}\mathbf{N_b} - k_{bn}\mathbf{N_a}}{|\mathbf{N_a} \times \mathbf{N_b}|} \tag{37a}$$

was provided by Faux and Pratt (1981, p. 262) and the second

$$k\mathbf{N_C} = \frac{k_{an} - k_{bn}\cos\varphi}{\sin^2\varphi}\mathbf{N_a} + \frac{k_{bn} - k_{an}\cos\varphi}{\sin^2\varphi}\mathbf{N_b} \tag{38}$$

by Ye and Maekawa (1999). Both reduce to Equ. (1) by squaring.

Surprisingly, the possibility of using these expressions to derive reduced formulae for the curvature of the intersection curve has not been exploited by their authors. We shall use Equ. (37a) for this purpose. It is more convenient to introduce into it the surface normal vectors $\mathbf{n_a}$, $\mathbf{n_b}$, through the expressions $\mathbf{N_a} = \mathbf{n_a}/|\mathbf{n_a}|$, $\boldsymbol{N_b} = \mathbf{n_b}/|\mathbf{n_b}|$ and write Equ. (37a) in the form

$$k\mathbf{B} = \frac{k_{an}|\mathbf{n_a}|\mathbf{n_b} - k_{bn}|\mathbf{n_b}|\mathbf{n_a}}{|\mathbf{t}|}, \quad |\mathbf{t}| = |\mathbf{n_a} \times \mathbf{n_b}| \tag{37b}$$

The convenience stems from the fact that introduction of the generic forms of the normal curvatures k_{an}, k_{bn} of S_α, S_b (Equ. (17)) simplifies this expression, as follows

$$k\mathbf{B} = \frac{c_{an}\mathbf{n_b} - c_{bn}\mathbf{n_a}}{|\mathbf{t}|^3} \tag{39a}$$

Equ. (39a) leads directly to reduced formulae for the curvature k of C and for its binormal vector $\mathbf{B}$, as follows

$$k = \frac{|c_{an}\mathbf{n_b} - c_{bn}\mathbf{n_a}|}{|\mathbf{t}|^3}, \qquad \mathbf{B} = \frac{c_{an}\mathbf{n_b} - c_{bn}\mathbf{n_a}}{|c_{an}\mathbf{n_b} - c_{bn}\mathbf{n_a}|} \tag{39b}$$

To generate a reduced curvature expression for the curvature k of the intersection curve C of two implicit surfaces S_f, S_g, for example, represented by $f(x, y, z) = 0$, $g(x, y, z) = 0$, we introduce into Equ. (39a) expressions for the normal curvature factors c_{fn}, c_{gn} of S_f, S_g (Equ. (3)) and the expressions $\mathbf{n_f} = \nabla\mathbf{f}$, $\mathbf{n_g} = \nabla\mathbf{g}$ for the surface normal vectors and $\mathbf{t} = \mathbf{n_f} \times \mathbf{n_g}$ for the tangent vector of C. We then have the following formulae for the curvature k and for the binormal vector $\mathbf{B}$ of C

$$k=\frac{((c_{fn}g_x-c_{gn}f_x)^2+(c_{fn}g_y-c_{gn}f_y)^2+(c_{fn}g_z-c_{gn}f_z)^2)^{1/2}}{(t_x^2+t_y^2+t_z^2)^{3/2}}$$
$$c_{fn}=-\big(f_{xx}t_x^2+f_{yy}t_y^2+f_{zz}t_z^2+2f_{xy}t_xt_y+2f_{xz}t_xt_z+2f_{yz}t_yt_z\big)$$
$$c_{gn}=-\big(g_{xx}t_x^2+g_{yy}t_y^2+g_{zz}t_z^2+2g_{xy}t_xt_y+2g_{xz}t_xt_z+2g_{yz}t_yt_z\big)$$
$$\mathbf{B}=\frac{[(c_{fn}g_x-c_{gn}f_x)(c_{fn}g_y-c_{gn}f_y)(c_{fn}g_z-c_{gn}f_z)]^{\mathrm{T}}}{((c_{fn}g_x-c_{gn}f_x)^2+(c_{fn}g_y-c_{gn}f_y)^2+(c_{fn}g_z-c_{gn}f_z)^2)^{1/2}} \tag{39c}$$

For two parametric surfaces $\mathrm{S_{r1}}$, $\mathrm{S_{r2}}$, represented by $\mathbf{r_1}(u,v)=[x_1(u,v)\ \ y_1(u,v)\ \ z_1(u,v)]^{\mathrm{T}}$, $\mathbf{r_2}(p,q)=[x_2(p,q)\ \ y_2(p,q)\ \ z_2(p,q)]^{\mathrm{T}}$ with normal vectors $\mathbf{n_1}=\mathbf{r_{1u}}\times\mathbf{r_{1v}}$, $\mathbf{n_2}=\mathbf{r_{2p}}\times\mathbf{r_{2q}}$, the tangent vector $\mathbf{t}=\mathbf{n_1}\times\mathbf{n_2}$ of their intersection curve C has tangential coordinates in the basis $\mathbf{r_{1u}}$, $\mathbf{r_{1v}}$ of the tangent plane of $\mathrm{S_{r1}}$ $a_1=-\mathbf{r_{1v}}\cdot\mathbf{n_2}$, $b_1=\mathbf{r_{1u}}\cdot\mathbf{n_2}$ and in the basis $\mathbf{r_{2p}}$, $\mathbf{r_{2q}}$ of the tangent plane of $\mathrm{S_{r2}}$ $a_2=-\mathbf{r_{2q}}\cdot\mathbf{n_1}$, $b_2=\mathbf{r_{2p}}\cdot\mathbf{n_1}$ (Equ. (20b)). Then, Equ. (39a) yields the formulae

$$k=\frac{((c_{r1}n_{2x}-c_{r2}n_{1x})^2+(c_{r1}n_{2y}-c_{r2}n_{1y})^2+(c_{r1}n_{2z}-c_{r2}n_{1z})^2)^{1/2}}{(t_x^2+t_y^2+t_z^2)^{3/2}}$$
$$c_{r1}=L_1'a_1^2+2M'a_1b_1+N_1'b_1^2,\quad L_1'=\mathbf{r_{1uu}n_1},\ M_1'=\mathbf{r_{1uv}n_1},\ N_1'=\mathbf{r_{1vv}n_1}$$
$$c_{r2}=L_2'a_2^2+2M'a_2b_2+N_2'b_2^2,\quad L_2'=\mathbf{r_{2pp}n_2},\ M_2'=\mathbf{r_{2pq}n_2},\ N_2'=\mathbf{r_{2qq}n_2}$$
$$\mathbf{B}=\frac{[(c_{r1}n_{2x}-c_{r2}n_{1x})(c_{r1}n_{2y}-c_{r2}n_{1y})(c_{r1}n_{2z}-c_{r2}n_{1z})]^{\mathrm{T}}}{((c_{r1}n_{2x}-c_{r2}n_{1x})^2+(c_{r1}n_{2y}-c_{r2}n_{1y})^2+(c_{r1}n_{2z}-c_{r2}n_{1z})^2)^{1/2}} \tag{39d}$$

The curvature formula for the intersection curve of a parametric with an implicit surface was derived in Section 1 (Equ. (6b)), but it could have more readily been derived from Equ. (39a).

Interest in the differential geometry of surface intersection curves has recently expanded to include intersection curves in R4 and even higher dimensions (Goldman, 2005; Aléssio, 2006; 2012; Düldül, 2010).

6. Normal and intersection curvatures of offset surfaces

Intersection curves of surfaces which are offsets of (also known as parallel to) given surfaces do arise in modern applications, justifying a revival of interest in their differential properties. For example, in the 3-axis CNC milling of free-form geometries, with a ball-end cutter of radius R, it is sometimes desirable to move the cutter with its ball-end being in simultaneous contact with two given surfaces, both of which must be machined or one must be machined, while the other is used for guiding the tool. Then, the center point of the ball-end of the cutter, whose motion is programmed, moves along the intersection of two surfaces which are offsets of the given surfaces, at distance R.

6.1. Normal curvature

Let $\mathrm{S_p}$ be an offset surface of a given surface S, at distance d. Classical texts focus on the fact that at corresponding points P, $\mathrm{P_p}$ on S, $\mathrm{S_p}$, that is points lying on a common normal line, the principal directions of the two surfaces are parallel and their principal curvatures k_{ni}, k_{pni}, $i=1,2$, are related by

$$k_{ni}=\frac{k_{pni}}{1+dk_{pni}},\quad k_{pni}=\frac{k_{ni}}{1-dk_{ni}},\ i=1,2 \tag{40}$$

They don't show, however, how the normal curvatures of S, $\mathrm{S_p}$ are related, in other directions. We shall investigate this matter, using Euler's equation (Struik, 1950), which expresses the normal curvature k_n at a surface point P, in a given tangent direction $\mathbf{t}$, in terms of the principal curvatures k_{n1}, k_{n2} of the surface at P and the angle φ of $\mathbf{t}$ with the direction of k_{n1}, as follows:

$$k_n=k_{n1}\cos^2\varphi+k_{n2}\sin^2\varphi=(k_{n1}-k_{n2})\cos^2\varphi+k_{n2} \tag{41a}$$

We apply Euler's equation to $\mathrm{S_p}$ and substitute in it the principal curvatures k_{pn1}, k_{pn2} of $\mathrm{S_p}$ by their expressions in terms of the principal curvatures of S (Equ. (40)), to obtain

$$\begin{aligned}k_{pn}&=(k_{pn1}-k_{pn2})\cos^2\varphi_p+k_{pn2}\\&=\left(\frac{k_{n1}}{1-dk_{n1}}-\frac{k_{n2}}{1-dk_{n2}}\right)\cos^2\varphi_p+\frac{k_{n2}}{1-dk_{n2}}\end{aligned} \tag{41b}$$

Since the principal directions on S, S_p are parallel, $\mathbf{t}$ makes equal angles with the directions of the principal curvatures k_{n1} of S and k_{pn1} of S_p. Thus, by applying Euler's equation to S, we express $\cos^2\varphi_p$ in terms of the principal curvatures of S

$$\cos^2\varphi_p = \frac{k_n - k_{n2}}{k_{n1} - k_{n2}} \tag{42}$$

and Equ. (41b) becomes

$$\begin{aligned} k_{pn} &= \left(\frac{k_{n1}}{1-dk_{n1}} - \frac{k_{n2}}{1-dk_{n2}}\right)\left(\frac{k_n - k_{n2}}{k_{n1}-k_{n2}}\right) + \frac{k_{n2}}{1-dk_{n2}} \\ &= \frac{k_n - dk_{n1}k_{n2}}{1 - d(k_{n1}+k_{n2}) + d^2k_{n1}k_{n2}} \\ &= \frac{k_n - dK}{1-2dH+d^2K} \end{aligned} \tag{41c}$$

The last expression of the normal curvature k_{pn} of S_p entails the normal curvature k_n of the base surface S in the tangent direction $\mathbf{t}$ and the Gaussian and mean curvatures K, H of S. Reduced formulae for these curvatures have been given in Section 2 (Equ. (15)–(17)). Substituting k_n, K, H by their generic expressions, we obtain

$$k_{ipn} = \frac{\frac{c_{in}}{|\mathbf{t}|^2|\mathbf{n_i}|} - \frac{dc_{iK}}{|\mathbf{n_i}|^4}}{1 - \frac{dc_{iH}}{|\mathbf{n_i}|^3} + \frac{d^2c_{iK}}{|\mathbf{n_i}|^4}} = \frac{c_{ipn}}{|\mathbf{t}|^2|\mathbf{n_i}|}, \qquad c_{ipn} = \frac{|\mathbf{n_i}|(c_{in}|\mathbf{n_i}|^3 - dc_{iK}|\mathbf{t}|^2)}{|\mathbf{n_i}|^4 - dc_{iH}|\mathbf{n_i}| + d^2c_{iK}} \tag{43}$$

6.2. Curvature of intersection curves of offset surfaces

Expressions derived by instantiating the normal curvature expression (43) for a surface S_{rp} offset of a given parametric surface S_r ($i=r$) and for a surface S_{fp} offset of a given implicit surface S_f ($i=f$), can now be combined in all three modes (implicit/implicit, parametric/parametric and parametric/implicit) of the given surfaces S_r, S_f, with the aid of Equ. (39a), to yield curvature formulae for the intersection curves of offset surfaces. It may also be desirable to compute the curvature of the intersection of a given surface S_a with a surface S_{bp} offset of a given surface S_b. In all these cases, Equ. (39a), (43) can be used to produce intersection curvature formulae for offset surfaces, as it was done for given implicit and parametric surfaces in Section 5, provided proper expressions for the curvature factors c_{rn}, c_{fn}, c_{rK}, c_{fK}, c_{rH}, c_{fH} of the given surfaces are utilized. For two surfaces S_{fp}, S_{gp} offset of given implicit surfaces S_f, S_g represented by $f(x,y,z)=0$, $g(x,y,z)=0$, for example, Equ. (39a) yields

$$\begin{aligned} k &= \frac{((c_{fpn}g_x - c_{gpn}f_x)^2 + (c_{fpn}g_y - c_{gpn}f_y)^2 + (c_{fpn}g_z - c_{gpn}f_z)^2)^{1/2}}{(t_x^2+t_y^2+t_z^2)^{3/2}} \\ c_{fpn} &= \frac{|\mathbf{n_f}|(c_{fn}|\mathbf{n_f}|^3 - dc_{fK}|\mathbf{t}|^2)}{|\mathbf{n_f}|^4 - dc_{fH}|\mathbf{n_f}| + d^2c_{fK}}, \qquad c_{gpn} = \frac{|\mathbf{n_g}|(c_{gn}|\mathbf{n_g}|^3 - dc_{gK}|\mathbf{t}|^2)}{|\mathbf{n_g}|^4 - dc_{gH}|\mathbf{n_g}| + d^2c_{gK}} \\ \mathbf{B} &= \frac{[(c_{fpn}g_x - c_{gpn}f_x)(c_{fpn}g_y - c_{gpn}f_y)(c_{fpn}g_z - c_{gpn}f_z)]^{\mathrm{T}}}{((c_{fpn}g_x - c_{gpn}f_x)^2 + (c_{fpn}g_y - c_{gpn}f_y)^2 + (c_{fpn}g_z - c_{gpn}f_z)^2)^{1/2}} \end{aligned} \tag{44}$$

with $\mathbf{n_f} = \nabla\mathbf{f}$, $\mathbf{n_g} = \nabla\mathbf{g}$, $\mathbf{t} = \mathbf{n_f}\times\mathbf{n_g}$, where the expressions for the normal curvature factors c_{fn}, c_{gn} are derived from Equ. (3) and those for c_{fK}, c_{fH}, c_{gK}, c_{gH} from Equ. (16).

Example 2. Given a spherical surface S_f of radius R and a circular conical surface S_g with a 90° aperture and its apex at the center of the sphere, implicitly represented by

$$f = x_f^2 + y_f^2 + z_f^2 - R^2 = 0, \qquad g = y_g^2 + z_g^2 - x_g^2 = 0$$

and offsets S_{fp}, S_{gp} of these surfaces at distance d (Fig. 2): a) Express, by means of geometry, the curvature k of the largest of the two circles of intersection of S_{fp}, S_{gp}, in terms of R, d. b) Verify the curvature expression found in (a) by means of the general curvature formula (44).

(a) In the x–z plane, the generatrix line OP of S_{gp} has equation $z = x + \sqrt{2}d$. It meets the circle $x^2+z^2=(R+d)^2$ at P, whose coordinates are found by solving the system of these two equations to be

$$x_P = \frac{\sqrt{2}(\sqrt{R^2+2dR}-d)}{2}, \qquad z_P = \frac{\sqrt{2}(\sqrt{R^2+2dR}+d)}{2}$$

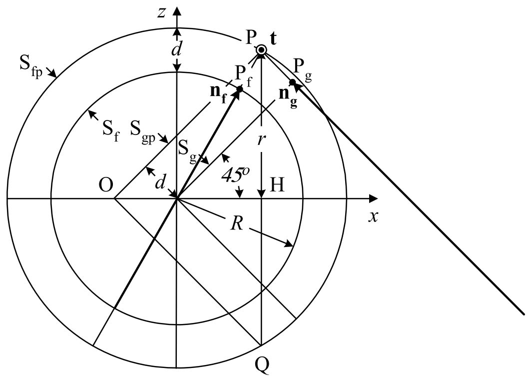

Fig. 2. *X*–*z* plane view of the given surfaces S_f, S_g, their offsets S_{fp}, S_{gp}, their surface normal vectors $\mathbf{n_f}$, $\mathbf{n_g}$ and the tangent vector $\mathbf{t}$ of the intersection circle of S_{fp}, S_{gp}.

Since z_P equals the radius r of the intersection circle of S_{fp}, S_{gp}, the curvature of this circle is

$$\boxed{k = \frac{1}{r} = \frac{\sqrt{2}}{(\sqrt{R^2 + 2dR} + d)}}$$

(b) Partial derivatives of the given surfaces S_f, S_g:

$$f_x = 2x_f, \quad f_y = 2y_f, \quad f_z = 2z_f, \quad f_{xx} = f_{yy} = f_{zz} = 2, \quad f_{xy} = f_{xz} = f_{yz} = 0$$
$$g_x = -2x_g, \quad g_y = 2y_g, \quad g_z = 2z_g, \quad g_{xx} = -2, \quad g_{yy} = g_{zz} = 2, \quad g_{xy} = g_{xz} = g_{yz} = 0$$

Surface normal vectors of S_f, S_g at P_f, P_g ($y_f = y_g = 0$):

$$\mathbf{n_f} = \nabla \mathbf{f} = [\, 2x_f \quad 0 \quad 2z_f \,]^T, \qquad \mathbf{n_g} = \nabla \mathbf{g} = [\, -2x_g \quad 0 \quad 2z_g \,]^T,$$
$$|\mathbf{n_f}| = 2R, \qquad |\mathbf{n_g}| = 2\left(x_g^2 + z_g^2\right)^{1/2} = 2(OP - d) = 2(\sqrt{2}r - d) = 2\sqrt{R^2 + 2dR}$$

Tangent vector of the intersection circle of S_{fp}, S_{gp} at P:

$$\mathbf{t_P} = \mathbf{n_f} \times \mathbf{n_g} = 4\left[0 \quad -(z_f x_g + x_f z_g) \quad 0\right]^T$$

To evaluate the norm of $\mathbf{t_P}$, we substitute the coordinates of $P_f(x_f, 0, z_f)$, $P_g(x_g, 0, z_g)$ in terms of coordinates of $P(x_P, 0, z_P)$, using the defining relations of the offset surfaces:

$$\mathbf{r_P} = \mathbf{r_f} + d\frac{\mathbf{n_f}}{|\mathbf{n_f}|} = \mathbf{r_g} + d\frac{\mathbf{n_g}}{|\mathbf{n_g}|}$$
$$|\mathbf{n_f}|x_f + 2dx_f = |\mathbf{n_f}|x_P \;\rightarrow\; x_f = \frac{R}{R+d}x_P, \qquad z_f = \frac{R}{R+d}z_P$$
$$|\mathbf{n_g}|x_g - 2dx_g = |\mathbf{n_g}|x_P \;\rightarrow\; x_g = \frac{|\mathbf{n_g}|x_P}{|\mathbf{n_g}| - 2d}, \qquad z_g = \frac{|\mathbf{n_g}|z_P}{|\mathbf{n_g}| + 2d}$$

Before applying curvature formula (44), it is convenient to express all quantities it entails in terms of R, d, $|\mathbf{n_g}|$, $|\mathbf{t_P}|$. Thus, we have

$$x_P = \frac{\sqrt{2}(|\mathbf{n_g}| - 2d)}{4}, \qquad r = z_P = \frac{\sqrt{2}(|\mathbf{n_g}| + 2d)}{4}$$
$$x_f = \frac{\sqrt{2}R(|\mathbf{n_g}| - 2d)}{4(R+d)}, \qquad y_f = \frac{\sqrt{2}R(|\mathbf{n_g}| + 2d)}{4(R+d)}$$
$$x_g = z_g = \frac{\sqrt{2}|\mathbf{n_g}|}{4}, \qquad |\mathbf{t_P}| = 4\left|(z_f x_g + x_f z_g)\right| = \frac{R|\mathbf{n_g}|^2}{R+d}$$

Curvature factors of S_{fp}, S_{gp} (Equ. (3), (17)):

$$c_{fn} = -2t_{Py}^2 = -2|\mathbf{t_P}|^2, \qquad c_{fK} = 16\left(x_f^2 + z_f^2\right) = 16R^2, \qquad c_{fH} = -16\left(x_f^2 + z_f^2\right) = -16R^2$$

$$c_{fpn} = \frac{|\mathbf{n_f}|(c_{fn}|\mathbf{n_f}|^3 - dc_{fK}|\mathbf{t_P}|^2)}{|\mathbf{n_f}|^4 - dc_{fH}|\mathbf{n_f}| + d^2 c_{fK}} = -\frac{2R|\mathbf{t_P}|^2}{R+d}$$

$$c_{gn} = -2t_{Py}^2 = -2|\mathbf{t_P}|^2, \qquad c_{gK} = 16\left(x_g^2 - z_g^2\right) = 0, \qquad c_{gH} = -16x_g^2 = -2|\mathbf{n_g}|^2$$

$$c_{gpn} = \frac{c_{gn}|\mathbf{n_g}|^3}{|\mathbf{n_g}|^3 - dc_{gH}} = -\frac{2|\mathbf{n_g}||\mathbf{t_P}|^2}{|\mathbf{n_g}| + 2d}$$

Curvature of the intersection circle of S_{fp}, S_{gp} (Equ. (44)):

$$k = \frac{((c_{fpn}g_x - c_{gpn}f_x)^2 + (c_{fpn}g_z - c_{gpn}f_z)^2)^{1/2}}{|\mathbf{t_P}|^3}$$

$$c_{fpn}g_x - c_{gpn}f_x = c_{fpn}(-2x_g) - c_{gpn}(2x_f) = \frac{2\sqrt{2}R|\mathbf{n_g}|^2|\mathbf{t_P}|^2}{(R+d)(|\mathbf{n_g}| + 2d)}$$

$$c_{fpn}g_z - c_{gpn}f_z = c_{fpn}(2z_g) - c_{gpn}(2z_f) = 0$$

$$k = \frac{|c_{fpn}g_x - c_{gpn}f_x|}{|\mathbf{t_P}|^3} = \frac{2\sqrt{2}}{|\mathbf{n_g}| + 2d} = \frac{\sqrt{2}}{(\sqrt{R^2 + 2dR} + d)}$$

7. Concluding remarks

The following properties of reduced curvature formulae can now be stated:

a) They are closed formulae, entailing only basic arithmetic operators (addition, subtraction, multiplication, division) and square root operators. They are thus suitable for casual users, whose skills do not extend beyond basic algebra and the extraction of function derivatives.
b) They are more efficient compared to alternative unreduced formulae (see Appendix A).

Although we have presented several reduced formulae, we have not exhausted all cases that may arise. We have not, for example, dealt with the curvature of curves defined by differential equations. Curves on offset surfaces may also arise in ways other than as intersections with other surfaces. An open problem is developing a curvature formula for the curve C_p traced by a point P_p on a surface S_p, offset of a given surface S, when its corresponding point P traces on S a given curve C. In this case, C and C_p do not have the same directions at corresponding points, so Euler's formula (Equ. (41a)) cannot be used to relate the curvatures of C, C_p. The solution of this problem would be of practical interest in the isoparametric 3-D machining of a surface patch S, with a ball-end cutter.

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Appendix A. Efficiency comparison of curvature formulae

We compare the cost of reduced curvature formulae to the cost of alternative unreduced formulae, in three cases, based on the following assumptions: The comparisons do not include the cost of computing function derivative values, which is the same for the compared formulae but case-dependent. These are assumed to be available. The cost of each formula is quantified by means of a multiplication count C_m, an addition count C_a and a square root count C_{sr}. In forming these counts, divisions are assumed equivalent to multiplications, small integer powers equivalent to repeated multiplications and subtractions and multiplications by 2 equivalent to additions. The cost of standard vector operations then is: For a scalar product $\mathbf{a} \cdot \mathbf{b}$, $C_m = 3$, $C_a = 2$, for a vector product $\mathbf{a} \times \mathbf{b}$, $C_m = 6$, $C_a = 3$, for a double vector product $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$, $C_m = 12$, $C_a = 6$.

Case 1. Computation of the tangential coordinates a, b of $\mathbf{t}$, when $\mathbf{t}$ is the tangent vector of the intersection curve of a parametric surface $\mathbf{r} = [x(u,v)\ \ y(u,v)\ \ z(u,v)]^{\mathrm{T}}$ by another surface with normal vector $\mathbf{n}$. Table 1 compares the cost of using the unreduced formulae of Ye and Maekawa (1999) for a, b (Equ. (20a)) to the cost of using our reduced formulae (Equ. (20b)).

Case 2. Computation of the curvature k of the curve of intersection of a parametric surface $\mathbf{r} = [x(u,v)\ \ y(u,v)\ \ z(u,v)]^{\mathrm{T}}$ by an implicit surface $f(x, y, z) = 0$. Table 2 compares the cost of using the formulae of Ye and Maekawa (1999), Equ. (20a), for computing the coordinates a, b required by formula (2), and formulae (1)–(3) for computing k to the cost of computing a, b by our reduced formulae (Equ. (20b)) and our reduced formula (Equ. (6b)) for k.

Table 1
Cost comparison of the unreduced and reduced formulae for a, b.

Ye and Maekawa (1999), Equ. (20a)	C_m	C_a	Our formulae, Equ. (20b)	C_m	C_a
E, F, G	9	6	a, b	6	4
$EG - F^2$	2	1			
$\mathbf{t} = (\mathbf{r_u} \times \mathbf{r_v}) \times \mathbf{n}$	12	6			
$\mathbf{t} \cdot \mathbf{r_u}$, $\mathbf{t} \cdot \mathbf{r_v}$	6	4			
a, b	6	2			
Total counts	35	19		6	4

Table 2
Cost comparison of unreduced and reduced formulae for computing the intersection curvature of a parametric with an implicit surface.

Ye and Maekawa (1999), Equ. (20a), (1)–(3)	C_m	C_a	C_{sr}	Our reduced formulae, Equ. (20b), (6b)	C_m	C_a	C_{sr}
A, b	35	19	0	a, b	6	4	0
L', M', N'	9	6	0	n_{rx}, n_{ry}, n_{rz}	6	3	0
c_{rn}	6	3	0	L', M', N'	9	6	0
$\lvert\mathbf{n_r}\rvert, \lvert\mathbf{n_f}\rvert, \lvert\mathbf{t}\rvert, \lvert\mathbf{t}\rvert^3$	11	6	3	c_{rn}	6	3	0
c_{fn}	9	8	0	t_x, t_y, t_z	6	3	0
$\cos\varphi, \sin\varphi$	7	2	0	$\lvert\mathbf{t}\rvert, \lvert\mathbf{t}\rvert^3$	5	2	1
k_{fn}, k_{rn}	4	0	0	c_{fn}	9	8	0
k	5	3	1	k	10	5	1
Total counts	86	47	4	Total counts	57	34	2

Case 3. We prove the equivalence of Goldman's formula for the curvature of the intersection curve of two implicit surfaces $f(x, y, z) = 0$, $g(x, y, z) = 0$ (Goldman, 2005)

$$k = \frac{|((\nabla f \times \nabla g) * \nabla(\nabla f \times \nabla g)) \times (\nabla f \times \nabla g)|}{|\nabla f \times \nabla g|^3} \tag{A.1a}$$

with our reduced formula for this curvature (Equ. (39c)) and, in the process, we compare the efficiencies of the two formulae. Since the denominator in both formulae is $|\mathbf{t}|^3$, it suffices to prove the equivalence of the numerators.

In Equ. (A.1a), $*$ is the matrix multiplication operator, with operands the curve's tangent vector $\mathbf{t} = \nabla f \times \nabla g$ and the matrix $M = \nabla\mathbf{t}$. This matrix is formed with columns the gradients of the components of $\mathbf{t}$, that is $M = [\nabla t_x \; \nabla t_y \; \nabla t_z]$. Thus, Equ. (A.1a) can be written

$$k = \frac{|[t_x \quad t_y \quad t_z] * [\nabla t_x \quad \nabla t_y \quad \nabla t_z] \times [t_x \quad t_y \quad t_z]|}{(t_x^2 + t_y^2 + t_z^2)^{3/2}} \tag{A.1b}$$

The columns of M are

$$\begin{aligned}
\nabla t_x &= \nabla(f_y g_z - f_z g_y) = \nabla f_y g_z + f_y \nabla g_z - \nabla f_z g_y - f_z \nabla g_y \\
\nabla t_y &= \nabla(f_z g_x - f_x g_z) = \nabla f_z g_x + f_z \nabla g_x - \nabla f_x g_z - f_x \nabla g_z \\
\nabla t_z &= \nabla(f_x g_y - f_y g_x) = \nabla f_x g_y + f_x \nabla g_y - \nabla f_y g_x - f_y \nabla g_x
\end{aligned} \tag{A.2}$$

Let $[X \; Y \; Z]$ be the vector $[t_x, t_y, t_z] * [\nabla t_x \; \nabla t_y \; \nabla t_z]$. Then, by matrix multiplication

$$\begin{aligned}
X &= (f_{xy} g_z + f_y g_{xz} - f_{xz} g_y - f_z g_{xy}) t_x + (f_{yy} g_z + f_y g_{yz} - f_{yz} g_y - f_z g_{yy}) t_y \\
&\quad + (f_{yz} g_z + f_y g_{zz} - f_{zz} g_y - f_z g_{yz}) t_z \\
Y &= (f_{xz} g_x + f_z g_{xx} - f_{xx} g_z - f_x g_{xz}) t_x + (f_{yz} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{yz}) t_y \\
&\quad + (f_{zz} g_x + f_z g_{xz} - f_{xz} g_z - f_x g_{zz}) t_z \\
Z &= (f_{xx} g_y + f_x g_{xy} - f_{xy} g_x - f_y g_{xx}) t_x + (f_{xy} g_y + f_x g_{yy} - f_{yy} g_x - f_y g_{xy}) t_y \\
&\quad + (f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz}) t_z
\end{aligned} \tag{A.3}$$

and the first component of the vector $[X \; Y \; Z] \times [t_x, t_y, t_z]$ is

$$\begin{aligned}
Y t_z - Z t_y &= \left(f_{yy} t_y^2 + f_{zz} t_z^2 + f_{xy} t_x t_y + f_{xz} t_x t_z + 2 f_{yz} t_y t_z\right) g_x \\
&\quad - \left(g_{yy} t_y^2 + g_{zz} t_z^2 + g_{xy} t_x t_y + g_{xz} t_x t_z + 2 g_{yz} t_y t_z\right) f_x - (f_{xx} t_x + f_{xy} t_y + f_{xz} t_z)(g_y t_y + g_z t_z) \\
&\quad + (g_{xx} t_x + g_{xy} t_y + g_{xz} t_z)(f_y t_y + f_z t_z)
\end{aligned} \tag{A.4a}$$

Table 3
Cost comparison of the unreduced Goldman's formula to our reduced formula for computing the intersection curvature of two implicit surfaces.

Goldman's formula, Equ. (A.1b)	C_m	C_a	C_{sr}	Our formula, Equ. (39c)	C_m	C_a	C_{sr}
t_x, t_y, t_z	6	3	0	t_x, t_y, t_z	6	3	0
$\|\mathbf{t}\|$, $\|\mathbf{t}\|^3$	5	2	1	$\|\mathbf{t}\|$, $\|\mathbf{t}\|^3$	5	2	1
Matrix $\nabla(\nabla\mathbf{t}) = [\nabla t_x\ \nabla t_y\ \nabla t_z]$	30	27	0	c_{fn}, c_{gn}	18	16	0
$[X\ Y\ Z] = [t_x, t_y, t_z] * [\nabla t_x\ \nabla t_y\ \nabla t_z]$	9	6	0	k	10	5	1
$[X\ Y\ Z] \times [t_x, t_y, t_z]$	6	3	0				
$\|[X\ Y\ Z] \times [t_x, t_y, t_z]\|$, k	4	2	1				
Total counts	60	43	2		39	26	2

which, by means of the normality conditions $f_y t_y + f_z t_z = -f_x t_x$, $g_y t_y + g_z t_z = -g_x t_x$, becomes

$$\begin{aligned} Yt_z - Zt_y &= \left(f_{xx}t_x^2 + f_{yy}t_y^2 + f_{zz}t_z^2 + 2f_{xy}t_xt_y + 2f_{xz}t_xt_z + 2f_{yz}t_yt_z\right)g_x \\ &\quad - \left(g_{xx}t_x^2 + g_{yy}t_y^2 + g_{zz}t_z^2 + 2g_{xy}t_xt_y + 2g_{xz}t_xt_z + 2g_{yz}t_yt_z\right)f_x \\ &= -(c_{fn}g_x - c_{gn}f_x) \end{aligned} \tag{A.4b}$$

Similarly, it can be shown that the second and third components of the vector $[X\ Y\ Z] \times [t_x, t_y, t_z]$ are $Zt_x - Xt_z = -(c_{fn}g_y - c_{gn}f_y)$ and $Xt_y - Yt_x = -(c_{fn}g_z - c_{gn}f_z)$. Thus, the numerator in Goldman's formula (Equ. (A.1b)) is

$$\begin{aligned} &\left|[t_x\quad t_y\quad t_z] * [\nabla t_x\quad \nabla t_y\quad \nabla t_z] \times [t_x\quad t_y\quad t_z]\right| \\ &= \left((c_{fn}g_x - c_{gn}f_x)^2 + (c_{fn}g_y - c_{gn}f_y)^2 + (c_{fn}g_z - c_{gn}f_z)^2\right)^{1/2} \end{aligned} \tag{A.5}$$

the same as in our formula (Equ. (39c)). Table 3 compares the cost of computing the intersection curvature of two implicit surfaces by Goldman's formula, Equ. (A.1b), to the cost of computing the same curvature by our reduced formula (39c).

Appendix B. Alternative proof of formulae (30), (34)

The curvature formula for a curve C, defined by a pair of parametric equations $x = x(u)$, $y = y(u)$ on an implicit surface S_f represented by $f(x, y, z) = 0$, was derived in Section 4 (Equ. (30)) from the curve representation $f(x(u), y(u), z) = 0$. As a correctness test, we shall derive this formula by considering C as the curve of intersection of S_f with a generalized cylindrical surface S_r with position vector $\mathbf{r} = [x(u)\ y(u)\ v]^T$ and using Equ. (6b) to obtain its curvature.

Normal vector of S_f: $\mathbf{n_f} = [f_x\ f_y\ f_z]^T$. Parametric derivatives and normal vector of S_r: $\mathbf{r_u} = [x_u\ y_u\ 0]^T$, $\mathbf{r_v} = [0\ 0\ 1]^T$, $\mathbf{r_{uu}} = [x_{uu}\ y_{uu}\ 0]^T$, $\mathbf{r_{uv}} = \mathbf{r_{vv}} = [0\ 0\ 0]^T$, $\mathbf{n_r} = \mathbf{r_u} \times \mathbf{r_v} = [y_u\ -x_u\ 0]^T$. Tangent vector of C: $\mathbf{t} = \mathbf{n_f} \times \mathbf{n_r} = [f_z x_u\ f_z y_u\ -z_p]^T$, $|\mathbf{t}| = (f_z^2(x_u^2 + y_u^2) + z_p^2)^{1/2}$, $z_p = f_x x_u + f_y y_u$.

Curvature factors of S_f, S_r (Equ. (2), (3)):

$$\begin{aligned} &c_{fn} = -\left(f_{xx}t_x^2 + f_{yy}t_y^2 + f_{zz}t_z^2 + 2f_{xy}t_xt_y + 2f_{xz}t_xt_z + 2f_{yz}t_yt_z\right) = z_{pp} + f_z^2(f_x x_{uu} + f_y y_{uu}) \\ &z_{pp} = -\left(f_z^2\left(f_{xx}x_u^2 + f_{yy}y_u^2 + 2f_{xy}x_u y_u + f_x x_{uu} + f_y y_{uu}\right) + 2f_z(f_{xz}x_u + f_{yz}y_u)z_p + f_{zz}z_p^2\right) \\ &L' = \mathbf{r_{uu}n_r} = x_{uu}y_u - y_{uu}x_u, \qquad M' = \mathbf{r_{uv}n_r} = 0, \qquad N' = \mathbf{r_{vv}n_r} = 0, \qquad a = -\mathbf{r_v n_f} = -f_z, \qquad b = \mathbf{r_u n_f} = z_p, \\ &c_{rn} = L'a^2 + 2M'ab + N'b^2 = f_z^2(x_{uu}y_u - y_{uu}x_u). \end{aligned}$$

Numerator terms of curvature formula (6b):

$$\begin{aligned} c_{rn}f_x - c_{fn}n_{rx} &= f_z^2(x_{uu}y_u - y_{uu}x_u)f_x - \left(z_{pp} + f_z^2(f_x x_{uu} + f_y y_{uu})\right)y_u \\ &= -\left(y_u z_{pp} - y_{uu}f_z^2 z_p\right) \\ c_{rn}f_y - c_{fn}n_{ry} &= f_z^2(x_{uu}y_u - y_{uu}x_u)f_y + \left(z_{pp} + f_z^2(f_x x_{uu} + f_y y_{uu})\right)x_u \\ &= -\left(f_z^2 z_p x_{uu} - z_{pp}x_u\right) \\ c_{rn}f_z - c_{fn}n_{rz} &= f_z^3(x_{uu}y_u - y_{uu}x_u) \end{aligned}$$

Formula (6b) for the intersection curvature of an implicit with a parametric surface:

$$\begin{aligned} k &= \frac{((c_{rn}f_x - c_{fn}n_{rx})^2 + (c_{rn}f_y - c_{fn}n_{ry})^2 + (c_{rn}f_z - c_{fn}n_{rz})^2)^{1/2}}{(t_x^2 + t_y^2 + t_z^2)^{3/2}} \\ &= \frac{((y_u z_{pp} - y_{uu}f_z^2 z_p)^2 + (f_z^2 z_p x_{uu} - z_{pp}x_u)^2 + f_z^6(x_u y_{uu} - x_{uu}y_u)^2)^{1/2}}{(f_z^2(x_u^2 + y_u^2) + z_p^2)^{3/2}} \end{aligned}$$

which is formula (30) with the parametric derivatives x', x'', y', y'' denoted here as x_u, x_{uu}, y_u, y_{uu}. Formula (34) for the curvature of a curve C defined implicitly as $g(x, y) = 0$ on an implicit surface $f(x, y, z) = 0$ can be proved in a similar manner, by considering C as the curve of intersection of two implicit surfaces $f(x, y, z) = 0$, $g(x, y) = 0$, the latter representing an implicitly defined generalized cylinder and using formula (39c).

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