Computer Aided Geometric Design 37 (2015) 1–8

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

# Convergence of univariate non-stationary subdivision schemes via asymptotic similarity ☆

C. Conti [a], N. Dyn [b], C. Manni [c], M.-L. Mazure [d,*]

[a] Dipartimento di Ingegneria Industriale, Univesità di Firenze, Italy
[b] School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, Israel
[c] Dipartimento di Matematica, Università di Roma "Tor Vergata", Italy
[d] Laboratoire Jean Kuntzmann, Université Joseph Fourier, Grenoble, France

ARTICLE INFO

*Article history:*
Received 30 March 2014
Received in revised form 1 June 2015
Accepted 2 June 2015
Available online 9 June 2015

*Keywords:*
Non-stationary subdivision schemes
Convergence
Reproduction of constants
Asymptotic equivalence
Asymptotic similarity

ABSTRACT

A new equivalence notion between non-stationary subdivision schemes, termed asymptotic similarity, which is weaker than asymptotic equivalence, is introduced and studied. It is known that asymptotic equivalence between a non-stationary subdivision scheme and a convergent stationary scheme guarantees the convergence of the non-stationary scheme. We show that for non-stationary schemes reproducing constants, the condition of asymptotic equivalence can be relaxed to asymptotic similarity. This result applies to a wide class of non-stationary schemes.



## 1. Introduction

This short paper studies univariate binary *non-stationary* uniform subdivision schemes. Such schemes are efficient iterative methods for generating smooth functions via the specification of an initial set of discrete data $\mathbf{f}^{[0]} := \{f_i^{[0]} \in \mathbb{R},\ i \in \mathbb{Z}\}$, and a set of refinement rules, mapping at each iteration the sequence of values $\mathbf{f}^{[k]} := \{f_i^{[k]} \in \mathbb{R},\ i \in \mathbb{Z}\}$ attached to the points of the grid $2^{-k}\mathbb{Z}$ into the sequence of values $\mathbf{f}^{[k+1]}$ attached to the points of $2^{-(k+1)}\mathbb{Z}$. At each level $k$, the refinement rule $S_{\mathbf{a}^{[k]}}$ is defined by a finitely supported *mask* $\mathbf{a}^{[k]} := \{a_i^{[k]},\ i \in \mathbb{Z}\}$, so that

$$\mathbf{f}^{[k+1]} := S_{\mathbf{a}^{[k]}}\mathbf{f}^{[k]} \quad \text{with} \quad \left(S_{\mathbf{a}^{[k]}}\mathbf{f}^{[k]}\right)_i := \sum_{j\in\mathbb{Z}} a_{i-2j}^{[k]} f_j^{[k]}. \tag{1}$$

Each subdivision scheme $\{S_{\mathbf{a}^{[k]}},\ k \ge 0\}$ we will deal with is assumed to be *local*, in the sense that there exists a positive integer $N$ such that $\text{supp}\,(\mathbf{a}^{[k]}) := \{i \in \mathbb{Z} \mid a_i^{[k]} \neq 0\} \subseteq [-N, N]$ for all $k \ge 0$.

☆ This paper has been recommended for acceptance by Ulrich Reif.

* Corresponding author.
*E-mail addresses:* costanza.conti@unifi.it (C. Conti), niradyn@post.tau.ac.il (N. Dyn), manni@axp.mat.uniroma2.it (C. Manni), Marie-Laurence.Mazure@imag.fr (M.-L. Mazure).

http://dx.doi.org/10.1016/j.cagd.2015.06.004
0167-8396/

The idea of proving the convergence of a non-stationary scheme by comparison with a convergent stationary one was first developed in Dyn and Levin (1995), via the notion of *asymptotic equivalence* between non-stationary schemes. Two subdivision schemes $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ and $\{S_{\mathbf{a}^{*[k]}}, k \geq 0\}$ are said to be asymptotically equivalent when

$$\sum_{k=0}^{\infty} \|S_{\mathbf{a}^{[k]}} - S_{\mathbf{a}^{*[k]}}\| < +\infty,$$

which holds if and only if $\sum_{k=0}^{\infty} \|\mathbf{a}^{[k]} - \mathbf{a}^{*[k]}\| < +\infty$. The main result of the present work is that for convergence analysis of non-stationary schemes reproducing constants, asymptotic equivalence can be replaced by the weaker notion of *asymptotic similarity*. We say that two schemes are asymptotically similar when

$$\lim_{k\to\infty} \|\mathbf{a}^{[k]} - \mathbf{a}^{*[k]}\| = 0. \tag{2}$$

The class of subdivision schemes to which our result applies is wide and important from the application point of view. For instance, this class contains all uniform subdivision schemes generating spaces of exponential polynomials with one exponent equal to zero, and in particular all subdivision schemes for uniform splines in such spaces (Conti and Romani, 2011; Dyn et al., 2003). Besides their classical interest in geometric modeling and approximation theory, uniform exponential B-splines are very useful in Signal Processing (Conti et al., 2015; Unser and Blu, 2005) and in Isogeometric Analysis (Manni et al., 2011, 2014). In the latter context, exponential B-splines based subdivision schemes permit to successfully address the difficult evaluation of these splines.

The article is organized as follows. In Sections 2 and 3 the analysis leading to the main result of this paper is presented. In Section 2 we derive a sufficient condition for the convergence of non-stationary schemes reproducing constants, in terms of difference schemes. This condition replaces the well-known necessary and sufficient condition for convergence in the stationary case. In Section 3 we introduce the asymptotic similarity relation (2) and develop some useful consequences for the analysis of non-stationary subdivision schemes. In particular, we show that, if two subdivision schemes reproduce constants, and if one of them satisfies the above-mentioned sufficient condition, so does the other. This fact is important for the proof of the convergence of non-stationary schemes reproducing constants by comparison (in the sense of (2)) with convergent stationary ones. Finally, in Section 4 we illustrate our result with non-stationary versions of the de Rham algorithm and of the four-point scheme.

Throughout the article the notation $\|\cdot\|$ refers to the sup-norm, for either operators, functions, or sequences in $\mathbb{R}^{\mathbb{Z}}$ and, in particular, we recall that $\|S_{\mathbf{a}^{[k]}}\| := \max\left(\sum_{i\in\mathbb{Z}} |a^{[k]}_{2i}|, \sum_{i\in\mathbb{Z}} |a^{[k]}_{2i+1}|\right)$.

## 2. A sufficient condition for convergence

Let $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ be a given subdivision scheme, defining successive $\mathbf{f}^{[k]}$, $k \geq 0$, via (1). At any level $k \geq 0$, we denote by $\mathcal{PL}(\mathbf{f}^{[k]})$ the piecewise linear function interpolating the sequence $\mathbf{f}^{[k]}$, *i.e.*, $\mathcal{PL}(\mathbf{f}^{[k]})(i2^{-k}) = f^{[k]}_i$ for all $i \in \mathbb{Z}$. The scheme is said to be *convergent* if, for any bounded $\mathbf{f}^{[0]}$, the sequence $\{\mathcal{PL}(\mathbf{f}^{[k]}),\ k \geq 0\}$ is uniformly convergent on $\mathbb{R}$. If so, the limit function is denoted by $S^{\infty}_{\{\mathbf{a}^{[k]},\, k\geq 0\}}\mathbf{f}^{[0]}$.

The subdivision scheme can equivalently be defined by its sequence of symbols, the *symbol* of the mask $\mathbf{a}^{[k]}$ of level $k$ being defined as the Laurent polynomial $a^{[k]}(z) := \sum_{i\in\mathbb{Z}} a^{[k]}_i z^i$. The scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is said to *reproduce constants* if $f^{[0]}_i = 1$ for all $i \in \mathbb{Z}$ implies $f^{[k]}_i = 1$ for all $i \in \mathbb{Z}$ and all $k \geq 0$, which holds if and only if

$$\sum_{i\in\mathbb{Z}} a^{[k]}_{2i} = \sum_{i\in\mathbb{Z}} a^{[k]}_{2i+1} = 1 \quad \text{for all } k \geq 0,$$

or if and only if the symbols satisfy

$$a^{[k]}(-1) = 0 \quad \text{and} \quad a^{[k]}(1) = 2 \quad \text{for all } k \geq 0. \tag{3}$$

If (3) holds, each symbol can be written as $a^{[k]}(z) = (1+z)q^{[k]}(z)$, where $q^{[k]}(z) := \sum_{i\in\mathbb{Z}} q^{[k]}_i z^i$ satisfies $q^{[k]}(1) = 1$, and we have

$$q^{[k]}_i = \sum_{j\leq i} (-1)^{i-j} a^{[k]}_j, \quad a^{[k]}_i = q^{[k]}_i + q^{[k]}_{i-1}, \quad i \in \mathbb{Z},\ k \geq 0. \tag{4}$$

From the right relation in (4) it is easily seen that the scheme $\{S_{\mathbf{q}^{[k]}}, k \geq 0\}$ permits the computation of all backward differences $\Delta f^{[k]}_i := f^{[k]}_i - f^{[k]}_{i-1}$, namely

$$\Delta\mathbf{f}^{[k+1]} = S_{\mathbf{q}^{[k]}} \Delta\mathbf{f}^{[k]}, \quad \text{with } \Delta\mathbf{f}^{[k]} := \{\Delta f^{[k]}_i, i \in \mathbb{Z}\}.$$

The non-stationary subdivision scheme $\{S_{\mathbf{q}^{[k]}}, k \geq 0\}$ is called the difference scheme of $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$.

The scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is *stationary* when its masks $\mathbf{a}^{[k]}$ do not depend on the level $k$, *i.e.*, $\mathbf{a}^{[k]} = \mathbf{a}$ for all $k \geq 0$. In that case we will use the simplified notation $\{S_{\mathbf{a}}\}$.

As is well known, reproduction of constants is necessary for convergence of stationary subdivision schemes. Let us recall also the following major convergence result about the stationary case (see *e.g.* Dyn, 2002).

**Theorem 1.** *Let $\{S_{\mathbf{a}}\}$ be a stationary subdivision scheme reproducing constants, with difference scheme $\{S_{\mathbf{q}}\}$. Then the scheme $\{S_{\mathbf{a}}\}$ converges if and only if there exists a positive integer n such that $\mu := \|(S_{\mathbf{q}})^n\| < 1$.*

A similar necessary and sufficient condition for the convergence of non-stationary subdivision schemes is not known. Nevertheless, a non-stationary version of the sufficient condition is given in Theorem 3 below.

**Definition 2.** We say that a subdivision scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$, assumed to reproduce constants, satisfies *Condition A*, when its difference scheme $\{S_{\mathbf{q}^{[k]}}, k \geq 0\}$ fulfills the following requirement:

$$\begin{aligned} &\text{there exist two integers } K \geq 0,\ n > 0,\ \text{such that} \\ &\mu := \sup_{k \geq K} \left\| S_{\mathbf{q}^{[k+n-1]}} \dots S_{\mathbf{q}^{[k+1]}} S_{\mathbf{q}^{[k]}} \right\| < 1. \end{aligned} \tag{5}$$

Let us recall that a scheme $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ is said to be *bounded*, if $\sup_{k \geq 0} \|S_{\mathbf{a}^{[k]}}\| < +\infty$, or, equivalently, due to locality, if $\sup_{k \geq 0} \|\mathbf{a}^{[k]}\| < +\infty$.

**Theorem 3.** *Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a bounded subdivision scheme reproducing constants and satisfying Condition A. Then, $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ converges. Moreover, there exists a positive number C, such that, for any initial $\mathbf{f}^{[0]}$,*

$$\|S^{\infty}_{\{\mathbf{a}^{[k]},\, k \geq 0\}} \mathbf{f}^{[0]} - \mathcal{PL}\left(\mathbf{f}^{[k]}\right)\| \leq C\, \widehat{\mu}^{k} \|\Delta \mathbf{f}^{[0]}\|, \quad k \geq 0, \quad \textit{with } \widehat{\mu} := \mu^{\frac{1}{n}}, \tag{6}$$

*where $\mu$ and n are provided by (5), and where $\{\mathbf{f}^{[k]},\ k \geq 0\}$ are the sequences generated by the subdivision scheme.*

Before proving the theorem we prove two lemmas. Below, as well as whenever we refer to a specific mask, we only indicate the non-zero elements.

**Lemma 4.** *Let $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ be a bounded subdivision scheme which reproduces constants, its locality being prescribed by the positive integer N. Let $\mathbf{h} := \left\{\frac{1}{2}, 1, \frac{1}{2}\right\}$ be the mask of the stationary linear B-spline subdivision scheme. The symbols of the masks $\{\mathbf{d}^{[k]} := \mathbf{a}^{[k]} - \mathbf{h}, k \geq 0\}$ can be written as*

$$d^{[k]}(z) = (1 - z^2) e^{[k]}(z), \tag{7}$$

*where, for each $k \geq 0$, the mask $\mathbf{e}^{[k]}$ satisfies*

$$e^{[k]}_i := \sum_{j \geq 0} d^{[k]}_{i-2j}, \quad \textit{for all } i \in \mathbb{Z}, \quad \operatorname{supp} \mathbf{e}^{[k]} \subset [-N, N-2]. \tag{8}$$

**Proof.** The factorization (7) is valid for the difference of any two subdivision schemes reproducing constants since their symbols take the same value at $-1$ and 1, see (3). The rest of the claim readily follows from (7). □

**Lemma 5.** *Under the assumptions of Theorem 3 there exists a positive constant $C_1$ such that*

$$\|\Delta \mathbf{f}^{[k]}\| \leq C_1\, \widehat{\mu}^{k} \|\Delta \mathbf{f}^{[0]}\|, \quad k \geq 0. \tag{9}$$

**Proof.** Select any integers $p$, $r$, with $p \geq 0$ and $0 \leq r \leq n-1$, where $n$ is given by (5). Repeated application of (5) yields:

$$\begin{aligned} \|\Delta \mathbf{f}^{[K+pn+r]}\| &\leq \mu^{p} \|\Delta \mathbf{f}^{[K+r]}\| \\ &\leq \widehat{\mu}^{K+pn+r} \frac{\|S_{\mathbf{q}^{[K+r-1]}} \dots S_{\mathbf{q}^{[1]}} S_{\mathbf{q}^{[0]}}\|}{\widehat{\mu}^{K+r}} \|\Delta \mathbf{f}^{[0]}\|. \end{aligned} \tag{10}$$

From (10) and from the fact that $\widehat{\mu} < 1$ it can easily be derived that (9) holds with

$$C_1 := \frac{1}{\widehat{\mu}^{K+n-1}} \max_{0 \leq k \leq K+n-1} \|S_{\mathbf{q}^{[k-1]}} \dots S_{\mathbf{q}^{[1]}} S_{\mathbf{q}^{[0]}}\|. \qquad \square$$

**Proof of Theorem 3.** By standard arguments it is sufficient to show that the sequence $\{F^{[k]} := \mathcal{PL}(\mathbf{f}^{[k]}),\ k \geq 0\}$, of piecewise linear interpolants satisfies

$$\|F^{[k+1]} - F^{[k]}\| \leq \Gamma\, \widehat{\mu}^k \|\Delta \mathbf{f}^{[0]}\|, \quad k \geq 0, \tag{11}$$

for some positive constant $\Gamma$. The constant $C$ in (6) can then be chosen as $C := \Gamma/(1-\widehat{\mu})$. With the help of the hat function

$$H(x) = \begin{cases} 1-|x|, & x \in (-1,1), \\ 0, & \text{otherwise}, \end{cases}$$

we can write $F^{[k+1]}$ and $F^{[k]}$ respectively as

$$F^{[k+1]}(x) = \sum_{i \in \mathbb{Z}} \left(S_{\mathbf{a}^{[k]}} \mathbf{f}^{[k]}\right)_i H(2^{k+1}x - i),$$

and

$$F^{[k]}(x) = \sum_{i \in \mathbb{Z}} \mathbf{f}_i^{[k]} H(2^k x - i) = \sum_{i \in \mathbb{Z}} \left(S_{\mathbf{h}} \mathbf{f}^{[k]}\right)_i H(2^{k+1}x - i),$$

where $S_{\mathbf{h}}$ is the subdivision scheme for linear B-splines recalled in Lemma 4. Hence, by the definition of $\mathbf{d}^{[k]}$ in Lemma 4, we obtain

$$F^{[k+1]}(x) - F^{[k]}(x) = \sum_{i \in \mathbb{Z}} g_i^{[k+1]} H(2^{k+1}x - i) \quad \text{with } \mathbf{g}^{[k+1]} := S_{\mathbf{d}^{[k]}} \mathbf{f}^{[k]}. \tag{12}$$

The left relations in (8) can be written as $d_i^{[k]} = e_i^{[k]} - e_{i-2}^{[k]}$ for all $i \in \mathbb{Z}$, implying that

$$g_i^{[k+1]} = \sum_{j \in \mathbb{Z}} e_{i-2j}^{[k]} \left(\Delta \mathbf{f}^{[k]}\right)_j, \quad i \in \mathbb{Z}. \tag{13}$$

Now, Lemma 4 and the boundedness assumption ensure that

$$\|\mathbf{e}^{[k]}\| \leq C_2 := N(\sup_{j \geq 0} \|\mathbf{a}^{[j]}\| + 1) < +\infty, \quad k \geq 0. \tag{14}$$

Gathering (14), (13), (12), (9) leads to (11), with $\Gamma := NC_1C_2$. $\square$

As in the stationary case, it can be proved that the limit function in Theorem 3 is Hölder continuous with exponent $|Log_2 \widehat{\mu}|$.

**Remark 6.** Different proofs of the fact that Condition A is sufficient for convergence already exist in the wider context of non-regular (*i.e.*, non-uniform, non-stationary) schemes, using non-regular grids, either nested (Maxim and Mazure, 2004) or non-nested (Mazure, 2005, 2006). Nevertheless, we did consider it useful to give a simplified proof in the context of uniform schemes and regular grids. Indeed, in that case the proof is made significantly more accessible by the use of the corresponding classical tools.

## 3. Asymptotically similar schemes

**Definition 7.** We say that two subdivision schemes $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ and $\{S_{\mathbf{a}^{*[k]}},\ k \geq 0\}$ are *asymptotically similar* if

$$\lim_{k \to \infty} \|\mathbf{a}^{[k]} - \mathbf{a}^{*[k]}\| = 0. \tag{15}$$

Clearly, asymptotic similarity is an equivalence relation between subdivision schemes, which is weaker than asymptotic equivalence. By the locality of the two schemes, proving their asymptotic similarity simply consists in checking that

$$\lim_{k \to \infty} (a_i^{[k]} - a_i^{*[k]}) = 0 \quad \text{for } -N \leq i \leq N,$$

where $[-N, N]$ contains the support of the masks $\mathbf{a}^{[k]}$, $\mathbf{a}^{*[k]}$ for $k \geq 0$. Note that (15) can be replaced by $\lim_{k\to\infty} \|S_{\mathbf{a}^{[k]}} - S_{\mathbf{a}^{*[k]}}\| = 0$ as well. If two subdivision schemes are asymptotically similar and if one of them is bounded, so is the other.

Depending on the properties of the schemes, asymptotic similarity can be expressed in different ways:

**Proposition 8.** *Given two subdivision schemes $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ and $\{S_{\mathbf{a}^{*[k]}},\ k \geq 0\}$ which both reproduce constants, the following properties are equivalent:*

(i) $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ *and* $\{S_{\mathbf{a}^{*[k]}},\ k \geq 0\}$ *are asymptotically similar;*
(ii) *the difference schemes* $\{S_{\mathbf{q}^{[k]}},\ k \geq 0\}$ *and* $\{S_{\mathbf{q}^{*[k]}},\ k \geq 0\}$ *are asymptotically similar.*

*If, in addition, one of the two subdivision schemes* $\{S_{\mathbf{a}^{*[k]}},\ k \geq 0\}$ *or* $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ *is bounded, then* (i) *is also equivalent to*

(iii) *for any fixed* $p \geq 0$, $\lim_{k\to\infty} \left\| S_{\mathbf{q}^{[k+p]}} \dots S_{\mathbf{q}^{[k]}} - S_{\mathbf{q}^{*[k+p]}} \dots S_{\mathbf{q}^{*[k]}} \right\| = 0.$

**Proof.** Without loss of generality we can assume that the locality of the two schemes is determined by the same positive integer $N$. Then, by application of (4) we can derive that $\operatorname{supp} \mathbf{q}^{[k]},\ \operatorname{supp} \mathbf{q}^{*[k]} \subset [-N, N-1]$, and that

$$\frac{1}{2}\|\mathbf{a}^{[k]} - \mathbf{a}^{*[k]}\| \leq \|\mathbf{q}^{[k]} - \mathbf{q}^{*[k]}\| \leq 2N\|\mathbf{a}^{[k]} - \mathbf{a}^{*[k]}\|\,.$$

The equivalence between (i) and (ii) follows. Clearly, (ii) is implied by (iii). As for the implication (ii) ⇒ (iii), it follows by induction from the equality

$$\begin{aligned} &S_{\mathbf{q}^{[k+p+1]}} S_{\mathbf{q}^{[k+p]}} \dots S_{\mathbf{q}^{[k]}} - S_{\mathbf{q}^{*[k+p+1]}} S_{\mathbf{q}^{*[k+p]}} \dots S_{\mathbf{q}^{*[k]}} \\ &\quad = \left(S_{\mathbf{q}^{[k+p+1]}} - S_{\mathbf{q}^{*[k+p+1]}}\right) S_{\mathbf{q}^{[k+p]}} \dots S_{\mathbf{q}^{[k]}} + S_{\mathbf{q}^{*[k+p+1]}} \left(S_{\mathbf{q}^{[k+p]}} \dots S_{\mathbf{q}^{[k]}} - S_{\mathbf{q}^{*[k+p]}} \dots S_{\mathbf{q}^{*[k]}}\right), \quad k \geq 0\,, \end{aligned}$$

and from the boundedness of the two schemes. □

**Proposition 9.** *Let* $\{S_{\mathbf{a}^{*[k]}}, k \geq 0\}$ *be a bounded subdivision scheme reproducing constants and satisfying Condition A. Then, any subdivision scheme* $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ *which reproduces constants and is asymptotically similar to* $\{S_{\mathbf{a}^{*[k]}}, k \geq 0\}$*, also satisfies Condition A.*

**Proof.** Using the notation of Proposition 8, from Condition A we derive the existence of two integers $K^*, n$ such that

$$\mu^* := \sup_{k \geq K^*} \left\| S_{\mathbf{q}^{*[k+n-1]}} \dots S_{\mathbf{q}^{*[k]}} \right\| < 1.$$

Select any $\mu \in (\mu^*, 1)$ and choose $\varepsilon > 0$ such that $\mu^* + \varepsilon < \mu$. The two schemes being asymptotically similar, and $\{S_{\mathbf{a}^{*[k]}}, k \geq 0\}$ being bounded, we know that (iii) of Proposition 8 holds. We can thus find $\widetilde{K} \geq 0$, such that

$$\left\| S_{\mathbf{q}^{[k+n]}} \dots S_{\mathbf{q}^{[k]}} - S_{\mathbf{q}^{*[k+n]}} \dots S_{\mathbf{q}^{*[k]}} \right\| \leq \varepsilon \quad \text{for all } k \geq \widetilde{K}.$$

Clearly, we have

$$\left\| S_{\mathbf{q}^{[k+n]}} \dots S_{\mathbf{q}^{[k]}} \right\| \leq \mu < 1, \quad \text{for each } k \geq K := \max(K^*, \widetilde{K}). \tag{16}$$

The claim is proved. □

**Remark 10.** We would like to draw the reader's attention to the fact that we have not proved that, when two bounded non-stationary subdivision schemes reproducing constants are asymptotically similar, convergence of one of them implies convergence of the other. Convergence of the second scheme is obtained only when convergence of the first one results from Condition A. This follows from Proposition 9 and Theorem 3. This is actually sufficient to prove Theorem 11 below, which is the main application of all previous results.

**Theorem 11.** *Let* $\{S_{\mathbf{a}^*}\}$ *be a convergent stationary subdivision scheme with* $\mu^* := \left\|(S_{\mathbf{q}^*})^n\right\| < 1$*. Let* $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ *be a non-stationary subdivision scheme reproducing constants which is asymptotically similar to* $\{S_{\mathbf{a}^*}\}$*. Then, the scheme* $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ *is convergent and for any* $\eta \in (\mu^{*\frac{1}{n}}, 1)$ *there exists a positive constant C such that, for any initial bounded* $\mathbf{f}^{[0]}$,

$$\|S^{\infty}_{\{\mathbf{a}^{[k]},\ k \geq 0\}} \mathbf{f}^{[0]} - \mathcal{PL}\left(\mathbf{f}^{[k+1]}\right)\| \leq C\, \eta^k \|\Delta \mathbf{f}^{[0]}\|, \quad k \geq 0.$$

**Proof.** The existence of a positive integer $n$ with $\mu^* := \left\|(S_{\mathbf{q}^*})^n\right\| < 1$ is due to the convergence of the stationary scheme $\{S_{\mathbf{a}^*}\}$, see Theorem 1. In other words, $\{S_{\mathbf{a}^*}\}$ satisfies Condition A. We also know that $\{S_{\mathbf{a}^*}\}$ reproduces constants. Accordingly, by application of Proposition 9, we can say that $\{S_{\mathbf{a}^{[k]}}, k \geq 0\}$ satisfies Condition A too. Furthermore, we know that we can apply Theorem 3 using any $\mu \in (\mu^*, 1)$ (see (16)). □

## 4. Illustrations

We conclude this work with two examples illustrating the convergence claim in Theorem 11, and with an example showing the necessity of reproduction of constants in this theorem. In what follows we only write the non-zero elements of the masks.

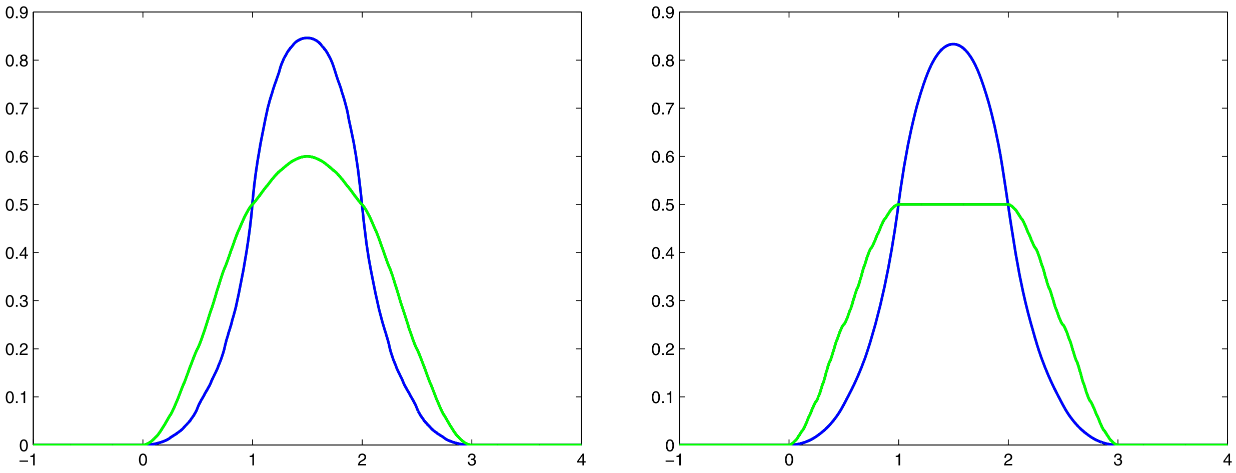


**Fig. 1.** Limit functions obtained via non-stationary de Rham schemes starting from $\mathbf{f}^{[1]} = \boldsymbol{\delta}$, with $\gamma_k = \gamma + \frac{\alpha}{k}$, $k \geq 1$. Left: $\gamma = 2$. Right: $\gamma = 1.5$. For both pictures the displayed functions correspond to $\alpha = 2.5$ (blue); $\alpha = -1.5$ (green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

### *4.1. First example*

The oldest instances of subdivision schemes are certainly the de Rham schemes (de Rham, 1956). Given a positive parameter $\gamma$, the associated de Rham scheme consists in placing on each segment of the polygonal line of any given level $k$, exactly two consecutive points of the next level, so that they divide this segment with ratios $1:\gamma:1$. This corner cutting geometric construction yields a stationary scheme $\{S_{\mathbf{a}^*}\}$, with mask

$$\mathbf{a}^* = \left\{ \frac{1}{2+\gamma}, \frac{1+\gamma}{2+\gamma}, \frac{1+\gamma}{2+\gamma}, \frac{1}{2+\gamma} \right\}, \tag{17}$$

known to converge for any value of $\gamma > 0$. In a very natural way, a more general geometric construction can be done using a sequence of positive parameters $\gamma_k$, $k \geq 0$, instead of a single parameter $\gamma$ for all levels. We now divide each segment of level $k$ with ratios $1:\gamma_k:1$. The scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ produced is a non-stationary version of the de Rham scheme. Its mask $\mathbf{a}^{[k]}$ of level $k$ is obtained by replacing $\gamma$ by $\gamma_k$ in (17). All masks have the same support and

$$a_i^* - a_i^{[k]} = \pm \frac{\gamma_k - \gamma}{(2+\gamma_k)(2+\gamma)} \quad \text{for all } i \in \operatorname{supp}(\mathbf{a}) \text{ and for all } k \geq 0.$$

Accordingly, for the scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ to be asymptotically similar to the classical stationary de Rham scheme $\{S_{\mathbf{a}^*}\}$ of parameter $\gamma$, it is necessary and sufficient to choose the sequence $\gamma_k$, $k \geq 0$, so that

$$\lim_{k\to\infty} \gamma_k = \gamma. \tag{18}$$

Since $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ obviously reproduces constants, we deduce from Theorem 11 that if the sequence $\{\gamma_k,\ k \geq 0\}$, converges to a limit in $(0, +\infty)$ the corresponding non-stationary de Rham scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ converges.

We illustrate this in Fig. 1, where, for $\gamma = 2$, and $\gamma = 1.5$, limit functions corresponding to two sequences $\gamma_k$, $k \geq 1$, are shown, starting from the initial sequence $\mathbf{f}^{[1]} := \boldsymbol{\delta} = \{\delta_{i,0},\ i \in \mathbb{Z}\}$. For $\gamma = 2$, $\{S_{\mathbf{a}^*}\}$ is simply the Chaikin algorithm with mask $\mathbf{a} = \{\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\}$. In either illustration, the non-stationary subdivision scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 1\}$ is not asymptotically equivalent to the corresponding de Rham scheme $\{S_{\mathbf{a}^*}\}$ since $\sum_{k\geq 1} \|S_{\mathbf{a}^{[k]}} - S_{\mathbf{a}^{*[k]}}\| = +\infty$.

### *4.2. Second example*

We now start with another famous example of stationary scheme, the four-point scheme $\{S_{\mathbf{a}^*}\}$ (Dyn et al., 1987). Given a fixed parameter $w$, at any level $k \geq 0$, the values at level $(k+1)$ are computed from those at level $k$ according to the following formulas:

$$f_{2i}^{[k+1]} = f_i^{[k]}, \quad f_{2i+1}^{[k+1]} = (\frac{1}{2} + w)(f_i^{[k]} + f_{i+1}^{[k]}) - w(f_{i-1}^{[k]} + f_{i+2}^{[k]}), \quad i \in \mathbb{Z}. \tag{19}$$

It is known that convergence holds when $w \in (-\frac{1}{2}, \frac{1}{2})$.

Here too a very natural approach consists in trying not to keep the same parameter $w$ at each level. This idea already appeared in Conti and Romani (2010), Levin (1999), Marinov et al. (2005), where even non-uniform versions of the four-point scheme were addressed. Let us limit ourselves to a uniform non-stationary four-point scheme, associated with a sequence

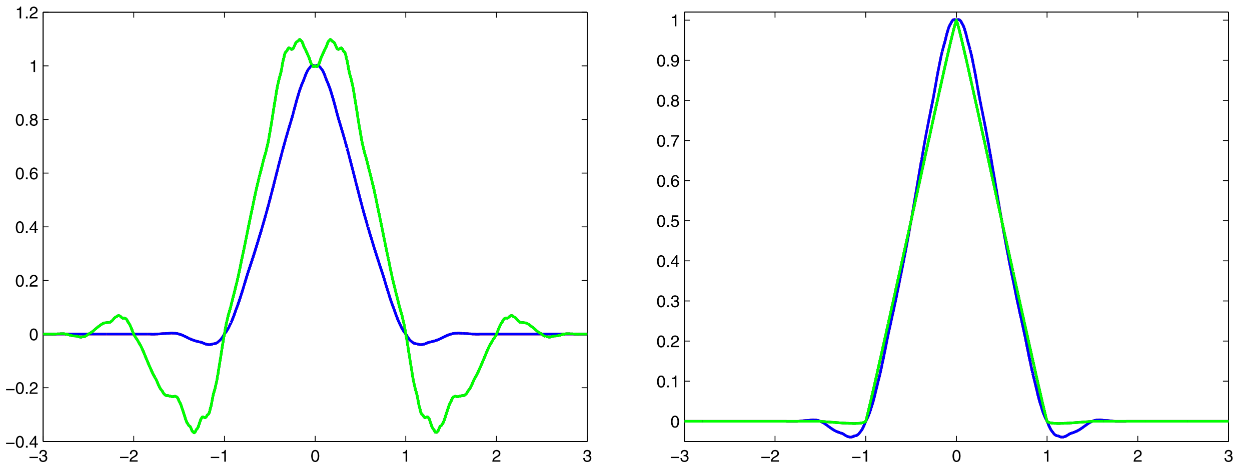


**Fig. 2.** Limit functions obtained via (19) starting from $\mathbf{f}^{[1]} = \boldsymbol{\delta}$. Left $w_k := w + \frac{\alpha}{k}$, $k \geq 1$, $w = \frac{1}{8}$ and $\alpha = -0.125$ (blue); $\alpha = 0.125$ (green). Right $w_k := w(1 - \frac{1}{k^\alpha})$, $k \geq 1$, $w = \frac{1}{8}$ and $\alpha = 1$ (blue), $\alpha = 0.1$ (green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

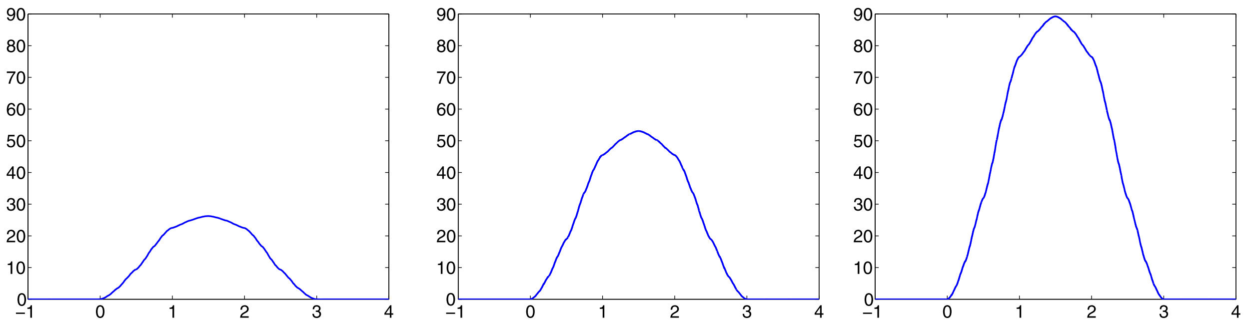


**Fig. 3.** From left to right: 8; 12; 16 iterations of (20) starting from $\mathbf{f}^{[1]} = \boldsymbol{\delta}$.

of parameters $w_k$, $k \geq 0$. The scheme $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$, is simply obtained by replacing $w$ by $w_k$ in (19) at any level $k \geq 0$. Clearly

$$\|\mathbf{a}^{[k]} - \mathbf{a}^*\| = |w_k - w|, \quad k \geq 0,$$

which means that $\{S_{\mathbf{a}^{[k]}},\ k \geq 0\}$ is asymptotically similar to $\{S_{\mathbf{a}^*}\}$ if and only if $\lim_{k\to\infty} w_k = w$, while it is asymptotically equivalent to $\{S_{\mathbf{a}^*}\}$ if and only if $\sum_{k\geq 0} |w_k - w| < +\infty$. Accordingly, by Theorem 11, we can state that a non-stationary four-point scheme converges as soon as its sequence of parameters converges to a limit in the interval $(-\frac{1}{2}, \frac{1}{2})$. The convergence is illustrated in Fig. 2 for two sequences of parameters $w_k$. In all these illustrations, the schemes are not asymptotically equivalent to the corresponding classical four-point scheme.

### *4.3. About reproduction of constants*

Let us now consider the family of masks $\{\mathbf{a}^{[k]},\ k \geq 1\}$, with

$$\mathbf{a}^{[k]} = \left\{ \frac{1}{4} + \frac{1}{k},\ \frac{3}{4} + \frac{1}{k},\ \frac{3}{4} + \frac{1}{k},\ \frac{1}{4} + \frac{1}{k} \right\}, \quad k \geq 1. \tag{20}$$

Fig. 3 shows the results after 8, 12, 16 iterations in the left, in the center and in the right, respectively. It clearly shows that the corresponding non-stationary scheme is not convergent. Still, it is asymptotically similar to the Chaikin scheme. This is not in contradiction with Theorem 11 since reproduction of constants is not satisfied. Indeed, $\sum_{i\in\mathbb{Z}} a_{2i}^{[k]} = \sum_{i\in\mathbb{Z}} a_{2i+1}^{[k]} = 1 + \frac{2}{k} \neq 1$ for all $k \geq 1$. This enhances the importance of all assumptions for the validity of Theorem 11.

## 5. Concluding remarks

Non-stationary subdivision schemes are harder to analyze than their stationary counterparts. Analyzing them by comparison with a simpler scheme is quite a natural idea. Up to now, the main tool for such a comparison was the asymptotic equivalence, as developed in Dyn and Levin (1995), see also Dyn et al. (2007). In the present article, we have replaced it by asymptotic similarity, a simpler and weaker equivalence relation between non-stationary schemes. Our work presents a twofold interest. It was clearly important on the theoretical side to point out that asymptotic equivalence may be a too demanding requirement. Indeed, provided that it reproduces constants, a non-stationary scheme which is asymptotically similar to a convergent stationary one is convergent. Certainly, many among the known converging non-stationary schemes are asymptotically equivalent to stationary analogues. Still, we have presented two simple examples which show that asymptotic similarity can also be useful in the design of new converging non-stationary schemes.

To enhance the significance of asymptotic similarity, we would like to mention that this notion was first introduced for non-regular (non-uniform, non-stationary) schemes in Mazure (in press) (simply named there *equivalence*). However, the non-uniformity made it necessary to define it locally. In this non-regular framework, it proved to be a forceful tool to analyze not only the convergence of a subdivision scheme but also the regularity of the limit functions it produces. As pointed out by D. Levin, in the uniform setting, smoothness of the generated limits can be proved by combining asymptotic similarity with the notion of smoothing factors introduced in Dyn and Levin (1995).

Further results concerning asymptotic similarity for uniform non-stationary schemes can be found in Charina et al. (2015) along with several important extensions of this notion.

## References


Charina, M., Conti, C., Guglielmi, N., Protasov, V., 2015. Regularity of non-stationary subdivision: a matrix approach. arXiv:1406.7131v2.

Conti, C., Romani, L., 2010. Affine combination of B-splines subdivision masks and its non-stationary counterparts. BIT Numer. Math. 50, 269–299.

Conti, C., Romani, L., 2011. Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction. J. Comput. Appl. Math. 236, 543–556.

Conti, C., Romani, L., Unser, M., 2015. Ellipse-preserving Hermite interpolation and subdivision. J. Math. Anal. Appl. 426, 211–227.

de Rham, G., 1956. Sur une courbe plane. J. Math. Pures Appl. 35, 25–42.

Dyn, N., 2002. Analysis of convergence and smoothness by the formalism of Laurent polynomials. In: Iske, A., Quak, E., Floater, M.S. (Eds.), Tutorials on Multiresolution in Geometric Modelling. Springer-Verlag, Heidelberg, pp. 51–68.

Dyn, N., Levin, D., 1995. Analysis of asymptotically equivalent binary subdivision schemes. J. Math. Anal. Appl. 193, 594–621.

Dyn, N., Levin, D., Gregory, J., 1987. A 4-point interpolatory subdivision scheme for curve design. Comput. Aided Geom. Des. 4, 257–268.

Dyn, N., Levin, D., Luzzatto, A., 2003. Exponentials reproducing subdivision schemes. Found. Comput. Math. 3, 187–206.

Dyn, N., Levin, D., Yoon, J., 2007. Analysis of univariate nonstationary subdivision schemes with application to Gaussian-based interpolatory schemes. SIAM J. Math. Anal. 39, 470–488.

Levin, D., 1999. Using Laurent polynomial representation for the analysis of non-uniform binary subdivision schemes. Adv. Comput. Math. 11, 41–54.

Manni, C., Pelosi, F., Sampoli, M.L., 2011. Generalized B-splines as a tool in isogeometric analysis. Comput. Methods Appl. Mech. Eng. 200, 867–881.

Manni, C., Pelosi, F., Speleers, H., 2014. Local hierarchical *h*-refinements in IgA based on generalized B-splines. In: Lect. Notes Comput. Sci., vol. 8177, pp. 341–363.

Marinov, M., Dyn, N., Levin, D., 2005. Geometrically controlled 4-point interpolatory scheme. In: Dodgson, N.A., Floater, M.S., Sabin, M.A. (Eds.), Advances in Multiresolution for Geometric Modelling. Springer-Verlag, pp. 301–315.

Maxim, V., Mazure, M.-L., 2004. Subdivision schemes and irregular grids. Numer. Algorithms 35, 1–28.

Mazure, M.-L., 2005. Subdivision schemes and non nested grids. In: Mache, D.H., Szabados, J., de Bruin, M.G. (Eds.), Trends and Applications in Constructive Approximation. In: Int. Ser. Numer. Math., vol. 151. Birkhäuser, pp. 135–163.

Mazure, M.-L., 2006. On Chebyshevian spline subdivision. J. Approx. Theory 143, 74–110.

Mazure, M.-L., in press. Lagrange interpolatory subdivision schemes in Chebyshev spaces. Found. Comput. Math. http://dx.doi.org/10.1007/s10208-014-9209-9.

Unser, M., Blu, T., 2005. Cardinal exponential splines: part I – Theory and filtering algorithms. IEEE Trans. Signal Process. 53, 1425–1438.