Computer Aided Geometric Design 40 (2015) 40–58

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Control point based exact description of curves and surfaces, in extended Chebyshev spaces ☆

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A R T I C L E I N F O

Article history:
Received 22 June 2015
Received in revised form 16 September 2015
Accepted 24 September 2015
Available online 23 October 2015

Keywords:
Extended Chebyshev vector spaces
Normalized B-basis functions
Basis transformation
Curves and surfaces
Control point based exact description

A B S T R A C T

Extended Chebyshev spaces that also comprise the constants represent large families of functions that can be used in real-life modeling or engineering applications that also involve important (e.g. transcendental) integral or rational curves and surfaces. Concerning CAGD, the unique normalized B-bases of such vector spaces ensure optimal shape preserving properties, important evaluation or subdivision algorithms and useful shape parameters. Therefore, we propose global explicit formulas for the entries of those transformation matrices that map these normalized B-bases to the traditional (or ordinary) bases of the underlying vector spaces. Then, we also describe general and ready to use control point configurations for the exact representation of those traditional integral parametric curves and (hybrid) surfaces that are specified by coordinate functions given as (products of separable) linear combinations of ordinary basis functions. The obtained results are also extended to the control point and weight based exact description of the rational counterpart of these integral parametric curves and surfaces. The universal applicability of our methods is presented through polynomial, trigonometric, hyperbolic or mixed extended Chebyshev vector spaces.

1. Introduction

Normalized B-bases (a comprehensive study of which can be found in Peña (1999) and the references therein) are normalized totally positive bases that imply optimal shape preserving properties for the representation of curves described as convex combinations of control points and basis functions. Similarly to the classical Bernstein polynomials of degree $n \in \mathbb{N}$ – that in fact form the normalized B-basis of the vector space of polynomials of degree at most n on the interval $[0, 1]$, cf. Carnicer and Peña (1993) – normalized B-bases provide shape preserving properties like closure for the affine transformations of the control points, convex hull, variation diminishing (which also implies the preservation of convexity of plane control polygons), endpoint interpolation, monotonicity preserving, hodograph and length diminishing, and a recursive corner cutting algorithm (also called B-algorithm) that is the analogue of the de Casteljau algorithm of Bézier curves. Among all normalized totally positive bases of a given vector space of functions the normalized B-basis is the least variation diminishing and the shape of the generated curve more mimics that of its control polygon. Important curve design algorithms like evaluation, subdivision, degree elevation or knot insertion are in fact corner cutting algorithms that can be treated in a unified way by means of B-algorithms induced by B-bases.

☆ This paper has been recommended for acceptance by Oleg Davydov.

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http://dx.doi.org/10.1016/j.cagd.2015.09.005

Curve and surface modeling tools based on non-polynomial normalized B-bases also ensure further advantages like: possible shape or design parameters; singularity free exact parametrization (e.g. parametrization of conic sections may correspond to natural arc-length parametrization); higher or even infinite order of precision concerning (partial) derivatives; ordinary (i.e., traditionally parametrized) integral curves and surfaces can exactly be described by means of control points without any additional weights (the calculation of which, apart of some simple cases, is cumbersome for the designer); important transcendental curves and surfaces which are of interest in real-life applications can also be exactly represented (the standard rational Bézier or NURBS models cannot encompass these geometric objects). Moreover, concerning condition numbers and stability, a normalized B-basis is the unique normalized totally positive basis that is optimally stable among all non-negative bases of a given vector space of functions, cf. (Peña, 1999, Corollary 3.4, p. 89). These advantageous properties make normalized B-bases ideal blending function system candidates for curve and surface modeling.

Besides their interest in the classical contexts of CAGD and approximation theory, normalized B-bases and their spline counterparts have also been used in isogeometric analysis recently (consider e.g. Manni et al., 2011 and the references therein). Compared with classical final element methods, isogeometric analysis provides several advantages when one describes the geometry by generalized B-splines and invokes an isoparametric approach in order to approximate the unknown solutions of differential equations (e.g. of Poisson type problems) or Dirichlet boundary conditions by the same type of functions.

Let $n \geq 1$ be a fixed integer and consider the extended Chebyshev (EC) system

$$\mathcal{F}_n^{\alpha,\beta} = \left\{\varphi_{n,i}(u) : u \in [\alpha,\beta]\right\}_{i=0}^{n}, \ \varphi_{n,0} \equiv 1, \ -\infty < \alpha < \beta < \infty \tag{1}$$

of basis functions in $C^n([\alpha,\beta])$, i.e., by definition (Karlin and Studden, 1966), for any integer $0 \leq r \leq n$, any strictly increasing sequence of knot values $\alpha \leq u_0 < u_1 < \ldots < u_r \leq \beta$, any positive integers (or multiplicities) $\{m_k\}_{k=0}^{r}$ such that $\sum_{k=0}^{r} m_k = n+1$, and any real numbers $\{\xi_{k,\ell}\}_{k=0,\ \ell=0}^{r,\ m_k-1}$ there always exists a unique function

$$f := \sum_{i=0}^{n} \lambda_{n,i}\varphi_{n,i} \in \mathbb{S}_n^{\alpha,\beta} := \left\langle \mathcal{F}_n^{\alpha,\beta} \right\rangle := \operatorname{span} \mathcal{F}_n^{\alpha,\beta}, \ \lambda_{n,i} \in \mathbb{R}, \ i = 0,1,\ldots,n \tag{2}$$

that satisfies the conditions of the Hermite interpolation problem

$$f^{(\ell)}(u_k) = \xi_{k,\ell}, \ \ell = 0,1,\ldots,m_k-1, \ k = 0,1,\ldots,r. \tag{3}$$

In what follows, we assume that the sign-regular determinant of the coefficient matrix of the linear system (3) of equations is strictly positive for any permissible parameter settings introduced above. Under these circumstances, the vector space $\mathbb{S}_n^{\alpha,\beta}$ of functions is called an EC space of dimension $n+1$. In terms of zeros, this definition means that any non-zero element of $\mathbb{S}_n^{\alpha,\beta}$ vanishes at most n times in the interval $[\alpha,\beta]$. Such spaces and their corresponding spline counterparts have been widely studied, consider e.g. articles by Lyche (1985), Carnicer and Peña (1994), Mazure (1999), (2001), Mainar et al. (2001), Lü et al. (2002), Carnicer et al. (2004), (2007), Mainar and Peña (2004), Costantini et al. (2005), Mainar and Peña (2010) and many other references therein.

Hereafter we will also refer to $\mathcal{F}_n^{\alpha,\beta}$ as the ordinary basis of $\mathbb{S}_n^{\alpha,\beta}$. Using (Carnicer and Peña, 1995, Theorem 5.1) it follows that the vector space $\mathbb{S}_n^{\alpha,\beta}$ also has a strictly totally positive basis, i.e., a basis such that all minors of all its collocation matrices are strictly positive. Since the constant function $1 \equiv \varphi_{n,0} \in \mathbb{S}_n^{\alpha,\beta}$, the aforementioned strictly totally positive basis is normalizable, therefore the vector space $\mathbb{S}_n^{\alpha,\beta}$ also has a unique non-negative normalized B-basis

$$\mathcal{B}_n^{\alpha,\beta} = \left\{b_{n,i}(u) : u \in [\alpha,\beta]\right\}_{i=0}^{n} \tag{4}$$

that in addition to the identity

$$\sum_{i=0}^{n} b_{n,i}(u) \equiv 1, \ \forall u \in [\alpha,\beta] \tag{5}$$

also fulfills the properties

$$b_{n,0}(\alpha) = b_{n,n}(\beta) = 1, \tag{6}$$

$$b_{n,i}^{(j)}(\alpha) = 0, \ j = 0,\ldots,i-1, \ b_{n,i}^{(i)}(\alpha) > 0, \tag{7}$$

$$b_{n,i}^{(j)}(\beta) = 0, \ j = 0,1,\ldots,n-1-i, \ (-1)^{n-i} b_{n,i}^{(n-i)}(\beta) > 0 \tag{8}$$

conform (Carnicer and Peña, 1995, Theorem 5.1) and (Mazure, 1999, Equation (3.6)).

Using the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$ of $\mathbb{S}_n^{\alpha,\beta}$, one of our objectives is to provide explicit closed formulas for the control point based exact description of integral curves that are specified with coordinate functions given in traditional parametric form in the ordinary basis $\mathcal{F}_n^{\alpha,\beta}$ of the same vector space. Based on homogeneous coordinates and central projection, we also propose an algorithm for the control point (and weight) based exact description of the rational counterpart of these ordinary integral curves. Results can easily be extended to the exact representation of families of (hybrid) integral and rational surfaces that are exclusively given in each of their variables by using ordinary EC basis functions of the type (1).

To the best of the author's knowledge, the coefficient based exact representation of ordinary (rational) functions, curves and surfaces by means of the (rational or spline counterpart of the) normalized B-basis of an arbitrary EC space (that also comprises the constant functions) was not considered in such a general unified context. Without providing an exhaustive survey, so far the presented problem appears in the literature for example in case of conversion algorithms related to Bernstein polynomials, monomials and the classical families of orthogonal Jacobi, Gegenbauer, Legendre, Chebyshev, Laguerre and Hermite polynomials (Cargo and Shisha, 1966; Barrio and Peña, 2004); in special lower dimensional vector spaces (e.g. in Zhang, 1996; Mainar et al., 2001; Carnicer et al., 2003, 2006; Romani et al., 2014); in case of conical and helical arcs, of catenaries, of patches on all types of quadrics and of helicoidal surfaces (e.g. in Pottmann and Wagner, 1994; Lü et al., 2002); of certain (rational) trigonometric curves of arbitrarily finite order like epi- and hypotrochoidal arcs (Sánchez-Reyes, 1999a), or segments of offset-rational sinusoidal spirals, arachnidas and epi spirals (Sánchez-Reyes, 1999b); or more recently, in case of arbitrary trigonometric and hyperbolic (rational) polynomials, curves, (hybrid) surfaces and volumes of finite order (Róth, 2015a).

The rest of the paper is organized as follows. Section 2 lists our main results, namely it describes closed formulas for the basis transformation that maps the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$ of the vector space $\mathbb{S}_n^{\alpha,\beta}$ to its ordinary basis $\mathcal{F}_n^{\alpha,\beta}$ and also specifies control point configurations for the exact representation of certain large classes of integral and rational curves and surfaces that are specified in traditional parametric form by means of ordinary bases like $\mathcal{F}_n^{\alpha,\beta}$. Although the presented results are mainly of theoretical interest, Section 2 also studies the computational complexity of the proposed basis conversion formulas and – compared with alternative cubic time numerical methods like curve interpolation or least square approximation – points out that these can more efficiently be implemented up to $n = 15$. Section 3 emphasizes the universal applicability of the general basis transformation described in Section 2 with examples that can be compared to presumably already existing results in the literature. This section considers EC vector spaces of functions that may be important in CAGD, in engineering, in (projective) geometry, in (numerical) analysis or in approximation theory. The proofs of all theoretical results stated in Sections 2 and 3 can be found in Section 4. In the end, Section 5 closes the paper with our final remarks.

2. Main results and remarks

At first, we provide explicit formulas for the transformation of the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$ of the vector space $\mathbb{S}_n^{\alpha,\beta}$ to its ordinary basis $\mathcal{F}_n^{\alpha,\beta}$.

Theorem 2.1 *(General basis transformation). The matrix form of the linear transformation that maps the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$ to the ordinary basis $\mathcal{F}_n^{\alpha,\beta}$ is*

$$\left[\varphi_{n,i}(u)\right]_{i=0}^{n} = \left[t_{i,j}^{n}\right]_{i=0,\ j=0}^{n,\ n} \cdot \left[b_{n,i}(u)\right]_{i=0}^{n},\ \forall u \in [\alpha,\beta], \tag{9}$$

where $t_{0,j}^n = 1,\ j = 0,1,\dots,n$ and $t_{i,0}^n = \varphi_{n,i}(\alpha),\ t_{i,n}^n = \varphi_{n,i}(\beta),\ i = 0,1,\dots,n$, while

$$\begin{aligned} t_{i,j}^n = \varphi_{n,i}(\alpha) - \frac{1}{b_{n,j}^{(j)}(\alpha)} \cdot \sum_{r=1}^{j-1} \frac{\varphi_{n,i}^{(r)}(\alpha)}{b_{n,r}^{(r)}(\alpha)} \Bigg(& b_{n,r}^{(j)}(\alpha) + \\ & + \sum_{\ell=1}^{j-r-1} (-1)^{\ell} \sum_{r<k_1<k_2<\ldots<k_\ell<j} \frac{b_{n,r}^{(k_1)}(\alpha)\, b_{n,k_1}^{(k_2)}(\alpha)\, b_{n,k_2}^{(k_3)}(\alpha) \ldots b_{n,k_{\ell-1}}^{(k_\ell)}(\alpha)\, b_{n,k_\ell}^{(j)}(\alpha)}{b_{n,k_1}^{(k_1)}(\alpha)\, b_{n,k_2}^{(k_2)}(\alpha) \ldots b_{n,k_\ell}^{(k_\ell)}(\alpha)} \Bigg) + \frac{\varphi_{n,i}^{(j)}(\alpha)}{b_{n,j}^{(j)}(\alpha)}, \\ & i = 1,2,\dots,n,\ j = 1,2,\dots,\left\lfloor \frac{n}{2} \right\rfloor, \end{aligned} \tag{10}$$

$$\begin{aligned} t_{i,n-j}^n = \varphi_{n,i}(\beta) - \frac{1}{b_{n,n-j}^{(j)}(\beta)} \cdot \sum_{r=1}^{j-1} \frac{\varphi_{n,i}^{(r)}(\beta)}{b_{n,n-r}^{(r)}(\beta)} \Bigg(& b_{n,n-r}^{(j)}(\beta) + \\ & + \sum_{\ell=1}^{j-r-1} (-1)^{\ell} \sum_{r<k_1<k_2<\ldots<k_\ell<j} \frac{b_{n,n-r}^{(k_1)}(\beta)\, b_{n,n-k_1}^{(k_2)}(\beta)\, b_{n,n-k_2}^{(k_3)}(\beta) \ldots b_{n,n-k_{\ell-1}}^{(k_\ell)}(\beta)\, b_{n,n-k_\ell}^{(j)}(\beta)}{b_{n,n-k_1}^{(k_1)}(\beta)\, b_{n,n-k_2}^{(k_2)}(\beta) \ldots b_{n,n-k_\ell}^{(k_\ell)}(\beta)} \Bigg) \\ & + \frac{\varphi_{n,i}^{(j)}(\beta)}{b_{n,n-j}^{(j)}(\beta)},\ i = 1,2,\dots,n,\ j = 1,2,\dots,\left\lfloor \frac{n}{2} \right\rfloor. \end{aligned} \tag{11}$$

Remark 2.1 *(Evaluation).* If in formulas (10) or (11), for some $\ell = 1, 2, \ldots, j - r - 1$ (with $r = 1, 2, \ldots, j - 1$ and $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$) there exist no integers $k_1, k_2, \ldots, k_\ell$ such that $r < k_1 < k_2 < \ldots < k_\ell < j$ then, by convention, the summation corresponding to ℓ equals 0. If $n = 2z \geq 2$, then for $j = z$ one can evaluate the entries $\left[t_{i,z}\right]_{i=1}^{n}$ of the middle column by using either of these formulas, since the zth coefficients of the ordinary basis functions (1) in the normalized B-basis (4) are unique.

Except some special but important cases, in general, one does not know the closed form of the normalized B-basis (4) of $\mathbb{S}_n^{\alpha,\beta}$. In case of EC spaces of traditional, trigonometric or hyperbolic polynomials of finite degree we have explicit closed formulas cf. Carnicer and Peña (1993), Sánchez-Reyes (1998) and Shen and Wang (2005), respectively; in case of a special class of mixed (e.g. algebraic trigonometric, algebraic hyperbolic, or both trigonometric and hyperbolic) EC spaces these functions appear in recursive integral form cf. Mainar and Peña (2010) and the references therein; while the most general (determinant based) formulas that can be applied in such spaces was published in Mazure (1999). Thus, concerning the evaluation of (10) and (11), in general, one can differentiate the formulas presented in Mazure (1999, Theorem 3.4, p. 658) in order to calculate the higher order derivatives of the normalized B-basis functions (4) at the endpoints of the interval $[\alpha, \beta]$. Namely, by using the function

$$\phi(u) := \left[\begin{array}{cccc} \varphi_{n,1}(u) & \varphi_{n,2}(u) & \ldots & \varphi_{n,n}(u) \end{array}\right]^T,\ u \in [\alpha, \beta],$$

one has to substitute the parameter values $u = \alpha$ and $u = \beta$ into the derivative formulas

$$b_{n,0}^{(j)}(u) = \frac{\det[\,\phi^{(1)}(\beta) \ \cdots \ \phi^{(n-1)}(\beta) \ \ \phi^{(j)}(u)\,]}{\det[\,\phi^{(1)}(\beta) \ \cdots \ \phi^{(n-1)}(\beta) \ \ \phi(\alpha) - \phi(\beta)\,]}, \tag{12}$$

$$b_{n,n}^{(j)}(u) = \frac{\det[\,\phi^{(1)}(\alpha) \ \cdots \ \phi^{(n-1)}(\alpha) \ \ \phi^{(j)}(u)\,]}{\det[\,\phi^{(1)}(\alpha) \ \cdots \ \phi^{(n-1)}(\alpha) \ \ \phi(\beta) - \phi(\alpha)\,]}, \tag{13}$$

$$\begin{aligned} b_{n,i}^{(j)}(u) = & \frac{\det[\,\phi^{(1)}(\alpha) \ \cdots \ \phi^{(i-1)}(\alpha) \ \ \phi^{(i)}(\alpha) \ \ \phi^{(1)}(\beta) \ \cdots \ \phi^{(n-i-1)}(\beta) \ \ \phi^{(n-i)}(\beta)\,]}{\det[\,\phi^{(1)}(\alpha) \ \cdots \ \phi^{(i-1)}(\alpha) \ \ \phi(\beta) - \phi(\alpha) \ \ \phi^{(1)}(\beta) \ \cdots \ \phi^{(n-i-1)}(\beta) \ \ \phi^{(n-i)}(\beta)\,]} \\ & \cdot \frac{\det[\,\phi(\beta) - \phi(\alpha) \ \ \phi^{(1)}(\alpha) \ \cdots \ \phi^{(i-1)}(\alpha) \ \ \phi^{(j)}(u) \ \ \phi^{(1)}(\beta) \ \cdots \ \phi^{(n-i-1)}(\beta)\,]}{\det[\,\phi(\beta) - \phi(\alpha) \ \ \phi^{(1)}(\alpha) \ \cdots \ \phi^{(i-1)}(\alpha) \ \ \phi^{(i)}(\alpha) \ \ \phi^{(1)}(\beta) \ \cdots \ \phi^{(n-i-1)}(\beta)\,]} \end{aligned} \tag{14}$$

for all $i = 1, 2, \ldots, n - 1$ and $j = 1, 2, \ldots, n$. However, as it is also mentioned in Mazure (1999), these general relations are difficult and computationally expensive to evaluate even in the most simple cases for either arbitrarily big or general values of the order n. Therefore, Section 3 provides explicit closed formulas for the required endpoint derivatives in several special cases. Due to properties (7) and (8), these expressions should only be used whenever one does not know the exact value of the required endpoint derivatives.

Another core result of the current section is presented in the next statement which is an immediate corollary of Theorem 2.1 that provides closed formulas for the exact description of ordinary integral curves as convex combinations of control points and normalized B-basis functions (4).

Corollary 2.1 *(Exact description of ordinary integral curves). The ordinary integral parametric curve*

$$\mathbf{c}(u) = \sum_{i=0}^{n} \boldsymbol{\lambda}_i \varphi_{n,i}(u),\ u \in [\alpha, \beta],\ \boldsymbol{\lambda}_i = \left[\lambda_i^\ell\right]_{\ell=1}^{\delta} \in \mathbb{R}^\delta,\ \delta \geq 2 \tag{15}$$

of order n can be written as an EC B-curve

$$\mathbf{c}(u) \equiv \sum_{j=0}^{n} \mathbf{p}_j b_{n,j}(u),\ \forall u \in [\alpha, \beta],\ \mathbf{p}_j = \left[p_j^\ell\right]_{\ell=1}^{\delta} \in \mathbb{R}^\delta, \tag{16}$$

of the same order, where $p_j^\ell = \sum_{i=0}^{n} \lambda_i^\ell t_{i,j}^n,\ j = 0, 1, \ldots, n,\ \ell = 1, 2, \ldots, \delta.$

Corollary 2.1 can easily be extended to the control point based exact description of those ordinary bi- or higher variate integral surfaces (volumes) that are specified in traditional parametric form with coordinate functions described as sums of separable products of ordinary linear combinations. Further details can be found in Section 2 of the extended version (Róth, 2015b) of the current manuscript.

If the denominator of the rational counterpart of the ordinary integral curve (15) is strictly positive, then, by means of control points and non-negative weights of rank 1, one can also exactly describe ordinary rational curves as it is illustrated in the steps of the next algorithm.

Algorithm 2.1 *(Exact description of ordinary rational curves).* Consider in $\mathbb{R}^{\delta}$ the rational curve

$$\mathbf{c}(u) = \frac{1}{c^{\delta+1}(u)} \left[c^{\ell}(u) \right]_{\ell=1}^{\delta}, \ u \in [\alpha, \beta] \tag{17}$$

given in ordinary parametric form, where

$$c^{\ell}(u) = \sum_{i=0}^{n} \lambda_i^{\ell} \varphi_{n,i}(u), \ \ell = 1, 2, \ldots, \delta + 1, \ c^{\delta+1}(u) > 0, \ \forall u \in [\alpha, \beta].$$

Using the rational counterpart of EC B-curves (16), the process that provides the control point and weight based exact representation

$$\mathbf{c}(u) \equiv \frac{\sum_{j=0}^{n} w_j \mathbf{p}_j b_{n,j}(u)}{\sum_{r=0}^{n} w_r b_{n,r}(u)}, \ \forall u \in [\alpha, \beta] \tag{18}$$

consists of the following steps:

- apply Theorem 2.1 to the higher dimensional pre-image $\mathbf{c}^{\wp}(u) = \left[c^{\ell}(u) \right]_{\ell=1}^{\delta+1}, \ u \in [\alpha, \beta]$, i.e., compute control points $\mathbf{p}_j^{\wp} = \left[p_j^{\ell} \right]_{\ell=1}^{\delta+1}, \ j = 0, 1, \ldots, n$ for the exact description of $\mathbf{c}^{\wp}$ in the pre-image space $\mathbb{R}^{\delta+1}$;
- project the obtained control points from the origin $\mathbf{0}_{\delta+1} \in \mathbb{R}^{\delta+1}$ onto the hyperplane $x^{\delta+1} = 1$ that results in the control points $\mathbf{p}_j = \frac{1}{p_j^{\delta+1}} \left[p_j^{\ell} \right]_{\ell=1}^{\delta} \in \mathbb{R}^{\delta}$ and weights $w_j = p_j^{\delta+1}$ needed for the rational representation (18);
- the above generation process does not necessarily ensure the non-negativity of all weights, since the last coordinate of some control points $\mathbf{p}_j^{\wp}$ in the pre-image space $\mathbb{R}^{\delta+1}$ can be negative; if this is the case, one should elevate the dimension (and consequently the order n of the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$) of the underlying EC space with an algorithm that generates a sequence of control polygons in $\mathbb{R}^{\delta+1}$ that converges to $\mathbf{c}^{\wp}$ which, by definition, is a geometric object of one branch that does not intersect the vanishing plane $x^{\delta+1} = 0$, since the $(\delta+1)$th coordinate of all its points is strictly positive; therefore, by using proper dimension elevation methods, it is guaranteed that exists a finite and minimal order for which all weights are non-negative.

Remark 2.2 *(About the last step of Algorithm 2.1).* If the pre-image $\mathbf{c}^{\wp}$ of (17) is described as B-curves of type (16) by means of the normalized B-bases of the EC spaces $\mathbb{S}_n^{\alpha,\beta} \subset \mathbb{S}_{n+1}^{\alpha,\beta}$, then

$$\mathbf{c}^{\wp}(u) = \sum_{j=0}^{n} \mathbf{p}_j^{\wp} b_{n,j}(u) \equiv \sum_{j=0}^{n+1} \mathbf{p}_{1,j}^{\wp} b_{n+1,j}(u), \ \forall u \in [\alpha, \beta],$$

where $\mathbf{p}_{1,0}^{\wp} \equiv \mathbf{p}_0^{\wp}$, $\mathbf{p}_{1,n+1}^{\wp} \equiv \mathbf{p}_n^{\wp}$, while $\mathbf{p}_{1,j}^{\wp} = \left(1 - \xi_j\right) \mathbf{p}_{j-1}^{\wp} + \xi_j \mathbf{p}_j^{\wp}$ for some real numbers $\xi_j \in (0, 1)$, $j = 1, \ldots, n$. Iterating this corner cutting based representation of $\mathbf{c}^{\wp}$ in the normalized B-bases of the nested EC spaces $\mathbb{S}_n^{\alpha,\beta} \subset \mathbb{S}_{n+1}^{\alpha,\beta} \subset \ldots \subset \mathbb{S}_{n+z}^{\alpha,\beta} \subset \ldots$, one obtains a sequence of control polygons which converges to a Lipschitz-continuous limit curve (de Boor, 1990) that, in general, does not necessarily coincide with $\mathbf{c}^{\wp}$. As it is pointed out in a unified manner in Róth (2015a, Remark 2.3, p. 76), in case of vector spaces of finite order trigonometric/hyperbolic polynomials, the sequence of order elevated control polygons always converges to the curve generated by the first term of the sequence. In case of traditional polynomials of finite degree, one can use the well-known degree elevation techniques of (rational) Bézier curves. However, in general, the initial ordinary basis $\mathcal{F}_n^{\alpha,\beta}$ can iteratively be appended by new linearly independent functions in infinitely many ways and not every choice of functions leads to a sequence of order elevated control polygons that fulfills the desired convergence property, e.g. in EC Müntz spaces a recent characterization of the required convergence of the dimension/order elevation process can be found in Ait-Hadou (2014). In order to illustrate the last step of Algorithm 2.1, Fig. 1 shows two different control point configurations for the exact representation of the rational trigonometric curve

$$\mathbf{c}(u) = \frac{1}{\frac{5}{8} - \frac{1}{2}\sin(2u)} \begin{bmatrix} \frac{1}{\sqrt{2}}(\sin(u) + \cos(u)) \\ \frac{3}{2}\cos(2u) \end{bmatrix}, \ u \in \left[0, \frac{\pi}{2}\right], \tag{19}$$

by means of second and fourth order normalized trigonometric basis functions of type (26).

Using multivariate rational EC B-surfaces, the steps of Algorithm 2.1 can also easily be extended to the control point and weight based exact description of certain large classes of ordinary rational surfaces and volumes. Further details can be found in Section 2 of Róth (2015b).

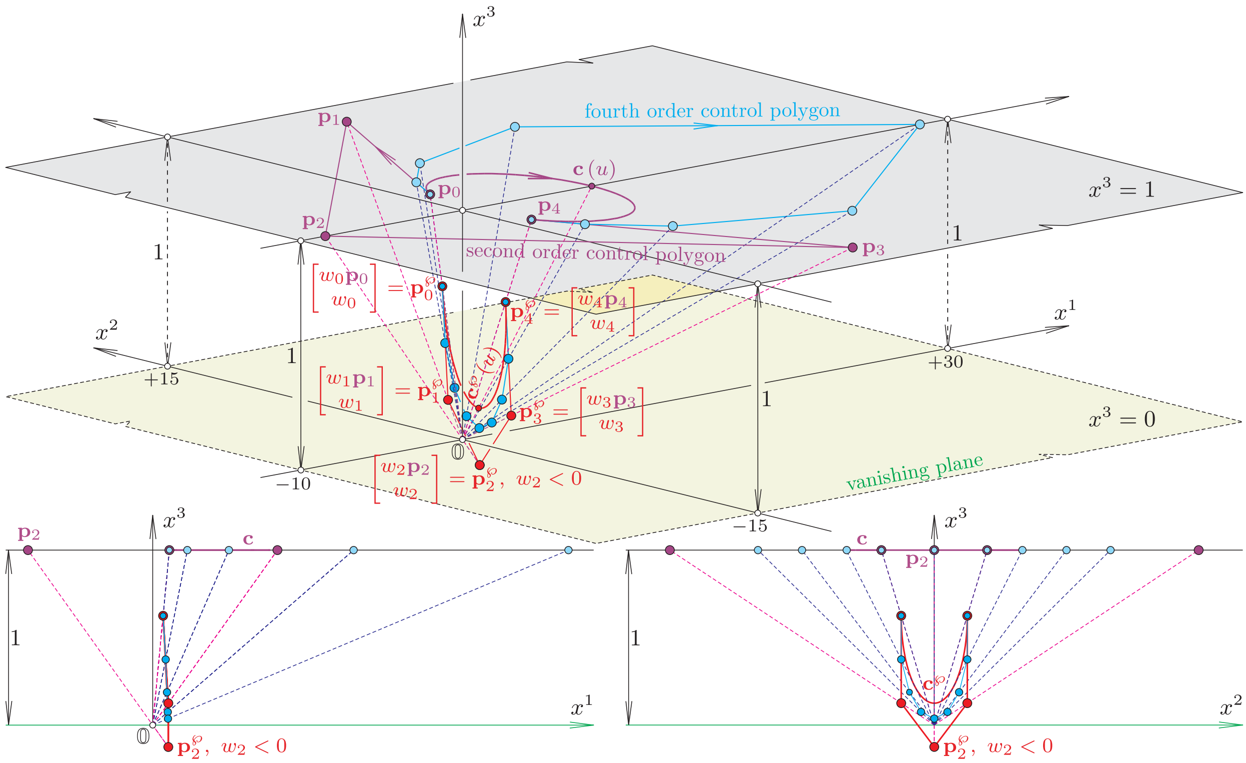

Fig. 1. Although the integral pre-image $\mathbf{c}^{\wp}$ of (19) is above the vanishing plane $x^3 = 0$, minimal (i.e., second) order trigonometric normalized B-basis functions of type (26) do not ensure the non-negativity of all weights required by the rational representation (18), since the third coordinate w_2 of the control point $\mathbf{p}_2^{\wp}$ is strictly negative. Such pathological cases usually violate the convex hull property of (18). Increasing the order of the applied normalized B-basis functions from 2 to 4, all control points needed for the exact representation of the pre-image $\mathbf{c}^{\wp}$ will be above the vanishing plane, i.e., in this case there exists a minimal order starting from which the applied dimension elevation technique ensures the non-negativity of all weights.

Theorem 2.2 *(Computational complexity). Provided that endpoint derivatives* $\{\varphi_{n,i}^{(j)}(\alpha), \varphi_{n,i}^{(j)}(\beta), b_{n,i}^{(j)}(\alpha), b_{n,i}^{(j)}(\beta)\}_{i=1,\,j=0}^{n,\,\lfloor \frac{n}{2}\rfloor}$ *are calculated and stored in advance in permanent lookup tables for a fixed value of* $n \geq 1$*, the total number of floating point operations (or shortly flops) that have to be performed for the evaluation of formulas* (10)–(11) *is*

$$\kappa(n) = \begin{cases} 2^{\lfloor \frac{n}{2}\rfloor+1}\left(\lfloor \frac{n}{2}\rfloor - 3\right) + 2\lfloor \frac{n}{2}\rfloor + 6 + 2n\lfloor \frac{n}{2}\rfloor\left(\lfloor \frac{n}{2}\rfloor + 1\right), & n\,(\mathrm{mod}\,2) = 1, \\ 2^{\frac{n}{2}-1}\left(\frac{3n}{2} - 10\right) + n + 5 + \frac{n^3}{2}, & n\,(\mathrm{mod}\,2) = 0. \end{cases} \tag{20}$$

Using the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$, the control point based exact description of the ordinary integral curve (15) can also be imagined either as a curve interpolation problem or as the least square approximation of the considered curve. Both of these alternative numeric methods can be reduced to the solutions of δ systems of linear equations that determine the unknown coordinates of the control points $\left[\mathbf{p}_j\right]_{j=0}^{n}$ appearing in the EC B-curve representation (16). In what follows, we neglect both the floating-point round-off errors and the computational cost of the construction of the regular main matrices of the size $(n+1)\times(n+1)$ of these alternative methods, and we compare the exponential computational cost (20) to the total work

$$\kappa_{LU}(n,\delta) = \frac{2}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 - \frac{1}{6}(n+1) + \left(2(n+1)^2 - (n+1)\right)\delta \tag{21}$$

of an LU decomposition based algorithm that efficiently solves δ systems of linear equations. (Expression (21) covers the cost of computing of all multipliers, of all row operations and of all forward and backward substitutions as well, Press et al. 2007, pp. 48–55, 106–108.) Naturally, as n tends to infinity, the growth rate of the exponential cost function (20) is substantially bigger than that of the cubic one (21), however if $n \in \{1, 2, \ldots, 15\}$ then (20) is less than (21) as it is illustrated in Fig. 2, i.e., compared with other cubic time numerical algorithms, the proposed general basis transformation can more efficiently be implemented up to 16-dimensional EC spaces despite the seemingly complicated nature of formulas (10)–(11).

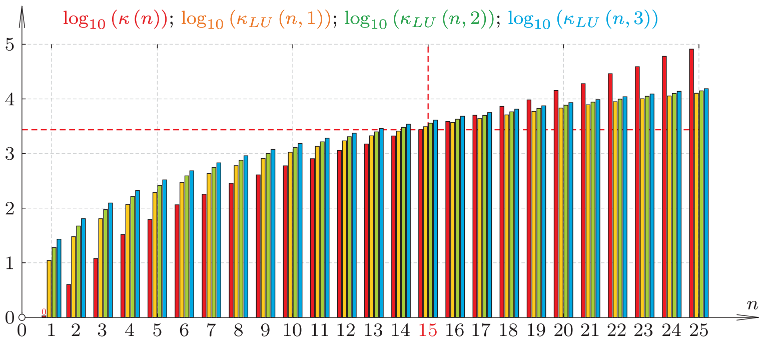

Fig. 2. Logarithmic scales of computational costs (20) and (21) for different values of $n \geq 1$ and $\delta \geq 1$.

3. Examples

This section applies closed formulas (10) and (11) in case of different vector spaces of functions that can be spanned by ordinary EC bases of the type (1). Our intention is only to emphasize the global applicability of the general basis transformation described in Theorem 2.1 with examples that can be compared to possible already existing results in the literature. Formulas (10) and (11) depend on the higher order endpoint derivatives of the ordinary and normalized B-basis of the underlying vector space. The following subsections specify these values in case of polynomial, trigonometric, hyperbolic and algebraic trigonometric vector spaces of functions. Although we consider several reflection invariant EC spaces, general formulas (10)–(11) are valid in not necessarily reflection invariant EC spaces as well.

3.1. Traditional polynomials

The system $\mathcal{B}_n^{0,1} = \left\{b_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i} : u \in [0,1]\right\}_{i=0}^n$ of Bernstein polynomials of degree n is the normalized B-basis of the EC space $\mathbb{S}_n^{0,1} = \left\langle \left\{\varphi_{n,i}(u) = u^i : u \in [0,1]\right\}_{i=0}^n \right\rangle$.

Proposition 3.1 *(Polynomial endpoint derivatives). In this case one has that*

$$\varphi_{n,i}^{(j)}(0) = \begin{cases} i!, & j = i, \\ 0, & j \neq i, \end{cases} \quad \varphi_{n,i}^{(j)}(1) = \begin{cases} \frac{i!}{(i-j)!}, & i \geq j \geq 0, \\ 0, & i < j \leq n, \end{cases} \tag{22}$$

$$b_{n,i}^{(j)}(0) = \begin{cases} 0, & i > j \geq 0, \\ (-1)^{j-i} \cdot j! \cdot \binom{n}{j} \cdot \binom{j}{i}, & i \leq j \leq n, \end{cases} \quad b_{n,i}^{(j)}(1) = (-1)^j \, b_{n,n-i}^{(j)}(0), \; i = 0, 1, \ldots, n \tag{23}$$

for all $j = 0, 1, \ldots, n$.

Example 3.1 *(Traditional polynomials of finite degree).* As it is proved in Appendix A of the extended version (Róth, 2015b) of the current manuscript, the substitution for $\alpha = 0$ and $\beta = 1$ of derivatives (22)–(23) into formulas (10) and (11) leads to the expected classical transformation matrix of entries

$$t_{i,j}^n = \begin{cases} \frac{\binom{j}{i}}{\binom{n}{i}}, & j = i, i+1, \ldots, n, \\ 0, & j = 0, 1, \ldots, i-1, \end{cases} \tag{24}$$

where $i = 0, 1, \ldots, n$.

3.2. Trigonometric polynomials

Let $\alpha = 0$ and $\beta \in (0, \pi)$ be fixed parameters and consider the ordinary basis

$$\mathcal{F}_{2n}^{0,\beta} = \left\{\varphi_{2n,0}(u) \equiv 1, \; \left\{\varphi_{2n,2i-1}(u) = \sin(iu), \; \varphi_{2n,2i}(u) = \cos(iu)\right\}_{i=1}^n : u \in [0,\beta]\right\} \tag{25}$$

of trigonometric polynomials of order at most n (degree $2n$). Using the results of Sánchez-Reyes (1998), the normalized B-basis of the vector space $\mathbb{S}_{2n}^{0,\beta} = \langle \mathcal{F}_{2n}^{0,\beta} \rangle$ can linearly be reparametrized into the form

$$\mathcal{B}_{2n}^{0,\beta} = \left\{b_{2n,i}(u) = c_{2n,i}^{\beta} \sin^{2n-i}\left(\frac{\beta-u}{2}\right) \sin^i\left(\frac{u}{2}\right) : u \in [0,\beta]\right\}_{i=0}^{2n}, \tag{26}$$

where

$$c^{\beta}_{2n,i} = c^{\beta}_{2n,2n-i} = \frac{1}{\sin^{2n}\left(\frac{\beta}{2}\right)} \sum_{r=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \binom{n}{i-r}\binom{i-r}{r}\left(2\cos\left(\frac{\beta}{2}\right)\right)^{i-2r},\ i=0,1,\ldots,n \tag{27}$$

are symmetric normalizing coefficients. It is obvious that

$$\varphi^{(j)}_{2n,2i-1}(0) = i^j \varphi_{2n,2i-1}\left(\frac{j\pi}{2}\right),\ \varphi^{(j)}_{2n,2i-1}(\beta) = i^j \varphi_{2n,2i-1}\left(\beta+\frac{j\pi}{2}\right),\ j=0,1,\ldots,n,\ i=1,2,\ldots,n,$$

$$\varphi^{(j)}_{2n,2i}(0) = i^j \varphi_{2n,2i}\left(\frac{j\pi}{2}\right),\ \varphi^{(j)}_{2n,2i}(\beta) = i^j \varphi_{2n,2i}\left(\beta+\frac{j\pi}{2}\right),\ j=0,1,\ldots,n,\ i=1,2,\ldots,n,$$

$$b_{2n,0}(0) = b_{2n,2n}(\beta) = 1,\ b_{2n,i}(0) = b_{2n,2n-i}(\beta) = 0,\ i=1,2,\ldots,2n,$$

while the higher order derivatives $\left\{b^{(j)}_{2n,i}(0),\ b^{(j)}_{2n,i}(\beta)\right\}_{i=0,\ j=1}^{2n,\ n}$ are specified by the next theorem.

Theorem 3.1 *(Trigonometric endpoint derivatives). For arbitrary derivative order $j=1,2,\ldots,n$ we have that*

$$\begin{aligned}\frac{b^{(j)}_{2n,2r+1}(0)}{c^{\beta}_{2n,2r+1}} &= \frac{1}{2^{2n-1}} \sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r} (-1)^{n+1-k-\ell}\binom{2(n-r-1)+1}{k}\binom{2r+1}{\ell} \\ &\cdot\left((n-k-\ell)^j - (n-k-2r+\ell-1)^j\right)\cos\left((2(n-r-k)-1)\frac{\beta}{2}-\frac{j\pi}{2}\right),\end{aligned} \tag{28}$$

for all $r=0,1,\ldots,n-1$ and

$$\begin{aligned}\frac{b^{(j)}_{2n,2r}(0)}{c^{\beta}_{2n,2r}} &= \frac{\binom{2n-2r}{n-r}}{2^{2n-1}}\sum_{\ell=0}^{r-1}(-1)^{r-\ell}\binom{2r}{\ell}(r-\ell)^j\cos\left(\frac{j\pi}{2}\right) \\ &+ \frac{\binom{2r}{r}}{2^{2n-1}}\sum_{k=0}^{n-r-1}(-1)^{n-r-k}\binom{2(n-r)}{k}(n-r-k)^j\cos\left((n-r-k)\beta-\frac{j\pi}{2}\right) \\ &+ \frac{1}{2^{2n-1}}\sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r-1}(-1)^{n-k-\ell}\binom{2(n-r)}{k}\binom{2r}{\ell} \\ &\cdot\left((n-k-\ell)^j + (n-k-2r+\ell)^j\right)\cos\left((n-r-k)\beta-\frac{j\pi}{2}\right),\end{aligned} \tag{29}$$

for all $r=0,1,\ldots,n$. At the same time $b^{(j)}_{2n,i}(\beta) = (-1)^j b^{(j)}_{2n,2n-i}(0),\ i=0,1,\ldots,2n,\ j=0,1,\ldots,n.$

Example 3.2 *(Second order trigonometric polynomials).* Consider the ordinary basis

$$\begin{aligned}\mathcal{F}^{0,\beta}_4 = \{&\varphi_{4,0}(u)=1,\ \varphi_{4,1}(u)=\sin(u),\ \varphi_{4,2}(u)=\cos(u), \\ &\varphi_{4,3}(u)=\sin(2u),\ \varphi_{4,4}(u)=\cos(2u) : u\in[0,\beta]\},\ \beta\in(0,\pi)\end{aligned}$$

of the vector space of trigonometric polynomials of order at most two (or degree 4) and its normalized B-basis

$$\mathcal{B}^{0,\beta}_4 = \left\{b_{4,i}(u) = c^{\beta}_{4,i}\sin^{4-i}\left(\frac{\beta-u}{2}\right)\sin^i\left(\frac{u}{2}\right) : u\in[0,\beta]\right\}_{i=0}^{4}, \tag{30}$$

where

$$c^{\beta}_{4,0} = c^{\beta}_{4,4} = \frac{1}{\sin^4\left(\frac{\beta}{2}\right)},\ c^{\beta}_{4,1} = c^{\beta}_{4,3} = \frac{4\cos\left(\frac{\beta}{2}\right)}{\sin^4\left(\frac{\beta}{2}\right)},\ c^{\beta}_{4,2} = \frac{2+4\cos^2\left(\frac{\beta}{2}\right)}{\sin^4\left(\frac{\beta}{2}\right)}.$$

Applying identities (10) and (11) for $n=4$ and $\alpha=0$, one obtains the transformation matrix

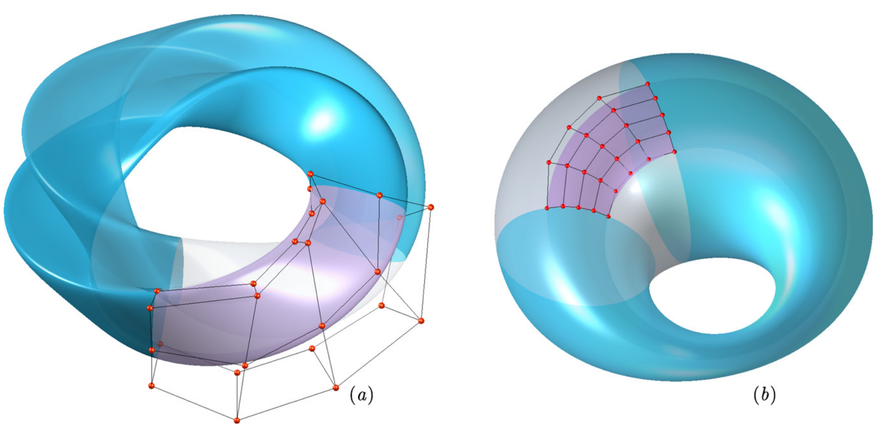

Fig. 3. Control point configurations for the exact description of patches of (a) a special integral variant (32) of Alfred Gray's non-orientable Klein Bottle and of (b) a rational ring Dupin cyclide.

$$\left[t_{i,j}^{4}\right]_{i=0,\ j=0}^{4,\ 4}=\begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 0 & \frac{1}{2}\tan\left(\frac{\beta}{2}\right) & \frac{3\sin(\beta)}{2+4\cos^2\left(\frac{\beta}{2}\right)} & \sin(\beta)-\frac{1}{2}\cos(\beta)\tan\left(\frac{\beta}{2}\right) & \sin(\beta)\\ 1 & 1 & \frac{3(1+\cos(\beta))}{2+4\cos^2\left(\frac{\beta}{2}\right)} & \cos(\beta)+\frac{1}{2}\sin(\beta)\tan\left(\frac{\beta}{2}\right) & \cos(\beta)\\ 0 & \tan\left(\frac{\beta}{2}\right) & \frac{6\sin(\beta)}{2+4\cos^2\left(\frac{\beta}{2}\right)} & \sin(2\beta)-\cos(2\beta)\tan\left(\frac{\beta}{2}\right) & \sin(2\beta)\\ 1 & 1 & \frac{6\cos(\beta)}{2+4\cos^2\left(\frac{\beta}{2}\right)} & \cos(2\beta)+\sin(2\beta)\tan\left(\frac{\beta}{2}\right) & \cos(2\beta)\end{bmatrix}, \tag{31}$$

based on which Fig. 3 shows control net configurations for the exact description of some patches of integral and rational trigonometric surfaces given by traditional parametric equations

$$\begin{cases} x(u,v)=3\cos(u)+\frac{1}{2}(1+\cos(2u))\sin(v)-\frac{1}{2}\sin(2u)\sin(2v),\\ y(u,v)=3\sin(u)+\frac{1}{2}\sin(2u)\sin(v)-\frac{1}{2}(1-\cos(2u))\sin(2v),\\ z(u,v)=\cos(u)\sin(2v)+\sin(u)\sin(v)\end{cases} \tag{32}$$

and

$$\begin{cases} x(u,v)=\dfrac{\mu(c-a\cos(u)\cos(v))+b^2\cos(u)}{a-c\cos(u)\cos(v)},\\ y(u,v)=\dfrac{b\sin(u)(a-\mu\cos(v))}{a-c\cos(u)\cos(v)},\\ z(u,v)=\dfrac{b\sin(v)(c\cos(u)-\mu)}{a-c\cos(u)\cos(v)},\end{cases} \tag{33}$$

respectively, where $(u,v)\in[0,2\pi]\times[0,2\pi]$ and $a=6,\ b=4\sqrt{2},\ c=2,\ \mu=3$.

3.3. Hyperbolic polynomials

Now, let $\alpha=0$ and $\beta>0$ be fixed parameters. Using hyperbolic sine and cosine functions in expressions (25)–(27) instead of the trigonometric ones, we obtain the vector space of hyperbolic polynomials of order at most n (or degree $2n$) the unique normalized B-basis of which was introduced in Shen and Wang (2005). In this case

$$\varphi_{2n,0}^{(j)}(u)=0,\ j=1,2,\ldots,n,$$
$$b_{2n,0}(0)=b_{2n,2n}(\beta)=1,\ b_{2n,i}(0)=b_{2n,2n-i}(\beta)=0,\ i=1,2,\ldots,n$$

and

$$\varphi_{2n,2i-1}^{(j)}(0)=\begin{cases}0, & j\,(\mathrm{mod}\,2)=0,\\ i^j, & j\,(\mathrm{mod}\,2)=1,\end{cases}\quad \varphi_{2n,2i-1}^{(j)}(\beta)=\begin{cases}i^j\sinh(i\beta), & j\,(\mathrm{mod}\,2)=0,\\ i^j\cosh(i\beta), & j\,(\mathrm{mod}\,2)=1,\end{cases}$$

$$\varphi_{2n,2i}^{(j)}(0)=\begin{cases} i^j, & j \,(\mathrm{mod}\, 2)=0,\\ 0, & j\,(\mathrm{mod}\,2)=1,\end{cases},\quad \varphi_{2n,2i}^{(j)}(\beta)=\begin{cases} i^j\cosh(i\beta), & j\,(\mathrm{mod}\,2)=0,\\ i^j\sinh(i\beta), & j\,(\mathrm{mod}\,2)=1\end{cases}$$

for all $i, j \in \{1, 2, \ldots, n\}$, while the higher order derivatives $\left\{b_{2n,i}^{(j)}(0),\ b_{2n,i}^{(j)}(\beta)\right\}_{i=0,\ j=1}^{2n,\ n}$ are specified by the next theorem.

Theorem 3.2 *(Hyperbolic endpoint derivatives). For arbitrary derivative order $j = 1, 2, \ldots, n$, one has that*

$$\frac{b_{2n,2r+1}^{(j)}(0)}{c_{2n,2r+1}^{\beta}}=\begin{cases}\frac{1}{2^{2n-1}}\sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r}(-1)^{n+((n-r-1)\bmod 2)+(r \bmod 2)-k-\ell-1}\binom{2(n-r-1)+1}{k}\binom{2r+1}{\ell}\\ \quad\cdot\left((n-k-2r+\ell-1)^j-(n-k-\ell)^j\right)\cosh\left((2(n-k-r)-1)\frac{\beta}{2}\right), & j\,(\mathrm{mod}\,2)=0,\\ \frac{1}{2^{2n-1}}\sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r}(-1)^{n+((n-r-1)\bmod 2)+(r \bmod 2)-k-\ell}\binom{2(n-r-1)+1}{k}\binom{2r+1}{\ell}\\ \quad\cdot\left((n-k-2r+\ell-1)^j-(n-k-\ell)^j\right)\sinh\left((2(n-k-r)-1)\frac{\beta}{2}\right), & j\,(\mathrm{mod}\,2)=1,\end{cases}$$

for all $r = 0, 1, \ldots, n-1$, while

$$\frac{b_{2n,2r}^{(j)}(0)}{c_{2n,2r}^{\beta}}$$

$$=\begin{cases}\frac{\binom{2(n-r)}{n-r}\binom{2r}{r}}{2^{2n}}+\frac{\binom{2(n-r)}{n-r}}{2^{2n-1}}\sum_{\ell=0}^{r+(r\bmod 2)-1}(-1)^{r+(r\bmod 2)-\ell}\binom{2r}{\ell}2^j(r-\ell)^j\\ \quad+\frac{\binom{2r}{r}}{2^{2n-1}}\sum_{k=0}^{(n-r)+((n-r)\bmod 2)-1}(-1)^{(n-r)+((n-r)\bmod 2)-k}\binom{2(n-r)}{k}(n-r-k)^j\cosh((n-r-k)\beta)\\ \quad+\frac{1}{2^{2n-1}}\sum_{k=0}^{(n-r)+((n-r)\bmod 2)-1}\sum_{\ell=0}^{r+(r\bmod 2)-1}(-1)^{n+((n-r)\bmod 2)+(r\bmod 2)-k-\ell}\binom{2(n-r)}{k}\binom{2r}{\ell}\\ \quad\cdot\left((n-k-3r+2\ell)^j+(n+r-k-2\ell)^j\right)\cosh((n-r-k)\beta), & j\,(\mathrm{mod}\,2)=0,\\ \frac{\binom{2(n-r)}{n-r}\binom{2r}{r}}{2^{2n}}+\frac{\binom{2(n-r)}{n-r}}{2^{2n-1}}\sum_{\ell=0}^{r+(r\bmod 2)-1}(-1)^{r+(r\bmod 2)-\ell}\binom{2r}{\ell}2^j(r-\ell)^j\sinh(2(r-\ell)\beta)\\ \quad+\frac{1}{2^{2n-1}}\sum_{k=0}^{(n-r)+((n-r)\bmod 2)-1}\sum_{\ell=0}^{r+(r\bmod 2)-1}(-1)^{n+((n-r)\bmod 2)+(r\bmod 2)-k-\ell}\binom{2(n-r)}{k}\binom{2r}{\ell}\\ \quad\cdot\left((n+r-k-2\ell)^j-(n-k-3r+2\ell)^j\right)\sinh(2\beta(r-\ell)), & j\,(\mathrm{mod}\,2)=1\end{cases}$$

for all $r = 1, 2, \ldots, n$ and $b_{2n,i}^{(j)}(\beta)=(-1)^j\, b_{2n,2n-i}^{(j)}(0)$, $i = 0, 1, \ldots, 2n$, $j = 0, 1, \ldots, n$.

Example 3.3 *(Second order hyperbolic polynomials).* Using hyperbolic sine, cosine and tangent functions instead of the trigonometric ones that appear in Example 3.2 and applying the second order hyperbolic normalized B-basis (Shen and Wang, 2005) with the shape parameter $\beta > 0$, one can easily construct the hyperbolic counterpart

$$\left[t_{i,j}^4\right]_{i=0,\ j=0}^{4,\ 4}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & \frac{1}{2}\tanh\left(\frac{\beta}{2}\right) & \frac{3\sinh(\beta)}{2+4\cosh^2\left(\frac{\beta}{2}\right)} & \sinh(\beta)-\frac{1}{2}\cosh(\beta)\tanh\left(\frac{\beta}{2}\right) & \sinh(\beta)\\ 1 & 1 & \frac{3(1+\cosh(\beta))}{2+4\cosh^2\left(\frac{\beta}{2}\right)} & \cosh(\beta)-\frac{1}{2}\sinh(\beta)\tanh\left(\frac{\beta}{2}\right) & \cosh(\beta)\\ 0 & \tanh\left(\frac{\beta}{2}\right) & \frac{6\sinh(\beta)}{2+4\cosh^2\left(\frac{\beta}{2}\right)} & \sinh(2\beta)-\cosh(2\beta)\tanh\left(\frac{\beta}{2}\right) & \sinh(2\beta)\\ 1 & 1 & \frac{6\cosh(\beta)}{2+4\cosh^2\left(\frac{\beta}{2}\right)} & \cosh(2\beta)-\sinh(2\beta)\tanh\left(\frac{\beta}{2}\right) & \cosh(2\beta)\end{bmatrix} \tag{34}$$

of the trigonometric basis transformation (31), the structurally difference of which consists in the highlighted operators.

Remark 3.1 *(Asymptotic behavior).* By means of Juhász and Róth (2014, Proposition 2.1) and Shen and Wang (2005, Proposition 5), we know that the linear reparametrization $\left\{b_{2n,i}(\beta x) : x \in [0,1]\right\}_{i=0}^{2n}$ of both of the trigonometric normalized B-basis (26) and its hyperbolic counterpart degenerate to the classical Bernstein polynomial basis of even degree $2n$ as the parameter β tends to 0 from above. This asymptotic behavior is also valid in case of higher order derivatives, i.e., $\lim_{\beta\to 0} b_{2n,i}^{(j)}(\beta x) = q_{2n,i}^{(j)}(x)$, $\forall x \in [0,1]$, $i = 0, 1, \ldots, 2n$, $j \geq 0$, where $q_{2n,i}(x) := \binom{2n}{i} x^i (1-x)^{2n-i}$.

3.4. A class of mixed spaces

Using the results of Carnicer et al. (2004), for appropriately fixed values of $\beta > 0$, one can also provide general formulas for the required endpoint derivatives of the normalized B-basis in case of a large family of reflection invariant mixed EC spaces $\mathbb{S}_n^{0,\beta}$ that are formed by all solutions of the constant-coefficient homogeneous linear differential equation

$$\sum_{i=0}^{n+1} \gamma_i v^{(i)}(u) = 0,\ \gamma_i \in \mathbb{R},\ u \in [0,\beta] \tag{35}$$

of order $n+1$, the characteristic polynomial $p_{n+1}(r)$, $r \in \mathbb{C}$ of which is an either even or odd function such that $r = 0$ is one of its (presumably higher order) zeros. Further details can be found in Proposition 3.1, Example 3.5 and Appendix B of the extended version (Róth, 2015b) of the current manuscript. Here, we list only a simple example.

Example 3.4 *(Quadratic algebraic trigonometric functions).* The (higher order) zeros of the odd characteristic polynomial $p_5(r) = r^3(r^2+1)$, $r \in \mathbb{C}$ generate the reflection invariant EC space $\mathbb{S}_4^{0,\beta} = \langle \mathcal{F}_4^{0,\beta} \rangle = \langle \{\varphi_{4,0}(u) = 1, \varphi_{4,1}(u) = u, \varphi_{4,2}(u) = u^2, \varphi_{4,3}(u) = \sin(u), \varphi_{4,4}(u) = \cos(u) : u \in [0,\beta]\}\rangle$ of algebraic trigonometric functions. If $\beta \in (0, \beta_4^\star := 2\pi)$, then the normalized B-basis of this space is

$$\begin{aligned}
\mathcal{B}_4^{0,\beta} = \Big\{ & b_{4,0}(u) = b_{4,4}(\beta-u),\ b_{4,1}(u) = b_{4,3}(\beta-u),\\
& b_{4,2}(u) = c_{4,2}^{\beta}\Big(2\beta(\sin(u)-\sin(\beta)) - 2\beta(1-\cos(\beta))u + \beta^2 + 2\beta\sin(\beta-u) - \beta^2\cos(\beta-u)\\
&\qquad + \beta^2(\cos(\beta)-\cos(u)) + 2(1-\cos(\beta))u^2 + \beta(\beta-u)u\sin(\beta)\Big),\\
& b_{4,3}(u) = c_{4,3}^{\beta}\Big(2(\beta-u) + 2(\sin(u)-\sin(\beta)) + 2(u\cos(\beta)-\beta\cos(u)) + 2\sin(\beta-u)\\
&\qquad + \beta^2(u-\sin(u)) - (\beta-\sin(\beta))u^2\Big),\\
& b_{4,4}(u) = c_{4,4}^{\beta}(2\cos(u) + u^2 - 2) : u \in [0,\beta]\Big\}
\end{aligned}$$

that can be constructed by using either the differential equation based iterative integral representation published in Mainar and Peña (2010) and the references therein or the determinant based formulas of Mazure (1999, Theorem 3.4). The critical length $\beta_4^\star = 2\pi$ was determined in Carnicer et al. (2004, Section 5) or Carnicer et al. (2007, Proposition 3), while positive scalars

$$\begin{aligned}
c_{4,2}^{\beta} &= \frac{4 - 4\cos(\beta) - 2\beta\sin(\beta)}{\left(\beta^2 - 4\cos(\beta) - 4\beta\sin(\beta) + \beta^2\cos(\beta) + 4\right)^2},\\
c_{4,3}^{\beta} &= \frac{2(\beta - \sin(\beta))}{\left(2\cos(\beta) + \beta^2 - 2\right)\left(\beta^2 - 4\cos(\beta) - 4\beta\sin(\beta) + \beta^2\cos(\beta) + 4\right)},\\
c_{4,4}^{\beta} &= \frac{1}{2\cos(\beta) + \beta^2 - 2}
\end{aligned}$$

are normalizing coefficients. Applying Theorem 2.1 with the settings above, one obtains the transformation matrix

$$\left[t_{i,j}^4\right]_{i=0,\ j=0}^{4,\ 4} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1\\
0 & \frac{2\cos\beta+\beta^2-2}{2(\beta-\sin\beta)} & \frac{(2-2\cos\beta-\beta\sin\beta)\beta}{4-4\cos\beta-2\beta\sin\beta} & \beta - \frac{2\cos\beta+\beta^2-2}{2(\beta-\sin\beta)} & \beta\\
0 & 0 & \frac{\beta^2-4\cos\beta-4\beta\sin\beta+\beta^2\cos\beta+4}{2-2\cos\beta-\beta\sin\beta} & \beta^2 - \frac{(2\cos\beta+\beta^2-2)\beta}{(\beta-\sin\beta)} & \beta^2\\
0 & \frac{2\cos\beta+\beta^2-2}{2(\beta-\sin\beta)} & \frac{(2-2\cos\beta-\beta\sin\beta)\beta}{4-4\cos\beta-2\beta\sin\beta} & \sin(\beta) - \frac{(2\cos\beta+\beta^2-2)\cos(\beta)}{2(\beta-\sin\beta)} & \sin(\beta)\\
1 & 1 & \frac{(2\sin\beta-\beta-\beta\cos\beta)\beta}{4-4\cos\beta-2\beta\sin\beta} & \cos(\beta) + \frac{(2\cos\beta+\beta^2-2)\sin(\beta)}{2(\beta-\sin\beta)} & \cos(\beta)
\end{bmatrix} \tag{36}$$

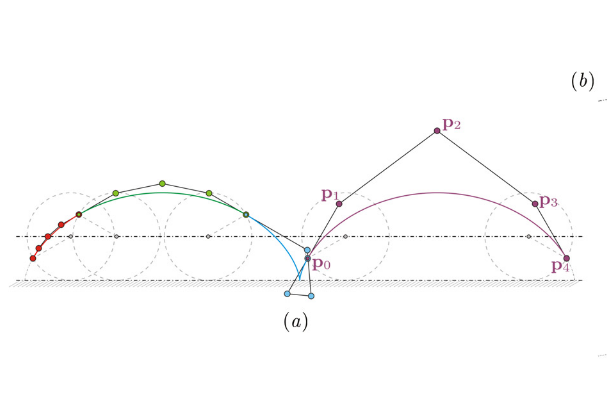

Fig. 4. (a) Cycloids and (b) helices of different shape parameters described by means of Theorem 2.1 and of the basis transformation (36) detailed in Example 3.4.

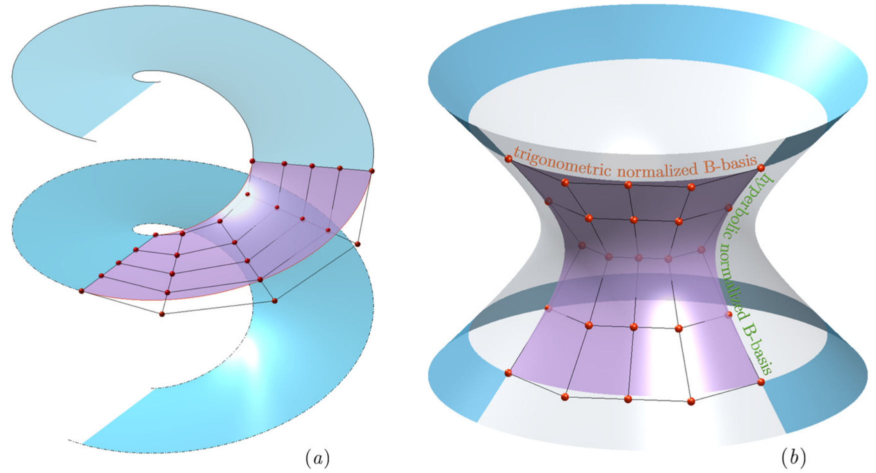

Fig. 5. Control point based exact description of (a) helicoidal and of (b) hyperboloidal patches, respectively. The control net of the patch (a) was constructed in both direction by means of basis transformations of the type (36) with different shape parameters. In case of patch (b) the control point configuration was obtained by using both the trigonometric and hyperbolic basis transformations (31) and (34), respectively.

that maps $\mathcal{B}_4^{0,\beta}$ to $\mathcal{F}_4^{0,\beta}$, based on which Fig. 4 illustrates the control point based exact description of the cycloidal arcs

$$\begin{aligned}
&\mathbf{c}_\ell(u) = \begin{bmatrix}\gamma_\ell\\0\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}\cdot u + \begin{bmatrix}-\cos(\gamma_\ell)\\ \sin(\gamma_\ell)\end{bmatrix}\cdot\sin(u) - \begin{bmatrix}\sin(\gamma_\ell)\\ \cos(\gamma_\ell)\end{bmatrix}\cdot\cos(u),\ u\in[\gamma_\ell,\gamma_{\ell+1}],\\
&\beta_\ell := \gamma_{\ell+1}-\gamma_\ell, \gamma_\ell := \frac{\pi}{6}\left(\ell^2+\ell+2\right),\ \ell = 0,1,2,3
\end{aligned} \tag{37}$$

and of helices

$$\mathbf{h}_\ell(u) = \begin{bmatrix}0\\0\\\frac{2}{\beta_\ell}\end{bmatrix}\cdot u + \begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\sin(u) + \begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\cos(u),\ u\in[0,\beta_\ell],\ \beta_\ell = \frac{\pi}{3}+\ell\cdot\frac{\pi}{6},\ \ell=0,1,2,3. \tag{38}$$

Fig. 5(a) shows the control point exact description of the patch

$$\begin{cases} x(u,v) = (2+u)\cdot\cos(v),\\ y(u,v) = (2+u)\cdot\sin(v), \quad (u,v)\in[0,2]\times\left[0,\frac{2\pi}{3}\right]\\ z(u,v) = v, \end{cases} \tag{39}$$

of a cylindrical helicoid.

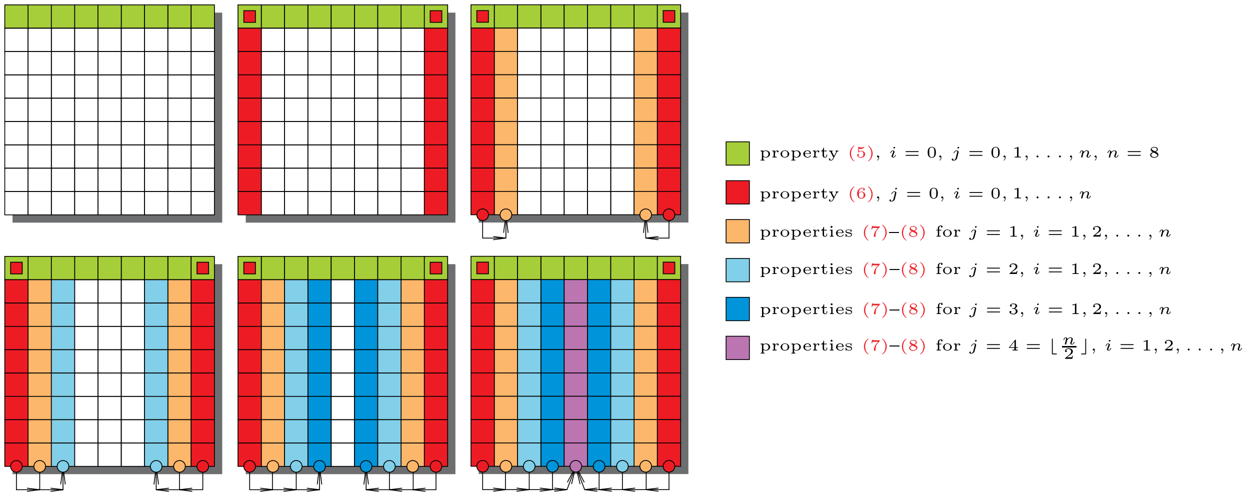

Fig. 6. Outline of the proof.

Remark 3.2 *(Hybrid EC B-surfaces).* Naturally, one can also combine different types of normalized B-basis functions in order to describe hybrid surfaces as it is shown in Fig. 5(b) that illustrates the control point based exact description of the hyperboloidal patch

$$\begin{cases} x(u,v) = \left(1+\cosh\left(u-\frac{3}{2}\right)\right)\cdot\sin(v), \\ y(u,v) = \left(1+\cosh\left(u-\frac{3}{2}\right)\right)\cdot\cos(v), \qquad (u,v)\in[0,3]\times\left[0,\frac{\pi}{2}\right]. \\ z(u,v) = \sinh\left(u-\frac{3}{2}\right), \end{cases} \tag{40}$$

4. Proof of main results

Proof of Theorem 2.1. The linear transformation $[t_{i,j}^n]_{i=0,\,j=0}^{n,\,n}$ that maps the normalized B-basis $\mathcal{B}_n^{\alpha,\beta}$ of the vector space $\mathbb{S}_n^{\alpha,\beta}$ to its ordinary basis $\mathcal{F}_n^{\alpha,\beta}$ will be constructed by mathematical induction on the column index j or $n-j$, where $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Using one of the properties (5)–(8), at each step j we will compare the left and right side of the jth order derivative of the matrix equality (9), thus obtaining an iterative process that is outlined in Fig. 6.

First of all, observe that

$$t_{0,j} = 1,\ \forall j = 0, 1, \ldots, n$$

and

$$t_{i,0}^n b_{n,i}(\alpha) = t_{i,0}^n = \varphi_{n,i}(\alpha),\quad t_{i,n}^n b_{n,n}(\beta) = t_{i,n}^n = \varphi_{n,i}(\beta),\ i = 0, 1, \ldots, n,$$

due to the partition of unity property (5) and to the endpoint interpolation property (6), respectively. Using forward substitutions, the elements of the columns $[t_{i,j}^n]_{i=1}^n$, $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ are iteratively determined by differentiating the matrix equality (9) with gradually increasing order and applying the Hermite conditions (7) at $u = \alpha$. In order to formulate a mathematical induction hypothesis, let us consider some special cases. When $j = 1$ one obtains that

$$\varphi_{n,i}^{(1)}(\alpha) = t_{i,0}^n b_{n,0}^{(1)}(\alpha) + t_{i,1}^n b_{n,1}^{(1)}(\alpha),\ i = 0, 1, \ldots, n,$$

where $b_{n,1}^{(1)}(\alpha) \neq 0$ and for the special subcase $i = 0$ one has that

$$b_{n,0}^{(1)}(\alpha) + b_{n,1}^{(1)}(\alpha) = \varphi_{n,0}^{(1)}(\alpha) = 0,$$

i.e.,

$$b_{n,0}^{(1)}(\alpha) = -b_{n,1}^{(1)}(\alpha),$$

$$t_{i,1}^n = \frac{1}{b_{n,1}^{(1)}(\alpha)}\left(\varphi_{n,i}^{(1)}(\alpha) - t_{i,0}^n b_{n,0}^{(1)}(\alpha)\right) = \frac{1}{b_{n,1}^{(1)}(\alpha)}\left(\varphi_{n,i}^{(1)}(\alpha) + \varphi_{n,i}(\alpha)\, b_{n,1}^{(1)}(\alpha)\right) = \varphi_{n,i}(\alpha) + \frac{\varphi_{n,i}^{(1)}(\alpha)}{b_{n,1}^{(1)}(\alpha)},$$
$$i = 1, 2, \ldots, n.$$

For $j=2$, we have that

$$\varphi_{n,i}^{(2)}(\alpha)=t_{i,0}^{n}b_{n,0}^{(2)}(\alpha)+t_{i,1}^{n}b_{n,1}^{(2)}(\alpha)+t_{i,2}^{n}b_{n,2}^{(2)}(\alpha),$$

where $b_{n,2}^{(2)}(\alpha)\neq 0$ and for the special subcase $i=0$ we obtain that

$$b_{n,0}^{(2)}(\alpha)+b_{n,1}^{(2)}(\alpha)+b_{n,2}^{(2)}(\alpha)=\varphi_{n,0}^{(2)}(\alpha)=0,$$

i.e.,

$$b_{n,0}^{(2)}(\alpha)+b_{n,1}^{(2)}(\alpha)=-b_{n,2}^{(2)}(\alpha),$$

$$\begin{aligned}
t_{i,2}^{n}&=\frac{1}{b_{n,2}^{(2)}(\alpha)}\left(\varphi_{n,i}^{(2)}(\alpha)-t_{i,0}^{n}b_{n,0}^{(2)}(\alpha)-t_{i,1}^{n}b_{n,1}^{(2)}(\alpha)\right)\\
&=\frac{1}{b_{n,2}^{(2)}(\alpha)}\left(\varphi_{n,i}^{(2)}(\alpha)-\varphi_{n,i}(\alpha)b_{n,0}^{(2)}(\alpha)-\left(\varphi_{n,i}(\alpha)+\frac{\varphi_{n,i}^{(1)}(\alpha)}{b_{n,1}^{(1)}(\alpha)}\right)b_{n,1}^{(2)}(\alpha)\right)\\
&=\varphi_{n,i}(\alpha)-\frac{1}{b_{n,2}^{(2)}(\alpha)}\cdot\frac{\varphi_{n,i}^{(1)}(\alpha)}{b_{n,1}^{(1)}(\alpha)}b_{n,1}^{(2)}(\alpha)+\frac{\varphi_{n,i}^{(2)}(\alpha)}{b_{n,2}^{(2)}(\alpha)},\ i=1,2,\ldots,n.
\end{aligned}$$

In case of $j=3$ one obtains that

$$\varphi_{n,i}^{(3)}(\alpha)=t_{i,0}^{n}b_{n,0}^{(3)}(\alpha)+t_{i,1}^{n}b_{n,1}^{(3)}(\alpha)+t_{i,2}^{n}b_{n,2}^{(3)}(\alpha)+t_{i,3}^{n}b_{n,3}^{(3)}(\alpha),\ i=0,1,\ldots,n,$$

where $b_{n,3}^{(3)}(\alpha)\neq 0$ and for the special subcase $i=0$, one has that

$$b_{n,0}^{(3)}(\alpha)+b_{n,1}^{(3)}(\alpha)+b_{n,2}^{(3)}(\alpha)+b_{n,3}^{(3)}(\alpha)=\varphi_{n,0}^{(3)}(\alpha)=0,$$

i.e.,

$$-b_{n,3}^{(3)}(\alpha)=b_{n,0}^{(3)}(\alpha)+b_{n,1}^{(3)}(\alpha)+b_{n,2}^{(3)}(\alpha),$$

$$\begin{aligned}
t_{i,3}^{n}&=\frac{1}{b_{n,3}^{(3)}(\alpha)}\left(\varphi_{n,i}^{(3)}(\alpha)-t_{i,0}^{n}b_{n,0}^{(3)}(\alpha)-t_{i,1}^{n}b_{n,1}^{(3)}(\alpha)-t_{i,2}^{n}b_{n,2}^{(3)}(\alpha)\right)\\
&=\frac{1}{b_{n,3}^{(3)}(\alpha)}\left(\varphi_{n,i}^{(3)}(\alpha)-\varphi_{n,i}(\alpha)b_{n,0}^{(3)}(\alpha)-\left(\varphi_{n,i}(\alpha)+\frac{\varphi_{n,i}^{(1)}(\alpha)}{b_{n,1}^{(1)}(\alpha)}\right)b_{n,1}^{(3)}(\alpha)\right.\\
&\quad\left.-\left(\varphi_{n,i}(\alpha)-\frac{1}{b_{n,2}^{(2)}(\alpha)}\cdot\frac{\varphi_{n,i}^{(1)}(\alpha)}{b_{n,1}^{(1)}(\alpha)}b_{n,1}^{(2)}(\alpha)+\frac{\varphi_{n,i}^{(2)}(\alpha)}{b_{n,2}^{(2)}(\alpha)}\right)b_{n,2}^{(3)}(\alpha)\right)\\
&=\varphi_{n,i}(\alpha)-\frac{1}{b_{n,3}^{(3)}(\alpha)}\left(\frac{\varphi_{n,i}^{(1)}(\alpha)}{b_{n,1}^{(1)}(\alpha)}\left(b_{n,1}^{(3)}(\alpha)-\frac{b_{n,1}^{(2)}(\alpha)b_{n,2}^{(3)}(\alpha)}{b_{n,2}^{(2)}(\alpha)}\right)+\frac{\varphi_{n,i}^{(2)}(\alpha)}{b_{n,2}^{(2)}(\alpha)}b_{n,2}^{(3)}(\alpha)\right)+\frac{\varphi_{n,i}^{(3)}(\alpha)}{b_{n,3}^{(3)}(\alpha)},\\
&\quad i=1,2,\ldots,n.
\end{aligned}$$

One can observe that expressions corresponding to these special cases are in accordance with formula (10). Now, fix the column index $j\in\left\{1,2,\ldots,\lfloor\frac{n}{2}\rfloor-1\right\}$ and assume that formula (10) is valid up to the selected index j and we will also prove it for $j+1$. We can proceed as follows:

$$\varphi_{n,i}^{(j+1)}(\alpha)=\sum_{\gamma=0}^{j+1}t_{i,\gamma}^{n}b_{n,\gamma}^{(j+1)}(\alpha),\ i=0,1,\ldots,n,$$

where $b_{n,j+1}^{(j+1)}(\alpha)\neq 0$ and for the special subcase $i=0$ we have that

$$\sum_{\gamma=0}^{j+1}b_{n,\gamma}^{(j+1)}(\alpha)=\varphi_{n,0}^{(j+1)}(\alpha)=0,$$

i.e.,

$$\sum_{\gamma=0}^{j} b_{n,\gamma}^{(j+1)}(\alpha) = -b_{n,j+1}^{(j+1)}(\alpha) \tag{41}$$

and

$$\begin{aligned}
t_{i,j+1}^{n} &= \frac{1}{b_{n,j+1}^{(j+1)}(\alpha)} \cdot \left(\varphi_{n,i}^{(j+1)}(\alpha) - \sum_{\gamma=0}^{j} t_{i,\gamma}^{n} b_{n,\gamma}^{(j+1)}(\alpha) \right) \\
&= \frac{\varphi_{n,i}^{(j+1)}(\alpha)}{b_{n,j+1}^{(j+1)}(\alpha)} - \frac{1}{b_{n,j+1}^{(j+1)}(\alpha)} \sum_{\gamma=0}^{j} b_{n,\gamma}^{(j+1)}(\alpha)\, t_{i,\gamma}^{n} \\
&= \frac{\varphi_{n,i}^{(j+1)}(\alpha)}{b_{n,j+1}^{(j+1)}(\alpha)} - \frac{1}{b_{n,j+1}^{(j+1)}(\alpha)} \cdot \sum_{\gamma=0}^{j} b_{n,\gamma}^{(j+1)}(\alpha) \left(\varphi_{n,i}(\alpha) + \frac{\varphi_{n,i}^{(\gamma)}(\alpha)}{b_{n,\gamma}^{(\gamma)}(\alpha)} - \frac{1}{b_{n,\gamma}^{(\gamma)}(\alpha)} \cdot \sum_{r=1}^{\gamma-1} \frac{\varphi_{n,i}^{(r)}(\alpha)}{b_{n,r}^{(r)}(\alpha)} \left(b_{n,r}^{(\gamma)}(\alpha) \right.\right. \\
&\quad \left.\left. + \sum_{\ell=1}^{\gamma-r-1} (-1)^{\ell} \sum_{r<k_{\gamma,1}<k_{\gamma,2}<\ldots<k_{\gamma,\ell}<\gamma} \frac{b_{n,r}^{(k_{\gamma,1})}(\alpha)\, b_{n,k_{\gamma,1}}^{(k_{\gamma,2})}(\alpha)\, b_{n,k_{\gamma,2}}^{(k_{\gamma,3})}(\alpha) \ldots b_{n,k_{\gamma,\ell-1}}^{(k_{\gamma,\ell})}(\alpha)\, b_{n,k_{\gamma,\ell}}^{(\gamma)}(\alpha)}{b_{n,k_{\gamma,1}}^{(k_{\gamma,1})}(\alpha)\, b_{n,k_{\gamma,2}}^{(k_{\gamma,2})}(\alpha) \ldots b_{n,k_{\gamma,\ell}}^{(k_{\gamma,\ell})}(\alpha)} \right) \right) \\
&\overset{(41)}{=} \frac{\varphi_{n,i}^{(j+1)}(\alpha)}{b_{n,j+1}^{(j+1)}(\alpha)} + \varphi_{n,i}(\alpha) - \frac{1}{b_{n,j+1}^{(j+1)}(\alpha)} \cdot \sum_{\gamma=0}^{j} b_{n,\gamma}^{(j+1)}(\alpha) \left(\frac{\varphi_{n,i}^{(\gamma)}(\alpha)}{b_{n,\gamma}^{(\gamma)}(\alpha)} - \frac{1}{b_{n,\gamma}^{(\gamma)}(\alpha)} \cdot \sum_{r=1}^{\gamma-1} \frac{\varphi_{n,i}^{(r)}(\alpha)}{b_{n,r}^{(r)}(\alpha)} \left(b_{n,r}^{(\gamma)}(\alpha) \right.\right. \\
&\quad \left.\left. + \sum_{\ell=1}^{\gamma-r-1} (-1)^{\ell} \sum_{r<k_{\gamma,1}<k_{\gamma,2}<\ldots<k_{\gamma,\ell}<\gamma} \frac{b_{n,r}^{(k_{\gamma,1})}(\alpha)\, b_{n,k_{\gamma,1}}^{(k_{\gamma,2})}(\alpha)\, b_{n,k_{\gamma,2}}^{(k_{\gamma,3})}(\alpha) \ldots b_{n,k_{\gamma,\ell-1}}^{(k_{\gamma,\ell})}(\alpha)\, b_{n,k_{\gamma,\ell}}^{(\gamma)}(\alpha)}{b_{n,k_{\gamma,1}}^{(k_{\gamma,1})}(\alpha)\, b_{n,k_{\gamma,2}}^{(k_{\gamma,2})}(\alpha) \ldots b_{n,k_{\gamma,\ell}}^{(k_{\gamma,\ell})}(\alpha)} \right) \right) \\
&= \varphi_{n,i}(\alpha) - \frac{1}{b_{n,j+1}^{(j+1)}(\alpha)} \cdot \sum_{r=1}^{j} \frac{\varphi_{n,i}^{(r)}(\alpha)}{b_{n,r}^{(r)}(\alpha)} \left(b_{n,r}^{(j+1)}(\alpha) \right. \\
&\quad \left. + \sum_{\ell=1}^{j-r} (-1)^{\ell} \sum_{r<k_1<k_2<\ldots<k_\ell<j+1} \frac{b_{n,r}^{(k_1)}(\alpha)\, b_{n,k_1}^{(k_2)}(\alpha)\, b_{n,k_2}^{(k_3)}(\alpha) \ldots b_{n,k_{\ell-1}}^{(k_\ell)}(\alpha)\, b_{n,k_\ell}^{(j+1)}(\alpha)}{b_{n,k_1}^{(k_1)}(\alpha)\, b_{n,k_2}^{(k_2)}(\alpha) \ldots b_{n,k_\ell}^{(k_\ell)}(\alpha)} \right) + \frac{\varphi_{n,i}^{(j+1)}(\alpha)}{b_{n,j+1}^{(j+1)}(\alpha)},
\end{aligned}$$

which means that our induction hypothesis is correct for all column indices $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Using a similar technique based on backward substitutions, the entries of the columns $[t_{i,n-j}^{n}]_{i=1}^{n}$, $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ can also iteratively be determined by differentiating the matrix equality (9) with gradually increasing order and applying the Hermite conditions (8) at $u = \beta$. After correct reformulations one obtains exactly the formula (11). $\square$

Proof of Theorem 2.2. Let $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, $r \in \{1, 2, \ldots, j-1\}$ and $\ell \in \{1, 2, \ldots, j-r-1\}$ be fixed indices at the moment and consider formula (10). There are $\binom{j-r-1}{\ell}$ pairwise distinct strictly increasing sequences $r < k_1 < k_2 < \ldots < k_\ell < j$ of length ℓ between r and j that can also be stored in permanent lookup tables, since they are independent of the applied normalized B-basis. In case of each of these sequences one has to evaluate the fraction

$$f_{j,r,\ell} := \frac{b_{n,r}^{(k_1)}(\alpha)\, b_{n,k_1}^{(k_2)}(\alpha)\, b_{n,k_2}^{(k_3)}(\alpha) \cdot \ldots \cdot b_{n,k_{\ell-1}}^{(k_\ell)}(\alpha)\, b_{n,k_\ell}^{(j)}(\alpha)}{b_{n,k_1}^{(k_1)}(\alpha)\, b_{n,k_2}^{(k_2)}(\alpha) \cdot \ldots \cdot b_{n,k_\ell}^{(k_\ell)}(\alpha)}$$

that includes 2ℓ flops (i.e., ℓ multiplications in the nominator, $\ell - 1$ multiplications in the denominator and 1 division). Thus, the total number of flops required for the evaluation of the summation

$$s_{j,r,\ell} := \sum_{r<k_1<k_2<\ldots<k_\ell<j} f_{j,r,\ell} \tag{42}$$

equals

$$2\ell \cdot \binom{j-r-1}{\ell} + \left(\binom{j-r-1}{\ell} - 1 \right)$$

for each fixed values of $\ell = 1, 2, \ldots, r-1$, where the last term in the parentheses appears due to additions that have to be performed in (42). If one considers all possible values of ℓ and observes that $(-1)^{\ell}$ is just an alternating sign (implying either addition or subtraction), the number of flops performed during the evaluation of the expression

$$g_{j,r} := \frac{b_{n,r}^{(j)}(\alpha) + \sum_{\ell=1}^{j-r-1} (-1)^{\ell} s_{j,r,\ell}}{b_{n,r}^{(r)}(\alpha)}$$

is

$$\sum_{\ell=1}^{j-r-1} \left(2\ell \cdot \binom{j-r-1}{\ell} + \binom{j-r-1}{\ell} - 1 \right) + [1 + (j-r-2)] + 1 = 2^{j-r-1} \cdot (j-r)$$

that consists of the evaluation cost of all $\{s_{j,r,\ell}\}_{\ell=1}^{j-r-1}$, of $1 + (j-r-2) = j-r-1$ additions and of 1 division.

Observe that values $\{g_{j,r}\}_{r=1}^{j-1}$ are independent of the row index i for all fixed column indices $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, i.e., they can be evaluated and stored in a temporary lookup table by performing

$$\sum_{r=1}^{j-1} 2^{j-r-1} \cdot (j-r) = 2^{j-1} \cdot j - 2^{j} + 1$$

flops and later they can be reused for the evaluation of all quantities

$$h_{i,j} := \sum_{r=1}^{j-1} \varphi_{n,i}^{(r)}(\alpha) \cdot g_{j,r},\ i = 1, 2, \ldots, n.$$

Thus, independently of i, the calculation of $h_{i,j}$ takes an additional $2j-3$ flops (i.e., $j-1$ multiplication and $j-2$ addition) for all fixed values of $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Finally, each of the column entries

$$t_{i,j}^{n} = \varphi_{n,i}(\alpha) - \frac{h_{i,j} - \varphi_{n,i}^{(j)}(\alpha)}{b_{n,j}^{(j)}(\alpha)},\ i = 1, 2, \ldots, n$$

can be evaluated by means of 3 additional flops for all fixed values of $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Thus, the total number of flops required for the evaluation of unknown entries $\left[t_{i,j}^{n}\right]_{i=1,\, j=1}^{n,\, \lfloor \frac{n}{2} \rfloor}$ of the general transformation matrix is

$$\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(2^{j-1} \cdot j - 2^{j} + 1 \right) + n \cdot \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} [(2j-3) + 3] = 2^{\lfloor \frac{n}{2} \rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor - 3 \right) + \left\lfloor \frac{n}{2} \right\rfloor + 3 + n \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right). \tag{43}$$

Since the structure of formula (11) is very similar to that of (10), one can conclude that for odd numbers n the total computational cost of all unknown entries of the general transformation matrix is twice of (43), while for even values of n the entries of the middle column do not have to be reevaluated by means of (11), i.e., in this latter case the partial computational cost (43) has to be increased by

$$\sum_{j=1}^{\frac{n}{2}-1} \left(2^{j-1} \cdot j - 2^{j} + 1 \right) + n \cdot \sum_{j=1}^{\frac{n}{2}-1} [(2j-3) + 3] = 2^{\frac{n}{2}-1} \left(\frac{n}{2} - 4 \right) + \frac{n}{2} + 2 + \frac{n^2}{2} \cdot \left(\frac{n}{2} - 1 \right),$$

which leads to the final expression (20). □

Proof of Theorem 3.1. In order to determine the higher order derivatives of normalized B-basis functions (26) at the endpoints of the interval $[0, \beta]$, we will make use of trigonometric identities

$$\sin^{2r+1}(\theta) = \frac{2}{2^{2r+1}} \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{2r+1}{\ell} \sin((2(r-\ell)+1)\theta),$$

$$\sin^{2r}(\theta) = \frac{1}{2^{2r}} \binom{2r}{r} + \frac{2}{2^{2r}} \sum_{\ell=0}^{r-1} (-1)^{r-\ell} \binom{2r}{\ell} \cos((2(r-\ell))\theta),$$

where $r \in \mathbb{N}$ and $\theta \in \mathbb{R}$. E.g. if $i = 2r+1$ $(r = 0, 1, \ldots, n-1)$, then

$$
\begin{aligned}
\frac{b_{2n,2r+1}(u)}{c^{\beta}_{2n,2r+1}} &= \sin^{2(n-r-1)+1}\left(\frac{\beta-u}{2}\right)\sin^{2r+1}\left(\frac{u}{2}\right)\\
&= \left(\frac{2}{2^{2(n-r-1)+1}}\sum_{k=0}^{n-r-1}(-1)^{n-r-1-k}\binom{2(n-r-1)+1}{k}\sin\left((2(n-r-k)-1)\frac{\beta-u}{2}\right)\right)\\
&\quad\cdot\left(\frac{2}{2^{2r+1}}\sum_{\ell=0}^{r}(-1)^{r-\ell}\binom{2r+1}{\ell}\sin\left((2(r-\ell)+1)\frac{u}{2}\right)\right)\\
&= \frac{1}{2^{2(n-1)}}\sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r}(-1)^{n+1-k-\ell}\binom{2(n-r-1)+1}{k}\binom{2r+1}{\ell}\\
&\quad\cdot\sin\left((2(n-r-k)-1)\frac{\beta-u}{2}\right)\sin\left((2(r-\ell)+1)\frac{u}{2}\right)\\
&= \frac{1}{2^{2n-1}}\sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r}(-1)^{n+1-k-\ell}\binom{2(n-r-1)+1}{k}\binom{2r+1}{\ell}\\
&\quad\cdot\left(\cos\left((n-k-\ell)u-(2(n-r-k)-1)\frac{\beta}{2}\right)\right.\\
&\quad\left.-\cos\left((n-k-2r+\ell-1)u-(2(n-r-k)-1)\frac{\beta}{2}\right)\right),
\end{aligned}
$$

from which follows that

$$
\begin{aligned}
\frac{b^{(j)}_{2n,2r+1}(u)}{c^{\beta}_{2n,2r+1}} &= \frac{1}{2^{2n-1}}\sum_{k=0}^{n-r-1}\sum_{\ell=0}^{r}(-1)^{n+1-(k+\ell)}\binom{2(n-r-1)+1}{k}\binom{2r+1}{\ell}\\
&\quad\cdot\left((n-(k+\ell))^j\cos\left((n-(k+\ell))u-(2(n-r-k)-1)\frac{\beta}{2}+\frac{j\pi}{2}\right)\right.\\
&\quad\left.-(n-k-2r+\ell-1)^j\cos\left((n-k-2r+\ell-1)u-(2(n-r-k)-1)\frac{\beta}{2}+\frac{j\pi}{2}\right)\right)
\end{aligned}
$$

for all $j \geq 0$. Substituting $u = 0$ into the last expression, one obtains exactly the formula (28). If $i = 2r$ ($r = 0, 1, \ldots, n$), then one can proceed analogously. □

Proof of Theorem 3.2. In order to determine the higher order derivatives of the hyperbolic counterpart of the normalized B-basis functions (26) (see also Subsection 3.3) at the endpoints of the interval $[0, \beta]$, one can follow the steps of the proof of Theorem 3.1 by applying the hyperbolic identities

$$
\sinh^{2r+1}(\theta) = \frac{2}{2^{2r+1}}\sum_{\ell=0}^{r}(-1)^{r+(r \bmod 2)-\ell}\binom{2r+1}{\ell}\sinh\left((2(r-\ell)+1)\theta\right),
$$

$$
\sinh^{2r}(\theta) = \frac{1}{2^{2r}}\binom{2r}{r} + \frac{2}{2^{2r}}\sum_{\ell=0}^{r+(r \bmod 2)-1}(-1)^{r+(r \bmod 2)-\ell}\binom{2r}{\ell}\cosh\left((2(r-\ell))\theta\right)
$$

and basic properties

$$
\sinh^{(j)}(\omega u) = \begin{cases}\omega^j\sinh(\omega u), & j\,(\mathrm{mod}\,2) = 0,\\ \omega^j\cosh(\omega u), & j\,(\mathrm{mod}\,2) = 1,\end{cases}
$$

$$
\cosh^{(j)}(\omega u) = \begin{cases}\omega^j\cosh(\omega u), & j\,(\mathrm{mod}\,2) = 0,\\ \omega^j\sinh(\omega u), & j\,(\mathrm{mod}\,2) = 1,\end{cases}
$$

$$
2\cosh(\theta_1)\cosh(\theta_2) = \cosh(\theta_1+\theta_2) + \cosh(\theta_1-\theta_2)
$$

of the hyperbolic sine and cosine functions, where $r \in \mathbb{N}$ and $\theta, \theta_1, \theta_2, \omega \in \mathbb{R}$. □

5. Final remarks

As listed in Section 1, concerning geometric modeling, the normalized B-bases (of EC spaces that also comprise the constant functions) ensure many optimal shape preserving properties and algorithms. Moreover, they may also provide useful design or shape parameters that can arbitrarily be specified by the user or the engineer. In Section 3, we have seen that polynomial, trigonometric, hyperbolic or mixed EC spaces allow us to obtain the control point based exact description of many (rational) curves and surfaces that are important in several areas of applied mathematics. The investigated large classes of vector spaces also ensure the description of famous geometrical objects (like ellipses; epi- and hypocycloids; Lissajous curves; torus knots; foliums; rose curves; the witch of Agnesi; the cissoid of Diocles; Bernoulli's lemniscate; Zhukovsky airfoil profiles; cycloids; hyperbolas; helices; catenaries; Archimedean and logarithmic spirals; ellipsoids; tori; hyperboloids; catenoids; helicoids; ring, horn and spindle Dupin cyclides; non-orientable surfaces such as Boy's and Steiner's surfaces and the Klein Bottle of Gray).

EC bases of type (1) represent a large family of vector spaces that can be used in real-world applications, e.g. besides of examples described in Section 3, general formulas of Theorem 2.1 can also be applied in the exponential space

$$\left\langle\left\{1, e^{\lambda_1 u}, e^{\lambda_2 u}, \ldots, e^{\lambda_n u} : u \in [\alpha, \beta]\right\}\right\rangle,\ 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n,\ \alpha < \beta,$$

or in the space

$$\left\langle\left\{1, u^{\lambda_1}, u^{\lambda_2}, \ldots, u^{\lambda_n} : u \in [\alpha, \beta]\right\}\right\rangle,\ 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n,\ [\alpha, \beta] \subset (0, \infty)$$

of restricted Müntz polynomials among many others.

Storing in permanent lookup tables the zeroth and higher order endpoint derivatives of the ordinary EC basis (1) and of the normalized B-basis (4) induced by it, general formulas (10)–(11) and the proposed control point based curve/surface modeling tools can efficiently be implemented up to $n = 15$, even by means of a sequential algorithm. Considering that, in practice, curves and surfaces are mostly composed of continuously joined lower order arcs and patches, even a sequential but clever implementation of Theorem 2.1 can be useful in case of real-life applications. (If one uses multi-threading, the value of n, for which one can provide an efficient implementation, can be higher.) Nevertheless, if $n > 15$, the presented results are mainly of theoretical interest.

Acknowledgements

The author would like to thank the anonymous reviewers for their careful reading of the manuscript. Their comments helped to improve the presentation of the paper. This research was supported by the European Union and the State of Hungary in the framework TÁMOP-4.2.4.A/2-11/1-2012-0001 'National Excellence Program'.

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