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Explicit μ-bases for conic sections and planar rational cubic curves ☆

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ABSTRACT

We derive explicit formulas for the μ-bases of conic sections and planar rational cubic curves. Using the μ-bases for planar rational cubic curves, we find explicit formulas for their implicit equations and double points. We also extend the explicit formula for the μ-bases of conic sections to μ-bases for rational curves of degree n in n-dimensions.

1. Introduction

μ-bases for rational curves are important since we can use μ-bases to implicitize rational planar curves (Chen and Wang, 2003). Moreover, μ-bases for rational curves in arbitrary dimensions can be applied to locate and analyze their singularities (Song et al., 2007; Jia and Goldman, 2009, 2012; Shi and Chen, 2010). There are fast algorithms to compute μ-bases for rational curves based mainly on Gaussian Elimination (Chen and Wang, 2003; Song and Goldman, 2009). However, we do not know what the μ-bases look like before we run the algorithm; this drawback prevents us from finding closed formulas for the singularities or the implicit equation for rational curves based on the method of μ-bases.

Here we give explicit formulas for the μ-bases of conic sections and planar rational cubic curves. As fundamental objects in Computer Aided Geometric Design and Computer Graphics, conic sections and planar rational cubic curves have a wide range of applications, for example, in animation control and font design. Implicit equations for planar rational cubic curves will also help us when we deal with intersection algorithms (Sederberg and Parry, 1986; Thomassen, 2005).

Special attention has been devoted to computing implicit equations for planar rational cubic curves. Sederberg et al. (1985) treat planar rational cubic curves as monoids, that is, degree n curves with a singular point of order $n-1$. This approach simplifies the derivation of the implicit equation. Floater (1995) proves that the implicit equation of every planar rational cubic curve can be represented by linear combinations of six basis functions. Using the results of

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Busé and Jouanolou (2003), one can also derive a closed formula for the implicit equation of a rational cubic curve $\mathbf{P}(t) = (x(t), y(t), w(t))$ as the determinant of 3×3 matrix by inspecting the Bezoutian matrix:

$$M = x \cdot Bez(y(t), w(t)) + y \cdot Bez(w(t), x(t)) + w \cdot Bez(x(t), y(t))$$

(see Section 4 for further details). In contrast, using our explicit formulas for μ-bases of planar rational cubic curves, we derive two closed formulas – one as the determinant of a sparse 3×3 Sylvester matrix and one as the determinant of a 2×2 Bezoutian matrix – for the implicit equations of planar rational cubic curves. Moreover, we also provide closed formulas for the double point of planar rational cubic curves. Explicit formulas for implicit equations of planar rational cubic Bézier curves are also presented recently in Barrowclough (2014). But our explicit formulas for these implicit equations based on μ-bases are more compact. Moreover, using μ-bases for planar rational cubic curves, we get explicit formulas for their double points in a simpler, more straightforward way.

This paper is structured in the following fashion. In Section 2 we review the notion of μ-bases for planar rational curves. We give the explicit formulas for the μ-bases and closed form expressions for the implicit equations of conic sections and planar rational cubic curves in Section 3 and Section 4. We also provide explicit formulas for the double point of planar rational cubic curves in Section 4. Section 5 extends the explicit formula for the μ-bases of conic sections to μ-bases for rational curves of degree n in n-dimensions. We close in Section 6 with some observations about our methods.

2. μ-bases for planar rational curves

In this section we briefly review the concept of μ-bases for planar rational curves. We begin with the notion of syzygies. Consider a real planar rational curve

$$\mathbf{P}(t) = \big(x(t), y(t), w(t)\big)$$

in homogeneous form, where $\gcd\big(x(t), y(t), w(t)\big) = 1$. A *syzygy* of $\mathbf{P}(t)$ is a vector of polynomials $\mathbf{L}(t) = \big(l_0(t), l_1(t), l_2(t)\big)$ such that $\mathbf{L}(t) \cdot \mathbf{P}(t) \equiv 0$. The set of all syzygies of $\mathbf{P}(t)$ is a free module with two generators over the ring $\mathbb{R}[t]$ of polynomials in t with real coefficients (Chen and Wang, 2003). A *μ-basis* for the planar rational curve $\mathbf{P}(t)$ is a special basis for this free module.

Definition 1. Two syzygies $\mathbf{u}_0(t)$, $\mathbf{u}_1(t)$ are a μ-basis for a rational curve $\mathbf{P}(t)$ if

1) $\deg(\mathbf{u}_0(t)) + \deg(\mathbf{u}_1(t)) = \deg(\mathbf{P}(t))$,
2) $\mathbf{u}_0(t) \times \mathbf{u}_1(t) = \lambda \mathbf{P}(t)$ for some nonzero constant λ.

There are also several other equivalent definitions of a μ-basis for the curve $\mathbf{P}(t)$ (see Chen and Wang, 2003 for more details). For example, 2) in Definition 1 can be replaced by

2′) $\mathbf{u}_0(t)$ and $\mathbf{u}_1(t)$ are $\mathbb{R}[t]$-linearly independent.

Every planar rational curve has a μ-basis. Moreover, there is an algorithm for computing μ-bases based on Gaussian elimination (Chen and Wang, 2003). In contrast, we shall find closed formulas for a μ-basis of conic sections and planar rational cubic curves.

One important property of μ-bases is that the implicit equation of the planar rational curve $\mathbf{P}(t)$ can be calculated as

$$f(x, y, w)^\sigma = \mathrm{Res}_t\big(\mathbf{u}_0(t) \cdot (x, y, w), \mathbf{u}_1(t) \cdot (x, y, w)\big) = 0, \tag{1}$$

where σ is the degree of the parametrization $\mathbf{P}$. Thus both the parametric and implicit equations of the curve $\mathbf{P}(t)$ can be retrieved from its μ-basis $\mathbf{u}_0(t), \mathbf{u}_1(t)$. Moreover, we can also compute the singular points of a planar rational curve $\mathbf{P}(t)$ from its μ-basis (Chen et al., 2008; Jia and Goldman, 2009).

From the viewpoint of geometry, if $\mathbf{l}(t) = (l_0(t), l_1(t), l_2(t))$ is a syzygy of $\mathbf{P}(t)$, we call

$$\mathbf{l}(t) \cdot (x, y, w) = l_0(t)x + l_1(t)y + l_2(t)w = 0 \tag{2}$$

a *moving line* that *follows* the curve $\mathbf{P}(t)$ because the point $\mathbf{P}(t) = (x(t), y(t), w(t))$ lies on the line $\mathbf{l}(t) \cdot (x, y, w) = 0$. The line (2) changes as the parameter t varies, thus (2) represents a family of lines. If all the lines in the family pass through a common point, the common point is called an *axis* and the moving line is called an *axial moving line*.

When $\deg(\mathbf{l}(t)) = 1$, the moving line (2) is a linear moving line. One simple case of an axial moving line is a linear moving line. Since a linear moving line $\mathbf{l}(t) \cdot (x, y, w) = 0$ can be rewritten as $\big(\mathbf{l}_0 + \mathbf{l}_1 t\big) \cdot (x, y, w) = 0$, it is easy to check that $\mathbf{l}_0 \times \mathbf{l}_1$ is the axis of this moving line. Thus every linear moving line is an axial moving line.

Let $\mathbf{l}(t) \cdot (x, y, w) = 0$ be an axial moving line for $\mathbf{P}(t)$ with an axis $A = (x_0, y_0, w_0)$. Then since for all t the axis A lies on the line $\mathbf{l}(t) \cdot (x, y, w) = 0$,

$$\mathbf{l}(t) \cdot A = \mathbf{l}(t) \cdot (x_0, y_0, w_0) \equiv 0.$$

On the other hand, $\mathbf{l}(t) \cdot \mathbf{P}(t) \equiv 0$. So the axis A is also a point on the curve $\mathbf{P}(t)$. Moreover, singular points of a planar rational curve are also related to its axial moving lines.

Theorem 2. *(See Song et al., 2007.) A is a singular point of order $k \geq 2$ of a planar rational curve $\mathbf{P}(t)$ of degree n if and only if there is a moving line of degree $n-k$ with axis A that follows the curve $\mathbf{P}(t)$.*

In this paper, our investigations of planar rational curves focus only on conic sections and planar rational cubic curves. Moreover, all the rational curves studied in this paper are assumed to be non-degenerate. It is well known that there are no singular points on non-degenerate conic sections and there is always exactly one double point (i.e., a singular point of order 2) on every non-degenerate planar rational cubic curve. So by Theorem 2, for every non-degenerate planar rational cubic curve there is a linear moving line that follows the curve, and the axis of this linear moving line is the unique double point of the cubic curve.

3. μ-bases for conic sections

In this section we shall find explicit formulas for a μ-basis of an arbitrary conic section. We shall then use this μ-basis to derive a closed formula for the implicit equation of a conic section from its rational quadratic parametrization.

Consider a non-degenerate conic in homogeneous form

$$\mathbf{P}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2, \tag{3}$$

where $\mathbf{p}_i \in \mathbb{R}^3$. The three coefficient vectors $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ must be linearly independent, that is, $(\mathbf{p}_0 \times \mathbf{p}_2) \cdot \mathbf{p}_1 \neq 0$. Otherwise there is a constant vector $\Gamma \neq 0$ such that

$$\Gamma \perp \mathbf{p}_i,\ i = 0, 1, 2.$$

So $\Gamma \cdot \mathbf{P}(t) \equiv 0$. Hence the conic $\mathbf{P}(t)$ degenerates into the line $\Gamma \cdot (x, y, w) = 0$.

Let $\mathbf{u}_0(t), \mathbf{u}_1(t)$ be a μ-basis for the conic $\mathbf{P}(t)$. From the definition of a μ-basis,

$$\deg(\mathbf{u}_0(t)) = \deg(\mathbf{u}_1(t)) = 1.$$

Next we shall show how to find explicit formulas for two degree one syzygies of the conic (3).

Computing the outer product of $\mathbf{p}_2$ and $\mathbf{P}(t)$, we get

$$\mathbf{p}_2 \times \mathbf{P}(t) = (\mathbf{p}_2 \times \mathbf{p}_0) + (\mathbf{p}_2 \times \mathbf{p}_1)t.$$

Moreover $\mathbf{p}_2 \times \mathbf{P}(t)$ follows $\mathbf{P}(t)$ since $(\mathbf{p}_2 \times \mathbf{P}(t)) \perp \mathbf{P}(t)$. Thus we have found an explicit formula for one degree one syzygy of the conic $\mathbf{P}(t)$. To get a second degree one syzygy for $\mathbf{P}(t)$, we compute the outer product of $\mathbf{p}_0$ and $\mathbf{P}(t)$. It is easy to see that

$$\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t} = (\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t.$$

Moreover $\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t}$ also follows $\mathbf{P}(t)$ since $\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t} \perp \mathbf{P}(t)$. Thus, $\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t}$ is also a degree one syzygy for $\mathbf{P}(t)$.

Next we shall show $\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t}$ and $\mathbf{p}_2 \times \mathbf{P}(t)$ are a μ-basis for the conic $\mathbf{P}(t)$.

Computing the outer product of $\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t}$ and $\mathbf{p}_2 \times \mathbf{P}(t)$ yields

$$\begin{aligned}
&\left(\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t}\right) \times \big(\mathbf{p}_2 \times \mathbf{P}(t)\big) \\
&\quad = \big((\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t\big) \times \big((\mathbf{p}_2 \times \mathbf{p}_0) + (\mathbf{p}_2 \times \mathbf{p}_1)t\big) \\
&\quad = ((\mathbf{p}_0 \times \mathbf{p}_2) \cdot \mathbf{p}_1)\mathbf{P}(t).
\end{aligned} \tag{4}$$

Let $|\mathbf{abc}|$ denote the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. It is well known that

$$|\mathbf{abc}| = \det(\mathbf{abc}),$$

where $\det(\mathbf{abc})$ is the determinant of the 3×3 matrix whose columns are the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Then (4) can be rewritten as

$$\left(\frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t}\right) \times \big(\mathbf{p}_2 \times \mathbf{P}(t)\big) = |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_1| \cdot \mathbf{P}(t).$$

Since $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ are linearly independent, $|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_1| \neq 0$. Therefore it follows from the definition of a μ-basis that

$$\begin{aligned}
&\tfrac{\mathbf{p}_0 \times \mathbf{P}(t)}{t} = (\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t \\
&\mathbf{p}_2 \times \mathbf{P}(t) = (\mathbf{p}_2 \times \mathbf{p}_0) + (\mathbf{p}_2 \times \mathbf{p}_1)t
\end{aligned} \tag{5}$$

are a μ-basis for the conic $\mathbf{P}(t)$. Moreover by Equation (1), a closed form expression for the implicit equation of the conic curve $\mathbf{P}(t)$ is

$$F(x, y, w) \equiv \det\begin{pmatrix} (\mathbf{p}_0 \times \mathbf{p}_1)\cdot\mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_2)\cdot\mathbf{X} \\ (\mathbf{p}_2 \times \mathbf{p}_0)\cdot\mathbf{X} & (\mathbf{p}_2 \times \mathbf{p}_1)\cdot\mathbf{X} \end{pmatrix} = 0,$$

where $\mathbf{X} = (x, y, w)$. Notice that this determinant is exactly the Bézout resultant of the rational parametrization in Equation (3).

Remark 1. By inspecting the matrix

$$M = x \cdot Bez(y(t), w(t)) + y \cdot Bez(w(t), x(t)) + w \cdot Bez(x(t), y(t)), \tag{6}$$

where $Bez(f(t), g(t))$ represents the Bezoutian matrix associated with the two polynomials $f(t), g(t)$, one can find a different way to get the implicitization matrix (Busé and Jouanolou, 2003):

$$\begin{pmatrix} (\mathbf{p}_0 \times \mathbf{p}_1)\cdot\mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_2)\cdot\mathbf{X} \\ (\mathbf{p}_2 \times \mathbf{p}_0)\cdot\mathbf{X} & (\mathbf{p}_2 \times \mathbf{p}_1)\cdot\mathbf{X} \end{pmatrix}. \tag{7}$$

However, it is not so obvious that one can rewrite the matrix M in the form (7) if there is no advance notice. On the other hand, the matrix (7) follows directly after we get the closed formula of the μ-basis in Equation (5).

Example 1. Consider a conic $\mathbf{P}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2$, where

$$\mathbf{p}_0 = (3, 2, -2), \mathbf{p}_1 = (0, 2, 1), \mathbf{p}_2 = (-3, 4, 1).$$

Then

$$(\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t = (6, -3, 6) + (10, 3, 18)t,$$
$$(\mathbf{p}_2 \times \mathbf{p}_0) + (\mathbf{p}_2 \times \mathbf{p}_1)t = (-10, -3, -18) + (2, 3, -6)t$$

are a μ-basis for $\mathbf{P}(t)$. Indeed

$$\big((\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t\big) \times \big((\mathbf{p}_2 \times \mathbf{p}_0) + (\mathbf{p}_2 \times \mathbf{p}_1)t\big)$$
$$= |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_1| \cdot \mathbf{P}(t) = 24\mathbf{P}(t).$$

Moreover,

$$F(x, y, w) \equiv \det\begin{pmatrix} 6x - 3y + 6w & 10x + 3y + 18w \\ -10x - 3y - 18w & 2x + 3y - 6w \end{pmatrix} = 0,$$

is the implicit equation of the conic $\mathbf{P}(t)$.

Remark 2. When the conic curve is expressed in terms of the Bernstein basis

$$\mathbf{P}(t) = \sum_{i=0}^{2} \mathbf{p}_i B_i^2(t) = \sum_{i=0}^{2} \mathbf{p}_i \frac{2!}{i!(2-i)!}(1-t)^{2-i} t^i,$$

let $\hat{\mathbf{p}}_i = \frac{2!}{i!(2-i)!}\mathbf{p}_i$, $i = 0, 1, 2$. Then

$$\frac{\hat{\mathbf{p}}_0 \times \mathbf{P}(t)}{t} = (\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_1)(1-t) + (\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_2)t,$$
$$\frac{\hat{\mathbf{p}}_2 \times \mathbf{P}(t)}{1-t} = (\hat{\mathbf{p}}_2 \times \hat{\mathbf{p}}_0)(1-t) + (\hat{\mathbf{p}}_2 \times \hat{\mathbf{p}}_1)t.$$

are a μ-basis for the conic $\mathbf{P}(t)$. Moreover, the implicit equation of the conic curve $\mathbf{P}(t)$ is

$$F(x, y, w) \equiv \det\begin{pmatrix} (\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_1)\cdot\mathbf{X} & (\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_2)\cdot\mathbf{X} \\ (\hat{\mathbf{p}}_2 \times \hat{\mathbf{p}}_0)\cdot\mathbf{X} & (\hat{\mathbf{p}}_2 \times \hat{\mathbf{p}}_1)\cdot\mathbf{X} \end{pmatrix} = 0,$$

where $\mathbf{X} = (x, y, w)$.

4. μ-bases for planar rational cubic curves

In this section we shall find explicit formulas for a μ-basis of an arbitrary planar rational cubic curve. We shall then use this μ-basis to find a closed formula for the double point and the implicit equation of the planar rational cubic curve.

Consider a non-degenerate planar rational cubic curve in homogeneous form

$$\mathbf{P}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2 + \mathbf{p}_3 t^3, \tag{8}$$

where $\mathbf{p}_i \in \mathbb{R}^3$. Notice that $\mathbf{p}_0 \neq 0$ and $\mathbf{p}_3 \neq 0$ otherwise the cubic curve $\mathbf{P}(t)$ will degenerate into a conic or a line. Set

$$M = (\mathbf{p}_0 \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3),$$

where M is the 3×4 matrix whose columns are the four vectors $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. For the rational cubic curve $\mathbf{P}(t)$ to be non-degenerate we must have

$$\text{Rank}(M) = 3. \tag{9}$$

If $\text{Rank}(M) < 3$, there is a constant vector $\Gamma \neq 0$ such that $\Gamma \cdot \mathbf{p}_i = 0,\ i = 0, \ldots, 3$. Hence $\Gamma \cdot \mathbf{P}(t) \equiv 0$. Thus the cubic curve will degenerate into the line $\Gamma \cdot (x, y, w) = 0$.

Again by inspecting the matrix (6) based on Busé and Jouanolou (2003), one can derive the following closed formula for the implicitization matrix of the cubic curve $\mathbf{P}(t)$:

$$\begin{pmatrix} (\mathbf{p}_0 \times \mathbf{p}_3) \cdot \mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_2) \cdot \mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_1) \cdot \mathbf{X} \\ (\mathbf{p}_1 \times \mathbf{p}_3) \cdot \mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_3 + \mathbf{p}_1 \times \mathbf{p}_2) \cdot \mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_2) \cdot \mathbf{X} \\ (\mathbf{p}_2 \times \mathbf{p}_3) \cdot \mathbf{X} & (\mathbf{p}_1 \times \mathbf{p}_3) \cdot \mathbf{X} & (\mathbf{p}_0 \times \mathbf{p}_3) \cdot \mathbf{X} \end{pmatrix}. \tag{10}$$

On the other hand, from (1) we can derive two different closed formulas for the implicitization matrices of the cubic curve $\mathbf{P}(t)$ based on an explicit μ-basis for the cubic curve $\mathbf{P}(t)$.

Let $\mathbf{u}_0(t), \mathbf{u}_1(t)$ be a μ-basis for the rational cubic curve $\mathbf{P}(t)$ with $\deg(\mathbf{u}_0(t)) \leq \deg(\mathbf{u}_1(t))$. From the definition of a μ-basis, we know that $\deg(\mathbf{u}_0(t)) + \deg(\mathbf{u}_1(t)) = \deg(\mathbf{P}(t))$. Thus for any μ-basis of the planar rational cubic curve $\mathbf{P}(t)$,

$$\deg(\mathbf{u}_0(t)) = 1,\ \deg(\mathbf{u}_1(t)) = 2.$$

Express $\mathbf{u}_0(t)$ and $\mathbf{u}_1(t)$ in the power basis:

$$\mathbf{u}_0(t) = \mathbf{u}_{00} + \mathbf{u}_{01} t,$$

$$\mathbf{u}_1(t) = \mathbf{u}_{10} + \mathbf{u}_{11} t + \mathbf{u}_{12} t^2.$$

Then

$$\mathbf{u}_0(t) \cdot \mathbf{X} = \mathbf{u}_{00} \cdot \mathbf{X} + (\mathbf{u}_{01} \cdot \mathbf{X}) t,$$

$$\mathbf{u}_1(t) \cdot \mathbf{X} = \mathbf{u}_{10} \cdot \mathbf{X} + (\mathbf{u}_{11} \cdot \mathbf{X}) t + (\mathbf{u}_{12} \cdot \mathbf{X}) t^2,$$

where $\mathbf{X} = (x, y, w)$.

Calculating the Sylvester matrix of $\mathbf{u}_0(t) \cdot \mathbf{X}$ and $\mathbf{u}_1(t) \cdot \mathbf{X}$ with respect to t yields an alternative form of the implicitization matrix for the cubic curve $\mathbf{P}(t)$:

$$\begin{pmatrix} \mathbf{u}_{00} \cdot \mathbf{X} & 0 & \mathbf{u}_{10} \cdot \mathbf{X} \\ \mathbf{u}_{01} \cdot \mathbf{X} & \mathbf{u}_{00} \cdot \mathbf{X} & \mathbf{u}_{11} \cdot \mathbf{X} \\ 0 & \mathbf{u}_{01} \cdot \mathbf{X} & \mathbf{u}_{12} \cdot \mathbf{X} \end{pmatrix}. \tag{11}$$

Notice that the matrix in (11) is sparser than the matrix in (10). Moreover, calculating the Bezoutian matrix of $\mathbf{u}_0(t) \cdot \mathbf{X}$ and $\mathbf{u}_1(t) \cdot \mathbf{X}$ with respect to t yields a different implicitization matrix with smaller size for the cubic curve $\mathbf{P}(t)$:

$$\begin{pmatrix} \mathbf{X}(\mathbf{u}_{00}^T \mathbf{u}_{12} - \mathbf{u}_{11}^T \mathbf{u}_{01}) \mathbf{X}^T & \mathbf{u}_{01} \mathbf{X}^T \\ -\mathbf{X}(\mathbf{u}_{10}^T \mathbf{u}_{01}) \mathbf{X}^T & \mathbf{u}_{00} \mathbf{X}^T \end{pmatrix}. \tag{12}$$

Next we shall show case by case how to find explicit μ-bases for non-degenerate planar rational cubic curves. We shall consider three distinct cases.

4.1. Case 1: $\mathbf{p}_0 \times \mathbf{p}_3 = 0$

Suppose $\mathbf{p}_0 \times \mathbf{p}_3 = 0$. Consider

$$\mathbf{p}_3 \times \mathbf{P}(t) = \big((\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2) t\big) t.$$

In this case $\mathbf{p}_3 \times \mathbf{p}_1 \neq 0$, $\mathbf{p}_3 \times \mathbf{p}_2 \neq 0$. Otherwise $\text{Rank}(M) < 3$ which contradicts condition (9). Thus when $\mathbf{p}_0 \times \mathbf{p}_3 = 0$,

$$\frac{\mathbf{p}_3 \times \mathbf{P}(t)}{t} = (\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2)t \tag{13}$$

is a degree one syzygy for $\mathbf{P}(t)$ because $\mathbf{p}_3 \times \mathbf{P}(t) \cdot \mathbf{P}(t) \equiv 0$.

On the other hand, $\mathbf{P}(t)$ can be rewritten as $(\mathbf{p}_0 + \mathbf{p}_1 t) + (\mathbf{p}_2 + \mathbf{p}_3 t)t^2$, so it is easy to check that

$$(\mathbf{p}_0 + \mathbf{p}_1 t) \times (\mathbf{p}_2 + \mathbf{p}_3 t) \tag{14}$$

also follows $\mathbf{P}(t)$. Expanding (14) yields

$$\begin{aligned}&(\mathbf{p}_0 + \mathbf{p}_1 t) \times (\mathbf{p}_2 + \mathbf{p}_3 t)\\ &\quad = \mathbf{p}_0 \times \mathbf{p}_2 + (\mathbf{p}_1 \times \mathbf{p}_2)t + (\mathbf{p}_1 \times \mathbf{p}_3)t^2.\end{aligned}$$

Computing the outer product of (13) and (14) and recalling that by assumption $\mathbf{p}_0 \times \mathbf{p}_3 = 0$, we get

$$\big((\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2)t\big) \times \big((\mathbf{p}_0 + \mathbf{p}_1 t) \times (\mathbf{p}_2 + \mathbf{p}_3 t)\big) = |\mathbf{p}_3\mathbf{p}_1\mathbf{p}_2| \cdot \mathbf{P}(t) \tag{15}$$

where $|\mathbf{p}_3\mathbf{p}_1\mathbf{p}_2|$ is a nonzero constant, otherwise $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly dependent and $\mathrm{Rank}(M) < 3$, which again contradicts condition (9). Thus from the definition of a μ-basis,

$$\begin{aligned}\mathbf{u}_0(t) &= (\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2)t,\\ \mathbf{u}_1(t) &= \mathbf{p}_0 \times \mathbf{p}_2 + (\mathbf{p}_1 \times \mathbf{p}_2)t + (\mathbf{p}_1 \times \mathbf{p}_3)t^2\end{aligned} \tag{16}$$

are a μ-basis for the rational cubic curve $\mathbf{P}(t)$ when $\mathbf{p}_0 \times \mathbf{p}_3 = 0$.

Hence by (11), the determinant of the Sylvester matrix

$$\begin{pmatrix} (\mathbf{p}_3 \times \mathbf{p}_1)\cdot\mathbf{X} & 0 & (\mathbf{p}_0 \times \mathbf{p}_2)\cdot\mathbf{X} \\ (\mathbf{p}_3 \times \mathbf{p}_2)\cdot\mathbf{X} & (\mathbf{p}_3 \times \mathbf{p}_1)\cdot\mathbf{X} & (\mathbf{p}_1 \times \mathbf{p}_2)\cdot\mathbf{X} \\ 0 & (\mathbf{p}_3 \times \mathbf{p}_2)\cdot\mathbf{X} & (\mathbf{p}_1 \times \mathbf{p}_3)\cdot\mathbf{X} \end{pmatrix}, \tag{17}$$

represents the implicit equation of the rational cubic curve $\mathbf{P}(t)$. In addition, by (12), the determinant of the Bezoutian matrix

$$\begin{pmatrix} \mathbf{X}\big((\mathbf{p}_3 \times \mathbf{p}_1)^T(\mathbf{p}_1 \times \mathbf{p}_3) - (\mathbf{p}_3 \times \mathbf{p}_2)^T(\mathbf{p}_1 \times \mathbf{p}_2)\big)\mathbf{X}^T & (\mathbf{p}_3 \times \mathbf{p}_2)\mathbf{X}^T \\ -\mathbf{X}\big((\mathbf{p}_0 \times \mathbf{p}_2)^T(\mathbf{p}_3 \times \mathbf{p}_2)\big)\mathbf{X}^T & (\mathbf{p}_3 \times \mathbf{p}_1)\mathbf{X}^T \end{pmatrix}, \tag{18}$$

also represents the implicit equation of the cubic curve $\mathbf{P}(t)$.

Since

$$\big((\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2)t\big) \cdot \mathbf{p}_3 \equiv 0,$$

it follows that $\mathbf{p}_3$ is the axial point of the linear moving line $\big((\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2)t\big) \cdot (x, y, w) = 0$. Hence by Theorem 2, $\mathbf{p}_3$ is the unique double point of the rational cubic curve $\mathbf{P}(t)$.

Example 2. Consider the planar rational cubic curve

$$\mathbf{P}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2 + \mathbf{p}_3 t^3,$$

where $\mathbf{p}_0 = (3, 2, -2)$, $\mathbf{p}_1 = (0, 2, 1)$, $\mathbf{p}_2 = (-3, 4, 1)$, $\mathbf{p}_3 = 2\mathbf{p}_0$. Clearly $\mathbf{p}_0 \times \mathbf{p}_3 = 0$. Moreover,

$$\begin{aligned}&\mathbf{p}_3 \times \mathbf{p}_1 = (12, -6, 12),\ \ \mathbf{p}_3 \times \mathbf{p}_2 = (20, 6, 36)\\ &\mathbf{p}_0 \times \mathbf{p}_2 = (10, 3, 18),\ \ \mathbf{p}_1 \times \mathbf{p}_2 = (-2, -3, 6),\ \ \mathbf{p}_1 \times \mathbf{p}_3 = (-12, 6, -12).\end{aligned}$$

So in this case, by (17) and (18), the determinants of the matrices

$$\begin{pmatrix} 12x - 6y + 12w & 0 & 10x + 3y + 18w \\ 20x + 6y + 36w & 12 - 6y + 12w & -2x - 3y + 6w \\ 0 & 20x + 6y + 36w & -12x + 6y - 12w \end{pmatrix}$$

and

$$\begin{pmatrix} -104x^2 + 216yx - 336xw - 18y^2 + 216yw - 360w^2 & 20x + 6y + 36w \\ 200x^2 + 120yx + 720xw + 18y^2 + 216yw + 648w^2 & 12x - 6y + 12w \end{pmatrix}$$

both represent the implicit equation of the cubic curve $\mathbf{P}(t)$.

In addition by (16),

$$\begin{aligned}&(\mathbf{p}_3 \times \mathbf{p}_1) + (\mathbf{p}_3 \times \mathbf{p}_2)t = (12 + 20t, -6 + 6t, 12 + 36t),\\ &(\mathbf{p}_0 + \mathbf{p}_1 t) \times (\mathbf{p}_2 + \mathbf{p}_3 t) = (10 - 2t - 12t^2, 3 - 3t + 6t^2, 18 + 6t - 12t^2)\end{aligned}$$

are a μ-basis for the cubic curve $\mathbf{P}(t)$. Indeed

$$\big((\mathbf{p}_3\times\mathbf{p}_1)+(\mathbf{p}_3\times\mathbf{p}_2)\big)\times\big((\mathbf{p}_0+\mathbf{p}_1 t)\times(\mathbf{p}_2+\mathbf{p}_3 t)\big)$$
$$=|\mathbf{p}_3\mathbf{p}_1\mathbf{p}_2|\cdot\mathbf{P}(t)=-48\mathbf{P}(t).$$

Finally, $\mathbf{p}_3=(6,4,-4)$ is the unique double point of the cubic curve $\mathbf{P}(t)$.

4.2. Case 2: $\mathbf{p}_0\times\mathbf{p}_3\neq 0$

Suppose $\mathbf{p}_0\times\mathbf{p}_3\neq 0$. To construct two syzygies for $\mathbf{P}(t)$, set

$$\mathbf{Q}(t)=\mathbf{p}_3\times\mathbf{P}(t)=(\mathbf{p}_3\times\mathbf{p}_0)+(\mathbf{p}_3\times\mathbf{p}_1)t+(\mathbf{p}_3\times\mathbf{p}_2)t^2, \tag{19}$$
$$\hat{\mathbf{Q}}(t)=\frac{\mathbf{p}_0\times\mathbf{P}(t)}{t}=(\mathbf{p}_0\times\mathbf{p}_1)+(\mathbf{p}_0\times\mathbf{p}_2)t+(\mathbf{p}_0\times\mathbf{p}_3)t^2. \tag{20}$$

It is easy to check that both $\mathbf{Q}(t)$ and $\hat{\mathbf{Q}}(t)$ follow $\mathbf{P}(t)$. Thus we get two syzygies for $\mathbf{P}(t)$. The relation between these two syzygies is explained in the following lemma.

Lemma 3. *Consider a non-degenerate planar rational cubic curve $\mathbf{P}(t)$. If $\mathbf{p}_0\times\mathbf{p}_3\neq 0$, $\mathbf{Q}(t)$ and $\hat{\mathbf{Q}}(t)$ are $\mathbb{R}[t]$-linearly independent.*

Proof. Calculating the outer product of $\mathbf{Q}(t)$ and $\hat{\mathbf{Q}}(t)$ yields

$$\mathbf{Q}(t)\times\hat{\mathbf{Q}}(t)=(\mathbf{p}_3\times\mathbf{P}(t))\times\Big(\frac{\mathbf{p}_0\times\mathbf{P}(t)}{t}\Big)$$
$$=\big(|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|+|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t\big)\mathbf{P}(t). \tag{21}$$

If $\mathbf{Q}(t)\times\hat{\mathbf{Q}}(t)\equiv 0$, $|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|=|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|=0$. But $|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|$ and $|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|$ cannot be zero simultaneously. Otherwise,

$$\mathbf{p}_2=\tau_0\mathbf{p}_0+\tau_1\mathbf{p}_3,\ \ \mathbf{p}_1=\tau_2\mathbf{p}_0+\tau_3\mathbf{p}_3,$$

for some constants $\tau_i, i=0,1,2,3$. Thus $\mathrm{Rank}(M)\le 2<3$ which contradicts condition (9). Therefore,

$$\mathbf{Q}(t)\times\hat{\mathbf{Q}}(t)\not\equiv 0.$$

Equivalently, $\mathbf{Q}(t),\hat{\mathbf{Q}}(t)$ are $\mathbb{R}[t]$-linearly independent. □

$\hat{\mathbf{Q}}(t)$ is a degree two syzygy for $\mathbf{P}(t)$ if $\mathbf{p}_0\times\mathbf{p}_3\neq 0$. But $\hat{\mathbf{Q}}(t)$ may sometimes have a linear factor.

Lemma 4. *Consider a non-degenerate planar rational cubic curve (8). Suppose $\mathbf{p}_0\times\mathbf{p}_3\neq 0$. Let $\hat{\mathbf{Q}}(t)$ be defined as in (20). Then $\hat{\mathbf{Q}}(t)$ can be factored into $\alpha(t)\mathbf{l}(t)$ for some linear polynomial $\alpha(t)$ if and only if*

$$\Delta=|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|^2-|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2|\cdot|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|=0.$$

Moreover, if $\Delta=0$, then $\alpha(t)=\gamma(|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|+|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t)$ for some nonzero constant γ.

Proof. ($\Rightarrow$) If $\hat{\mathbf{Q}}(t)=\alpha(t)\mathbf{l}(t)$ for some degree one polynomial $\alpha(t)$,

$$\hat{\mathbf{Q}}(t_0)=(\mathbf{p}_0\times\mathbf{p}_1)+(\mathbf{p}_0\times\mathbf{p}_2)t_0+(\mathbf{p}_0\times\mathbf{p}_3)t_0^2=0,$$

where t_0 is the root of $\alpha(t)$. Calculating the dot product of $\hat{\mathbf{Q}}(t_0)$ with $\mathbf{p}_3$ and $\mathbf{p}_2$ yields

$$\hat{\mathbf{Q}}(t_0)\cdot\mathbf{p}_3=|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|+|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t_0=0, \tag{22}$$
$$\hat{\mathbf{Q}}(t_0)\cdot\mathbf{p}_2=|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2|-|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t_0^2=0. \tag{23}$$

From (22), we get

$$|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t_0=-|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| \tag{24}$$

Substituting (24) into (23) yields

$$|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2|+|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|t_0=0 \tag{25}$$

Since t_0 satisfies (22) and (25) simultaneously, it follows that $\Delta=0$.

($\Leftarrow$) Suppose $\Delta=0$. Then

$$|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|\neq 0.$$

In other words, $\mathbf{p}_0, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent. Otherwise $|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3| = |\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| = 0$. Thus both $\mathbf{p}_1$ and $\mathbf{p}_2$ can be represented by linear combinations of $\mathbf{p}_0, \mathbf{p}_3$ in the case of $\mathbf{p}_0 \times \mathbf{p}_3 \neq 0$. So Rank$(M) = 2$ which contradicts condition (9).

But if $\Delta = 0$, there is a real parameter t_0 that simultaneously satisfies (22) and (25). Hence the real parameter t_0 also satisfies (23), since (23) can be expressed by a linear combination of (22) and (25) as

$$\hat{\mathbf{Q}}(t_0) \cdot \mathbf{p}_2 = \big(|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2| + |\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|t_0\big) - t_0(|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| + \det(\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3)t_0).$$

Thus there is a real parameter t_0 that satisfies (22) and (23) simultaneously. From (22) and (23), we get $\hat{\mathbf{Q}}(t_0) \perp \mathbf{p}_2, \hat{\mathbf{Q}}(t_0) \perp \mathbf{p}_3$. On the other hand, from the definition of $\hat{\mathbf{Q}}(t)$, $\hat{\mathbf{Q}}(t_0) \perp \mathbf{p}_0$. But $\mathbf{p}_0, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent, so $\hat{\mathbf{Q}}(t_0) = 0$. Thus $\hat{\mathbf{Q}}(t)$ has a linear factor $\gamma_1(t - t_0)$ for some nonzero constant γ_1.

The parameter t_0 also satisfies (22), so

$$|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| + |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t = |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|(t - t_0).$$

Hence

$$\alpha(t) = \gamma(|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| + |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t),$$

where $\gamma = \gamma_1/|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|$. □

4.2.1. Case 2.1: $\mathbf{p}_0 \times \mathbf{p}_3 \neq 0$ *and* $\Delta = 0$

If $\Delta = 0$, then it follows from Lemma 4 that $\hat{\mathbf{Q}}(t)$ can be expressed as

$$\hat{\mathbf{Q}}(t) = (|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| + |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t) \cdot (\mathbf{l}_0 + \mathbf{l}_1 t),$$

where $\mathbf{l}_0, \mathbf{l}_1 \in \mathbb{R}^3$. Recall that $\hat{\mathbf{Q}}(t)$ is defined as

$$\hat{\mathbf{Q}}(t) = (\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t + (\mathbf{p}_0 \times \mathbf{p}_3)t^2.$$

So

$$\mathbf{l}_0 = \frac{\mathbf{p}_0 \times \mathbf{p}_1}{|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|},\ \mathbf{l}_1 = \frac{\mathbf{p}_0 \times \mathbf{p}_3}{|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|}.$$

Thus

$$\mathbf{L}_0(t) = \hat{\mathbf{Q}}(t)/(|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| + |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|t) = \mathbf{l}_0 + \mathbf{l}_1 t \tag{26}$$

is a degree one syzygy for $\mathbf{P}(t)$. Since by Lemma 3, $\mathbf{Q}(t), \hat{\mathbf{Q}}(t)$ are $\mathbb{R}[t]$-linearly independent, it is easy to show that $\mathbf{L}_0(t), \mathbf{Q}(t)$ are also $\mathbb{R}[t]$-linearly independent. Thus in the case of $\Delta = 0$,

$$\mathbf{u}_0(t) = \mathbf{L}_0(t),\quad \mathbf{u}_1(t) = \mathbf{Q}(t)$$

are a μ-basis for $\mathbf{P}(t)$.

Again by (11) and (12), the determinant of the Sylvester matrix

$$\begin{pmatrix} \mathbf{l}_0 \cdot \mathbf{X} & 0 & (\mathbf{p}_3 \times \mathbf{p}_0) \cdot \mathbf{X} \\ \mathbf{l}_1 \cdot \mathbf{X} & \mathbf{l}_0 \cdot \mathbf{X} & (\mathbf{p}_3 \times \mathbf{p}_1) \cdot \mathbf{X} \\ 0 & \mathbf{l}_1 \cdot \mathbf{X} & (\mathbf{p}_3 \times \mathbf{p}_2) \cdot \mathbf{X} \end{pmatrix}, \tag{27}$$

and the determinant of the Bezoutian matrix

$$\begin{pmatrix} \mathbf{X}\big(\mathbf{l}_0^T(\mathbf{p}_3 \times \mathbf{p}_2) - (\mathbf{p}_3 \times \mathbf{p}_1)^T\mathbf{l}_1\big)\mathbf{X}^T & \mathbf{l}_1\mathbf{X}^T \\ -\mathbf{X}\big(\mathbf{l}_1^T(\mathbf{p}_3 \times \mathbf{p}_0)\big)\mathbf{X}^T & \mathbf{l}_0\mathbf{X}^T \end{pmatrix}, \tag{28}$$

both represent the implicit equation of the cubic curve $\mathbf{P}(t)$.

Using (26), it is easy to check that

$$\mathbf{L}_0(t) \cdot \mathbf{p}_0 \equiv 0,$$

so $\mathbf{p}_0$ is the axis of the linear moving line $\mathbf{L}_0(t) \cdot (x, y, w) = 0$. Thus again by Theorem 2, $\mathbf{p}_0$ is the unique double point of the rational cubic curve $\mathbf{P}(t)$.

Example 3. Consider the planar rational cubic curve

$$\mathbf{P}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2 + \mathbf{p}_3 t^3,$$

where

$$\mathbf{p}_0 = (3, 2, -2), \mathbf{p}_1 = (14, 0, -2), \mathbf{p}_2 = (-3, 4, 1), \mathbf{p}_3 = (-2, -2, 0).$$

After a straightforward calculation, we find that

$$|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3| = 52,\ |\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2| = -104,\ |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3| = -26.$$

Hence $\Delta = |\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|^2 - |\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2||\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3| = 0$. Now

$$\mathbf{p}_3 \times \mathbf{p}_0 = (4, -4, 2), \mathbf{p}_3 \times \mathbf{p}_1 = (4, -4, 28), \mathbf{p}_3 \times \mathbf{p}_2 = (-2, 2, -14),$$
$$\mathbf{l}_0 = \frac{\mathbf{p}_0 \times \mathbf{p}_1}{|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|} = \left(-\frac{1}{13}, -\frac{11}{26}, -\frac{7}{13}\right),\ \mathbf{l}_1 = \frac{\mathbf{p}_0 \times \mathbf{p}_3}{|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|} = \left(\frac{2}{13}, -\frac{2}{13}, \frac{1}{13}\right).$$

So in this case, we can get a degree one syzygy

$$\mathbf{L}_0(t) = \mathbf{l}_0 + \mathbf{l}_1 t = \left(-\frac{1}{13} + \frac{2}{13}t, -\frac{11}{26} - \frac{2}{13}t, -\frac{7}{13} + \frac{1}{13}t\right)$$

and a degree two syzygy

$$\begin{aligned}\mathbf{Q}(t) &= \mathbf{p}_3 \times \mathbf{p}_0 + (\mathbf{p}_3 \times \mathbf{p}_1)t + (\mathbf{p}_3 \times \mathbf{p}_2)t^2 \\ &= (4 + 4t - 2t^2, -4 - 4t + 2t^2, 2 + 28t - 14t^2).\end{aligned}$$

It is also easy to check that

$$\mathbf{L}_0(t) \times \mathbf{Q}(t) = -\mathbf{P}(t).$$

Hence $\mathbf{L}_0(t), \mathbf{Q}(t)$ are a μ-basis for $\mathbf{P}(t)$. Clearing the denominator in $\mathbf{L}_0(t)$, we get

$$\tilde{\mathbf{L}}_0(t) = 26\mathbf{L}_0(t) = (4t - 2, -4t - 11, 2t - 14).$$

Hence $\tilde{\mathbf{L}}_0(t), \mathbf{Q}(t)$ are also a μ-basis for $\mathbf{P}(t)$.

By (11) and (12), the determinants of the matrices

$$\begin{pmatrix} -2x - 11y - 14w & 0 & 4x - 4y + 2w \\ 4x - 4y + 2w & -2x - 11y - 14w & 4x - 4y + 28w \\ 0 & 4x - 4y + 2w & -2x + 2y - 14w \end{pmatrix}$$

and

$$\begin{pmatrix} -12x^2 + 50xy - 64xw - 38y^2 + 246yw + 140w^2 & 4x - 4y + 2w \\ -16x^2 + 32xy - 16xw - 16y^2 + 16yw - 4w^2 & -2x - 11y - 14w \end{pmatrix}$$

both represent the implicit equation of the cubic curve $\mathbf{P}(t)$.

In addition, $\mathbf{p}_0 = (3, 2, -2)$ is the unique double point of the rational cubic curve $\mathbf{P}(t)$.

4.2.2. *Case 2.2:* $\mathbf{p}_0 \times \mathbf{p}_3 \neq 0$ *and* $\Delta \neq 0$

If $\Delta \neq 0$, we cannot get a degree one syzygy directly from $\hat{\mathbf{Q}}(t)$ as in (26). Instead set

$$\mathbf{L}_1(t) = (\mathbf{p}_0 \times \mathbf{q}_0) + (\mathbf{p}_3 \times \mathbf{q}_1)t, \tag{29}$$

where

$$\mathbf{q}_0 = \lambda_0\mathbf{p}_1 + \lambda_1\mathbf{p}_2 + \lambda_2\mathbf{p}_3,\ \mathbf{q}_1 = -\lambda_1\mathbf{p}_0 - \lambda_2\mathbf{p}_1 + \lambda_3\mathbf{p}_2,$$
$$\lambda_0 = |\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2|,\ \lambda_1 = |\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|,\ \lambda_2 = -|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|,\ \lambda_3 = -|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2|. \tag{30}$$

Now it is straightforward to check that

$$\mathbf{L}_1(t) \cdot \mathbf{P}(t) \equiv 0.$$

Thus $\mathbf{L}_1(t)$ follows $\mathbf{P}(t)$. Calculating the outer product of $\mathbf{L}_1(t)$ and $\hat{\mathbf{Q}}(t)$, we find that

$$\begin{aligned}&\big((\mathbf{p}_0 \times \mathbf{q}_0) + (\mathbf{p}_3 \times \mathbf{q}_1)t\big) \times \big((\mathbf{p}_0 \times \mathbf{p}_1) + (\mathbf{p}_0 \times \mathbf{p}_2)t + (\mathbf{p}_0 \times \mathbf{p}_3)t^2\big) \\ &= \Delta \cdot \mathbf{P}(t).\end{aligned}$$

Since by assumption $\Delta \neq 0$, it follows from the definition of a μ-basis that

$$\mathbf{u}_0(t) = \mathbf{L}_1(t),\quad \mathbf{u}_1(t) = \hat{\mathbf{Q}}(t) \tag{31}$$

are a μ-basis for $\mathbf{P}(t)$.

Again by (11) and (12), the determinant of the Sylvester matrix

$$\begin{pmatrix} (\mathbf{p}_0\times\mathbf{q}_0)\cdot\mathbf{X} & 0 & (\mathbf{p}_0\times\mathbf{p}_1)\cdot\mathbf{X} \\ (\mathbf{p}_3\times\mathbf{q}_1)\cdot\mathbf{X} & (\mathbf{p}_0\times\mathbf{q}_0)\cdot\mathbf{X} & (\mathbf{p}_0\times\mathbf{p}_2)\cdot\mathbf{X} \\ 0 & (\mathbf{p}_3\times\mathbf{q}_1)\cdot\mathbf{X} & (\mathbf{p}_0\times\mathbf{p}_3)\cdot\mathbf{X} \end{pmatrix} \tag{32}$$

and the determinant of Bezoutian matrix

$$\begin{pmatrix} \mathbf{X}\big((\mathbf{p}_0\times\mathbf{q}_0)^T(\mathbf{p}_0\times\mathbf{p}_3)-(\mathbf{p}_0\times\mathbf{p}_2)^T(\mathbf{p}_3\times\mathbf{q}_1)\big)\mathbf{X}^T & (\mathbf{p}_3\times\mathbf{q}_1)\mathbf{X}^T \\ -\mathbf{X}\big((\mathbf{p}_3\times\mathbf{q}_1)^T(\mathbf{p}_0\times\mathbf{p}_1)\big)\mathbf{X}^T & (\mathbf{p}_0\times\mathbf{q}_0)\mathbf{X}^T \end{pmatrix} \tag{33}$$

both represent the implicit equation of the cubic curve $\mathbf{P}(t)$.

In addition, since $\big((\mathbf{p}_0\times\mathbf{q}_0)\times(\mathbf{p}_3\times\mathbf{q}_1)\big)\cdot\mathbf{L}_1(t)\equiv 0$, it follows that

$$(\mathbf{p}_0\times\mathbf{q}_0)\times(\mathbf{p}_3\times\mathbf{q}_1)$$

is the axis of the linear moving line $\mathbf{L}_1(t)\cdot(x,y,w)=0$. Thus by Theorem 2, $\big((\mathbf{p}_0\times\mathbf{q}_0)\times(\mathbf{p}_3\times\mathbf{q}_1)\big)$ is the unique double point of the rational cubic curve $\mathbf{P}(t)$.

Example 4. Consider the planar rational cubic curve

$$\mathbf{P}(t)=\mathbf{p}_0+\mathbf{p}_1t+\mathbf{p}_2t^2+\mathbf{p}_3t^3,$$

where $\mathbf{p}_0=(3,2,-2)$, $\mathbf{p}_1=(0,2,1)$, $\mathbf{p}_2=(-3,4,1)$, $\mathbf{p}_3=(-2,-2,0)$. After a straightforward calculation, we get

$$\lambda_0=|\mathbf{p}_1\mathbf{p}_3\mathbf{p}_2|=-10,\ \lambda_1=|\mathbf{p}_0\mathbf{p}_2\mathbf{p}_3|=-26,$$
$$\lambda_2=-|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_3|=6,\ \ \lambda_3=-|\mathbf{p}_0\mathbf{p}_1\mathbf{p}_2|=24,$$

so $\Delta=-588\neq 0$. Hence

$$\begin{aligned} &\mathbf{q}_0=\lambda_0\mathbf{p}_1+\lambda_1\mathbf{p}_2+\lambda_2\mathbf{p}_3=(66,-136,-36),\\ &\mathbf{q}_1=-\lambda_1\mathbf{p}_0-\lambda_2\mathbf{p}_1+\lambda_3\mathbf{p}_2=(6,136,-34),\\ &\mathbf{p}_0\times\mathbf{q}_0=(-344,-24,-540),\ \ \mathbf{p}_3\times\mathbf{q}_1=(68,-68,-260),\\ &\mathbf{p}_0\times\mathbf{p}_1=(6,-3,6),\ \ \mathbf{p}_0\times\mathbf{p}_2=(10,3,18),\ \ \mathbf{p}_0\times\mathbf{p}_3=(-4,4,-2). \end{aligned}$$

Thus by (32) and (33), the determinants of the matrices

$$\begin{pmatrix} -344x-24y-540w & 0 & 6x-3y+6w \\ 68x-68y-260w & -344x-24y-540w & 10x+3y+18w \\ 0 & 68x-68y-260w & -4x+4y-2w \end{pmatrix}$$

and

$$\begin{pmatrix} 696x^2-804xy+4224xw+108y^2-108yw+5760w^2 & 68x-68y-260w \\ -408x^2+612xy+1152xw-204y^2-372yw+1560w^2 & -344x-24y-540w \end{pmatrix}$$

both represent the implicit equation of the cubic curve $\mathbf{P}(t)$.

In addition by (31),

$$\begin{aligned} \mathbf{L}_1(t)&=(\mathbf{p}_0\times\mathbf{q}_0)+(\mathbf{p}_3\times\mathbf{q}_1)t=(-344+68t,-24-68t,-540-260t)\\ \hat{\mathbf{Q}}(t)&=(\mathbf{p}_0\times\mathbf{p}_1)+(\mathbf{p}_0\times\mathbf{p}_2)t+(\mathbf{p}_0\times\mathbf{p}_3)t^2\\ &=(6+10t-4t^2,-3+3t+4t^2,6+18t-2t^2) \end{aligned}$$

are a μ-basis for $\mathbf{P}(t)$. Computing the outer product of $\mathbf{L}_1(t)$ and $\hat{\mathbf{Q}}(t)$ yields

$$\mathbf{L}_1(t)\times\hat{\mathbf{Q}}(t)=\Delta\cdot\mathbf{P}(t)=-588\mathbf{P}(t).$$

Finally, $(\mathbf{p}_0\times\mathbf{q}_0)\times(\mathbf{p}_3\times\mathbf{q}_1)=16(-1905,-7885,1564)$. So the point $(-1905,-7885,1564)$ is the unique double point of the rational cubic curve $\mathbf{P}(t)$.

Remark 3. When the planar rational cubic curve is expressed in terms of the Bernstein basis

$$\mathbf{P}(t)=\sum_{i=0}^{3}\mathbf{p}_iB_i^3(t)=\sum_{i=0}^{3}\mathbf{p}_i\frac{3!}{i!(3-i)!}(1-t)^{3-i}t^i,$$

let $\hat{\mathbf{p}}_i = \frac{3!}{i!(3-i)!}\mathbf{p}_i$, $i = 0, \ldots, 3$ and set

$$\mathbf{Q}(t) = \frac{\mathbf{p}_3 \times \mathbf{P}(t)}{1-t} = (\mathbf{p}_3 \times \mathbf{p}_0)(1-t)^2 + (\mathbf{p}_3 \times \mathbf{p}_1)(1-t)t + (\mathbf{p}_3 \times \mathbf{p}_2)t^2,$$

$$\hat{\mathbf{Q}}(t) = \frac{\mathbf{p}_0 \times \mathbf{P}(t)}{t} = (\mathbf{p}_0 \times \mathbf{p}_1)(1-t)^2 + (\mathbf{p}_0 \times \mathbf{p}_2)(1-t)t + (\mathbf{p}_0 \times \mathbf{p}_3)t^2.$$

We can prove in a similar fashion that

- If $\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_3 = 0$, then

  $$(\hat{\mathbf{p}}_3 \times \hat{\mathbf{p}}_1)(1-t) + (\hat{\mathbf{p}}_3 \times \hat{\mathbf{p}}_2)t,\ (\hat{\mathbf{p}}_0(1-t) + \hat{\mathbf{p}}_1 t) \times (\hat{\mathbf{p}}_2(1-t) + \hat{\mathbf{p}}_3 t)$$

  are a μ-basis for the planar rational cubic curve $\mathbf{P}(t)$. Moreover, $\mathbf{p}_3$ is the unique double point of $\mathbf{P}(t)$.
- If $\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_3 \neq 0$ and $\hat{\Delta} = |\hat{\mathbf{p}}_0\hat{\mathbf{p}}_1\hat{\mathbf{p}}_3|^2 - |\hat{\mathbf{p}}_0\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2| \cdot |\hat{\mathbf{p}}_0\hat{\mathbf{p}}_2\hat{\mathbf{p}}_3| = 0$, then

  $$\mathbf{L}_0(t) = \hat{\mathbf{Q}}(t)/(|\hat{\mathbf{p}}_0\hat{\mathbf{p}}_1\hat{\mathbf{p}}_3|(1-t) + |\hat{\mathbf{p}}_0\hat{\mathbf{p}}_2\hat{\mathbf{p}}_3|t), \mathbf{Q}(t)$$

  are a μ-basis for the planar rational cubic curve $\mathbf{P}(t)$. Moreover, $\mathbf{p}_0$ is the unique double point of $\mathbf{P}(t)$.
- If $\hat{\mathbf{p}}_0 \times \hat{\mathbf{p}}_3 \neq 0$ and $\hat{\Delta} \neq 0$, then

  $$\mathbf{L}_1(t) = (\hat{\mathbf{p}}_0 \times \hat{\mathbf{q}}_0)(1-t) + (\hat{\mathbf{p}}_3 \times \hat{\mathbf{q}}_1)t, \hat{\mathbf{Q}}(t)$$

  are a μ-basis for the planar rational cubic curve $\mathbf{P}(t)$, where

  $$\hat{\mathbf{q}}_0 = \lambda_0\hat{\mathbf{p}}_1 + \lambda_1\hat{\mathbf{p}}_2 + \lambda_2\hat{\mathbf{p}}_3,\ \hat{\mathbf{q}}_1 = -\lambda_1\hat{\mathbf{p}}_0 - \lambda_2\hat{\mathbf{p}}_1 + \lambda_3\hat{\mathbf{p}}_2,$$

  $$\lambda_0 = |\hat{\mathbf{p}}_1\hat{\mathbf{p}}_3\hat{\mathbf{p}}_2|,\ \lambda_1 = |\hat{\mathbf{p}}_0\hat{\mathbf{p}}_2\hat{\mathbf{p}}_3|,\ \lambda_2 = -|\hat{\mathbf{p}}_0\hat{\mathbf{p}}_1\hat{\mathbf{p}}_3|,\ \lambda_3 = -|\hat{\mathbf{p}}_0\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2|.$$

  Moreover, $(\hat{\mathbf{p}}_0 \times \hat{\mathbf{q}}_0) \times (\hat{\mathbf{p}}_3 \times \hat{\mathbf{q}}_1)$ is the unique double point of $\mathbf{P}(t)$.

5. μ-bases for rational space curves

In this section, we shall generalize our result about μ-bases for conics in 2-dimensions to degree n ($n \geq 3$) rational curves in n-dimensions by constructing explicit μ-bases for degree n rational curves in n-dimensions (e.g., rational cubic curves in 3-dimensions).

5.1. Definition of μ-bases for rational space curves

Since in Section 2 we gave the definition of μ-bases only for planar rational curves, we need to recall here the concept of μ-bases for rational space curves. Consider a real rational space curve

$$\mathbf{F}(t) = \big(f_0(t), f_1(t), \ldots, f_n(t)\big), (n \geq 3),$$

in homogeneous form, where $\gcd\big(f_0(t), f_1(t), \ldots, f_n(t)\big) = 1$. The curve $\mathbf{F}(t)$ is a rational curve in n-dimensions. A *syzygy* of $\mathbf{F}(t)$ is a vector of polynomials $\mathbf{L}(t) = \big(l_0(t), l_1(t), \ldots, l_n(t)\big)$ such that $\mathbf{L}(t) \cdot \mathbf{F}(t) \equiv 0$. The set of all syzygies of $\mathbf{F}(t)$ is once again a free module over $\mathbb{R}[t]$ with a basis consisting of n generators (Cox et al., 1998). There are also several equivalent definitions for a μ-basis of a rational space curve $\mathbf{F}(t)$ (Song and Goldman, 2009). One simple definition of μ-bases for rational space curves is as follows.

Definition 5. (See Song and Goldman, 2009.) Consider a rational curve $\mathbf{F}(t)$ in n-dimensions. A set of n syzygies $\mathbf{u}_0(t), \ldots, \mathbf{u}_{n-1}(t)$ of $\mathbf{F}(t)$ is a μ-basis for $\mathbf{F}(t)$ if

1) $\deg(\mathbf{u}_0(t)) + \cdots + \deg(\mathbf{u}_{n-1}(t)) = \deg(\mathbf{F}(t))$,
2) $[\mathbf{u}_0(t), \ldots, \mathbf{u}_{n-1}(t)] = \lambda\mathbf{F}(t)$ for some nonzero constant λ.

Here the outer product $[\mathbf{u}_0(t), \ldots, \mathbf{u}_{n-1}(t)]$ of n vectors $\mathbf{u}_0(t), \ldots, \mathbf{u}_{n-1}(t)$ is the $(n+1)$-tuple defined by

$$[\mathbf{u}_0(t), \ldots, \mathbf{u}_{n-1}(t)] = \det\begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ u_{0,0} & u_{0,1} & \cdots & u_{0,n} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n-1,0} & u_{n-1,1} & \cdots & u_{n-1,n} \end{pmatrix},$$

where $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ are the unit vectors along the coordinate axes, and $u_{i,j}$ denotes the j-th element of $\mathbf{u}_i(t)$.

Every rational space curve has a μ-basis. Moreover, the method to compute μ-bases for planar rational curves, which is based on Gaussian Elimination (Chen and Wang, 2003), has also been generalized to compute μ-bases for rational space curves in n-dimensions (Song and Goldman, 2009).

5.2. μ-bases for degree n rational space curves in n-dimensional space

Next we shall show how to find μ-bases for degree n rational space curves in n-dimensions.

Consider a non-degenerate degree n rational curve in n-dimensions

$$\mathbf{F}(t) = \big(f_0(t), f_1(t), \dots, f_n(t)\big), \ (n \geq 3),$$

in homogeneous form. The curve $\mathbf{F}(t)$ can be expressed in terms of the power basis

$$\mathbf{F}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \dots + \mathbf{p}_n t^n,$$

where $\mathbf{p}_i \in \mathbb{R}^{n+1}$. Let

$$\tilde{M} = (\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n), \tag{34}$$

be the $(n+1) \times (n+1)$ matrix whose columns are the $n+1$ vectors $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n$. For $\mathbf{F}(t)$ to be a non-degenerate rational space curve we must have $\text{Rank}(\tilde{M}) = n+1$. That is,

$$\det(\tilde{M}) \neq 0. \tag{35}$$

Otherwise if $\det(\tilde{M}) = 0$, there is a constant vector $\Gamma \neq 0$ such that $\Gamma \cdot \mathbf{p}_i = 0, \ i = 0, \dots, n$. Hence $\Gamma \cdot \mathbf{F}(t) \equiv 0$. Thus the rational space curve $\mathbf{F}(t)$ will degenerate into a rational curve that lies in the $(n-1)$-dimensional hyperplane perpendicular to the vector Γ.

Notice that $\mathbf{F}(t) = (f_0(t), f_1(t), \dots, f_n(t))$ can be rewritten as

$$(f_0(t), f_1(t), \dots, f_n(t)) = (1, t, t^2, \dots, t^n)\tilde{M}^T,$$

where $\tilde{M}^T$ is the transpose of the matrix $\tilde{M}$ (34). Moreover the $(n+1) \times n$ matrix

$$R = \begin{pmatrix} t & 0 & \cdots & 0 \\ -1 & t & \cdots & 0 \\ 0 & -1 & & \vdots \\ \vdots & \vdots & & t \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

satisfies

$$(1, t, t^2, \dots, t^n)R = 0.$$

Thus the columns of the matrix R are orthogonal to $(1, t, \dots, t^n)$ and form a μ-basis for the normal curve $(1, t, \dots, t^n)$. Similarly, the columns of the matrix $(\tilde{M}^T)^{-1}R$ are orthogonal to $\mathbf{F}(t)$ and form a μ-basis for the curve $\mathbf{F}(t)$. There are several efficient methods to compute the inverse of the matrix $\tilde{M}^T$ in practice, e.g., solving the linear system corresponding to $\tilde{M}^T$.

Our goal here, however, is to find an explicit formula for a μ-basis of the curve $\mathbf{F}(t)$. To find such an explicit formula, we can simply compute the adjoint matrix of $\tilde{M}^T$. It is easy to verify that the adjoint matrix of $\tilde{M}^T$ is

$$(\tilde{M}^T)^* = \big(\mathbf{c}_0, -\mathbf{c}_1, \dots, (-1)^i\mathbf{c}_i, \dots, (-1)^n\mathbf{c}_n\big),$$

where

$$\mathbf{c}_i = [\mathbf{p}_0, \dots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \mathbf{p}_{i+2}, \dots, \mathbf{p}_n], \ i = 0, \dots, n.$$

Hence

$$(\tilde{M}^T)^* R = (\mathbf{c}_1 + \mathbf{c}_0 t, \mathbf{c}_2 + \mathbf{c}_1 t, \dots, \mathbf{c}_n + \mathbf{c}_{n-1} t).$$

Therefore,

$$\mathbf{c}_1 + \mathbf{c}_0 t, \mathbf{c}_2 + \mathbf{c}_1 t, \dots, \mathbf{c}_n + \mathbf{c}_{n-1} t \tag{36}$$

is an explicit μ-basis for the curve $\mathbf{F}(t)$.

Alternatively, we can also reconstruct an explicit formula for a μ-basis of the rational space curve $\mathbf{F}(t)$ as we did for conic curves in Section 3. Set

$$\mathbf{L}_i(t) = \frac{[\mathbf{p}_0, \dots, \mathbf{p}_{i-1}, \mathbf{F}(t), \mathbf{p}_{i+2}, \dots, \mathbf{p}_n]}{t^i}, \ i = 0, \dots, n-1.$$

$\mathbf{L}_i(t)$ follows $\mathbf{F}(t)$ since

$$[\mathbf{p}_0, \dots, \mathbf{p}_{i-1}, \mathbf{F}(t), \mathbf{p}_{i+2}, \dots, \mathbf{p}_n] \perp \mathbf{F}(t).$$

After a simple computation, it is easy to show that $\mathbf{L}_i(t)$ can also be expressed by

$$\mathbf{L}_i(t) = \mathbf{c}_{i+1} + \mathbf{c}_i t,\ i = 0, \ldots, n-1.$$

Obviously, $\mathbf{L}_0(t), \ldots, \mathbf{L}_{n-1}(t)$ are equal to (36).

Example 5. Consider the rational cubic space curve $\mathbf{F}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2 + \mathbf{p}_3 t^3$, where

$$\mathbf{p}_0 = (1, 3, 3, -2), \mathbf{p}_1 = (1, 2, -2, -1), \mathbf{p}_2 = (0, 0, 2, 1), \mathbf{p}_3 = (-1, -3, 1, -2).$$

$\mathbf{F}(t)$ is a rational curve in 3-dimensions. After a straightforward calculation, we find that

$$\det(\tilde{M}) = \det\left((\mathbf{p}_0^T, \mathbf{p}_1^T, \mathbf{p}_2^T, \mathbf{p}_3^T)\right) = 12.$$

Set

$$\mathbf{L}_0(t) = [\mathbf{F}(t), \mathbf{p}_2, \mathbf{p}_3] = (-36 - 10t, 12 + 5t, t, -2t),$$
$$\mathbf{L}_1(t) = \frac{[\mathbf{p}_0, \mathbf{F}(t), \mathbf{p}_3]}{t} = (44 - 36t, -16 + 12t, 4, 4),$$
$$\mathbf{L}_2(t) = \frac{[\mathbf{p}_0, \mathbf{p}_1, \mathbf{F}(t)]}{t^2} = (-14 + 44t, 7 - 16t, -1 + 4t, 2 + 4t).$$

By construction, $\mathbf{L}_i(t) \cdot \mathbf{F}(t) \equiv 0, i = 0, 1, 2$. On the other hand,

$$[\mathbf{L}_0(t), \mathbf{L}_1(t), \mathbf{L}_2(t)] = \det(\tilde{M})^2 \mathbf{F}(t) = 144\mathbf{F}(t).$$

Thus $\mathbf{L}_0(t), \mathbf{L}_1(t), \mathbf{L}_2(t)$ are indeed a μ-basis for the rational cubic space curve $\mathbf{F}(t)$.

Remark 4. Conics discussed in Section 3 are a special case of degree n rational curves in n-dimensions where $n = 2$.

Remark 5. When a degree n rational curve in n-dimensions ($n \geq 2$) is expressed in terms of the Bernstein basis

$$\mathbf{F}(t) = \sum_{i=0}^{n} \mathbf{p}_i B_i^n(t) = \sum_{i=0}^{n} \mathbf{p}_i \frac{n!}{i!(n-i)!}(1-t)^{n-i} t^i,$$

where $\mathbf{p}_i \in \mathbb{R}^{n+1}$. Set $\hat{\mathbf{p}}_i = \frac{n!}{i!(n-i)!}\mathbf{p}_i, i = 0, \ldots, n$, and let

$$\mathbf{c}_i = [\hat{\mathbf{p}}_0, \ldots, \hat{\mathbf{p}}_{i-1}, \hat{\mathbf{p}}_{i+1}, \ldots, \hat{\mathbf{p}}_n],\ i = 0, \ldots, n.$$

Then once again we can prove that

$$\mathbf{L}_i(t) = \mathbf{c}_{i+1}(1-t) + \mathbf{c}_i t,\ i = 0, \ldots, n-1,$$

are a μ-basis for $\mathbf{F}(t)$.

6. Conclusion

We have constructed explicit formulas for μ-bases of planar rational cubic curves and degree n rational curves in n-dimensions ($n \geq 2$). But the formulas for the μ-bases of planar rational cubic curves are not so concise as the formulas for the μ-bases of conic sections. We need to treat several different cases of the planar rational cubic curve. So we can imagine that it will be harder and harder to write down explicit formulas for the μ-bases of rational planar curves of arbitrary degree: there are more and more cases and many more conditions to consider.

We also generalized our results to these rational curves expressed in terms of the Bernstein basis, so we can also write down their μ-bases directly without messy conversions between the power basis and the Bernstein basis.

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