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Uniform approximation of a circle by a parametric polynomial curve ☆

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ABSTRACT

In the paper two new approaches for construction of parametric polynomial approximants of a unit circle are presented. The obtained approximants have better approximation properties than those given by other methods, i.e., smaller radial error, symmetry, and exponential error decay.

1. Introduction

A circle arc is a basic object in CAGD and in many applications. Conics are the oldest curves, and are used in architecture, robotics, and in many other fields. The unit circle has a nice parameterization

$$\boldsymbol{c}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi).$$

In CAGD, parametric polynomial and rational curves and splines are fundamental objects. A circle arc does not have a parametric polynomial representation, however, it can be represented by using a quadratic rational Bézier curve. The whole circle can thus be represented by a quadratic rational spline, e.g. as a NURBS. A natural question is, whether it is possible to obtain a good parametric polynomial approximation of a circular arc.

A lot of papers study good approximation of circular segments with the radial error as the parametric distance. Quadratic Bézier approximants are considered in Mørken (1991), and their generalizations to the cubic case can be found in Dokken et al. (1990) and Goldapp (1991). The quartic case is systematically studied in Ahn and Kim (1997), Kim and Ahn (2007) and Hur and Kim (2011), and quintic Bézier approximants are derived in Fang (1998, 1999). Recently, quartic G^1 approximants were analyzed in Kovač and Žagar (2014).

General results on Hermite type approximation of conic sections by parametric polynomial curves of odd degree are given in Floater (1995, 1997). The results hold true only asymptotically, i.e., for small segments of a particular conic section. Hermite approximation of ellipse segments by cubic parametric Bézier curves is studied in Dokken (2003) and also in Dokken (1997).

In recent years some quite surprisingly good approximations of the whole circle were obtained. An approach based on Taylor approximation was improved by idea of geometric interpolation and a construction of polynomial approximants for

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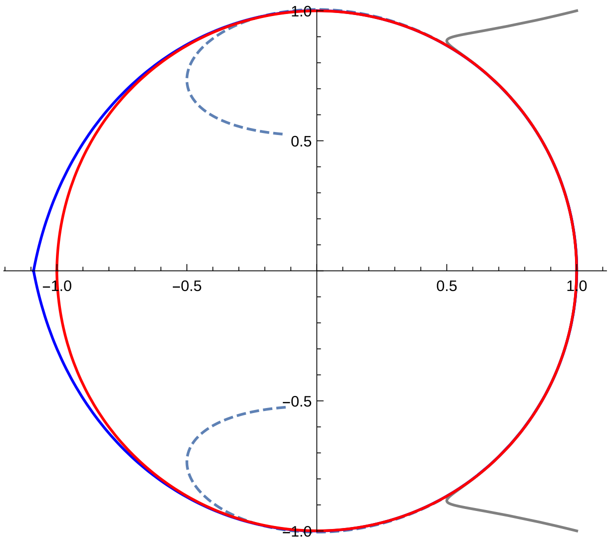

Fig. 1. Quintic parametric polynomial approximant from Lyche and Mørken (1994) (gray), parametric approximant of the same degree given in Jaklič et al. (2013) (blue), quintic Taylor approximant (blue dashed), and the new quintic approximant (red). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

all odd degrees was obtained in Lyche and Mørken (1994). The construction that covered also even degrees was presented in Jaklič et al. (2007a). By looking at the problem from a different perspective, it turned out that the obtained construction was just one of several solutions of a nonlinear problem, and that there exist better solutions. In Jaklič et al. (2013), the best such solution was presented, which gives a good approximation of a conic section. It has many nice properties, it is symmetric, shape preserving, it gives a high order approximation of the whole circle $\boldsymbol{c}$, and for higher degrees it circles the circle several times before it deviates from $\boldsymbol{c}$. Furthermore, it is given in a closed form.

In this paper, a novel parametric polynomial approximant for the whole circle is presented, that gives a better approximation. It has many desired properties, such as symmetry and high order approximation. Another approach is presented that yields even (slightly) better results. Since solving of nonlinear equations and systems are involved, unfortunately the solutions could not be given exactly.

As a motivation, consider Fig. 1. Here a parametric quintic polynomial approximant of the unit circle, given in Lyche and Mørken (1994), Jaklič et al. (2007a) as

$$\begin{pmatrix} 1-2t^2+2t^4 \\ 2t-2t^3+t^5 \end{pmatrix}$$

is shown together with the quintic approximant from Jaklič et al. (2013) (with radial error 0.08999)

$$\begin{pmatrix} 1-(3+\sqrt{5})t^2+(1+\sqrt{5})t^4 \\ (1+\sqrt{5})t-(3+\sqrt{5})t^3+t^5 \end{pmatrix},$$

and the quintic Taylor approximant, together with the new quintic approximant

$$\begin{pmatrix} 0.99947004-3.87624490\,t^2+1.87807588\,t^4 \\ 2.79286220\,t-3.43427162\,t^3+0.69240320\,t^5 \end{pmatrix},$$

which is visually indistinguishable from the unit circle (the radial error is 0.00052). Note that also the new quartic approximant gives a better result than the quintic approximant from Jaklič et al. (2013) with radial error 0.0109 (see Fig. 2).

Our goal is to construct parametric polynomials x_n and y_n of degree $\le n$, which yield a good approximation of the whole unit circle

$$\cos^2(t)+\sin^2(t)=1.$$

Let us consider the expression

$$x_n^2(t)+y_n^2(t)=1+\epsilon(t). \tag{1}$$

Here, ϵ is a polynomial of degree $\le 2n$. Since the circle does not have a parametric polynomial parameterization, every polynomial approximation (x_n, y_n) yields a deviation ϵ from the unit circle with the implicit equation

$$x^2+y^2=1.$$

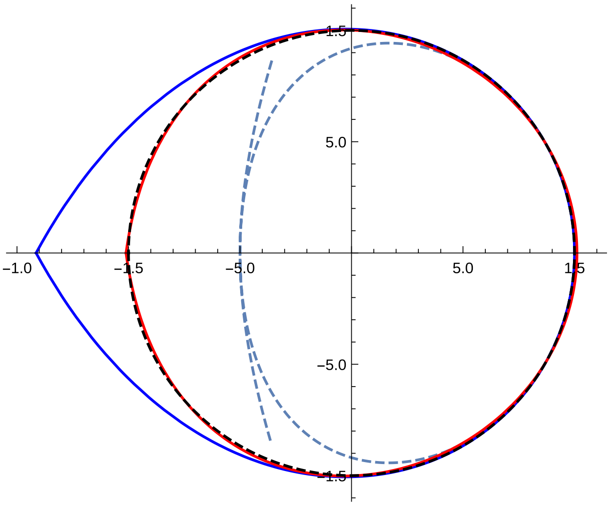

Fig. 2. Quartic parametric polynomial approximant from Jaklič et al. (2013) (blue), quartic Taylor approximant (blue dashed), and the new quartic approximant (red). The circle is drawn black-dashed. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

So, in some sense, ϵ should be small to obtain a good approximation. Let us be more precise. We will consider the radial error

$$e(t) := \sqrt{x_n^2(t) + y_n^2(t)} - 1$$

on some interval $[-t^*, t^*]$. There are several problems:

- the choice of ϵ,
- how to obtain x_n and y_n from the equation (1),
- how to compute an appropriate t^*.

Furthermore, the solution should be symmetric, have a nice shape resembling the circle, and the error should be as small as possible. In Jaklič et al. (2013), the choice $\epsilon(t) = t^{2n}$ was analyzed and a nice approximation in a closed form was obtained, which enables precise error analysis and applications. Its disadvantage is, that it is a one-sided approximation of the circle, and has only one (multiple) contact with the circle at $t = 0$, and is thus Taylor-like in its nature.

There are some more natural choices. An idea right at hand would be to take x_n and y_n as Taylor expansion of cos and sin up to the degree n. This yields a good local approximation near 0, but the approximation quickly deviates from the circle (see Fig. 2 and Jaklič et al., 2007b).

A quite similar expression to (1) is connected to Pell's equation

$$T_n^2(t) - (t^2 - 1)U_{n-1}^2(t) = 1,$$

where T_n and U_n are Chebyshev polynomials of the first and second kind. Thus

$$T_n^2(t) + U_{n-1}^2(t) = 1 + t^2 U_{n-1}^2(t).$$

Unfortunately this solution does not give a good approximation of the circle (see Fig. 3).

One of promising ways is to use the equation of an ellipse in the polar form

$$r(t,a) = \frac{1}{1 + a\cos t}$$

and take $\epsilon(t)$ as Taylor expansion of $\sqrt{r(t,a)} - 1$ up to degree $2n$. The results of such a choice are good, but another choice will give even better results.

In this paper, we will tackle the case $\epsilon(t) = a\,T_{2n}(t)$, where $0 < a < 1$ is an unknown parameter. Best solutions for $n \le 9$ are obtained. They can be given in a closed form, but computation of the optimal parameter a requires solving a nonlinear equation. It will be shown, that the solutions can be improved. A way how to choose an optimal polynomial ϵ will be shown, and thus slightly better approximations for small n will be presented.

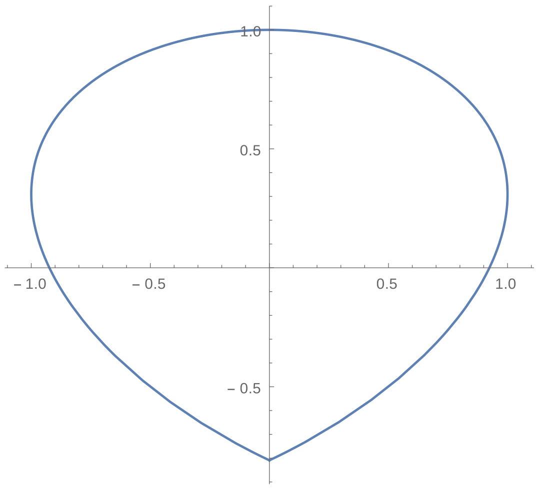

Fig. 3. Parametric polynomial approximant given by the solution of Pell's equation for $n = 5$.

2. Solutions

Firstly, let us recall the main idea for solving the equation (1) in Jaklič et al. (2013). Solving the equation (1) is equivalent to solving

$$x_n^2(t) + y_n^2(t) = 1$$

in the factorial ring $\mathbb{R}[t]/\epsilon(t)$. But since there are additional restrictions on degrees, the problem cannot be tackled by classical algebraic tools.

But the special form of the equation enables an approach that straightforwardly yields all the solutions.

The equation

$$x_n^2(t) + y_n^2(t) = 1 + t^{2n}$$

can be rewritten as

$$(x_n(t) + \mathrm{i}\, y_n(t))\,(x_n(t) - \mathrm{i}\, y_n(t)) = \prod_{k=0}^{2n-1} \left(t - e^{\mathrm{i}\,\frac{2k+1}{2n}\pi}\right), \quad e^{\mathrm{i}\,\varphi} := \cos\varphi + \mathrm{i}\sin\varphi, \tag{2}$$

where the right-hand side is the factorization of $1 + t^{2n}$ over $\mathbb{C}$. From the uniqueness of the polynomial factorization over $\mathbb{C}$ up to a constant factor, and from the fact that the factors in (2) appear in conjugate pairs, it follows that

$$x_n(t) + \mathrm{i}\, y_n(t) = \gamma \prod_{k=0}^{n-1} \left(t - e^{\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi}\right), \quad \gamma \in \mathbb{C}, \quad |\gamma| = 1,$$

where $\sigma_k = \pm 1$. In order to interpolate the point $(1, 0)$, γ must be chosen as

$$\gamma := (-1)^n \prod_{k=0}^{n-1} e^{-\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi},$$

which implies

$$x_n(t) + \mathrm{i}\, y_n(t) = (-1)^n \prod_{k=0}^{n-1} \left(t\, e^{-\mathrm{i}\,\sigma_k \frac{2k+1}{2n}\pi} - 1\right) =: p_e(t; \boldsymbol{\sigma}),$$

with $\boldsymbol{\sigma} = (\sigma_k)_{k=0}^{n-1} \in \{-1, 1\}^n$.

The best solution is (see Jaklič et al., 2013, Thm. 1)

$$x_n(t) = \mathrm{Re}\left(p_e\left(t; \boldsymbol{\sigma}^*\right)\right), \quad y_n(t) = \mathrm{Im}\left(p_e\left(t; \boldsymbol{\sigma}^*\right)\right), \quad \boldsymbol{\sigma}^* = (1)_{k=0}^{n-1}.$$

The polynomial x_n is an even function and y_n is an odd one. Their coefficients can be given in a closed form and possess a particular symmetry (see Jaklič et al., 2013). It was shown in Jaklič et al. (2013) that the radial error decreases exponentially with n.

3. Better approximant

Our goal is to consider the equation

$$x_n^2(t) + y_n^2(t) = 1 + a\,T_{2n}(t),$$

where T_n is the Chebyshev polynomial of the first kind. A motivation for such a choice is straightforward. Among polynomials of degree $2n$ with leading coefficient 1, the polynomial

$$\frac{1}{2^{2n-1}}\,T_{2n}(t)$$

has minimal infinity norm

$$\frac{1}{2^{2n-1}}$$

on the interval $[-1, 1]$, and it is reached exactly $2n+1$ times at points $\cos\frac{k\pi}{n}$, $k = 0, 1, \ldots, 2n$. This gives $2n$ points (zeros of T_{2n}), where a parametric polynomial approximation (x_n, y_n) would intersect the unit circle.

Solving the equation

$$1 + a\,T_{2n}(t) = 0$$

is equivalent to finding values of the Chebyshev polynomial

$$T_{2n}(t) = -\frac{1}{a} < -1.$$

Thus

$$\cosh(2n\ \mathrm{arcCosh}(-t)) = -\frac{1}{a}.$$

Since

$$\mathrm{arcCosh}\, z = \log\left(z + \sqrt{z+1}\,\sqrt{z-1}\right) + 2k\pi\,\mathrm{i},$$

$$2n\ \mathrm{arcCosh}(-t) = \log\left(-\frac{1}{a} + \sqrt{-\frac{1}{a}+1}\,\sqrt{-\frac{1}{a}-1}\right) + 2k\pi\,\mathrm{i}\,.$$

Hence

$$\mathrm{sol}(k) := t = -\cosh\left(\frac{1}{2n}\log\left(-\frac{1}{a} + \sqrt{\frac{1}{a^2}-1}\right) + \frac{k\pi i}{n}\right), k = 0, \ldots, 2n-1. \tag{3}$$

Following the approach from (2),

$$1 + a\,T_{2n}(t) = a\,2^{2n-1}\prod_{k=0}^{2n-1}(t - \mathrm{sol}(k)),$$

where $\mathrm{sol}(k)$ are roots of the polynomial $1 + a\,T_{2n}(t)$, obtained in (3).

Recall

$$z\,\bar{z} = (x_n + \mathrm{i}\,y_n)(x_n - \mathrm{i}\,y_n) = x_n^2 + y_n^2.$$

From the uniqueness of the polynomial factorization over $\mathbb{C}$ up to a constant factor, and from the fact that the factors appear in conjugate pairs, it follows that each such choice of subsets of $\{0, 1, \ldots, 2n-1\}$ of size n, yields a solution of the problem, i.e., a parametric polynomial approximant of the unit circle. Thus $\binom{2n}{n}$ solutions are obtained. The choice

$$p(t) = \mathrm{i}^n\sqrt{a\,2^{2n-1}}\prod_{k=0}^{n-1}(t - \mathrm{sol}(k)),$$

$$x_n(t) = \mathrm{Re}(p(t)),$$

$$y_n(t) = \mathrm{Im}(p(t)),$$

yields the best approximation of the unit circle for a given parameter a.

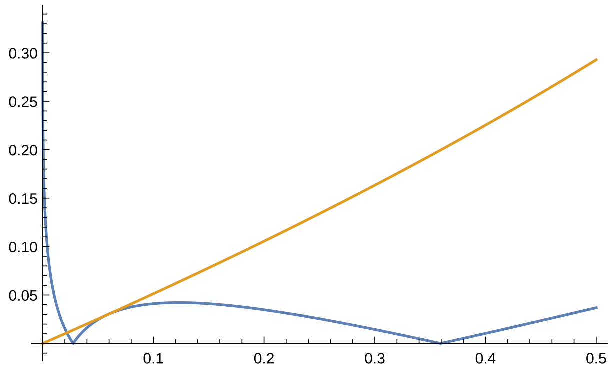

Fig. 4. Error functions in (6) for $n = 4$.

The radial error is

$$e(t) = \sqrt{x_n^2(t) + y_n^2(t)} - 1 = \sqrt{1 + a\,T_{2n}(t)} - 1. \tag{4}$$

We would like to obtain good approximant of the whole circle, thus we need to consider the following:

- the error will be small for small $a > 0$,
- the largest error will be obtained at extrema of Chebyshev polynomials and at boundary values $t = 0$ and t^*, where $y_n(t^*) = 0$,
- we would like the solution to resemble the circle, be symmetric, and give the best possible approximation of such kind.

Derivation of the radial error (4) yields

$$e'(t) = \frac{a\,n\,U_{2n-1}(t)}{\sqrt{1 + a\,T_{2n}(t)}},$$

thus extremal values are obtained at $t_k = \cos\frac{k\pi}{2n}$, $k = 1, 2, \ldots, 2n-1$ and

$$e(t_k) = \sqrt{1 + (-1)^k a} - 1, \tag{5}$$

since $U_{2n-1}(t_k) = 0$. Clearly they are of alternating signs and $\sqrt{1-a} - 1$ is greater by absolute value than the alternative.

We are interested in the solution on the interval $[-t^*, t^*]$, where $t^* > 0$ is the smallest value, such that $y_n(t^*) = 0$. Hence

$$e(t^*) = |x_n(t^*)| - 1.$$

Note that

$$e(0) = \sqrt{1 + a\,T_{2n}(0)} - 1 = \sqrt{1 + (-1)^n a} - 1$$

is equal to one of the extremal values (5). Thus we need to minimize the expression

$$|\sqrt{1 + a\,T_{2n}(t^*)} - 1|,$$

where t^* depends on a. Since also the boundary value needs to be considered, one has to find the smallest positive solution of the equation

$$|\sqrt{1 + a\,T_{2n}(t^*)} - 1| = 1 - \sqrt{1-a}.$$

This yields the final equation

$$||x_n(t^*)| - 1| = 1 - \sqrt{1-a}. \tag{6}$$

In Fig. 4, the functions on the left- and right-hand side of (6) are shown for $n = 4$. The smallest positive intersection yields the parameter a for the best polynomial approximant. This requires solving a nonlinear equation, and high precision arithmetic is needed for larger n.

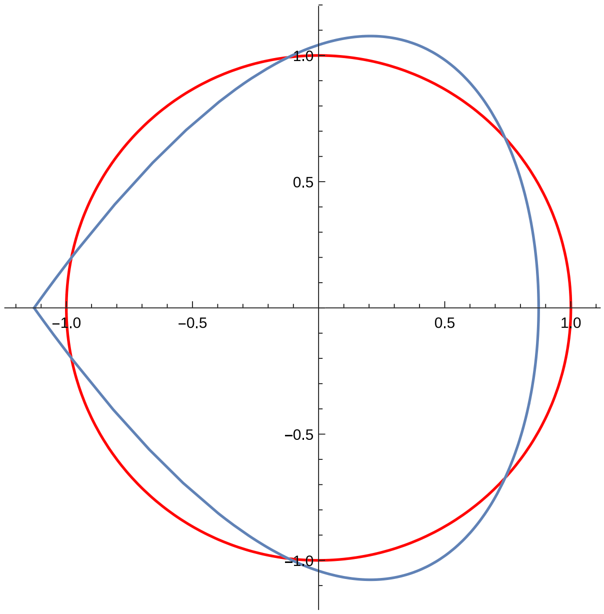

Fig. 5. Best parametric polynomial approximant of degree 3.

4. Uniform approximation of the unit circle

In the presented approach, we exploited nice approximation properties of Chebyshev polynomials. However, here the radial error does not reach the same absolute value at extremal points (so-called equioscillation). A better approximation can be obtained by using geometric interpolation in the following way.

Since a symmetric solution is sought, let us take

$$p(t) = (t^2 - t_1^2)(t^2 - t_2^2)\cdots(t^2 - t_n^2),$$

where $t_1, t_2, \ldots, t_n$ are unknown nonnegative numbers. Now let us consider the error function

$$e(t,a) = \sqrt{1 + a\,p(t)} - 1.$$

Its derivative

$$e'(t,a) = \frac{a\,p'(t)}{2\sqrt{1 + a\,p(t)}}$$

has $2n-1$ zeros t_i, one of them is 0. The extremal values of the error function, obtained at t_i, should oscillate in sign and have the same absolute value M_1. This yields a system of nonlinear equations for given a and t_n. Each such choice yields the appropriate values of $t_1, t_2, \ldots, t_{n-1}$ as the solution of the nonlinear system. By using the method, introduced in Section 3, we obtain the best parametric polynomial approximant $(x_n(t), y_n(t))$ among many possible. The first positive zero t^* of y_n determines the value $M_2 = ||x_n(t^*)| - 1|$. By considering

$$\min_{t_n, a} \max\{M_1, M_2\},$$

we can obtain the best possible Chebyshev-like solution.

This approach requires solving a system of nonlinear equations, solving a nonlinear equation, and bivariate minimization. Best approximants for small n will be given in the next section.

A natural question is whether the radial error is the best choice. Since there is not an equivalent theory of polynomials of best approximation in the parametric setting as far as I know, some other measure of error could perhaps yield better results.

5. Numerical examples

First, let us consider the case $n = 3$. For $a = 1/5$, the solution simplifies to

$$x_3(t) = \sqrt{\frac{2}{5}}\left(8\,t^2 \sinh\left(\frac{1}{6}\log\left(5 - 2\sqrt{6}\right)\right) - \sinh\left(\frac{1}{2}\log\left(5 - 2\sqrt{6}\right)\right)\right),$$

$$y_3(t) = \sqrt{\frac{2}{5}}\,t\left(-4t^2 - 1 + 4\cosh\left(\frac{1}{3}\log\left(5 - 2\sqrt{6}\right)\right)\right).$$

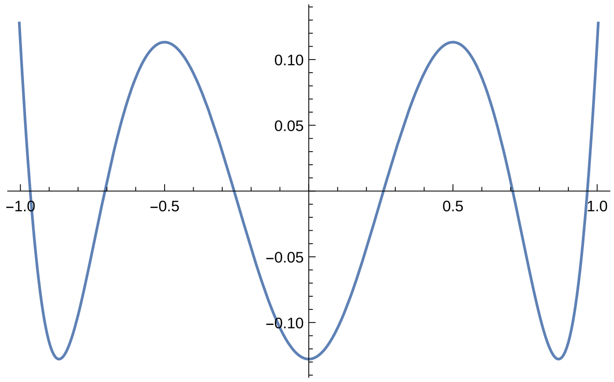

Fig. 6. Radial error for the best cubic approximant.

Table 1
Comparison of parametric polynomial approximations of the unit circle.

n	Approximant (Jaklič et al., 2013)	New approximant		
	Error	t_n^*	a	Error
3	2	1.00371	0.23921	0.127767
4	0.41421	1.000172	0.021873	0.010997
5	0.08999	1.0000056049	0.00105983	0.000530072
6	0.01389	1.0000030931	0.0000318193	0.0000159689
7	0.00138	1.0000008556	$6.60953 \cdot 10^{-7}$	$3.30532 \cdot 10^{-7}$
8	$9.5 \cdot 10^{-5}$	0.9999980693	$1.01285 \cdot 10^{-8}$	$5.06425 \cdot 10^{-9}$
9	$4.8 \cdot 10^{-6}$	0.9384805459	$4.0061 \cdot 10^{-10}$	$2.00305 \cdot 10^{-10}$

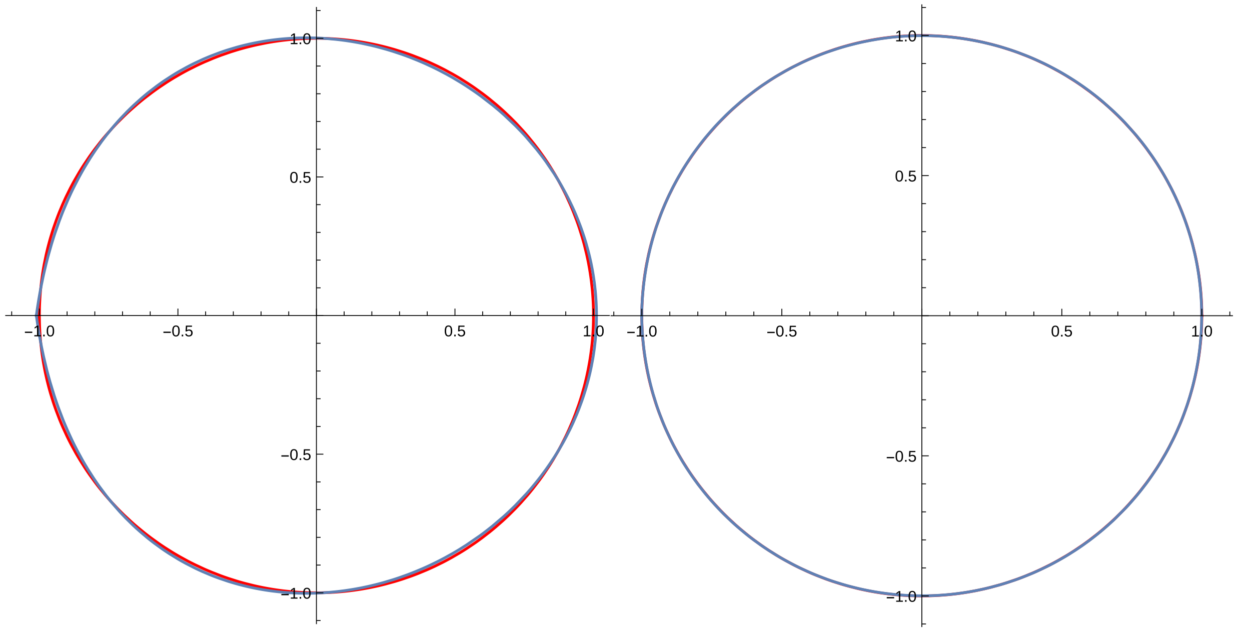

Fig. 7. Polynomial approximants of the unit circle for $n = 4, 5$.

The parameter value $a = 0.2392102070552632$ gives the best solution

$$x_3(t) = 0.872233 - 1.98524\,t^2, \quad y_3(t) = 2.78729\,t - 2.76672\,t^3,$$

with radial error 0.127767. It is shown in Fig. 5. Its radial error is presented in Fig. 6.

Note that an approximation with uniformly oscillating error $x_3^2 + y_3^2 - 1$ can be obtained at $a = 0.3253559144748783$. Here, the radial error is larger, 0.151241.

In Table 1, a comparison of parametric polynomial approximations of the unit circle is presented.

In Fig. 7, approximant for $n = 4$ and $n = 5$ are shown, and in Fig. 8, their curvature profiles are presented. The latter approximant is indistinguishable from the circle.

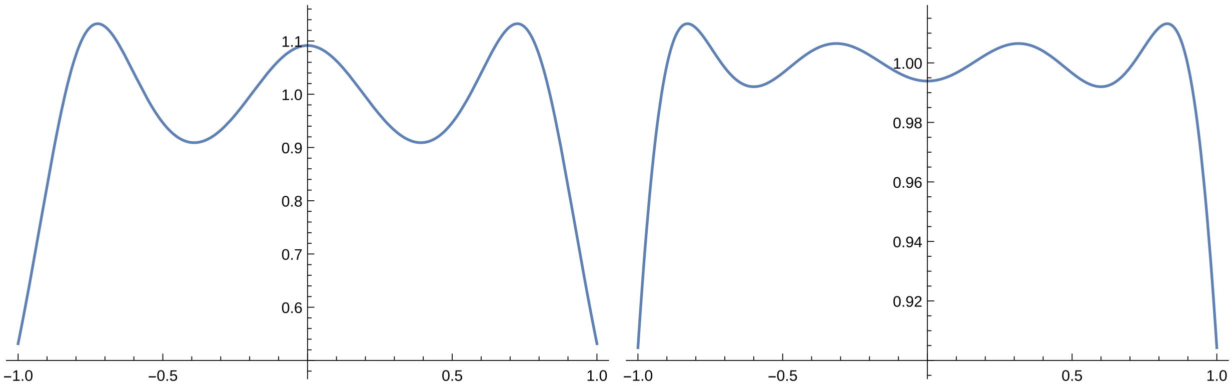

Fig. 8. Curvatures of the polynomial approximants of the unit circle for $n = 4, 5$.

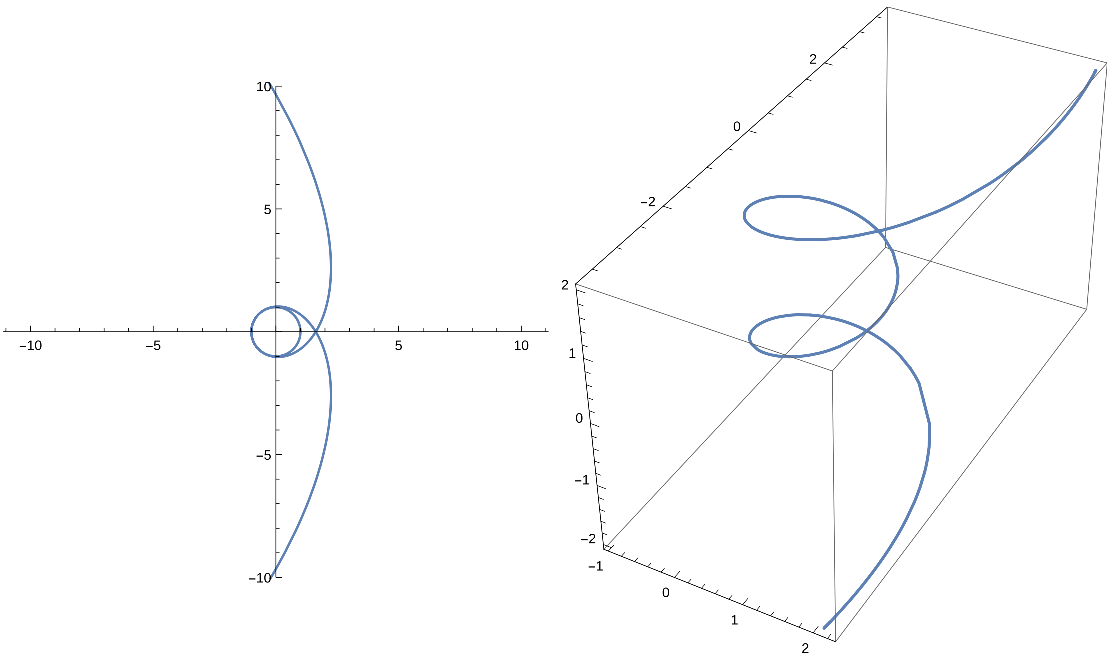

Fig. 9. Parametric polynomial approximant for $n = 7$ circles the unit circle 3 times, before deviating from it. Only two such circuits are shown, since the last one gives poor approximation (intersection with x-axis at -753.3). On the right-hand side a lift into space of the same curve $(x_7(t), y_7(t), t)$ is presented.

Table 2
Parametric polynomial approximants of the unit circle.

n	x_n and y_n
3	$0.87223264840565 - 1.9852407885334258\,t^2$ $2.787286743125938\,t - 2.7667176628214927\,t^3$
4	$1.0108773400989846 - 3.694997713373956\,t^2 + 1.6732434314389113\,t^4$ $2.602007499051689\,t - 2.6011101753794694\,t^3$
5	$0.9994699437191019 - 3.97346869128647\,t^2 + 1.9734689724376546\,t^4$ $2.8276698243105867\,t - 3.564283164798727\,t^3 + 0.7366367806515868\,t^5$
6	$1.0000158504 - 4.2541722246\,t^2 + 2.5094165602\,t^4 - 0.2552760363\,t^6$ $2.9165370429\,t - 3.9584900628\,t^3 + 1.0419646163\,t^5$
7	$0.9999996695 - 4.4230792820\,t^2 + 2.8461963497\,t^4 - 0.4231170676\,t^6$ $2.9742596413\,t - 4.2459402631\,t^3 + 1.3452671531\,t^5 - 0.0735834916\,t^7$
8	$1.00000000 - 4.53727730\,t^2 + 3.09276164\,t^4 - 0.57370221\,t^6 + 0.01821786\,t^8$ $3.01239993\,t - 4.44369639\,t^3 + 1.57120297\,t^5 - 0.13991316\,t^7$

Table 3
Uniform parametric polynomial approximants of the unit circle.

n	x_n and y_n	Error
3	$-0.87499994 + 1.04342054\,t^2$ $2.04295938\,t - 1.06583300\,t^3$	0.125
4	$1.01096 - 3.12912\,t^2 + 1.2\,t^4$ $-2.3944\,t + 2.02664\,t^3$	0.0109596
5	$0.99947004 - 3.87624490\,t^2 + 1.87807588\,t^4$ $2.79286220\,t - 3.43427162\,t^3 + 0.69240320\,t^5$	0.000529953
6	$1.00001591 - 4.18171979\,t^2 + 2.42466949\,t^4 - 0.24245463\,t^6$ $-2.89159481\,t + 3.85779676\,t^3 - 0.99816597\,t^5$	0.0000159103
7	$0.99999967 - 4.36763892\,t^2 + 2.77529322\,t^4 - 0.40740526\,t^6$ $2.95556067\,t - 4.16636090\,t^3 + 1.30350764\,t^5 - 0.07040567\,t^7$	$3.30481 \cdot 10^{-7}$
8	$1.00000001 - 4.49366852\,t^2 + 3.03359680\,t^4$ $-0.55731864\,t^6 + 0.01752749\,t^8$ $-2.99788856\,t + 4.37978650\,t^3 - 1.53372158\,t^5 + 0.13526280\,t^7$	$5.06408 \cdot 10^{-9}$

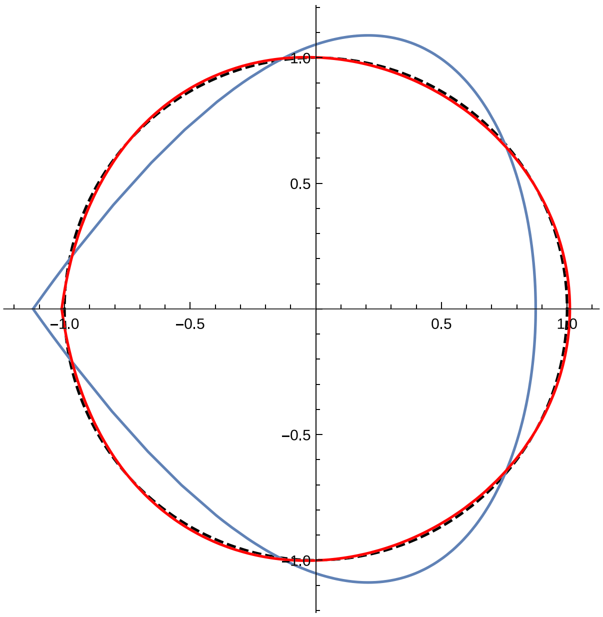

Fig. 10. Best uniform approximants for $n = 3$ (blue) and $n = 4$ (red), together with the unit circle (black, dashed). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The parametric polynomial approximant circles around the unit circle several times, before deviating from it for larger degree n, see Fig. 9. Such a behavior was observed also for approximants in Jaklič et al. (2013).

The best parametric polynomial approximants of the circle of low degree are presented in Table 2.

Now let us consider uniform approximation. Solutions for $n = 3, 4, \ldots, 8$ are given in Table 3, together with radial error. Parameter values for the best solution of the previous approach give good initial values for numerical computations that need to be done. In particular, we used the choice $t_n = 1$ and $a = a' \cdot 2^{2n-1}$, where a' is the value of a from Table 1. It turns out that the best solution is not unique, there are infinitely many of them.

For $n = 3$ and $n = 4$ the solutions are shown in Fig. 10. In Fig. 11, radial error for the approximant for $n = 8$ is shown. Note that for larger n, this approach yields solutions with error just slightly smaller that in the previously presented approach. Thus the previous approach could be used for theoretical study of error decay.

The results improve approximations of the unit sphere in Jaklič et al. (2012) (see Fig. 12, where quintic approximants are used for construction of parametric polynomial approximation of the unit sphere, and yield the radial error 0.000529953).

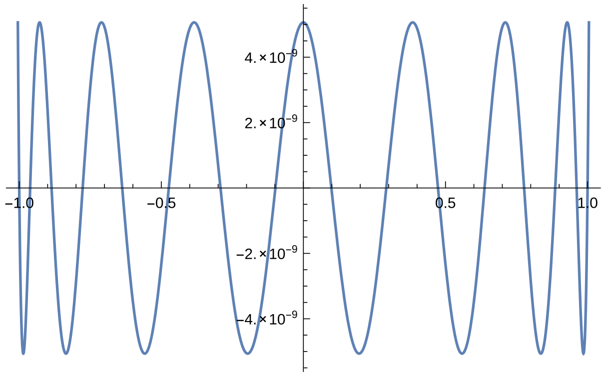

Fig. 11. Radial error for the best uniform approximant for $n = 8$.

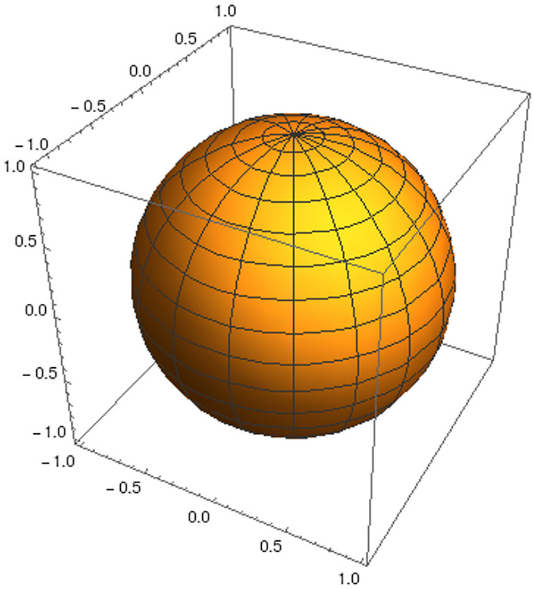

Fig. 12. Parametric polynomial approximation of the unit sphere by using quintic approximants x_5 and y_5.

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