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Computing μ-bases from algebraic ruled surfaces ☆

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ABSTRACT

We find a μ-basis for a rational ruled surface, starting from its implicit representation. A parametrization for this ruled surface is then deduced form this μ-basis. This parametrization does not have any non-generic base points and its directrix has the lowest possible degree. A complete interchange graph is built for the algebraic equation, the μ-basis and the parametric equation of a rational ruled surface.

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1. Introduction

A ruled surface is generated by sweeping a line along a directrix curve. Ruled surfaces are widely used in computer aided geometric design, geometric modeling and computer numerical control. There are many papers discussing ruled surfaces and their applications (see Johnstone, 1993; Chen, 2003; Chen and Wang, 2003a; Fioravanti et al., 2006; Sprott and Ravani, 2008; Liu et al., 2006; Shen and Yuan, 2010).

Parametric and implicit forms are the two main representations of curves and surfaces. The parametric and implicit forms each have their own specific advantages; a natural problem is to convert the forms from one to the other. Converting from the parametric form to the implicit one is the implicitization problem. Some typical methods focused on the implicitization problem include Gröbner bases, resultants and μ-bases. Among the most efficient of these techniques are μ-bases, since μ-bases permit us to compute with polynomials of lowest possible degree (Chen et al., 2005; Deng et al., 2005).

The parametrization problem is more difficult. To meet practical demands, people have designed parametrization algorithms for some commonly used surfaces, such as quadric algebraic surfaces (Berry and Patterson, 2001) and cubic algebraic surfaces (Bajaj et al., 1998). Recently, the author and his coauthor showed how to determine whether a given algebraic surface is a ruled surface and then parameterized this rational ruled surface based on algebraic computations (Shen and Pérez-Díaz, 2014).

A μ-basis can serve as a link between the implicit form and the parametric form of a curve or surface. We can recover the parametric equation as well as derive the implicit equation of the curve or surface from its μ-bases (Chen et al., 2005). There are special algorithms to compute a μ-basis from the parametrization of a ruled surfaces (Chen and Wang, 2003a) and general algorithms to compute a μ-basis from the parametrization of an arbitrary rational surface (Deng et al., 2005).

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However, computing a μ-basis from the implicit equation of a rational surface is still an open problem. μ-Bases can only be computed from the implicit equation for some special surfaces such as rational quadratics (Wang et al., 2008) and Dupin cyclides (Jia, 2014). In this paper, we investigate ruled surfaces, which are widely used in CAGD. We design an algorithm to find a μ-basis for a rational ruled surface directly from its implicit representation and then use this μ-basis to compute a parametrization for the ruled surface. This algorithm gives an alternatively simpler way than Shen and Pérez-Díaz (2014) to determine if an implicit algebraic equation represents a rational ruled surface and then to parametrize this ruled surface.

The paper is organized as follows. First, some preliminaries about μ-bases are presented in Section 2. In Section 3, we compute a μ-basis from the implicit equation of a rational ruled surface by finding a planar directrix and solving a system of equations generated from the conditional equation of the generators. Also connections for the interchange graph are discussed. In Section 4, an algorithm is given is given to find a μ-basis from the implicit equation of a rational ruled surface and an example is presented to illustrate the algorithm. In Section 5, we conclude the paper and suggest some problems for future research.

2. Preliminaries

A standard parametrization of a rational ruled surface $\mathcal{P}$ in the homogenous form is

$$(x, y, z, w) = \mathbf{P}(s,t) = \mathbf{P}_1(s) + t\mathbf{P}_2(s) \in \mathbb{R}[s,t]^4 \tag{1}$$

$\mathbf{P}_1(s)$ is called a directrix and $\mathbf{P}_2(s)$ is called a indicatrix of $\mathcal{P}$. We assume that the rational parametrization (1) is not a space curve. A line of (1) generated by fixing $s = s_0$ is called a generator (or ruling) of the ruled surface. A ruled surface is called doubly ruled if there are at least two generators through almost every point of the ruled surface, otherwise a ruled surface is called singly ruled. There are only three doubly ruled algebraic surfaces: the plane, the hyperbolic paraboloid, and the hyperboloid of one sheet (Hilbert and Cohn-Vossen, 1952).

A rational ruled surface can also be defined by an algebraic variety

$$f(x, y, z, w) = 0,\ f(x, y, z, w) \in \mathbb{R}[x, y, z, w]. \tag{2}$$

Here, we assume $f(x, y, z, w)$ is irreducible and nonlinear and we focus on irreducible non-planar varieties.

A moving plane $\mathbf{L}(s,t) := (A(s,t), B(s,t), C(s,t), D(s,t))$ is a family of planes $L(s,t) := A(s,t)x + B(s,t)y + C(s,t)z + D(s,t)w = 0$ with parameters s, t. A moving plane $\mathbf{L}(s,t)$ is said to follow a rational surface $\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)) \in \mathbb{R}[s,t]^4$ if

$$\mathbf{L}(s,t) \cdot \mathbf{P}(s,t) = A(s,t)a(s,t) + B(s,t)b(s,t) + C(s,t)c(s,t) + D(s,t)d(s,t) \equiv 0. \tag{3}$$

Three moving planes $\mathbf{p}(s,t), \mathbf{q}(s,t)$ and $\mathbf{r}(s,t)$ following $\mathbf{P}(s,t)$ form a *μ-basis* of $\mathbf{P}(s,t)$ if $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\mathbf{P}$ where k is a nonzero constant and $[\cdot]$ is the outer product.

For a rational ruled surface $\mathbf{P}(s,t)$, two moving planes $\mathbf{p}(s)$ and $\mathbf{q}(s)$ form a *fundamental μ-basis* of $\mathbf{P}(s,t)$ if $f(x, y, z, w) = k\mathrm{Res}(\mathbf{p} \cdot \mathbf{X}, \mathbf{q} \cdot \mathbf{X}, s)$, where k is a nonzero constant and $\mathbf{X} = (x, y, z, w)$. A fundamental μ-basis can be extended to a μ-basis.

Proposition 1. *For a ruled surface parametrically defined by $\mathbf{P}(s,t)$, there is a μ-basis of $\mathbf{P}(s,t)$ formed by three moving planes $\mathbf{p}(s), \mathbf{q}(s), \mathbf{r}(s,t) = \mathbf{u}(s) + t\mathbf{v}(s), \deg(\mathbf{p}) \le \deg(\mathbf{q})$. Furthermore,*

1. $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(f)$, *where $f(x, y, z, w) = 0$ is the implicit equation of $\mathbf{P}(s,t)$.*
2. $f(x, y, z, w) = k\mathrm{Res}(\mathbf{p} \cdot \mathbf{X}, \mathbf{q} \cdot \mathbf{X}, s)$, *where k is a nonzero constant.*
3. $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = k\mathbf{P}$ *where k is a nonzero constant.*
4. *The intersection line of the moving planes $\mathbf{p} \cdot \mathbf{X} = 0$ and $\mathbf{q} \cdot \mathbf{X} = 0$ are generators of $\mathbf{P}(s,t)$.*

Proof. Statements 1, 2 and 3 can be found in Chen and Wang (2003a). Statement 4 follows from statement 2. □

There are similar results for rational space curves and we here just give some propositions needed later in this paper.

Proposition 2. *A rational curve $\mathcal{C}$ parametrically defined by $\mathbf{P}(s) \in \mathbb{R}[s]^4$ has a μ-basis formed by three moving planes $\mathbf{p}(s), \mathbf{q}(s), \mathbf{r}(s) \in \mathbb{R}[s]^4$, $\deg(\mathbf{p}(s)) \le \deg(\mathbf{q}(s)) \le (\mathbf{r}(s))$. Furthermore,*

1. $\deg(\mathbf{p}(s)) + \deg(\mathbf{q}(s)) + \deg(\mathbf{r}(s)) = \deg(\mathbf{P}(s))$.
2. $[\mathbf{p}(s), \mathbf{q}(s), \mathbf{r}(s)] = k\mathbf{P}(s)$ *where k is a nonzero constant.*
3. *If $\mathcal{C}$ is planar, then $\mathbf{p}(s) = \mathbf{p}$ is a constant plane. Furthermore, the moving lines $\mathbf{p} \bigcap \mathbf{q}(s), \mathbf{p} \bigcap \mathbf{r}(s)$ form a μ-basis of $\mathcal{C}$ on the plane $\mathbf{p}$ and the algebraic representation of $\mathcal{C}$ is $\{\mathbf{X} | \mathbf{p} \cdot \mathbf{X} = 0, \mathrm{Res}(\mathbf{q} \cdot \mathbf{X}, \mathbf{r} \cdot \mathbf{X}, s) = 0\}$.*

Proof. Statements 1 and 2 can be found in Chen and Wang (2003b). For statement 3, the moving lines $\mathbf{l}_1(s) = \mathbf{p} \bigcap \mathbf{q}(s)$ and $\mathbf{l}_2(s) = \mathbf{p} \bigcap \mathbf{r}(s)$ naturally form a μ-basis of $\mathcal{C}$ since $\mathbf{p}, \mathbf{q}(s), \mathbf{r}(s)$ already form a μ-basis. The situation degenerates to the planar case and the algebraic variety in statement 3 is proposed by the propositions in Chen and Wang (2003b). □

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3. Parametrization and μ-basis for algebraic ruled surfaces

3.1. Planar directrix of ruled surface

We first show that there always exist planar directrices for ruled surfaces and we use this fact to reduce the computation for a μ-basis of the ruled surface.

Theorem 1. *A rational ruled surface has a parametrization of the form (1) such that its directrix is a planar curve in $\mathbb{R}^3$.*

Proof. Suppose a rational parametrization of the surface is

$$\mathbf{P}(s,t) = \mathbf{P}_1(s) + t\mathbf{P}_2(s) = (p_{11}(s), p_{12}(s), p_{13}(s), p_{14}(s)) + t(p_{21}(s), p_{22}(s), p_{23}(s), p_{24}(s)).$$

Let $L(x, y, z, w) \equiv ax + by + cz + dw = 0$ be a plane where a, b, c, d are generic coefficients. Consider the intersection curve between the ruled surface $\mathbf{P}(s,t)$ and the plane $L(x, y, z, w)$. Since the intersection equation $L(\mathbf{P}(s,t)) = 0$ is linear in t and we can solve for t as a function of s

$$t(s) = -\frac{ap_{11}(s) + bp_{12}(s) + cp_{13}(s) + dp_{14}}{ap_{21}(s) + bp_{22}(s) + cp_{23}(s) + dp_{24}} = \frac{t_n(s)}{t_d(s)} \in \mathbb{R}(s). \tag{4}$$

Hence, the planar intersection curve has a rational parametrization

$$\begin{cases} x = p_{11}(s) + t(s)p_{21}(s) \\ y = p_{12}(s) + t(s)p_{22}(s) \\ z = p_{13}(s) + t(s)p_{23}(s) \\ w = p_{14}(s) + t(s)p_{24}(s) \end{cases} \tag{5}$$

Since the curve is given in homogenous form, we can transform these rational functions to polynomials by multiplying by the denominator of $t(s)$. Reparameterizing $\mathbf{P}(s,t)$ with the birational transformation

$$\begin{cases} t = \tilde{t} + t(\tilde{s}) \\ s = \tilde{s} \end{cases} \tag{6}$$

we obtain a reparameterization with new parameters $\tilde{s}, \tilde{t}$

$$\tilde{\mathbf{P}}(\tilde{s}, \tilde{t}) = \tilde{\mathbf{P}}_1(\tilde{s}) + \tilde{t}\tilde{\mathbf{P}}_2(\tilde{s})$$

where $\tilde{\mathbf{P}}_1(\tilde{s}) = \mathbf{P}_1(\tilde{s}) + t(\tilde{s})\mathbf{P}_2(\tilde{s})$ and $\tilde{\mathbf{P}}_2(\tilde{s}) = \mathbf{P}_2(\tilde{s})$. By (5), the directrix curve $\tilde{\mathbf{P}}_1(\tilde{s})$ is actually a planar curve. □

In the proof of Theorem 1, the planar section is non-degenerated for a generic plane. But the intersection curve may degenerate for a specialized plane, for instance, the intersection of a cone and a plane may be the apex of the cone. It is known that the singular points are included in $S = \{f = 0, f_x = 0, f_y = 0, f_z = 0, f_w = 0\}$ where $f = 0$ is the implicit equation of the surface, and singular points form a one-dimensional variety since f is irreducible. A directrix should not be a singular subset and this assumption can be guaranteed by selecting a generic plane intersecting with S in only finitely many points.

Another unwanted situation is that the intersection curve may consist of some generators: for instance, when the ruled surface is a conical or a cylindrical surface, and the plane is a tangent plane of the surface. To get a directrix that does not consist of generators, the plane should not be parallel to almost all the generators of the ruled surface. In other words, the denominator $t_d(s)$ of equation (4) should not be zero. The only case where an arbitrary plane must be parallel to generators of an algebraic ruled surface is where the ruled surface is also a plane.

The next task is to find a rational parametrization $\mathbf{C}(s)$ of the intersection curve of the ruled surface and the plane. In our computation, the parametrization can be reduced to the planar situation by considering the problem in the intersecting plane. To find a generic point on $\mathbf{C}(s)$, the curve $\mathbf{C}(s)$ should not be a singular subset of the surface. Otherwise, the generators from a point on the directrix may not be determined; again an example is when we choose the directrix as the apex of a cone.

Proposition 3. *Let $\mathcal{P}$ be a non-planar ruled surface and let $\mathcal{L}$ be a generic plane. Their intersection curve is a directrix if it is not a subset of the union of singular points and generators of $\mathcal{P}$.*

Proof. By the preceding discussion, the intersection is a real curve and for the generic points we can find the generators covering $\mathcal{P}$, hence the intersection curve is a directrix. □

Remark 1. Proposition 3 provides a directrix for conveniently computing in the subsequent sections. The condition of Proposition 3 is sufficient but not necessary. A directrix of a ruled surface can be singular in certain parametrizations: for instance, when $\mathbf{P}(s) + t\mathbf{P}'(s)$ defines a tangential developable surface with the cuspidal edge $\mathbf{P}(s)$ which is certainly singular.

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3.2. Computing a μ-basis of an algebraic ruled surface

We now proceed to find the moving planes $\mathbf{p}(s)$ and $\mathbf{q}(s)$ guaranteed to exist by statement 4 of Proposition 1 and Proposition 2.

Therefore by statement 1 of Proposition 2, a rational space curve $\mathbf{C}(s)$ has a μ-basis formed by three moving planes $\mathbf{p}_c(s), \mathbf{q}_c(s), \mathbf{r}_c(s)$. Note that the planar section $\mathbf{C}(s)$ is a planar curve. By statement 2 of Proposition 2, the moving plane $\mathbf{p}_c(s)$ must be a constant plane $\mathbf{p}_c$ equal to the intersecting plane and $\mathbf{l}_1(s) = \mathbf{p}_c \bigcap \mathbf{q}_c(s)$ and $\mathbf{l}_2(s) = \mathbf{p}_c \bigcap \mathbf{r}_c(s)$ are two moving lines following $\mathbf{C}(s)$. The moving lines $\mathbf{l}_1(s)$ and $\mathbf{l}_2(s)$ form a μ-basis for the plane curve $\mathbf{C}(s)$ since their intersection points cover $\mathbf{C}(s)$ (see statement 3 of Proposition 2). Suppose the generators associated with $\mathbf{C}(s)$ are $\mathbf{g}(s)$; we have the following proposition.

Theorem 2. *Consider two planes, $\mathbf{p}(s)$ defined by $\mathbf{l}_1(s)$ and $\mathbf{g}(s)$, and $\mathbf{q}(s)$ defined by $\mathbf{l}_2(s)$ and $\mathbf{g}(s)$. Then $\mathbf{p}(s)$ and $\mathbf{q}(s)$ form a fundamental μ-basis of the ruled surface $f = 0$.*

Proof. Since the moving line $\mathbf{l}_1(s)$ follows the directrix $\mathbf{C}(s)$, the moving plane $\mathbf{p}(s)$ defined by $\mathbf{l}_1(s)$ and $\mathbf{g}(s)$ naturally follows the ruled surface $\mathcal{P}$. Similarly, $\mathbf{q}(s)$ is another moving plane that follows the ruled surface $\mathcal{P}$. Since their intersection line is exactly the generator covering $\mathcal{P}$ and the resultant of $\mathbf{p}(s)\cdot\mathbf{X}$ and $\mathbf{q}(s)\cdot\mathbf{X}$ give the intersection line of $\mathbf{p}(s)\cdot\mathbf{X} = 0$ and $\mathbf{q}(s)\cdot\mathbf{X} = 0$, we have $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, s) = kf$ where k is a nonzero constant. Therefore $\mathbf{p}(s)$ and $\mathbf{q}(s)$ form a fundamental μ-basis. □

Theorem 2 constructs a fundamental μ-basis from a μ-basis of the directrix and the generators. However, we do not need to compute the generators explicitly. We can determine the moving planes $\mathbf{p}(s)$ and $\mathbf{q}(s)$ from the following conditions

$$\begin{cases} \text{1) } \mathbf{p}(s) \text{ and } \mathbf{q}(s) \text{ pass through } \mathbf{l}_1(s) \text{ and } \mathbf{l}_2(s); \\ \text{2) the intersection line of } \mathbf{p}(s) \text{ and } \mathbf{q}(s) \text{ is included in } \mathcal{P}. \end{cases} \tag{7}$$

From the first condition, we can set

$$\mathbf{p}(s) = \alpha_1(s)\mathbf{p}_c(s) + \alpha_2(s)\mathbf{q}_c(s), \quad \mathbf{q}(s) = \beta_1(s)\mathbf{p}_c(s) + \beta_2(s)\mathbf{r}_c(s) \tag{8}$$

where α_i, β_i are indeterminants. For the second condition, let $\mathbf{v}_i(\alpha_1, \alpha_2, \beta_1, \beta_2; s), i = 0, 1$ be two points on the intersection line of $\mathbf{p}(s)$ and $\mathbf{q}(s)$. Then this intersection line can certainly be written as

$$(x, y, z, w) = \mathbf{v}_0(\alpha_1, \alpha_2, \beta_1, \beta_2; s) + \lambda\mathbf{v}_1(\alpha_1, \alpha_2, \beta_1, \beta_2; s) \tag{9}$$

where $\mathbf{v}_i, i = 1, 2$ are linear with respect to α_j and β_j, $j = 1, 2$. Substituting this line into the surface equation $f(x, y, z, w) = 0$ and expanding $f(\alpha_1, \alpha_2, \beta_1, \beta_2, s, \lambda)$ with respect to λ, the coefficients of α_j and β_j, $j = 1, 2$ from a system of equations, since f must be identically zero for any λ. To solve this system of equations, we can apply Wu's zero decomposition (see Wu, 2000). The solution of this system of equations will give explicit expressions for the moving planes in (8). By construction and statement 4 of Proposition 1, $\mathbf{p}(s)$ and $\mathbf{q}(s)$ form a fundamental μ-basis of the ruled surface, since their intersection lines cover the surface by moving along the directrix.

In practical computations, we can decrease the number of variables by setting $\alpha_2 = 1, \beta_2 = 1$. This simplification works because that the planes are in homogenous form and we can divide by $\alpha_2(s)$ and $\beta_2(s)$ since these parameters are not zero. Otherwise if either of $\alpha_2(s)$ or $\beta_2(s)$ is zero, then at least one moving plane that follows the surface is constant plane, which can only happen when the surface is a plane and this case is trivial.

From the fundamental μ-basis, we can recover the algebraic equation $f(x, y, z, w)$ by statement 2 of Proposition 1. To get a complete μ-basis as well as a parametrization of $\mathcal{P}$, we need to perform additional computations based on techniques in Chen (2003) and Chen and Wang (2003b). Let

$$\bar{\mathbf{P}}(s,t) = \mathbf{p}(s) + t\mathbf{q}(s) \tag{10}$$

be a dual ruled surface of $\mathcal{P}$. Then we can compute a fundamental μ-basis $\tilde{\mathbf{p}}(s), \tilde{\mathbf{q}}(s)$ for $\bar{\mathbf{P}}(s,t)$. Construct the ruled surface

$$\tilde{\mathbf{P}}(s,t) = \tilde{\mathbf{p}}(s) + t\tilde{\mathbf{q}}(s). \tag{11}$$

$\tilde{\mathbf{P}}(s,t)$ is doubly dual and exactly a parametrization of $\mathcal{P}$. Furthermore, this parametrization has no non-generic base points and its directrix, $\tilde{\mathbf{p}}(s)$, has lowest degree (Chen, 2003). As a final step, we need to compute a third moving plane $\mathbf{r}(s,t)$ to extend $\mathbf{p}$ and $\mathbf{q}$ to a μ-basis. This computation can directly follow the algorithm in Chen and Wang (2003b) since we have the parametrization of $\mathcal{P}$ and its fundamental μ-basis.

For a generic point $\mathbf{C}$ on the directrix, we can find a generator passing through this point and these generators will give a parametrization of the ruled surface, a similar approach is proposed in Johnstone (1993). From the parametric equation, one can find a μ-basis by the methods proposed in Chen and Wang (2003a). Following the discussion in subsection 3.2, we can compute a fundamental μ-basis directly from the algebraic surface. We can then compute a parametrization with better properties as well as a μ-basis. For implicitization, a preferred approach is based on the resultant computation in Shen and Yuan (2010) since we need only to compute a univariate resultant. Now we can give a complete interchange graph for the different representations of a rational ruled surface (see Fig. 1).

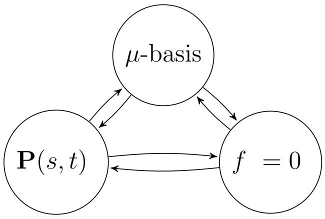

Fig. 1. Interchange graph for the different representations of a rational ruled surface.

4. The algorithm and example

Now we are ready to give an algorithm to compute a μ-basis for algebraic ruled surfaces.

Algorithm 1. Computing a μ-basis for an algebraic ruled surface
Input: An algebraic surface $\mathcal{P}$ defined by $f(x,y,z,w)=0$
Output: A parametrization $\mathbf{P}$ (without non-generic base points, with a directrix having lowest degree) and a mu-basis of $f(x,y,z,w)=0$ if $\mathcal{P}$ is ruled.

1. Select a nonsingular point of $f(x,y,z,w)=0$ and a plane $\mathcal{L}$ passing through the point this is not a tangential plane.
2. Find a rational parametrization $\mathbf{C}(s)$ of the intersection curve of $\mathcal{P}$ and $\mathcal{L}$ if it exists, otherwise return "$\mathcal{P}$ is not a ruled surface".
3. Compute a μ-basis $\mathbf{p_c}, \mathbf{q_c}, \mathbf{r_c}$ of $\mathbf{C}(s)$, where $\mathbf{p_c}$ is a representation of $\mathcal{L}$.
4. Let $\mathbf{p}(s)=\alpha\mathbf{p_c}+\mathbf{q_c}$ and $\mathbf{q}(s)=\beta\mathbf{p_c}+\mathbf{r_c}$. Solve for α,β from $f(\mathbf{p}(s)\bigcap\mathbf{q}(s))\equiv 0$ and update $\mathbf{p}(s),\mathbf{q}(s)$. Otherwise if there is no solution for α,β then return "$\mathcal{P}$ is not a ruled surface".
5. Compute a μ-basis $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ of the dual ruled surface $\mathbf{p}+t\mathbf{q}$ and compute the moving plane $\mathbf{r}(s,t)$ of the ruled surface $\tilde{\mathbf{p}}+t\tilde{\mathbf{q}}$. Output the parametrization $\tilde{\mathbf{p}}+t\tilde{\mathbf{q}}$ and its μ-basis $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$.

Remark 2. Although a random plane can generate a directrix for a non-planar ruled surface since the singular set is lower dimensional, we still have to check the validity of the intersection curve. Hence, in step 1, we prefer to give a specific plane, the intersection must be a directrix since it cannot be a subset consisting of singularities and generators.

We now give an example to illustrate our algorithm.

Example 1. Consider an algebraic surface $\mathcal{P}$ defined by

$$\begin{aligned} f(x,y,z,w) = {} & 11\,w^4 - 36\,w^3x - 12\,w^3y + 16\,zw^3 + 42\,w^2x^2 + 36\,w^2xy - 36\,w^2xz + 12\,w^2y^2 - 48\,w^2yz \\ & - 4\,z^2w^2 - 20\,wx^3 - 36\,wx^2y + 24\,wx^2z - 24\,wxy^2 + 72\,wxyz + 48\,wy^2z + 3\,x^4 + 12\,x^3y \\ & - 4\,x^3z + 12\,x^2y^2 - 24\,x^2yz - 48\,xy^2z - 32\,y^3z = 0. \end{aligned}$$

We find a nonsingular point $(-1,1,-4,1)$ on the surface and a plane

$$\mathcal{L}: x - z + 32w = 0$$

passing through the point but not tangent to $\mathcal{P}$.

We compute a parametrization of the intersection curve

$$\mathbf{C}(s) = \begin{pmatrix} -256\,s^3 - 792\,s^2 - 816\,s - 344 \\ -33\,s^4 + 206\,s^2 + 248\,s + 107 \\ -792\,s^2 - 1584\,s - 856 \\ 8\,s^3 - 24\,s - 16 \end{pmatrix}$$

which can be a directrix of $\mathcal{P}$. Since this curve is planar in the known plane $\mathcal{L}$, we can conveniently parameterize this curve in $\mathcal{L}$ with the command `parametrization` using the Maple software and project back to three dimension.

Using the algorithm from Deng et al. (2005), we compute a μ-basis of $\mathbf{C}(s)$ listed as columns in

$$(\mathbf{p_c}, \mathbf{q_c}(s), \mathbf{r_c}(s)) = \begin{pmatrix} -1 & -2\,s^2 + 4\,s + 2 & s^2 - 2\,s + 1 \\ 0 & 8\,s & 8 \\ 1 & 0 & 0 \\ -32 & -31\,s^2 - 70\,s - 43 & 32\,s^2 + 68\,s + 32 \end{pmatrix}.$$

One can see that $\mathbf{p}$ is exactly the plane $\mathcal{L}$.

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Let $\mathbf{p}(s) = \alpha(s)\mathbf{p_c}(s) + \mathbf{q_c}(s)$ and $\mathbf{q}(s) = \beta(s)\mathbf{p_c}(s) + \mathbf{r_c}(s)$. Finding $\alpha(s)$ and $\beta(s)$ by solving the equation system satisfying the conditions in (7) (see details in subsection 3.2), we get $\alpha = s^2 + 2s + 1$, $\beta = -s^2 - 2s - 1$ and then

$$(\mathbf{p}(s), \mathbf{q}(s)) = \begin{pmatrix} -s^2 + 6s + 3 & -4s \\ 8s & 8 \\ -s^2 - 2s - 1 & s^2 + 2s + 1 \\ s^2 - 6s - 11 & 4s \end{pmatrix}$$

form a fundamental μ-basis of $\mathcal{P}$.

Computing the μ-basis $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{q}}$ of the dual ruled surface $\mathbf{p} + t\mathbf{q}$ by the PMU-BASIS algorithm and $\mathbf{r}(s, t)$ by the MU-BASIS algorithm (Chen and Wang, 2003a), we get

$$(\tilde{\mathbf{p}}(s), \tilde{\mathbf{q}}(s), \mathbf{r}(s,t)) = \begin{pmatrix} 6s^2 + 4s + 14 & 8s + 8 & -s^2t + s^2 + 6st + 2s \\ -s^2 + 10s + 3 & s^2 - 2s - 3 & 8st - 8s - 16 \\ -24 & 24 & -s^2t - s^2 - st - 3s \\ 6s^2 + 12s + 6 & 0 & s^2t - s^2 - 6st - 2s - 8t + 8 \end{pmatrix}.$$

$\tilde{\mathbf{p}} + t\tilde{\mathbf{q}}$ is a parametrization of $\mathcal{P}$ and $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is a μ-basis of $\mathcal{P}$.

5. Conclusion and future work

In recent years, μ-bases have been shown to have interesting properties. Once a μ-basis of a rational curve or surface is obtained, we can efficiently compute both the implicit equation and a parametrization. Other researchers achieve some additional important results including theorems, algorithms and applications for μ-bases (Chen, 2003; Chen et al., 2005; Chen and Wang, 2003a, 2003b; Deng et al., 2005; Jia, 2014). Nevertheless, there are still open problems, for instance, computing a μ-basis from an implicit equation. This motivates us to consider the first problem on ruled surfaces which are popular in practical applications.

The current results are almost all for symbolic computations. However, in applications, the main computations are numerical computations. There are gaps between symbolic algorithms and numerical implements. In the future we hope to fill in some of these gaps with between symbolic and numeric computations.

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