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Algebro-geometric analysis of bisectors of two algebraic plane curves

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ABSTRACT

In this paper, a general theoretical study, from the perspective of the algebraic geometry, of the untrimmed bisector of two real algebraic plane curves is presented. The curves are considered in $\mathbb{C}^2$, and the real bisector is obtained by restriction to $\mathbb{R}^2$. If the implicit equations of the curves are given, the equation of the bisector is obtained by projection from a variety contained in $\mathbb{C}^7$, called the incidence variety, into $\mathbb{C}^2$. It is proved that all the components of the bisector have dimension 1. A similar method is used when the curves are given by parametrizations, but in this case, the incidence variety is in $\mathbb{C}^5$. In addition, a parametric representation of the bisector is introduced, as well as a method for its computation. Our parametric representation extends the representation in Farouki and Johnstone (1994b) to the case of rational curves.

1. Introduction

Given two geometric objects, their bisector is often defined as the geometric locus of the points which are equidistant from both objects. Examples of bisectors in the Euclidean plane are the perpendicular bisector of two points (or a segment), the angle bisector (the equidistant half-line from the sides of the angle), and the parabola, which is the equidistant curve between a straight line and a point external to the line. Subjects of particular interest are the study of the bisector of two curves, in the plane or in 3-dimensional space, and of the bisector of two surfaces. The bisector of two curves is sometimes called the equidistant curve. The untrimmed bisector is the locus of the centers of all the circles which are tangent to both curves. The untrimmed bisector contains the bisector as defined above, and a trimming method is a procedure to eliminate from it the parts that are not contained in the bisector.

Bisectors have been studied in the context of Computational Geometry because they play an important role in the construction of Voronoi diagrams (see Boissonnat et al., 2006; Aurenhammer and Klein, 2000; Devadoss and O'Rourke, 2011). Various papers on bisectors of algebraic curves have been written in the context of CAGD, starting with the articles (Farouki and Johnstone, 1994a) and (Farouki and Johnstone, 1994b) where the notion of untrimmed bisector is considered, for pairs of regular polynomial or rational curves, and a trimming procedure is presented. In Hoffmann and Vermeer (1991) a system of equations for the untrimmed bisector is proposed, together with the elimination of certain extraneous components. Elber and Kim (1998) consider C^1-continuous plane rational curves, and present a method of elimination to obtain a representation of the bisector in terms of the parameters of the initial curves. Some geometric and algebraic properties

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of the bisector of two curves, a curve and a surface, and two surfaces, are studied in Peternell (2000). In the thesis of Adamou (2013), a method for the parametrization of bisectors of rational curves is presented. Several approximate or interpolation methods for the computation of bisectors have been proposed (see, for example Farouki and Ramamurthy, 1998; Omirou and Demosthenous, 2006 or Oliveira and De Figueiredo, 2003).

In this paper, a general theoretical treatment, from the perspective of the algebraic geometry, of the untrimmed bisector of two real algebraic plane curves is presented. Similar analyses to other geometric objects, as offsets or conchoids, can be found in Arrondo et al. (1997), Sendra and Sendra (1997) and Sendra and Sendra (2008). The curves are considered in $\mathbb{C}^2$, and the equation of the bisector is obtained by projection from a variety $\mathcal{A}$ contained in $\mathbb{C}^7$, called the incidence variety, into $\mathbb{C}^2$. Each element of $\mathcal{A}$ is composed by one point (in complex coordinates) from each curve, one point in the bisector and an auxiliary variable. They must obey suitable equations. If the coordinates are restricted to $\mathbb{R}^2$, the real bisector is obtained. In this way, the three objects involved in the construction, namely the two original curves, $\mathcal{C}_1, \mathcal{C}_2$, and the bisector, $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$, are connected via rational maps (see Diagram (2)). Therefore, one may study how the properties of the two original curves can be translated to the bisector. Examples of this assertion could be the study of the rationality and of the genus of the bisector; this was, indeed, the crucial idea in Arrondo et al. (1999) for the case of offsets. Using the corresponding diagrams of the generic offset and of the bisector (see Diagram (3)), we relate these two geometric constructions, and we prove that all components of the bisector have dimension 1, by using that this property holds for non-degenerate offsets.

A similar method is applied to the case where both curves, or one of them, are given parametrically (see Diagram (6)). In this case, the incidence varieties are contained in $\mathbb{C}^5$ or $\mathbb{C}^6$, respectively. The bisector of two curves, although being a curve, turns to be a more complicated object. For instance, there is an explosion of the degree (see Remark 2.10), and the genus (see e.g. Example 2.6). From the point of view of applications, this is a serious obstacle since the implicit equation can be huge, and hence hard to manage. On the other hand, as pointed above, the genus is not invariant under the bisector operation, and thus the component of the bisector usually has positive genus. Therefore, in general, there do not exist rational parametrizations. In Farouki and Johnstone (1994b), an alternative representation for irreducible bisectors, based on the parameters space, is introduced. They assume the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ to be rational, regular and C^1-continuous. In this paper, we formally extend this representation to the general case.

The structure of the manuscript is the following. In section 2, the definition of untrimmed bisector using an incidence variety is presented, for the case of implicit curves, and some related theorems are proved. The characterization of the bisector as the intersection of offset curves at variable distance is analyzed in this context. In section 3, the incidence variety is introduced for the case of two parametric curves, assuming the parametrizations are normal, which is not much restrictive. The combination of one implicit and one parametric curves is also considered. A method to get a parametric representation of the bisector is presented in section 4. Several examples are presented in the three sections. In section 5 we discuss how the ideas and results in the paper can be generalized to hypersurfaces, and we point out where the difficulties are. In the last section some conclusions are stated, and directions in which to extend this research are presented.

2. Untrimmed bisectors: implicit case

We start this section analyzing the notion of bisector of two algebraic curves. Intuitively speaking, the (trimmed) bisector of two curves is the geometric locus of those points being at the same (Hausdorff) distance from the two curves. We recall that the distance from a point P to a non-empty subset A of a metric space, under a distance d, is

$$\mathrm{d}(P, A) = \inf\{\mathrm{d}(P, Q) \mid Q \in A\}.$$

In our case, A will be a real algebraic affine plane curve, and hence a set with infinitely many points. In order to deal algebraically with this concept, we would like to somehow skip the infimum in our definition. This leads to the notion of (untrimmed) bisector that corresponds with the geometric locus of those points that, being on the normal lines to both curves, are at the same distance from the two footpoints in the intersection of each curve with the corresponding normal line. In other words, the points in the untrimmed bisector are the centers of the circles which are tangent to both curves. Note that the untrimmed bisector is a superset of the trimmed bisector. In Example 2.4 the untrimmed bisector of two concentric circles of radii 1 and 2 has two components: the circle with the same center and radius 3/2, which is the trimmed bisector, and the circle of radius 1/2 (see Fig. 1).

In the following we analyze the notion of untrimmed bisector. The idea, as stated above, is to define it as an algebraic set. When describing this algebraic set, and in some degenerate cases, extraneous components might be introduced. Our goal is to guarantee that none of these extraneous factors is the full plane since, in that case, the untrimmed bisector would be the plane and would provide no information (see Theorem 2.12). For this purpose, throughout this paper, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two different real irreducible affine curves defined by $f_1(x, y)$ and $f_2(x, y)$, respectively. We use the notation $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \mathbf{z} = (z_1, z_2)$. Although we are interested in the case of real curves and real bisector, it is convenient to work with complex coordinates. Afterwards, $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ will be restricted to $\mathbb{R}^2$ to obtain the real bisector.

In order to deal algebraically with the notion of bisector, we need to combine, in the same affine space, the two original curves $\mathcal{C}_1$ and $\mathcal{C}_2$ as well as their bisector; for this purpose, we use incidence varieties as in Arrondo et al. (1997), Sendra and Sendra (2008). Since the curves are plane, and since their bisector is a geometric object in the plane, we need to work, at least, in $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$. Moreover, since we want to exclude some degenerate situations in the bisector construction as, for instance, the singular points on the original curves, we introduce an additional coordinate where these circumstances

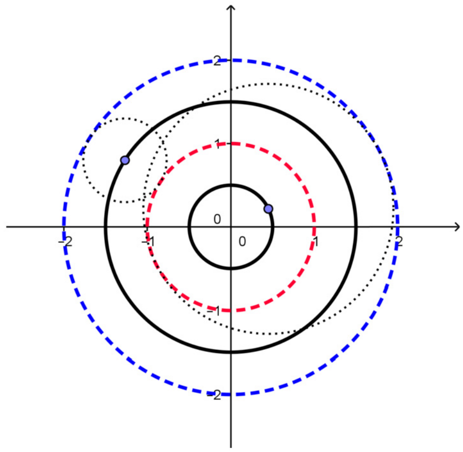

Fig. 1. Two concentric circles of radii 1 and 2 (dashed lines), and their bisector.

are forbidden. So, our working ambient for describing algebraically the bisector construction is the affine space $\mathbb{C}^7 = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}$. Now, we introduce an algebraic set $\mathcal{A}$, composed by elements of the form $(\mathbf{x}, \mathbf{y}, \mathbf{z}, W)$, where the element $\mathbf{x} \in \mathcal{C}_1, \mathbf{y} \in \mathcal{C}_2$, $\mathbf{z}$ belongs to the untrimmed bisector, and W is an auxiliary variable; this variety $\mathcal{A}$ is called an incidence variety. Then, the untrimmed bisector will be the projection of the incidence variety on the set of $\mathbf{z}$ coordinates. As incidence variety, we consider the set

$$\mathcal{A} = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}, W) \in \mathbb{C}^7 \;\middle|\; \begin{array}{ll} f_1(\mathbf{x}) = 0 & \text{[Equation of } \mathcal{C}_1] \\ f_2(\mathbf{y}) = 0 & \text{[Equation of } \mathcal{C}_2] \\ \left.\begin{array}{l} \operatorname{rank}\begin{pmatrix} \mathbf{z}-\mathbf{x} \\ \nabla f_1(\mathbf{x}) \end{pmatrix} = 1 \\ \operatorname{rank}\begin{pmatrix} \mathbf{z}-\mathbf{y} \\ \nabla f_2(\mathbf{y}) \end{pmatrix} = 1 \\ \|\mathbf{x}-\mathbf{z}\|^2 = \|\mathbf{y}-\mathbf{z}\|^2 \end{array}\right\} & \begin{array}{l}\text{[Equations of the} \\ \text{geometric} \\ \text{construction]}\end{array} \\ \|\nabla f_1(\mathbf{x})\|^2 \|\nabla f_2(\mathbf{y})\|^2 W = 1 & \begin{array}{l}\text{[Equation} \\ \text{avoiding} \\ \text{degenerations]}\end{array} \end{array} \right\}. \tag{1}$$

In this situation, we consider the projection map

$$\begin{array}{rccc} \pi_{\mathbf{z}}: & \mathcal{A} \subset \mathbb{C}^7 & \rightarrow & \mathbb{C}^2 \\ & (\mathbf{x}, \mathbf{y}, \mathbf{z}, W) & \mapsto & \mathbf{z} \end{array}$$

Then, we have the following definition (for $S \subset \mathbb{C}^2$ we denote by $\overline{S}$ its Zariski closure).

Definition 2.1. We define the (untrimmed) bisector of $\mathcal{C}_1, \mathcal{C}_2$ as the Zariski closure of $\pi_{\mathbf{z}}(\mathcal{A})$, and we represent it by $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$; i.e. $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_{\mathbf{z}}(\mathcal{A})}$.

So, if $\pi_{\mathbf{x}}, \pi_{\mathbf{y}}$ are the projections on the $\mathbf{x}$ and $\mathbf{y}$ coordinates, respectively, we have the following diagram connecting $\mathcal{C}_1, \mathcal{C}_2, \operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$.

$$\begin{tikzcd} \mathcal{C}_1 \subset \mathbb{C}^2 & \mathcal{A} \subset \mathbb{C}^7 \arrow[l, "\pi_{\mathbf{x}}"'] \arrow[r, "\pi_{\mathbf{y}}"] \arrow[d, "\pi_{\mathbf{z}}"'] & \mathcal{C}_2 \subset \mathbb{C}^2 \\ & \operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2) & \end{tikzcd} \tag{2}$$

Let us interpret the set $\mathcal{A}$. Suppose $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0, W_0) \in \mathcal{A}$. The first two equations imply that $\mathbf{x}_0 \in \mathcal{C}_1$ and $\mathbf{y}_0 \in \mathcal{C}_2$. The third equations (the rank conditions) ensure that $\mathbf{z}_0$ is on the normal line to $\mathcal{C}_1$ at $\mathbf{x}_0$, and on the normal line to $\mathcal{C}_2$ at $\mathbf{y}_0$. The fourth equation means that the distances (for $\mathbf{x}_0, \mathbf{y}_0$ real points) between $\mathbf{z}_0$ and $\mathbf{x}_0$ and between $\mathbf{z}_0$ and $\mathbf{y}_0$ are equal. The last equation implies that $\mathbf{x}_0$ and $\mathbf{y}_0$ are not singular points of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. But why do we need the last equation? If $\mathbf{x}_0 = \mathbf{y}_0 \in \mathcal{C}_1 \cap \mathcal{C}_2$ and it is a singular point on each curve, then $\|\mathbf{x}_0 - \mathbf{z}\|^2 = \|\mathbf{y}_0 - \mathbf{z}\|^2$ holds for all $\mathbf{z} \in \mathbb{C}^2$. Moreover, both rank conditions are trivial. So $\mathbf{x}_0, \mathbf{y}_0$ would generate in $\mathcal{A}$ a plane, namely $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z})$, and hence

$\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ would be the plane $\mathbb{C}^2$. Even if $\mathbf{x}_0 \neq \mathbf{y}_0$, extraneous components may appear because all points $\mathbf{z}_0$ satisfying $\|\mathbf{x}_0 - \mathbf{z}_0\|^2 = \|\mathbf{y}_0 - \mathbf{z}_0\|^2$ form a line which satisfies the equations. The last equation also guarantees that $\mathbf{x}_0, \mathbf{y}_0$ are not isotropic, what is used in the proof of Theorem 2.8 to relate bisectors and offsets; see also Remark 2.13. We recall that a point P on a curve $g(x, y) = 0$ is isotropic if

$$\frac{\partial g}{\partial x}(P)^2 + \frac{\partial g}{\partial y}(P)^2 = 0.$$

Observe that not only singularities are isotropic; $(i, -1/2)$ is isotropic and regular on the parabola $y = x^2/2$.

Remark 2.2. Some authors (see e.g. Hoffmann and Vermeer, 1991) choose to avoid the situations where there are infinitely many points in the bisector corresponding to the same footpoint. They happen when an element of $\mathcal{A}$ has $\mathbf{x} = \mathbf{y}$, (see the points $\mathbf{x} = (0, 0) = \mathbf{y}$ in Examples 2.5 and 2.7). This sort of points could be avoided by replacing the last equation by $\|\nabla f_1(\mathbf{x})\|^2 \|\nabla f_2(\mathbf{y})\|^2 \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right) W = 1$. However, we decided not to do so, because Theorem 2.8 below would not be true. On the other hand, the extraneous components arising from the cases $\mathbf{x} = \mathbf{y}$ can be removed in the trimming process. $\square$

Remark 2.3 *(Computational issues: first part).* Our main goal is the establishment of a theoretical algebro-geometric frame to analyze the bisectors. Nevertheless, in this remark, we discuss some of the computational issues associated to our construction. Taking into account that the untrimmed bisector is a projection, it can be obtained as follows. Let I be the ideal in $\mathbb{C}[\mathbf{x}, \mathbf{y}, \mathbf{z}, W]$ generated by the polynomials defining $\mathcal{A}$. Then, by the Closure Theorem (see Cox et al., 1997, p. 122), one has that the untrimmed bisector is the variety defined by $I \cap \mathbb{C}[\mathbf{z}]$. Hence elimination theory techniques, such as Gröbner bases, provide a method to compute the untrimmed bisector. Therefore, the complexity of the method is dominated by the computation of a Gröbner basis of an ideal depending on 7 variables. This, in general, may imply heavy computations.

Another computational issue is the determination of the components of the bisector. Since, in Theorem 2.12, we prove that all components have dimension 1, and since the bisector is an algebraic variety in $\mathbb{C}^2$, the irreducible decomposition of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ can be carried out computing the absolute factorization of a bivariate polynomial. Algorithms for this purpose can be found in Section 5.5 in Winkler (1996). $\square$

We illustrate the definition by means of some examples.

Example 2.4. We start with a simple example. Let $\mathcal{C}_1$ be the circle $x_1^2 + x_2^2 = 4$ and $\mathcal{C}_2$ the circle $y_1^2 + y_2^2 = 1$. Then, the incidence variety is defined by the polynomials

$$\{x_1^2 + x_2^2 - 4,\ y_1^2 + y_2^2 - 1, 2z_1x_2 - 2z_2x_1, 2z_1y_2 - 2z_2y_1,$$
$$(z_1 - x_1)^2 + (z_2 - x_2)^2 - (z_1 - y_1)^2 - (z_2 - y_2)^2, (4x_1^2 + 4x_2^2)(4y_1^2 + 4y_2^2)W - 1\}.$$

Considering $W > x_1 > x_2 > y_1 > y_2 > z_1 > z_2$, and computing a Gröbner basis w.r.t. the lex order, we get

$$\{16z_1^4 + 32z_1^2z_2^2 + 16z_2^4 - 40z_1^2 - 40z_2^2 + 9, -4z_1^2z_2 - 4z_2^3 + 3y_2 + 7z_2, -4z_1^3 - 4z_1z_2^2 + 3y_1 + 7z_1,$$
$$4z_1^2z_2 + 4z_2^3 + 3x_2 - 13z_2, 4z_1^3 + 4z_1z_2^2 + 3x_1 - 13z_1, 64W - 1\},$$

and hence $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the union of the two circles $4z_1^2 + 4z_2^2 = 1$ and $4z_1^2 + 4z_2^2 = 9$; see Fig. 1.

Example 2.5. Let $\mathcal{C}_1$ be the parabola $x_2 = x_1^2$ and $\mathcal{C}_2$ the line $y_2 = 0$ (see Fig. 2). Then, the incidence variety is defined by the polynomials

$$\{-{x_1}^2 + x_2, y_2, z_1 - x_1 + 2\,(z_2 - x_2)\,x_1, z_1 - y_1,$$
$$(z_1 - x_1)^2 + (z_2 - x_2)^2 - (z_1 - y_1)^2 - (z_2 - y_2)^2\,, (4x_1^2 + 1)W - 1\}\,.$$

Considering $W > x_1 > x_2 > y_1 > y_2 > z_1 > z_2$, and computing a Gröbner basis w.r.t. the lex order, we get that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the quintic defined by

$$z_1\left(16\,{z_1}^4 - 32\,{z_1}^2{z_2}^2 + 16\,{z_2}^4 - 40\,{z_1}^2z_2 - 24\,{z_2}^3 + {z_1}^2 + 12\,{z_2}^2 - 2\,z_2\right).$$

We observe that, in this case, the genus of the quartic in $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is 0. On the other hand, if one had introduced the condition $\mathbf{x} \neq \mathbf{y}$, then the line $z_1 = 0$ would have been removed from the bisector (see Remark 2.2).

Example 2.6. Let $\mathcal{C}_1$ be the parabola $x_2^2 - x_1 = 0$ and $\mathcal{C}_2$ the parabola $-y_1^2 + y_2 = 0$ (see Fig. 3). Applying the ideas above, one gets that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is a curve of degree 15 and its defining polynomial has 114 nonzero terms. $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ factors into the line $z_1 = z_2$ and a 14th-degree curve of genus 4.

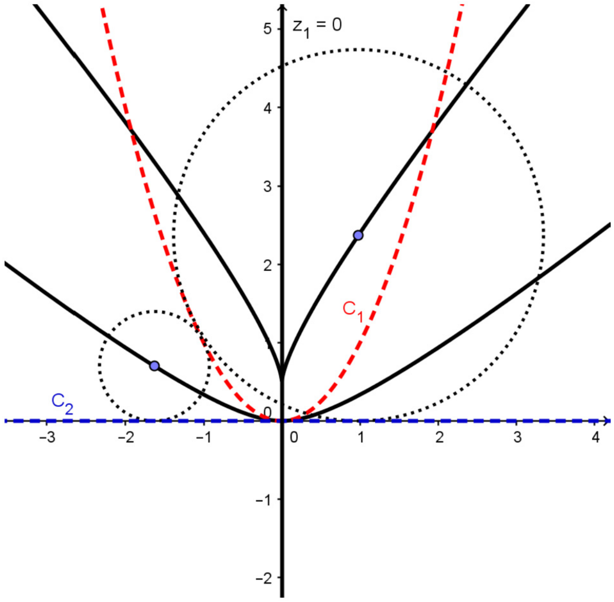

Fig. 2. Parabola $x_2 = x_1^2$, line $y_2 = 0$ (dashed lines), and their bisector.

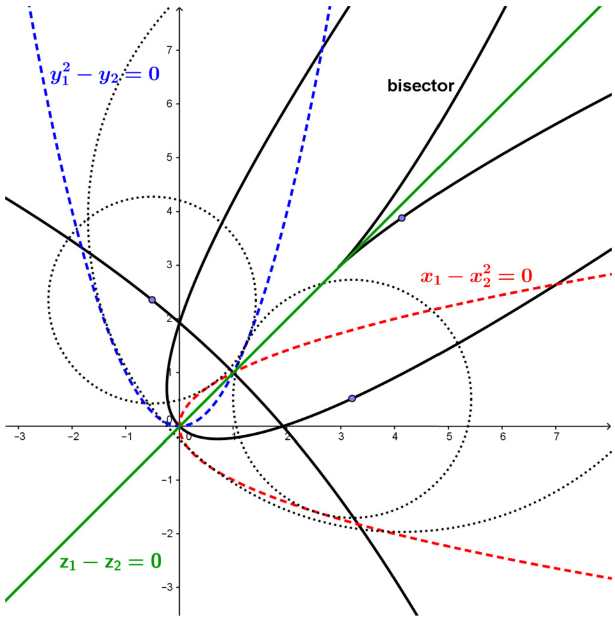

Fig. 3. Parabolas $x_1 = x_2^2$, $y_2 = y_1^2$ (dashed lines), and their bisector.

Example 2.7. Let $\mathcal{C}_1$ be the parabola $x_2 - x_1^2 = 0$, and $\mathcal{C}_2$ the cubic $y_2 - y_1^3 = 0$ (see Fig. 4). Applying the ideas above, one gets that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is a curve of degree 24, whose defining polynomial has 228 nonzero terms, and factors as the line $z_1 = 0$ and a 23rd-degree curve. This whole line correspond to the single common footpoint at the origin (see Remark 2.2).

Alternatively, one may relate bisectors to offsets. For this purpose, let

$$\mathcal{B}_i = \left\{ (\mathbf{x}, \mathbf{z}, d, W) \in \mathbb{C}^6 \;\middle|\; \begin{array}{l} f_i(\mathbf{x}) = 0, \\ \mathrm{rank}\begin{pmatrix} \mathbf{z} - \mathbf{x} \\ \nabla f_i(\mathbf{x}) \end{pmatrix} = 1, \\ \|\mathbf{x} - \mathbf{z}\|^2 = d^2, \\ \|\nabla f_i(\mathbf{x})\|^2 W = 1 \end{array} \right\}, \; i = 1, 2,$$

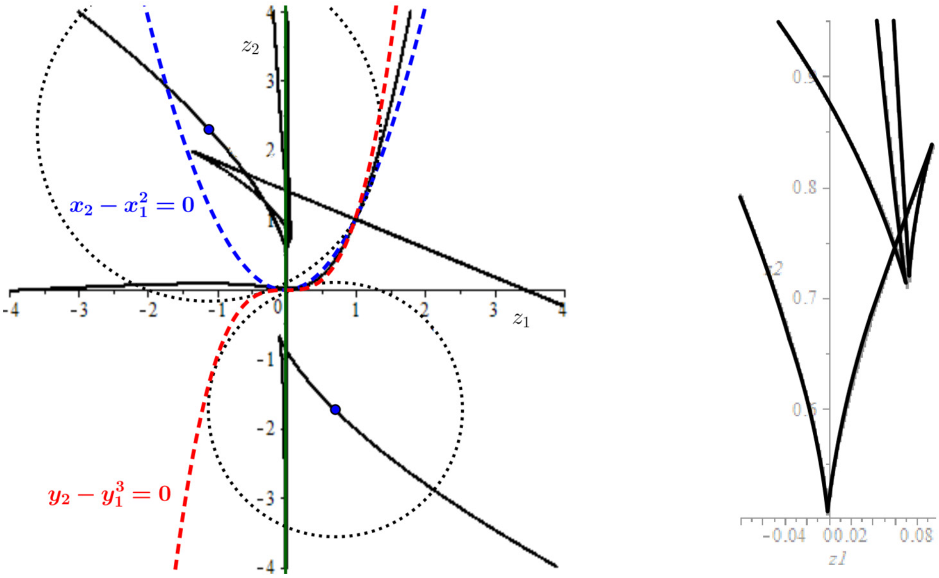

Fig. 4. Left: Parabola $x_2 = x_1^2$, cubic $y_2 = y_1^3$ (dashed lines), and their bisector. Right: Detail of the bisector for $z_1 \in (-0.05, 0.1)$.

$\pi_{\mathbf{z},d} : \mathbb{C}^6 \to \mathbb{C}^3$, $\pi_{\mathbf{z},d}(\mathbf{x}, \mathbf{z}, d, W) = (\mathbf{z}, d)$, and $\pi_{\mathbf{z}} : \mathbb{C}^3 \to \mathbb{C}^2$, $\pi_{\mathbf{z}}(\mathbf{z}, d) = \mathbf{z}$. We say that $\overline{\pi_{\mathbf{z},d}(\mathcal{B}_i)}$ is the generic offset of $\mathcal{C}_i$ (see Definition 1 in San Segundo and Sendra, 2009). In this situation, we have the following theorem.

Theorem 2.8. $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))}$.

Proof. Let $\mathbf{c} \in \pi_{\mathbf{z}}(\mathcal{A})$ (recall the definition of $\mathcal{A}$ in (1)). Then, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ and $w \in \mathbb{C}$ such that $(\mathbf{a}, \mathbf{b}, \mathbf{c}, w) \in \mathcal{A}$. Because of the first, second, and last equations defining $\mathcal{A}$, we know that $\mathbf{a} \in \mathcal{C}_1$, $\mathbf{b} \in \mathcal{C}_2$, and they are not singular points, neither isotropic points on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. Therefore, $\|\nabla f_1(\mathbf{a})\| \neq 0$, $\|\nabla f_2(\mathbf{b})\| \neq 0$. So, by the third and fourth equations we have that

$$\mathbf{c} = \mathbf{a} + \frac{\|\mathbf{c}-\mathbf{a}\|}{\|\nabla f_1(\mathbf{a})\|}\nabla f_1(\mathbf{a}) = \mathbf{b} + \frac{\|\mathbf{c}-\mathbf{b}\|}{\|\nabla f_2(\mathbf{b})\|}\nabla f_2(\mathbf{b}) \text{ and } \|\mathbf{c}-\mathbf{a}\| = \|\mathbf{c}-\mathbf{b}\|.$$

Therefore,

$$(\mathbf{a}, \mathbf{c}, \|\mathbf{c}-\mathbf{a}\|, 1/\|\nabla f_1(\mathbf{a})\|^2) \in \mathcal{B}_1, \ \ (\mathbf{b}, \mathbf{c}, \|\mathbf{c}-\mathbf{a}\|, 1/\|\nabla f_2(\mathbf{b})\|^2) \in \mathcal{B}_2.$$

So $\mathbf{c} \in \pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))$, and taking closures, $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \subset \overline{\pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))}$. Conversely, let $\mathbf{c} \in \pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))$. Then, there exists $d_0 \in \mathbb{C}$ such that $(\mathbf{c}, d_0) \in \pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2)$. So, there exist $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ and $W_1, W_2 \in \mathbb{C}$ such that $(\mathbf{a}, \mathbf{c}, d_0, W_1) \in \mathcal{B}_1$, $(\mathbf{b}, \mathbf{c}, d_0, W_2) \in \mathcal{B}_2$. Then, $(\mathbf{a}, \mathbf{b}, \mathbf{c}, W_1 W_2) \in \mathcal{A}$, whence $\mathbf{c} \in \pi_{\mathbf{z}}(\mathcal{A})$. Taking closures one gets the other inclusion. $\square$

Corollary 2.9. *Let $\mathcal{O}_i$ be the generic offset of $\mathcal{C}_i$ then* $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \subset \overline{\pi_{\mathbf{z}}(\mathcal{O}_1 \cap \mathcal{O}_2)}$.

Proof. Applying Theorem 2.8, one gets

$$\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_{\mathbf{z}}(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))} \subset \overline{\pi_{\mathbf{z}}(\overline{\pi_{\mathbf{z},d}(\mathcal{B}_1)} \cap \overline{\pi_{\mathbf{z},d}(\mathcal{B}_2)})} = \overline{\pi_{\mathbf{z}}(\mathcal{O}_1 \cap \mathcal{O}_2)}. \qquad \square$$

In the following diagram, we illustrate the combination of the offset and bisector constructions.

$$\begin{tikzcd}
\mathcal{B}_1 \subset \mathbb{C}^6 \arrow[d, "\pi_{\mathbf{x}}"'] \arrow[dr, "\pi_{\mathbf{z},d}"] & & \mathcal{B}_2 \subset \mathbb{C}^6 \arrow[d, "\pi_{\mathbf{x}}"] \arrow[dl, "\pi_{\mathbf{z},d}"'] \\
\mathcal{C}_1 \subset \mathbb{C}^2 & \mathcal{O}_1 \cap \mathcal{O}_2 \subset \mathbb{C}^3 \arrow[d, "\pi_{\mathbf{z}}"] & \mathcal{C}_2 \subset \mathbb{C}^2 \\
\mathcal{A} \subset \mathbb{C}^7 \arrow[u, "\pi_{\mathbf{x}}"] \arrow[r, "\pi_{\mathbf{z}}"] & \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \subset \mathbb{C}^2 & \mathcal{A} \subset \mathbb{C}^7 \arrow[u, "\pi_{\mathbf{y}}"'] \arrow[l, "\pi_{\mathbf{z}}"']
\end{tikzcd} \tag{3}$$

Remark 2.10 *(Bounding the degree of the bisector).* Corollary 2.9 can be used to analyze the degree and the genus of the bisector by means of the degree and the genus of the offsets (see San Segundo and Sendra, 2005, 2009 and Arrondo et al., 1999, respectively). In this remark we see how to derive a first bound for the degree of the bisector using the results of San Segundo and Sendra (2005, 2009). For the genus, the application is not so direct and needs a deeper analysis; and we leave this as future research.

Let $\mathcal{O}_i$ be the generic offset of $\mathcal{C}_i$, and let $O_i(\mathbf{z}, d)$ be the defining polynomial of $\mathcal{O}_i$. We note that $O_i \in \mathbb{C}[\mathbf{z}, d^2]$ (see Proposition 26 in San Segundo and Sendra, 2009). Therefore, by Corollary 2.9, it holds that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is included in the variety defined by $\mathrm{Res}_{d^2}(O_1(\mathbf{z}, d), O_2(\mathbf{z}, d))$, where Res_z denotes the resultant w.r.t. z. Therefore, taking into account the Sylvester expression of the resultant we have that

$$\deg(\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)) \leq \frac{1}{2}\left(\deg_d(O_2)\deg_{\mathbf{z}}(O_1) + \deg_d(O_1)\deg_{\mathbf{z}}(O_2)\right). \tag{4}$$

Finally, note that the results in San Segundo and Sendra (2005, 2009) provide the exact value of $\deg_{\mathbf{z}}(O_i)$ and $\deg_d(O_i)$, and hence upper bounds for the total degree of the bisector. □

Remark 2.11 *(Computational issues: second part).* Corollary 2.9 and Remark 2.10 provide two different alternatives to determine $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$. For instance, one may use interpolation techniques in combination with the degree bounds given in Remark 2.10. Also, one may compute $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ by determining the resultant $\mathrm{Res}_{d^2}(O_1(\mathbf{z}, d), O_2(\mathbf{z}, d))$ (where we use the notation introduced in Remark 2.10). Nevertheless, for this purpose, the generic offsets must be computed first. □

Theorem 2.12. *If $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is not empty, then all its components have dimension 1.*

Proof. From Lemma 3 in San Segundo and Sendra (2009), the generic offset $\mathcal{O}_i$ of $\mathcal{C}_i$ is a surface in $\mathbb{C}^3$. So, each irreducible component of $\pi_{\mathbf{z},d}(\mathcal{B}_i)$ is a quasiprojective variety of dimension 2. Furthermore, since $\mathcal{C}_1$ and $\mathcal{C}_2$ are irreducible and different, $\pi_{\mathbf{z},d}(\mathcal{B}_1)$ and $\pi_{\mathbf{z},d}(\mathcal{B}_2)$ have no common component. So, applying Corollary 1, page 75 in Shafarevich (1994) to each component of $\pi_{\mathbf{z},d}(\mathcal{B}_1)$ and each component of $\pi_{\mathbf{z},d}(\mathcal{B}_2)$, we get that either $\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2) = \emptyset$ or all its components have dimension 1. However, by Theorem 2.8, if $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) \neq \emptyset$ then $\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2) \neq \emptyset$. Now, let us prove that for every $\mathbf{c} \in \pi_z(\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2))$, $\pi_{\mathbf{z}}^{-1}(\mathbf{c})$ is finite. Indeed, if $\mathrm{card}(\pi^{-1}(\mathbf{c})) = \infty$ then there exist infinitely many $d_i \in \mathbb{C}$ such that $\mathcal{O}_1(\mathbf{c}, d_i) = \mathcal{O}_2(\mathbf{c}, d_i) = 0$. But this implies that $\mathbf{c}$ belongs to the offset of $\mathcal{C}_i$ for almost all distances, which is impossible (see Lemma 4 in San Segundo and Sendra, 2005). Therefore, by Theorem 11.12 in Harris (1995), the dimension of the components of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ and of $\pi_{\mathbf{z},d}(\mathcal{B}_1) \cap \pi_{\mathbf{z},d}(\mathcal{B}_2)$ is the same. □

Remark 2.13. If we allow the curves not to be real, and we exclude the isotropic condition in the incidence variety $\mathcal{A}$, the dimension of the bisector may drop to 0. For instance, let $\mathcal{C}_1$ be the parabola defined by $f_1 = x_2^2 - ix_1$ and $\mathcal{C}_2$ be the line defined by $f_2 = y_2 + iy_1 + 1$. Since $\|\nabla f_2\| = \|(i, 1)\| = 0$, applying Definition 2.1, we get that $\mathcal{A} = \emptyset$, and hence $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \emptyset$. Let us see what happens if we do not consider the last equation in the definition of $\mathcal{A}$. Then, the variety $\mathcal{A}$ is

$$\left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{C}^6 \;\middle|\; \begin{array}{l} x_2^2 - ix_1 = 0 \\ y_2 + iy_1 + 1 = 0 \\ -i\,x_2 + i\,z_2 - 2x_2x_1 + 2x_2z_1 = 0. \\ i\,y_2 - i\,z_2 - y_1 + z_1 = 0. \\ (x_1 - z_1)^2 + (x_2 - z_2)^2 = (y_1 - z_1)^2 + (y_2 - z_2)^2. \end{array} \right\}.$$

Computing a suitable Gröbner basis, one gets that the $\mathbf{z}$-elimination ideal of $\mathcal{A}$ is

$$< i\,z_1 + z_2 + 1, -21\,i\,z_1^4 + 8z_1^5 + 31\,i\,z_1^2 - 34z_1^3 - 5\,i + 18z_1 >$$

Therefore, the bisector is

$$\begin{aligned} \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2) &= \left\{ \mathbf{z} \in \mathbb{C}^2 \;\middle|\; \begin{array}{l} i\,z_1 + z_2 + 1 = 0 \\ -21\,i\,z_1^4 + 8z_1^5 + 31\,i\,z_1^2 - 34z_1^3 - 5\,i + 18z_1 = 0 \end{array} \right\} \\ &= \left\{ \left(\frac{5}{8}i, -\frac{3}{8}\right), \left(\frac{i}{2} \pm \frac{1}{2}\sqrt{3}, -\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right) \right\} \end{aligned}$$

Thus, if we exclude the last equation of the definition of $\mathcal{A}$, then $\dim(\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)) = 0$. We observe that $\left(\frac{i}{2} \pm \frac{1}{2}\sqrt{3}, -\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right) \in \mathcal{C}_1 \cap \mathcal{C}_2$, and that $P := \left(\frac{5}{8}i, -\frac{3}{8}\right) \in \mathcal{C}_2 \setminus \mathcal{C}_1$, but there exists a point in $\mathcal{C}_1$, namely $Q := \left(-\frac{i}{4}, \frac{1}{2}\right)$ such that $\|P - Q\|^2 = 0$. □

3. Untrimmed bisectors: parametric case

In Definition 2.1, we have introduced the notion of bisector of two plane curves, independently of the representation. Nevertheless, in many situations, the used curves are rational, and hence admit a rational parametric representation. In the following we see how to adapt the incidence variety $\mathcal{A}$ to the parametric case, such that the bisector computation is simplified.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be rational, and $\mathcal{P}_1(t_1)$ and $\mathcal{P}_2(t_2)$ be (non-necessarily proper) rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We assume that $\mathcal{P}_1(t_1)$ and $\mathcal{P}_2(t_2)$ are expressed as

$$\mathcal{P}_1(t_1)=\left(\frac{a_1(t_1)}{c(t_1)},\frac{a_2(t_1)}{c(t_1)}\right),\ \mathcal{P}_2(t_2)=\left(\frac{b_1(t_2)}{d(t_2)},\frac{b_2(t_2)}{d(t_2)}\right),$$

where $\gcd(a_1,a_2,c)=1$ and $\gcd(b_1,b_2,d)=1$. Besides ∇f_i, we consider

$$\mathcal{T}_1=\left(-\frac{\partial f_1}{\partial x_2},\frac{\partial f_1}{\partial x_1}\right),\quad \mathcal{T}_2=\left(-\frac{\partial f_2}{\partial y_2},\frac{\partial f_2}{\partial y_1}\right).$$

Associated with $\mathcal{P}_i$, we introduce the incidence variety $\mathcal{A}_{\mathcal{P}}$ defined as (we write $\mathbf{t}=(t_1,t_2)$)

$$\mathcal{A}_{\mathcal{P}}=\left\{(\mathbf{t},\mathbf{z},W)\in\mathbb{C}^5\ \middle|\ \begin{array}{l}(z_1\,c(t_1)-a_1(t_1),z_2\,c(t_1)-a_2(t_1))\cdot\mathcal{T}_1(\mathcal{P}_1(t_1))=0,\\ (z_1\,d(t_2)-b_1(t_2),z_2\,d(t_2)-b_2(t_2))\cdot\mathcal{T}_2(\mathcal{P}_2(t_2))=0,\\ \mathrm{num}(\|\mathbf{z}-\mathcal{P}_1(t_1)\|^2-\|\mathbf{z}-\mathcal{P}_2(t_2)\|^2)=0,\\ \Delta(\mathbf{t})W=1\end{array}\right\}, \tag{5}$$

where $\mathrm{num}(R)$ denotes the numerator of the rational function R and where

$$\Delta(\mathbf{t})=c(t_1)\,d(t_2)\,\mathrm{num}(\|\nabla f_1(\mathcal{P}_1(t_1))\|^2)\,\mathrm{num}(\|\nabla f_2(\mathcal{P}_2(t_2))\|^2)\,.$$

Denoting $\pi_{t_i}(\mathbf{t},\mathbf{z},W)=t_i$, we have the following diagram.

$$\begin{tikzcd}
\mathcal{C}_1\subset\mathbb{C}^2 & \mathbb{C} \arrow[l, "\mathcal{P}_1"'] & & \mathbb{C} \arrow[r, "\mathcal{P}_2"] & \mathcal{C}_2\subset\mathbb{C}^2 \\
& & \mathcal{A}_{\mathcal{P}} \arrow[ul, "\pi_{t_1}"] \arrow[ur, "\pi_{t_2}"'] \arrow[d, "\pi_{\mathbf{z}}"] & & \\
& & \mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2) & &
\end{tikzcd} \tag{6}$$

In the following theorem we assume that the parametrizations are normal. We recall that a parametrization is normal if the map that it defines is surjective over the curve, that is, if all (affine) points are reachable by at least one parameter value. There are two remarkable facts: any rational affine curve can always be parametrized normally over the algebraic closure of the ground field; and, if the parametrization is not normal, there is only one missing point that is, indeed, the point corresponding via the parametrization to the infinity of the parameter line (see Section 6.3 in Sendra et al., 2007 for details on normal parametrizations).

Theorem 3.1. *Let $\mathcal{P}_1(t_1)$, $\mathcal{P}_2(t_2)$ be normal. Then* $\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)=\overline{\pi_z(\mathcal{A}_{\mathcal{P}})}$.

Proof. Consider the rational map

$$\begin{array}{rccc}\varphi: & \mathcal{A}_{\mathcal{P}} & \to & \mathbb{C}^7\\ & (\mathbf{t},\mathbf{z},W) & \mapsto & \left(\mathcal{P}_1(t_1),\mathcal{P}_2(t_2),\mathbf{z},\dfrac{W}{c(t_1)d(t_2)}\right).\end{array}$$

Observe that, because of the last equation of $\mathcal{A}_{\mathcal{P}}$, φ is well-defined on all points of $\mathcal{A}_{\mathcal{P}}$. Moreover, $\varphi(\mathcal{A}_{\mathcal{P}})\subset\mathcal{A}$ (see the definition of $\mathcal{A}$ (1)). So, since the $\mathbf{z}$ component is invariant under φ, $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}})=\pi_{\mathbf{z}}(\varphi(\mathcal{A}_{\mathcal{P}}))\subset\pi_{\mathbf{z}}(\mathcal{A})$. Conversely, let $\mathbf{z}_0\in\pi_{\mathbf{z}}(\mathcal{A})$. Then, there exist $\mathbf{x}_0,\mathbf{y}_0,W_0$ such that $(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0,W_0)\in\mathcal{A}$. Moreover, since $\mathcal{P}_i$ are normal, there exist t_0,h_0 such that $\mathbf{x}_0=\mathcal{P}_1(t_0),\mathbf{y}_0=\mathcal{P}_2(h_0)$. Furthermore, since $\mathbf{x}_0,\mathbf{y}_0$ are not isotropic and their first component is nonzero, $\Delta(t_0,h_0)$ is not identically zero. Therefore, $(t_0,h_0,\mathbf{z}_0,1/\Delta(t_0,h_0))\in\mathcal{A}_{\mathcal{P}}$. So, $\mathbf{z}_0\in\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}})$. Thus, $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}})=\pi_{\mathbf{z}}(\mathcal{A})$. $\square$

Remark 3.2. We observe the following

1. Every rational curve can be parametrized properly and normally (see Theorem 6.26 in Sendra et al., 2007).
2. If we use non-normal parametrizations, it may happen that $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) \subsetneq \pi_{\mathbf{z}}(\mathcal{A})$, and hence the parametrization may not compute the whole bisector (see Example 3.3). If this happens, the missing subset is included in the union of the normal lines to $\mathcal{C}_1$ and $\mathcal{C}_2$ at the critical points of the parametrizations (see Corollary 6.23 in Sendra et al., 2007 for the notion of critical point).

   Proof. If $\mathcal{P}_1(t_1)$ is not normal (similarly for $\mathcal{P}_2(t_2)$) then exactly one point on $\mathcal{C}_1$ is not reachable by $\mathcal{P}_1$, namely, the critical point. So, the argument of the inclusion $\pi_{\mathbf{z}}(\mathcal{A}) \subset \pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}})$, in the proof of Theorem 3.1, may fail, and one can only ensure that $\pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) \subset \pi_{\mathbf{z}}(\mathcal{A})$. Indeed, the argument fails if for $\mathbf{z}_0 \in \operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ it holds that $\pi_{\mathbf{x}}(\pi_{\mathbf{z}}^{-1}(\mathbf{z})) = \emptyset$ for almost all $\mathbf{z}$ in the components of $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ that contains $\mathbf{z}_0$. In this situation, if $\mathbf{a}$ is the critical point of $\mathcal{P}_1$, by the definition of $\mathcal{A}$, almost all points, in those components of $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$, belong to the normal line to $\mathcal{C}_1$ at $\mathbf{a}$. Now, the result follows using that $\dim(\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)) = 1$ (see Theorem 2.12) and that the normal line is unique. □

3. If one does not want to use normal parametrizations, because of the previous remark, one can directly check whether the normal lines to the critical points are included in $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$.
4. In the definition of $\mathcal{A}_{\mathcal{P}}$ one can replace $\nabla f_i(\mathcal{P}_i)$ by $(\mathcal{P}_i)'$, where $(\mathcal{P}_i)'$ denotes the velocity vector. However, in this case a similar phenomenon to the normality can happen if the parametrization has singular points not being singular points of the curve. Nevertheless, this case may be avoided by checking whether the corresponding normal lines are in the bisector.

Example 3.3. Let $\mathcal{C}_1$ be the circle $x_1^2 + x_2^2 = 1$ and $\mathcal{C}_2$ be the line $y_1 + 1 = 0$. Using directly the definition of bisector, we get that $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is the line and the parabola defined as $z_2(-z_2^2 + 4z_1 + 4) = 0$. We consider now the parametrizations

$$\mathcal{P}_1(t_1) = \left(\frac{-t_1^2 + 1}{t_1^2 + 1}, \frac{2t_1}{t_1^2 + 1} \right), \; \mathcal{P}_2(t_2) = (-1, t_2).$$

Note that $\mathcal{P}_1(t_1)$ is not normal, and its critical point is $(-1, 0)$; $\mathcal{P}_2(t_2)$ is normal. The computation returns the correct answer. However, if we take $\mathcal{P}_2$ as the non-normal parametrization $\mathcal{P}_2(t_2) = (-1, 1/t_2)$, the computation returns the parabola $-z_2^2 + 4z_1 + 4 = 0$, and the line $z_2 = 0$ is missed. Note that the missing line is the normal line to both, $\mathcal{C}_1, \mathcal{C}_2$, at $(-1, 0)$.

Analogously, if only one of the curves, say $\mathcal{C}_1$, is expressed parametrically, we can consider the incidence variety

$$\mathcal{A}_{\mathcal{P}}^{I} = \left\{ (t, \mathbf{y}, \mathbf{z}, W) \in \mathbb{C}^6 \;\middle|\; \begin{array}{l} (z_1\, c(t_1) - a_1(t_1), z_2\, c(t_1) - a_2(t_1)) \cdot \mathcal{T}_1(\mathcal{P}_1(t)) = 0 \\ (\mathbf{z} - \mathbf{y}) \cdot \mathcal{T}_2(\mathbf{y}) = 0 \\ \operatorname{num}(\|\mathbf{z} - \mathcal{P}_1(t)\|^2 - \|\mathbf{z} - \mathbf{y}\|^2) = 0 \\ \Delta(t) W = 1 \end{array} \right\} \subset \mathbb{C}^5,$$

where $\Delta(t) = c(t)\, \operatorname{num}(\|\nabla f_1(\mathcal{P}_1(t))\|^2)\, \|\nabla f_2(\mathbf{y})\|^2$. Reasoning similarly, we have the following theorem.

Theorem 3.4. *Let $\mathcal{P}_1(t)$ be normal. Then* $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2) = \overline{\pi_z(\mathcal{A}_{\mathcal{P}}^{I})}$.

4. Parametric representation of the untrimmed bisector

Throughout this section, we consider that $\mathcal{C}_1, \mathcal{C}_2$ are rational, and we keep the notation used in Section 2. It is clear that, using the fact that $\operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is algebraic, the untrimmed bisector can be represented by means of its implicit equations. Nevertheless, these equations can be huge, and hence hard to manage (see e.g. Examples 2.6, 2.7). An alternative would be to use rational parametric representations of the bisector. However, in general, the bisector can be reducible (see e.g. Examples 2.4, 2.5, 2.6, 2.7). Furthermore, some of the bisector components have positive genus (see e.g. Example 2.6). In Elber and Kim (1998), an alternative representation for irreducible bisectors, based on the parameters space, is introduced. They assume the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ to be rational, regular and C^1-continuous. Here, in this section, we formally extend this representation to the general case.

In this situation, let us consider the following diagram (see (5) in Section 3 for the definition of $\mathcal{A}_{\mathcal{P}}$)

$$\begin{tikzcd} & \mathcal{A}_{\mathcal{P}} \arrow[dl, "\pi_{\mathbf{z}}"'] \arrow[dr, "\pi_{\mathbf{t}}"] & \\ \mathbb{C}^2 \supset \operatorname{Bis}(\mathcal{C}_1, \mathcal{C}_2) \supset \pi_{\mathbf{z}}(\mathcal{A}_{\mathcal{P}}) & & \pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}}) \subset \mathbb{C}^2 \end{tikzcd} \tag{7}$$

where $\pi_{\mathbf{z}}(\mathbf{t},\mathbf{z},W)=\mathbf{z}$, and $\pi_{\mathbf{t}}(\mathbf{t},\mathbf{z},W)=\mathbf{t}$. We denote by $\mathcal{M}_{\mathcal{P}}$ the Zariski closure of $\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})$, that is

$$\mathcal{M}_{\mathcal{P}} := \overline{\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})}.$$

In the following theorem we analyze the previous diagram. We prove that $\pi_{\mathbf{z}}$ is birational over each irreducible component. Also, we analyze the cardinality and dimension of the fiber of $\pi_{\mathbf{t}}$; the underlying geometric idea is that if two curves touch non-transversally at a common point, then there exists a linear family of circles touching both curves at that point.

Theorem 4.1. *Let $\Gamma \subset \mathcal{M}_{\mathcal{P}}$ be an irreducible component. Then,*

1. *If $\dim(\Gamma)>0$ then $\pi_{\mathbf{t}}:\overline{\pi_{\mathbf{t}}^{-1}(\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})\cap\Gamma)} \to \Gamma$ is a birational map.*
2. *If $\dim(\Gamma)=0$, say $\Gamma=\{(\alpha,\beta)\}$ then*
   (a) *If $\mathcal{P}_1(\alpha)\neq\mathcal{P}_2(\beta)$, $\pi_{\mathbf{t}}^{-1}((\alpha,\beta))$ has cardinality 1.*
   (b) *If $\mathcal{P}_1(\alpha)=\mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel, $\pi_{\mathbf{t}}^{-1}((\alpha,\beta))$ has cardinality 1.*
   (c) *If $\mathcal{P}_1(\alpha)=\mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel, then $\pi_{\mathbf{t}}^{-1}((\alpha,\beta))$ has dimension 1.*

Proof. Let $\dim(\Gamma)>0$. Since $\pi_{\mathbf{t}}$ is rational, we only need to prove that the generic fiber of $\pi_{\mathbf{t}}$ on Γ has cardinality 1. For this purpose, we consider the set Σ of all $\mathbf{t}$ such that $\mathcal{P}_1(t_1)=\mathcal{P}_2(t_2)$. Since $\mathcal{C}_1\neq\mathcal{C}_2$, Σ is finite, and since $\dim(\Gamma)>0$ and irreducible, then $\Gamma^* := (\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})\cap\Gamma)\setminus\Sigma$ is non-empty and dense in Γ. Let us study $\pi_{\mathbf{t}}^{-1}(\mathbf{t}_0)$, for $\mathbf{t}_0=(\alpha,\beta)\in\Gamma^*$. Since $\Gamma^*\subset\pi_{\mathbf{t}}(\mathcal{A}_{\mathcal{P}})$, the fiber is non-empty. Moreover, since W only appears in one equation, and with degree 1, the cardinality $\pi_{\mathbf{t}}^{-1}(\mathbf{t}_0)$ is equal to the number of $\mathbf{z}$ solutions of the first three equations. Because of the last equation of $\mathcal{A}_{\mathcal{P}}$, $\|\nabla f_1(\mathcal{P}_1(\alpha))\|^2\neq 0$, $\|\nabla f_2(\mathcal{P}_2(\beta))\|^2\neq 0$. So, the first and second equations of $\mathcal{A}_{\mathcal{P}}$ imply that the corresponding $\mathbf{z}$ in the fiber are in the intersection of the normal lines to $\mathcal{C}_1$ at $\mathcal{P}_1(\alpha)$ and to $\mathcal{C}_2$ at $\mathcal{P}_2(\beta)$. If $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel, the two lines intersect at a point, and hence the fiber has cardinality 1. Let us assume that $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel. We know that the fiber is not empty, hence both normal lines must be equal. So, since $\mathcal{P}_1(\alpha)\neq\mathcal{P}_2(\beta)$, the only $\mathbf{z}=\mathcal{P}_1(\alpha)+\lambda\mathbf{n}=\mathcal{P}_2(\beta)+\mu\mathbf{n}$, such that $\|\mathbf{z}-\mathcal{P}_1(\alpha)\|^2=\|\mathbf{z}-\mathcal{P}_2(\beta)\|^2$, where $\mathbf{n}=\nabla f_1(\mathcal{P}_1(\alpha))/\|\nabla f_1(\mathcal{P}_1(\alpha))\|$, is the point $\mathbf{z}=\mathcal{P}_1(\alpha)+\frac{1}{2}(\mathcal{P}_2(\beta)-\mathcal{P}_1(\alpha))$.

Now, let $\Gamma=\{(\alpha,\beta)\}$. If $\mathcal{P}_1(\alpha)\neq\mathcal{P}_2(\beta)$, then $\Gamma^*=\Gamma$ and the above reasoning is valid, and hence the fiber has cardinality 1. If $\mathcal{P}_1(\alpha)=\mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel, again, the reasoning is also valid. However, if $\mathcal{P}_1(\alpha)=\mathcal{P}_2(\beta)$ and $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel then $\pi_1^{-1}((\alpha,\beta))$ have dimension 1; namely the lifting to $\mathcal{A}_{\mathcal{P}}$ of the normal line at $\mathcal{P}_1(\alpha)$. □

Remark 4.2. In Theorem 4.1 we have considered zero-dimension components of $\mathcal{M}_{\mathcal{P}}$. However, in all checked examples $\mathcal{M}_{\mathcal{P}}$ has none zero-dimensional component. □

Based on the previous theorem, we introduce the following definition.

Definition 4.3. Let $\Gamma\subset\mathcal{M}_{\mathcal{P}}$ be irreducible of positive dimension. We associate with Γ the rational map $\Upsilon_\Gamma:\Gamma\to\mathrm{Bis}(\mathcal{C}_1,\mathcal{C}_2)$, where

$$\Upsilon_\Gamma=\pi_{\mathbf{z}}\circ(\pi_{\mathbf{t}}|_\Gamma)^{-1}.$$

Remark 4.4. If Γ is a rational curve, and $\mathcal{Q}$ is a parametrization, then $\Gamma\circ\mathcal{Q}$ is a parametrization of the component $\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma))$ of the bisector. □

Remark 4.5. From the computational point of view, if $\dim(\Gamma)=1$ and its defining polynomial is $\gamma(\mathbf{t})$, then $\mathcal{T}_1(\mathcal{P}_1(t_1))$, $\mathcal{T}_2(\mathcal{P}_2(t_2))$ are parallel in $\mathbb{C}(\Gamma)^2$ iff γ divides the numerator of the determinant of the matrix whose rows are $\mathcal{T}_i(\mathcal{P}_i(t_i))$. □

Remark 4.6. If Γ is an irreducible component of $\mathcal{M}_{\mathcal{P}}$ with $\dim(\Gamma)=0$, say $\Gamma=\{(\alpha,\beta)\}$, then

1. if $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are not parallel,

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma))=\left\{\begin{pmatrix}\mathcal{T}_1(\mathcal{P}_1(\alpha))\\ \mathcal{T}_2(\mathcal{P}_2(\beta))\end{pmatrix}^{-1}\begin{pmatrix}\mathcal{P}_1(\alpha)\cdot\mathcal{T}_1(\mathcal{P}_1(\alpha))\\ \mathcal{P}_2(\beta)\cdot\mathcal{T}_h(\mathcal{P}_2(\beta))\end{pmatrix}\right\}.$$

2. If $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel, and $\mathcal{P}_1(\alpha)\neq\mathcal{P}_2(\beta)$ then

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma))=\left\{\mathcal{P}_1(\alpha)+\frac{1}{2}(\mathcal{P}_2(\beta)-\mathcal{P}_1(\alpha))\right\}.$$

3. If $\nabla f_1(\mathcal{P}_1(\alpha)), \nabla f_2(\mathcal{P}_2(\beta))$ are parallel, and $\mathcal{P}_1(\alpha) = \mathcal{P}_2(\beta)$ then

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma)) = \{\mathcal{P}_1(\alpha) + \lambda \nabla f_1(\mathcal{P}_1(\alpha)) \mid \lambda \in \mathbb{C}\}. \qquad \square$$

Based on the previous result we introduce the following representation of the bisector.

Definition 4.7. We define the parametric representation of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ as the set

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \bigcup_{\Gamma \in \mathcal{M}_{\mathcal{P}}^+} \{(\Gamma, \Upsilon_\Gamma(\mathbf{t}))\} \bigcup_{\Gamma \in \mathcal{M}_{\mathcal{P}}^0} \{(\Gamma, \pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}(\Gamma)))\}$$

where $\mathcal{M}_{\mathcal{P}}^+$ denotes the set of all irreducible components of positive dimension of $\mathcal{M}_{\mathcal{P}}$, and $\mathcal{M}_{\mathcal{P}}^0$ denotes the set of all irreducible 0-dimensional components of $\mathcal{M}_{\mathcal{P}}$.

If a component Γ is rational, and $\mathcal{Q}(h)$ a parametrization of Γ (see Remark 4.4), we will write $(\mathbb{C}, \Upsilon_\Gamma(\mathcal{Q}(h)))$, instead of $(\Gamma, \Upsilon_\Gamma(\mathbf{t}))$.

Remark 4.8. We observe that $\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2)$ is a finite union of pairs; say that (A, B) is one of these pairs. A will be either an irreducible plane curve or a finite collection of points. Let us describe each case

- If A is a finite collection of points, then B is also a finite collection of points; namely, the points on the bisector associated to the points in A.
- If A is a curve, then B is a rational map from A on the bisector. So, if A is a curve, then B is a pair of rational functions. In addition, we may distinguish two cases:
  - If A is not rational, then B is defined by means of a pair of bivariate rational functions.
  - If A is rational, then A is birational to $\mathbb{C}$. So, we replace A by $\mathbb{C}$, and we replace B by the composition of B with any rational parametrization of A. Therefore, in this case, B is a pair of univariate rational functions. $\square$

Remark 4.9. If a normal to the original curves appears in the bisector, this line may be lost in the parametric representation; see Example 4.11. The reason is that $\pi_{\mathbf{t}} \circ \pi_{\mathbf{z}}^{-1}$ may send the whole line onto a point on a 1-dimensional component of $\mathcal{M}_{\mathcal{P}}$. Nevertheless, analyzing the points of $\mathcal{M}_{\mathcal{P}}$ where the tangent vectors $\mathcal{T}_i(\mathcal{P}_i)$ are parallel, and applying Remark 4.6, one can reach these lines; see again Examples 4.11 and 4.13. $\square$

We illustrate these ideas with some examples.

Example 4.10. We consider Example 2.4. We take

$$\mathcal{P}_1(t_1) = \left(\frac{4t_1}{t_1^2+1}, \frac{2(t_1^2-1)}{t_1^2+1}\right), \quad \mathcal{P}_2(t_2) = \left(\frac{2t_2}{t_2^2+1}, \frac{t_2^2-1}{t_2^2+1}\right).$$

Then, $\mathcal{M}_{\mathcal{P}} = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \equiv t_1 t_2 + 1 = 0$ and $\Gamma_2 \equiv t_1 - t_2 = 0$. On the other hand,

$$\det \begin{pmatrix} \mathcal{T}_1(\mathcal{P}_1(t_1)) \\ \mathcal{T}_2(\mathcal{P}_2(t_2)) \end{pmatrix} = \frac{-16(t_1t_2+1)(-t_2+t_1)}{(t_1^2+1)(t_2^2+1)}$$

that is zero in $\mathbb{C}(\Gamma_1)$ and in $\mathbb{C}(\Gamma_2)$. So, $\mathcal{T}_1(\mathcal{P}_1(t_1)), \mathcal{T}_2(\mathcal{P}_2(t_2))$ are parallel on both components. Therefore,

$$\Upsilon_{\Gamma_i} : \Gamma_i \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \mathbf{t} \mapsto \mathcal{P}_1(t_1) + \frac{1}{2}(\mathcal{P}_2(t_2) - \mathcal{P}_1(t_1)).$$

Furthermore, since Γ_i is rational, composing Υ_{Γ_i} with the parametrization $\mathcal{Q}_1(h) = (h, 1/h)$ of Γ_1 and $\mathcal{Q}_2(h) = (h, h)$, we get (see Fig. 5)

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \left\{\left(\mathbb{C}, \left(\frac{h}{h^2+1}, \frac{1}{2}\frac{h^2-1}{h^2+1}\right)\right), \left(\mathbb{C}, \left(\frac{3h}{h^2+1}, \frac{3}{2}\frac{h^2-1}{h^2+1}\right)\right)\right\}.$$

Recall that $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ is defined by $(4z_1^2 + 4z_2^2 - 1)(4z_1^2 + 4z_2^2 - 9) = 0$.

Example 4.11. We consider Example 2.5. Taking $\mathcal{P}_1(t_1) = (t_1, t_1^2)$ and $\mathcal{P}_2(t_2) = (t_2, 0)$, we get that $\mathcal{M}_{\mathcal{P}}$ is the rational curve Γ defined as $t_1^4 + t_1 t_2 - t_2^2 = 0$. On the other hand, the determinant of the tangent vectors is $2t_1$ that is not zero over $\mathbb{C}(\Gamma)$. So, they are not parallel. Therefore,

$$\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \mathbf{t} \mapsto \left(t_2, \frac{2t_1^3 + t_1 - t_2}{2t_1}\right).$$

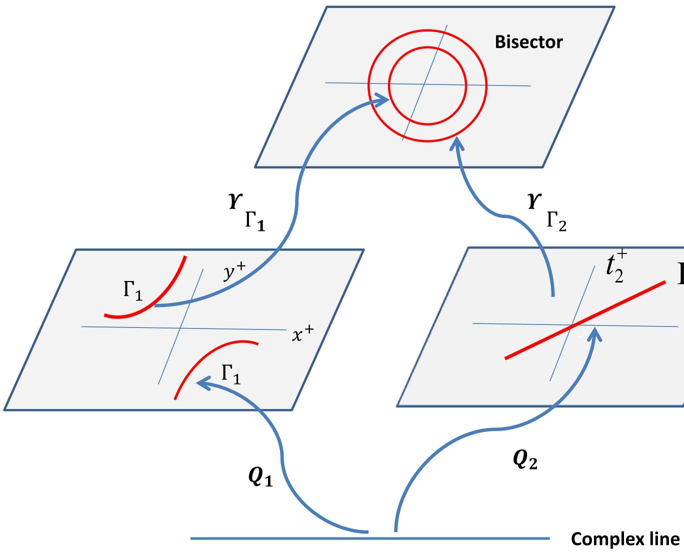

Fig. 5. $\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2)$ in Example 4.10.

Furthermore, since Γ is rational, composing Υ_Γ with the parametrization

$$\mathcal{Q}(h) = \left(\frac{h}{h^2-1}, \frac{-h}{h^4-2h^2+1}\right)$$

of Γ we get

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \left\{\left(\mathbb{C}, \left(\frac{-h}{h^4-2h^2+1}, \frac{(h^2+1)h^2}{2(h^4-2h^2+1)}\right)\right)\right\}.$$

See the implicit equation of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ in Example 2.5. One may observe that the parametric representation is missing the line $z_1 = 0$ that is the normal line at $(0, 0)$ of both initial curves $\mathcal{C}_1$ and $\mathcal{C}_2$. What happens is that $\pi_{\mathbf{t}}^{-1}(\pi_{\mathbf{z}}(0, \lambda)) = \{(0, 0)\} \subset \Gamma$; compare to Remark 4.9. Now, we consider the intersection of Γ with the determinant of the tangent vectors $\mathcal{T}_i(\mathcal{P}_i)$, namely $2t_1$. This gives, precisely the point $(\alpha, \beta) = (0, 0)$, and applying Remark 4.6, we get

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}((0,0))) = \{\mathcal{P}_1(0) + \lambda \nabla f_1(\mathcal{P}_1(0)) \mid \lambda \in \mathbb{C}\} = \{(0, \lambda) \mid \lambda \in \mathbb{C}\}.$$

So, we have

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \left\{\left(\mathbb{C}, \left(\frac{-h}{h^4-2h^2+1}, \frac{(h^2+1)h^2}{2(h^4-2h^2+1)}\right)\right), (\mathbb{C}, (0, h))\right\}.$$

Example 4.12. We consider Example 2.6. Taking $\mathcal{P}_1(t_1) = (t_1^2, t_1)$ and $\mathcal{P}_2(t_2) = (t_2, t_2^2)$, we get that $\mathcal{M}_{\mathcal{P}} = \Gamma_1 \cup \Gamma_2$ where Γ_1 is the line $t_1 = t_2$ and Γ_2 is the genus 4, 5-degree, curve defined as (observe that one component of $\mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2)$ has genus 4)

$$4t_1^4 t_2 + 4t_1^3 t_2^2 + 4t_1^2 t_2^3 + 4t_1 t_2^4 - 4t_1^2 t_2^2 - 3t_1^3 - 3t_1^2 t_2 - 3t_1 t_2^2 - 3t_2^3 + 2t_1 t_2 - t_1 - t_2 = 0$$

On the other hand, the determinant of the tangent vectors is $4t_1 t_2 - 1$ that is not zero over $\mathbb{C}(\Gamma_i)$. So, they are not parallel. Therefore,

$$\Upsilon_{\Gamma_i} : \Gamma_i \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2); \mathbf{t} \mapsto \left(\frac{t_2\left(4{t_1}^3 - 2{t_2}^2 + 2t_1 - 1\right)}{4t_2 t_1 - 1}, \frac{-t_1\left(-4{t_2}^3 + 2{t_1}^2 - 2t_2 + 1\right)}{4t_2 t_1 - 1}\right).$$

Furthermore, since Γ_1 can be parametrized by (h, h), we can take $\Upsilon_{\Gamma_1}(h, h)$ that is

$$\left(\frac{(2h+1)h^2}{2h^2+1}, \frac{(2h+1)h^2}{2h^2+1}\right) \sim (h, h).$$

Thus (see Fig. 6),

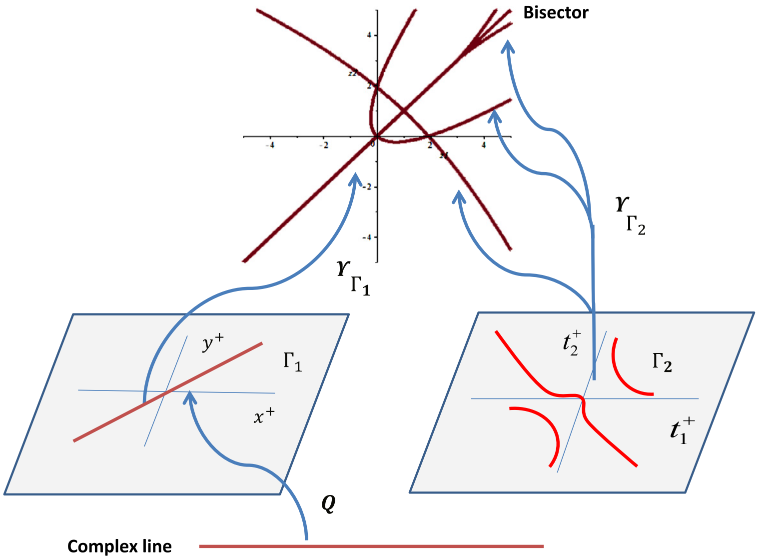

Fig. 6. PBis$(\mathcal{P}_1, \mathcal{P}_2)$ in Example 4.12.

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \left\{ (\mathbb{C}, (h, h)), \left(\Gamma_2, \left(\frac{{t_2}^2 \left(4{t_1}^3 + 2{t_1}^2 - 2t_2 - 1\right)}{4{t_2}^2{t_1}^2 - 1}, -\frac{{t_1}^2 \left(-4{t_2}^3 - 2{t_2}^2 + 2t_1 + 1\right)}{4{t_2}^2{t_1}^2 - 1} \right)\right) \right\}.$$

Observe that this representation is much simpler than the implicit equation of Bis$(\mathcal{C}_1, \mathcal{C}_2)$ (see details in Example 2.6).

Example 4.13. We consider Example 2.7. Taking $\mathcal{P}_1(t_1) = (t_1, t_1^2)$ and $\mathcal{P}_2(t_2) = (t_2, t_2^3)$, we get that $\mathcal{M}_{\mathcal{P}} = \Gamma$ where Γ is the genus 10, 8th-degree, curve defined as

$$3{t_2}^8 + 6{t_1}^2{t_2}^5 - 10t_1{t_2}^6 - 9{t_1}^4{t_2}^2 + 8{t_1}^3{t_2}^3 + 2{t_1}^5 - 3{t_1}^2{t_2}^2 + 4t_1{t_2}^3 - {t_2}^4 + 2{t_1}^2t_2 - 2t_1{t_2}^2 = 0$$

On the other hand, the determinant of the tangent vectors is $3t_2^2 - 2t_1$ that is not zero over $\mathbb{C}(\Gamma)$. So, they are not parallel. Therefore,

$$\Upsilon_\Gamma : \Gamma \to \mathrm{Bis}(\mathcal{C}_1, \mathcal{C}_2);\, \mathbf{t} \mapsto \left(-\frac{t_1t_2\left(-6{t_2}^4 + 6{t_1}^2t_2 + 3t_2 - 2\right)}{-3{t_2}^2 + 2t_1}, \frac{-3{t_2}^5 + 2{t_1}^3 + t_1 - t_2}{-3{t_2}^2 + 2t_1} \right).$$

One may observe that the parametric representation is missing the line $z_1 = 0$ that is the normal line at $(0, 0)$ of both initial curves $\mathcal{C}_1$ and $\mathcal{C}_2$. The reason for this is that $\pi_{\mathbf{t}}^{-1}(\pi_{\mathbf{z}}(0, \lambda)) = \{(0, 0), (0, (1/3)3^{3/4}), (0, (1/3i)3^{3/4}), (0, -(1/3)3^{3/4}), (0, -(1/3i)3^{3/4})\} \subset \Gamma$; compare with Remark 4.9. Now, we consider the intersection of Γ with the determinant of the tangent vectors $\mathcal{T}_i(\mathcal{P}_i)$, namely $3t_2^2 - 2t_1$. This gives, 7 points in Γ of the form $(0, \beta)$, and applying Remark 4.6, we get

$$\pi_{\mathbf{z}}(\pi_{\mathbf{t}}^{-1}((0, \beta))) = \{\mathcal{P}_1(0) + \lambda \nabla f_1(\mathcal{P}_1(0)) \mid \lambda \in \mathbb{C}\} = \{(0, \lambda) \mid \lambda \in \mathbb{C}\}.$$

So, we have

$$\mathrm{PBis}(\mathcal{P}_1, \mathcal{P}_2) = \left\{ \left(\Gamma, \left(-\frac{t_1t_2\left(-6{t_2}^4 + 6{t_1}^2t_2 + 3t_2 - 2\right)}{-3{t_2}^2 + 2t_1}, \frac{-3{t_2}^5 + 2{t_1}^3 + t_1 - t_2}{-3{t_2}^2 + 2t_1} \right)\right), (\mathbb{C}, (0, h)) \right\}.$$

Observe that this representation is much simpler than the implicit equation of Bis$(\mathcal{C}_1, \mathcal{C}_2)$

5. Generalization to hypersurfaces

In this section we comment on the generalization of the ideas and results in this paper to the case of hypersurfaces. The definition of (untrimmed) bisector given in Definition 2.1 generalizes directly to hypersurfaces. In this new setting, let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n), \mathbf{z} = (z_1, \ldots, z_n)$ and let $\mathcal{H}_1, \mathcal{H}_2$ be the real hypersurfaces defined by $f_1(\mathbf{x}) = 0$ and $f_2(\mathbf{y}) = 0$. Then, the incidence variety would be defined as

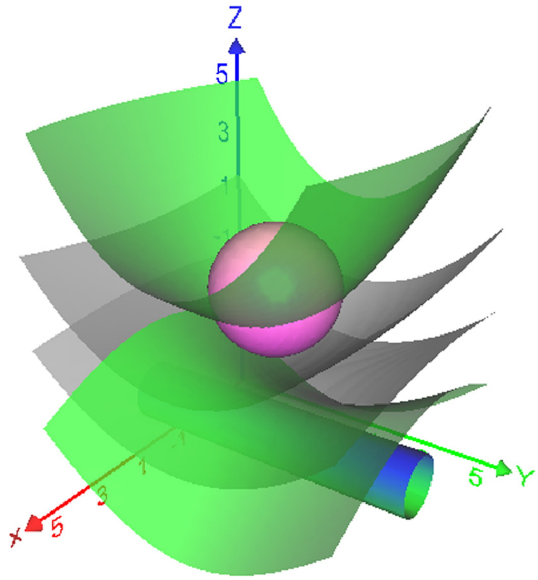

Fig. 7. Offset surface of a sphere and a cylinder.

$$\mathcal{A} = \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}, W) \in \mathbb{C}^{3n+1} \;\middle|\; \begin{array}{l} f_1(\mathbf{x}) = 0 \\ f_2(\mathbf{y}) = 0 \\ \operatorname{rank}\begin{pmatrix} \mathbf{z}-\mathbf{x} \\ \nabla f_1(\mathbf{x}) \end{pmatrix} = 1 \\ \operatorname{rank}\begin{pmatrix} \mathbf{z}-\mathbf{y} \\ \nabla f_2(\mathbf{y}) \end{pmatrix} = 1 \\ \|\mathbf{x}-\mathbf{z}\|^2 = \|\mathbf{y}-\mathbf{z}\|^2 \\ \|\nabla f_1(\mathbf{x})\|^2 \|\nabla f_2(\mathbf{y})\|^2 W = 1 \end{array} \right\}.$$

Let $\pi_{\mathbf{z}} : \mathcal{A} \subset \mathbb{C}^{3n+1} \to \mathbb{C}^n$; $(\mathbf{x}, \mathbf{y}, \mathbf{z}, W) \mapsto \mathbf{z}$. Then, the (untrimmed) bisector of $\mathcal{H}_1, \mathcal{H}_2$ is the Zariski closure of $\pi_{\mathbf{z}}(\mathcal{A})$, i.e. $\overline{\pi_{\mathbf{z}}(\mathcal{A})}$. The computation of the hypersurface bisector works identically as in the curve case (see Remark 2.3), as a consequence of the Closure Theorem. The following example illustrates how it works for $n = 3$.

Example 5.1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the sphere and cylinder defined by $x_1^2 + x_2^2 + x_3^2 = 4$ and $(y_3 + 5)^2 + y_1^2 = 1$, respectively (see Fig. 7). The incidence variety is defined by

$$\begin{aligned} &\{x_1^2 + x_2^2 + x_3^2 - 4, (y_3 + 5)^2 + y_1^2 - 1, 2z_1x_2 - 2z_2x_1, 2z_1x_3 - 2z_3x_1, 2z_2x_3 - 2z_3x_2, \\ &\quad -2(z_2 - y_2)y_1, (z_1 - y_1)(2y_3 + 10) - 2(z_3 - y_3)y_1, (z2 - y2)(2y_3 + 10), \\ &\quad (z_1 - x_1)^2 + (z_2 - x_2)^2 + (z_3 - x_3)^2 - (z_1 - y_1)^2 - (z_2 - y_2)^2 - (z_3 - y_3)^2, \\ &\quad (4x_1^2 + 4x_2^2 + 4x_3^2)(4y_1^2 + (2y_3 + 10)^2)W - 1\}. \end{aligned}$$

Considering $W > x_1 > x_2 > x_3 > y_1 > y_2 > y_3 > z_1 > z_2 > z_3$, and computing a Gröbner basis w.r.t. the lex order, we obtain

$$\begin{aligned} &(-z_2^4 + 20z_2^2z_3 + 36z_1^2 + 68z_2^2 - 64z_3^2 - 320z_3 - 256) \\ &\quad (-z_2^4 + 20z_2^2z_3 + 4z_1^2 + 52z_2^2 - 96z_3^2 - 480z_3 - 576). \end{aligned}$$

From the four sheets that one can see in Fig. 7, only one is the set of points equidistant from the sphere and the cylinder. It can be easily checked that three of these components, either touch or intersect the sphere or the cylinder. A method for computing parameterizations of the four components, containing square roots, is presented in Adamou (2013), section 3.4.

Note that in this example it is easy to find the equations of the generic offsets $\mathcal{O}_i(\mathbf{z}, d)$ and see that they have degree 2 in z_i and d. Hence the degree bound in (4) is verified.

The proofs of the Theorem 2.8 and Corollary 2.9 hold true similarly. However, the proof of Theorem 2.12 uses the fact that there is no point belonging to almost all offsets (i.e. for almost all values of the distance variable) of a curve. This is proved for curves in San Segundo and Sendra (2005). This, up to our knowledge, is an open question even for surfaces. Nevertheless, since Corollary 2.9 holds, the bound of the bisector given in Remark 2.10 is valid; note that formulas for computing the degree of the offset to a (rational) surface can be found in San Segundo and Sendra (2012).

The concepts and results in Section 3 are also valid for hypersurfaces, but one needs to work with normal hypersurfaces, and this is not a trivial property (for the case of surfaces, see e.g. Sendra et al., 2014, 2015). The results in Section 4, although theoretically correct for hypersurfaces, need a deeper analysis since the irreducible composition of the variety $\mathcal{M}_{\mathcal{P}}$ will include, in general, components whose whole dimension varies in $\{1, \ldots, n\}$.

6. Conclusions and future work

While all previous work about bisectors is mainly motivated by applications, in this article a general theoretical study of the untrimmed bisector of two real algebraic plane curves has been presented. It remains as an open question to prove that the parametric representation of the untrimmed bisector presented in Section 4 does not produce isolated points ever.

In the near future, we aim to devise a trimming method within the framework of the present paper.

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