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A quaternion approach to polynomial PN surfaces

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ABSTRACT

In this paper a new approach for a construction of polynomial surfaces with rational field of unit normals (PN surfaces) is presented. It is based on bivariate polynomials with quaternion coefficients. Relations between these coefficients are derived that allow one to construct PN surfaces of general odd and even degrees. For low degree PN surfaces the theoretical results are supplemented with algorithms and illustrated with numerical examples.

1. Introduction

Curves and surfaces which possess rational offsets are important for many practical applications in robotics, CAD/CAM systems, animations, manufacturing, etc. In the curve case, rational offsets can be assigned to the so-called Pythagorean hodograph (PH) curves. These curves have first been introduced in Farouki and Sakkalis (1990) and have been widely examined since then (see Farouki, 2008 and the references therein). The condition that characterizes a PH curve is a (piecewise) polynomial norm of its hodograph. Although, this condition connects the coefficients of the polynomial curve in a nonlinear way, an elegant construction that uses univariate polynomials with quaternion (complex) coefficients in a spatial (planar) case enables us to construct PH curves in a simple way. Moreover, interpolation schemes with these curves are easier to handle if the quaternion (complex) representation is used (see, e.g. Farouki, 1994; Farouki and Neff, 1995; Farouki et al., 2003, 2002; Pelosi et al., 2005; Kwon, 2010; Han, 2008; Choi et al., 2008; Bastl et al., 2014b, 2014a).

Surfaces with rational offsets are much less investigated than their curve counterparts. A surface with a rational field of unit normal vectors is called a Pythagorean normal vector (PN) surface, and such a surface clearly has rational offsets. Based on a dual approach PN surfaces were derived in Pottmann (1995) as the envelope of a two-parametric family of tangent planes with unit rational normals. Unfortunately, dual construction leads in general to rational surfaces and no algebraic criteria for a reduction of rational PN surfaces to polynomial ones is known yet. Also, to design a curve from its dual representation is not very intuitive and it is hard to avoid singularities and points at infinity.

One way to construct surfaces with rational offsets is to use surfaces with linear field of normal vectors, the so-called LN surfaces (see, e.g. Jüttler, 1998; Sampoli and Jüttler, 2000; Peternell and Odehnal, 2008a, 2008b). It is shown in Jüttler (1998) that one can find a rational reparameterization that converts the linear field of normal vectors to the one satisfying the PN property. The disadvantage of this approach is that we are limited to only linear normals and that the reparameterization

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raises the degree of the surface and its offset. In Lávička and Bastl (2007) it was shown that all non-developable quadratic triangular Bézier surfaces admit a rational convolution with any arbitrary rational surface. One direct consequence is that the offset surfaces of quadratic triangular Bézier surfaces are rational surfaces. An algorithm for computing the parameterization of these offsets can be found in Bastl et al. (2008). Moreover, in Peternell and Odehnal (2008a) it is proven with geometric reasons that the quadratic triangular Bézier surfaces are LN-surfaces, and the way to reparameterize the surfaces so that the normals obtain linear coordinate functions is demonstrated. For polynomial PN surfaces not much results can be found in the literature (see, e.g. Lávička and Vršek, 2012; Ueda, 1998; Lávička and Bastl, 2008). A family of cubic polynomial PN surfaces has been derived in Lávička and Vršek (2012). The authors introduce three different cubic surfaces as the generators of the whole family of cubic PN surfaces with birational Gauss mapping up to the translation, rotation and linear reparameterization. Two of these generators turn out to present a parameterization of the well known Enneper minimal surface with orthogonal parameter lines. The third one is a rotational surface based on a Tschirnhausen cubic curve.

A wide class of PN surfaces are the isothermal surfaces, defined by the property that the coefficients E, F, G of the first fundamental form satisfy $E = G$, $F = 0$. Isothermal surfaces belong to a family of scaled Pythagorean-hodograph preserving mappings distinguished by the property that for every PH curve in the surface domain the image curve is a PH curve too (Kim and Lee, 2008). A simply connected surface $\boldsymbol{S} : \Omega \to \mathbb{R}^3$ with zero mean curvature is called a *minimal surface* and it is known that it can be represented by the Enneper–Weierstrass parameterization which includes such a surface into the class of isothermal ones (see, e.g. Oprea, 2007). The Enneper–Weierstrass parameterization is of the form

$$\boldsymbol{S}(u,v) = \left(\mathrm{Re}\int f(1-g^2)\,dz,\ \mathrm{Re}\int \mathrm{i}\, f(1+g^2)\,dz,\ \mathrm{Re}\int 2fg\,dz\right)^T,$$

where $z = u + \mathrm{i}\, v$, f and fg^2 are holomorphic and g is meromorphic on Ω. As in the cubic case the Enneper–Weierstrass parameterization could also be used to compute different generators for polynomial PN surfaces of higher degrees. Unfortunately, such a representation might not be suitable if PN surfaces are to be used for design purposes.

In this paper a new approach to derive polynomial PN surfaces, which is based on bivariate polynomials with quaternion coefficients, is presented. It shares many similarities with the curve case but also has some important differences. One of them is that the quaternion coefficients may not be chosen completely free but are connected with particular relations. A simple closed form solution is derived that offers as much as possible degrees of freedom for both even and odd degrees. Namely, an explicit construction of polynomial PN surfaces of degrees $2n + 1$ and $2n + 2$, $n \geq 1$, is presented based on bivariate polynomials of degree n with quaternion coefficients. One of the main advantages of this new representation is that it can be applied to design approximation and interpolation schemes with PN surfaces in a simple way. Beside interpolation with LN surfaces (Sampoli and Jüttler, 2000) no other scheme involving surfaces with rational offsets can be found in the literature. The examination of curvature properties shows that the members of the derived family of PN surfaces have a vanishing mean curvature. For degrees three and four also the PN surfaces with nonzero mean curvature are constructed, but they offer less parameters of freedom and involve more complicated expressions. For higher degrees no such PN surfaces were found and their existence remains an open problem.

The paper is organized as follows. In Section 2 some basic properties of PN surfaces and quaternions are outlined. Section 3 presents a general approach to construct polynomial PN surfaces from bivariate polynomials with quaternion coefficients. In Section 4 and Section 5 PN surfaces of an odd and even degree are considered and some examples are presented. Section 6 reveals the curvature properties of the derived surfaces. In the last two sections a simple interpolation scheme and some future work considerations are given.

2. Preliminaries

Consider a surface given by a parametric representation

$$\boldsymbol{S} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \boldsymbol{S}(u,v).$$

The surface normal vector field is equal to

$$\boldsymbol{N} = \frac{\boldsymbol{S}_u \times \boldsymbol{S}_v}{\|\boldsymbol{S}_u \times \boldsymbol{S}_v\|},$$

where $\|\cdot\|$ denotes the Euclidean norm, and $\boldsymbol{S}_u$, $\boldsymbol{S}_v$ are partial derivatives with respect to parameters u and v. Rational (polynomial) surface $\boldsymbol{S}$ is called a Pytha gorean normal (or shortly PN) surface if its normal vector field $\boldsymbol{N}$ is rational in u and v. Such a surface has a property that its offset surface has a rational parametric representation, given by

$$\boldsymbol{S}_\delta = \boldsymbol{S} + \delta \boldsymbol{N}, \quad \delta \in \mathbb{R}.$$

Rational PN surfaces are constructed in Pottmann (1995) using the dual approach. Namely, a non-developable surface $\boldsymbol{S}$ is determined as the envelope of the two-parametric family of tangent planes

$$\boldsymbol{N}(u,v) \cdot \boldsymbol{S}(u,v) - f(u,v) = 0,$$

where $\boldsymbol{N} = (N_1, N_2, N_3)$ is a unit normal vector field and f, called a support function, denotes at each (u, v) the signed distance of the tangent plane from the origin. From a dual representation $\boldsymbol{L} = (-f, N_1, N_2, N_3)$ one obtains a homogeneous point representation by computing a wedge product (Kozak et al., 2014)

$$\boldsymbol{P} = (P_0, P_1, P_2, P_3) = \boldsymbol{L} \wedge \boldsymbol{L}_u \wedge \boldsymbol{L}_v,$$

and a surface follows as

$$\boldsymbol{S} = \frac{1}{P_0}(P_1, P_2, P_3).$$

This construction naturally leads to rational PN surfaces provided $\boldsymbol{N}$ and f are rational. The most common way to obtain rational vector field $\boldsymbol{N}$ is to use stereographic projection (see Pottmann, 1995). But another approach, described in this paper, is to use bivariate polynomials with quaternion coefficients.

Space of quaternions $\mathbb{H}$ is a 4-dimensional vector space with a standard basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$,

$$\mathbf{1} = \left(1, (0,0,0)^T\right), \mathbf{i} = \left(0, (1,0,0)^T\right), \mathbf{j} = \left(0, (0,1,0)^T\right), \mathbf{k} = \left(0, (0,0,1)^T\right).$$

Quaternions can be written as $\mathcal{A} = (a, \boldsymbol{a})$, where the first component is called a scalar part, and the remaining three components form a vector part of the quaternion, i.e.,

$$\operatorname{scal}(\mathcal{A}) = a, \quad \operatorname{vec}(\mathcal{A}) = \boldsymbol{a}.$$

A quaternion with a zero scalar part is called a pure quaternion, and such quaternions are identified with vectors in $\mathbb{R}^3$, i.e., $\mathcal{A} \equiv \boldsymbol{a}$ for $\mathcal{A} = (0, \boldsymbol{a})$. A quaternion sum and product are defined as

$$\mathcal{A} + \mathcal{B} = (a + b, \boldsymbol{a} + \boldsymbol{b}), \quad \mathcal{A}\mathcal{B} = (ab - \boldsymbol{a} \cdot \boldsymbol{b}, a\boldsymbol{b} + b\boldsymbol{a} + \boldsymbol{a} \times \boldsymbol{b}),$$

where $\mathcal{B} := (b, \boldsymbol{b})$. Equipped with these two operations $\mathbb{H}$ becomes an algebra. Every nonzero quaternion has its inverse, which is equal to

$$\mathcal{A}^{-1} = \frac{1}{\|\mathcal{A}\|^2}\overline{\mathcal{A}},$$

where $\overline{\mathcal{A}} = (a, -\boldsymbol{a})$ denotes the conjugate of $\mathcal{A}$, and $\|\mathcal{A}\| = \sqrt{\mathcal{A}\overline{\mathcal{A}}}$ is its norm.

Bivariate polynomials with coefficients in $\mathbb{H}$ form a ring denoted by $\mathbb{H}[u, v]$. The elements of this ring will shortly be called quaternion polynomials. Let

$$\mathcal{A}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \mathcal{A}_{i,j} u^i v^j, \quad \mathcal{A}_{i,j} \in \mathbb{H}, \tag{1}$$

be a quaternion polynomial of degree n. Then $\mathcal{A}$ is associated with three mappings $\boldsymbol{e}_i : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\boldsymbol{e}_i = \frac{\boldsymbol{h}_i}{\|\mathcal{A}\|^2}, \quad i = 1, 2, 3, \quad \boldsymbol{h}_1 := \mathcal{A}\, \mathbf{i}\, \overline{\mathcal{A}}, \quad \boldsymbol{h}_2 := \mathcal{A}\, \mathbf{j}\, \overline{\mathcal{A}}, \quad \boldsymbol{h}_3 := \mathcal{A}\, \mathbf{k}\, \overline{\mathcal{A}}. \tag{2}$$

Note that the multiplication in (2) yields pure quaternions which are then considered as vectors. Moreover it follows that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function.

The next section reveals the construction of polynomial PN surfaces from a quaternion polynomial (1) with some additional conditions on its coefficients.

3. General approach to polynomial PN surfaces

Suppose that a quaternion polynomial $\mathcal{A}$ of the form (1) is given. The goal is to construct a polynomial parametric surface $\boldsymbol{S} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, such that its normal $\boldsymbol{N}$ is equal to

$$\boldsymbol{N} = \boldsymbol{e}_3,$$

where $\boldsymbol{e}_3$ is defined in (2). If such a surface exists it is a PN surface and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the so-called *adapted moving frame* of the surface $\boldsymbol{S}$. Further, the tangent plane of $\boldsymbol{S}$ is at each parameter (u, v) spanned by the vectors $\boldsymbol{h}_1(u, v)$ and $\boldsymbol{h}_2(u, v)$. Therefore $\boldsymbol{S}_u$ and $\boldsymbol{S}_v$ are of the form

$$\boldsymbol{S}_u = \varphi_1 \boldsymbol{h}_1 + \varphi_2 \boldsymbol{h}_2 =: \boldsymbol{g}_1, \quad \boldsymbol{S}_v = \varphi_3 \boldsymbol{h}_1 + \varphi_4 \boldsymbol{h}_2 =: \boldsymbol{g}_2 \tag{3}$$

for some polynomial functions φ_i or for some rational functions φ_i for which $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ reduce to polynomials. Moreover, the condition

$$\frac{\partial \boldsymbol{g}_1}{\partial v} = \frac{\partial \boldsymbol{g}_2}{\partial u} \tag{4}$$

is clearly satisfied. In the next lemma, it is shown that the condition (4) is also sufficient for a polynomial PN surface to exist.

Lemma 1. *Suppose that two polynomial mappings $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are given such that (4) holds true. Then a surface $\boldsymbol{S}$ with the property that $\boldsymbol{S}_u = \boldsymbol{g}_1$ and $\boldsymbol{S}_v = \boldsymbol{g}_2$ exists and is of the form*

$$\boldsymbol{S}(u,v) = \frac{1}{2}\int_0^u \left(\boldsymbol{g}_1(u,v) + \boldsymbol{g}_1(u,0)\right) du + \frac{1}{2}\int_0^v \left(\boldsymbol{g}_2(u,v) + \boldsymbol{g}_2(0,v)\right) dv + \boldsymbol{S}(0,0). \tag{5}$$

Proof. From the condition $\boldsymbol{S}_u = \boldsymbol{g}_1$ it follows that

$$\boldsymbol{S}(u,v) = \int_0^u \boldsymbol{g}_1(u,v) du + C(v),$$

where C is a univariate function of v. If we differentiate this equation with respect to v and use the assumption (4) we obtain

$$\boldsymbol{S}_v(u,v) = \int_0^u \frac{\partial \boldsymbol{g}_1}{\partial v}(u,v) du + C'(v) = \int_0^u \frac{\partial \boldsymbol{g}_2}{\partial u}(u,v) du + C'(v) =$$
$$= \boldsymbol{g}_2(u,v) - \boldsymbol{g}_2(0,v) + C'(v).$$

Since $\boldsymbol{S}_v = \boldsymbol{g}_2$ it follows that

$$C(v) = \int_0^v \boldsymbol{g}_2(0,v) dv + \text{const},$$

and thus

$$\boldsymbol{S}(u,v) = \int_0^u \boldsymbol{g}_1(u,v) du + \int_0^v \boldsymbol{g}_2(0,v) dv + \text{const}. \tag{6}$$

Similarly, by starting from $\boldsymbol{S}_v = \boldsymbol{g}_2$ we derive that

$$\boldsymbol{S}(u,v) = \int_0^v \boldsymbol{g}_2(u,v) dv + \int_0^u \boldsymbol{g}_1(u,0) du + \text{const}. \tag{7}$$

Formula (5) follows then by averaging the equations (6) and (7), which concludes the proof. □

The next lemma reveals the closed form expression of the surface $\boldsymbol{S}$ defined in Lemma 1.

Lemma 2. *Suppose that*

$$\boldsymbol{g}_\ell(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} \boldsymbol{B}_{i,j}^{[\ell]} u^i v^j, \quad \ell = 1,2. \tag{8}$$

Then (4) holds true iff

$$(j+1)\boldsymbol{B}_{i,j+1}^{[1]} = (i+1)\boldsymbol{B}_{i+1,j}^{[2]}, \quad i = 0,1,\ldots,m-1, \; j = 0,1,\ldots,m-1-i, \tag{9}$$

and the surface $\boldsymbol{S}$, defined in (5), is equal to

$$\boldsymbol{S}(u,v) = \sum_{i=0}^{m+1} \sum_{j=0}^{m+1-i} \boldsymbol{S}_{i,j} u^i v^j, \tag{10}$$

with

$$\begin{aligned}\boldsymbol{S}_{i,0} &= \frac{1}{i}\boldsymbol{B}^{[1]}_{i-1,0}, \quad \boldsymbol{S}_{0,i} = \frac{1}{i}\boldsymbol{B}^{[2]}_{0,i-1}, \quad i = 1,2,\ldots,m+1,\\ \boldsymbol{S}_{i,j} &= \frac{1}{i}\boldsymbol{B}^{[1]}_{i-1,j} = \frac{1}{j}\boldsymbol{B}^{[2]}_{i,j-1}, \quad i = 1,2,\ldots,m, \; j = 1,2,\ldots,m+1-i,\end{aligned} \tag{11}$$

and $\boldsymbol{S}_{0,0}$ *an arbitrary constant.*

Proof. The proof of (9) is straightforward. By computing the integrals and changing the summation indexes we obtain from (5) that

$$\begin{aligned}\boldsymbol{S}(u,v) &= \boldsymbol{S}(0,0) + \frac{1}{2}\sum_{i=1}^{m+1}\sum_{j=0}^{m+1-i}\frac{1}{i}\boldsymbol{B}^{[1]}_{i-1,j}u^i v^j + \frac{1}{2}\sum_{i=1}^{m+1}\frac{1}{i}\boldsymbol{B}^{[1]}_{i-1,0}u^i +\\ &+ \frac{1}{2}\sum_{i=0}^{m}\sum_{j=1}^{m+1-i}\frac{1}{j}\boldsymbol{B}^{[2]}_{i,j-1}u^i v^j + \frac{1}{2}\sum_{j=1}^{m+1}\frac{1}{j}\boldsymbol{B}^{[2]}_{0,j-1}v^j = \boldsymbol{S}(0,0) +\\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m+1-i}\frac{1}{2}\left(\frac{1}{i}\boldsymbol{B}^{[1]}_{i-1,j} + \frac{1}{j}\boldsymbol{B}^{[2]}_{i,j-1}\right)u^i v^j + \sum_{i=1}^{m+1}\frac{1}{i}\boldsymbol{B}^{[1]}_{i-1,0}u^i + \sum_{j=1}^{m+1}\frac{1}{j}\boldsymbol{B}^{[2]}_{0,j-1}v^j,\end{aligned}$$

which by considering (9) completes the proof. □

Lemma 1 and Lemma 2 provide a way to construct a polynomial PN surface from a quaternion polynomial (1) provided (4) is fulfilled. The degree of such a surface is

$$2\deg(\mathcal{A}) + \max_{i=1,2,3,4}(\deg(\varphi_i)) + 1.$$

In particular, choosing φ_i as constants implies polynomial PN surfaces of degree $2\deg(\mathcal{A})+1$. Note that the same relation between degrees holds in the curve case since degree n quaternion polynomials of one variable imply spatial PH curves of degree $2n+1$. Moreover, if φ_i are constants, then any coordinate curve on the surface is a PH curve. More precisely, for $\boldsymbol{r}(u) = S(u, v_0)$, $v_0 = \text{const}$, the norm of the hodograph is equal to

$$\|\boldsymbol{r}'(u)\| = \|\varphi_1\boldsymbol{h}_1(u,v_0) + \varphi_2\boldsymbol{h}_2(u,v_0)\| = \sqrt{\varphi_1^2+\varphi_2^2}\,\|\mathcal{A}(u,v_0)\|^2,$$

which is a polynomial expression for constant φ_i, $i = 1,2$. Similarly for curves of the form $\boldsymbol{r}(v) = S(u_0, v)$, $u_0 = \text{const}$. To construct polynomial PN surfaces of an even degree one must choose φ_i as polynomials of an odd degree. In particular, linear functions φ_i imply polynomial PN surfaces of degree $2\deg(\mathcal{A})+2$.

Unfortunately, the condition (4) is not satisfied for arbitrary functions φ_i and arbitrary quaternion polynomials (1). In the following sections the conditions on quaternion coefficients that imply (4) to be true if φ_i are constants or linear functions are presented, which gives the construction of polynomial PN surfaces of an arbitrary degree.

4. Construction of polynomial PN surfaces of odd degree

Suppose that a quaternion polynomial $\mathcal{A}$ is given by (1) and $\boldsymbol{h}_i$, $i = 1,2,3$, are the associated mappings defined by (2). Let us define two quaternions

$$\mathcal{U}_1 := \varphi_1\mathbf{i} + \varphi_2\mathbf{j}, \quad \mathcal{U}_2 := \varphi_3\mathbf{i} + \varphi_4\mathbf{j}, \tag{12}$$

and let us assume first that

$$\varphi_i(u,v) = \alpha_i, \quad \alpha_i \in \mathbb{R}, \quad i = 1,2,3,4.$$

Then the partial derivatives from (3) are expressed as

$$\boldsymbol{g}_1 = \varphi_1\boldsymbol{h}_1 + \varphi_2\boldsymbol{h}_2 = \mathcal{A}\mathcal{U}_1\overline{\mathcal{A}}, \quad \boldsymbol{g}_2 = \varphi_3\boldsymbol{h}_1 + \varphi_4\boldsymbol{h}_2 = \mathcal{A}\mathcal{U}_2\overline{\mathcal{A}}, \tag{13}$$

which follows from (2) and some basis properties of the quaternions. Moreover, $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ can be expressed in a standard basis as in (8) with $m = 2n$ and

$$\boldsymbol{B}^{[\ell]}_{i,j} := \sum_{k=\max\{0,i-n\}}^{\min\{i,n\}}\;\sum_{r=\max\{0,j+i-n-k\}}^{\min\{j,n-k\}} \mathcal{A}_{k,r}\mathcal{U}_\ell\overline{\mathcal{A}}_{i-k,j-r}, \quad \ell = 1,2.$$

By Lemma 1 and Lemma 2 the quaternion polynomial $\mathcal{A}$ defines a PN surface if $n(2n+1)$ vector equations (9) with $m = 2n$ are satisfied. These equations present nonlinear relations between $\binom{n+2}{2}$ quaternion coefficients and four parameters α_i,

$i = 1, 2, 3, 4$. It seems that for $n \geq 2$ no solution exists since we have $3n(2n+1)$ scalar equations, but only $4\binom{n+2}{2} + 4$ free parameters. However, it turns out that some solutions can still be found. Note that the mixed derivatives of the surface to be constructed are of the form

$$\frac{\partial \boldsymbol{g}_1}{\partial v} = \mathcal{A}_v \mathcal{U}_1 \overline{\mathcal{A}} + \mathcal{A} \mathcal{U}_1 \overline{\mathcal{A}}_v = 2 \operatorname{vec}\left(\mathcal{A}_v \mathcal{U}_1 \overline{\mathcal{A}}\right),$$
$$\frac{\partial \boldsymbol{g}_2}{\partial u} = \mathcal{A}_u \mathcal{U}_2 \overline{\mathcal{A}} + \mathcal{A} \mathcal{U}_2 \overline{\mathcal{A}}_u = 2 \operatorname{vec}\left(\mathcal{A}_u \mathcal{U}_2 \overline{\mathcal{A}}\right),$$

and the condition (4) is fulfilled iff

$$\operatorname{vec}\left(\left(\mathcal{A}_v \mathcal{U}_1 - \mathcal{A}_u \mathcal{U}_2\right) \overline{\mathcal{A}}\right) = \boldsymbol{0}. \tag{14}$$

The relation (14) thus holds true iff

$$\mathcal{A}_v \mathcal{U}_1 - \mathcal{A}_u \mathcal{U}_2 = \psi \mathcal{A} \tag{15}$$

for some scalar function ψ. Since the left hand side of (15) is a quaternion polynomial of degree $n-1$, function ψ must be rational with the degree of the numerator being one less than the degree of the denominator. A particular case $\psi \equiv 0$ provides a nice closed form solution given in the following theorem.

Theorem 1. *If the coefficients of the quaternion polynomial (1) satisfy*

$$(j+1)\mathcal{A}_{i,j+1}\mathcal{U}_1 = (i+1)\mathcal{A}_{i+1,j}\mathcal{U}_2, \quad i = 0, 1, \ldots, n-1, \; j = 0, 1, \ldots, n-1-i, \tag{16}$$

then a parametric surface $\boldsymbol{S}$ *defined by (5) and (13) is a PN surface of degree* $2n+1$ *with*

$$\boldsymbol{S}_u \times \boldsymbol{S}_v = (\alpha_1\alpha_4 - \alpha_2\alpha_3)\, \|\mathcal{A}\|^2 \boldsymbol{h}_3, \qquad \boldsymbol{N} = \boldsymbol{e}_3.$$

Proof. The conditions (16) follow directly from

$$\mathcal{A}_v \mathcal{U}_1 - \mathcal{A}_u \mathcal{U}_2 = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \left((j+1)\mathcal{A}_{i,j+1}\mathcal{U}_1 - (i+1)\mathcal{A}_{i+1,j}\mathcal{U}_2\right) u^i v^j$$

and (15) by choosing $\psi \equiv 0$. Then (4) is fulfilled and the surface (5) has partial derivatives $\boldsymbol{S}_u = \boldsymbol{g}_1$ and $\boldsymbol{S}_v = \boldsymbol{g}_2$. Moreover

$$\boldsymbol{S}_u \times \boldsymbol{S}_v = (\alpha_1\alpha_4 - \alpha_2\alpha_3)\, \boldsymbol{h}_1 \times \boldsymbol{h}_2 = (\alpha_1\alpha_4 - \alpha_2\alpha_3)\, \|\mathcal{A}\|^2 \boldsymbol{h}_3$$

which completes the proof. □

Theorem 1 provides only the sufficient conditions for a PN surface to exist. But these conditions turn out to be the most simple ones and clearly provide as much degrees of freedom as possible. Furthermore, the computations show that choosing

$$\psi(u, v) = \frac{1}{\psi_{0,0} + \psi_{1,0} u + \psi_{0,1} v}$$

gives no solution for $n = 1$ and $n = 2$.

In the following lemma the explicit solution of equations (16) is provided. The proof follows by some straightforward computation.

Lemma 3. *The conditions (16) are fulfilled iff*

$$\mathcal{A}_{\lceil \frac{m}{2} \rceil + \ell, \lfloor \frac{m}{2} \rfloor - \ell} = \frac{\left(\lceil \frac{m}{2} \rceil\right)! \left(\lfloor \frac{m}{2} \rfloor\right)!}{\left(\lfloor \frac{m}{2} \rfloor - \ell\right)! \left(\lceil \frac{m}{2} \rceil + \ell\right)!} \mathcal{Q}_m \left(\mathcal{U}_1 \mathcal{U}_2^{-1}\right)^{\ell}, \; \ell = -\left\lceil \frac{m}{2} \right\rceil, \ldots, \left\lfloor \frac{m}{2} \right\rfloor,$$

for $m = 1, 2, \ldots, n$, *where* $\mathcal{Q}_m$ *are arbitrary quaternions and* $0! := 1$.

In practical applications, one tends to use low degree polynomial objects. In the following subsections PN surfaces of degrees 3 and 5 are considered in more detail and illustrated by some numerical examples.

4.1. Cubic polynomial PN surfaces

To construct a cubic polynomial PN surface we have to choose a linear quaternion polynomial $\mathcal{A}$ defined in (1) for $n=1$ with control points defined in Lemma 3. For a symmetry reasons, we redefine a free quaternion $\mathcal{Q}_1$ to $\mathcal{Q}_1\mathcal{U}_2^{-1}$ and obtain that the control points must be equal to

$$\mathcal{A}_{0,0}, \quad \mathcal{A}_{1,0}=\mathcal{Q}_1\mathcal{U}_2^{-1}, \quad \mathcal{A}_{0,1}=\mathcal{Q}_1\mathcal{U}_1^{-1}. \tag{17}$$

Then $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are of the form (8) with $m=2$,

$$\begin{aligned}
\boldsymbol{B}_{0,0}^{[\ell]} &= \mathcal{A}_{0,0}\mathcal{U}_\ell\overline{\mathcal{A}}_{0,0},\\
\boldsymbol{B}_{1-k,k}^{[\ell]} &= 2\operatorname{vec}\left(\mathcal{A}_{0,0}\mathcal{U}_\ell\overline{\mathcal{A}}_{1-k,k}\right),\\
\boldsymbol{B}_{1,1}^{[\ell]} &= 2\operatorname{vec}\left(\mathcal{A}_{0,1}\mathcal{U}_\ell\overline{\mathcal{A}}_{1,0}\right),\\
\boldsymbol{B}_{2-2k,2k}^{[\ell]} &= \mathcal{A}_{1-k,k}\mathcal{U}_\ell\overline{\mathcal{A}}_{1-k,k},
\end{aligned} \tag{18}$$

for $k=0,1$, $\ell=1,2$, and a cubic PN surface is given in Lemma 2. The complete construction is summarized in Algorithm 1:

Algorithm 1 Construction of a cubic PN surface.

Input: Quaternions $\mathcal{A}_{0,0}$, $\mathcal{Q}_1$, parameters $(\alpha_i)_{i=1}^4$, and $\boldsymbol{P}\in\mathbb{R}^3$.
Output: A cubic PN surface $\boldsymbol{S}$.
1: Set $\mathcal{U}_1=\left(0,(\alpha_1,\alpha_2,0)^T\right)$, $\mathcal{U}_2=\left(0,(\alpha_3,\alpha_4,0)^T\right)$;
2: Compute $\mathcal{A}_{1,0}$ and $\mathcal{A}_{0,1}$ by (17);
3: Compute $\boldsymbol{B}_{i,j}^{[\ell]}$ for $\ell=1,2$ and $0\le i+j\le 2$ by (18);
4: Set $\boldsymbol{S}_{0,0}=\boldsymbol{P}$, $m=2$;
5: Compute $\boldsymbol{S}_{i,j}$ for $1\le i+j\le 3$ by (11);
6: Compute $\boldsymbol{S}$ by (10).

Note that there are 15 free parameters in Algorithm 1, but a family of the derived cubic PN surfaces is only 14-parametric. Namely, by multiplying a quaternion polynomial by some constant factor, we can fix one of the (nonzero) parameters α_i to a fixed value.

As an example let us choose

$$\mathcal{A}_{0,0}=\left(1,(0,0,0)^T\right), \quad \mathcal{Q}_1=\left(-\frac{1}{2},\left(1,-\frac{1}{4},\frac{2}{3}\right)^T\right),$$
$$\alpha_1=\frac{1}{4}, \quad \alpha_2=-\frac{3}{4}, \quad \alpha_3=\frac{1}{2}, \quad \alpha_4=-\frac{1}{2}.$$

Then

$$\mathcal{A}(u,v)=\frac{1}{60}\left(3(25u+14v+20),(2(-5u-18v),2(-35u-26v),3(15u+22v))^T\right)$$

and

$$\boldsymbol{g}_1(u,v)=\begin{pmatrix}\frac{33u^2}{32}+\frac{83uv}{72}+\frac{7u}{4}+\frac{7v^2}{72}+2v+\frac{1}{4}\\ -\frac{341u^2}{288}-\frac{5uv}{72}-\frac{3u}{2}+\frac{107v^2}{120}-\frac{v}{2}-\frac{3}{4}\\ \frac{55u^2}{24}+\frac{29uv}{6}+\frac{5u}{6}+\frac{61v^2}{30}+\frac{4v}{3}\end{pmatrix},$$
$$\boldsymbol{g}_2(u,v)=\begin{pmatrix}\frac{83u^2}{144}+\frac{7uv}{36}+2u-\frac{11v^2}{36}+\frac{9v}{5}+\frac{1}{2}\\ -\frac{5u^2}{144}+\frac{107uv}{60}-\frac{u}{2}+\frac{1309v^2}{900}+\frac{2v}{5}-\frac{1}{2},\\ \frac{29u^2}{12}+\frac{61uv}{15}+\frac{4u}{3}+\frac{33v^2}{25}+\frac{22v}{15}\end{pmatrix}.$$

The PN surface (see Fig. 1) equals

$$\boldsymbol{S}(u,v)=\begin{pmatrix}\frac{11u^3}{32}+\frac{83u^2v}{144}+\frac{7u^2}{8}+\frac{7uv^2}{72}+2uv+\frac{u}{4}-\frac{11v^3}{108}+\frac{9v^2}{10}+\frac{v}{2}\\ -\frac{341u^3}{864}-\frac{5u^2v}{144}-\frac{3u^2}{4}+\frac{107uv^2}{120}-\frac{uv}{2}-\frac{3u}{4}+\frac{1309v^3}{2700}+\frac{v^2}{5}-\frac{v}{2}\\ \frac{55u^3}{72}+\frac{29u^2v}{12}+\frac{5u^2}{12}+\frac{61uv^2}{30}+\frac{4uv}{3}+\frac{11v^3}{25}+\frac{11v^2}{15}\end{pmatrix} \tag{19}$$

and

$$\|(\boldsymbol{S}_u\times\boldsymbol{S}_v)(u,v)\|=\frac{\left(1265u^2+2024uv+900u+1012v^2+504v+360\right)^2}{518400},$$

which confirms the PN property.

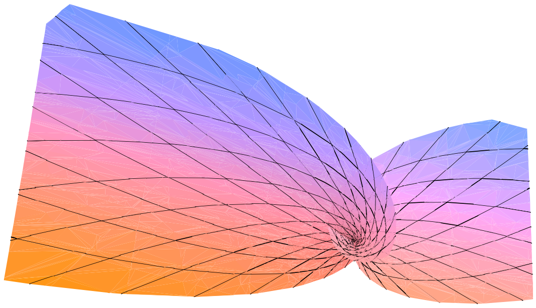

Fig. 1. A cubic PN surface (19) for $(u, v) \in [-3, 3] \times [-3, 3]$.

Construction of cubic PN surfaces has already been considered in Lávička and Vršek (2012). The authors derived the family of PN surfaces of the form

$$\boldsymbol{S}(u, v) = c_1 \boldsymbol{P}_1(u, v) + c_2 \boldsymbol{P}_2(u, v) + c_3 \boldsymbol{P}_3(u, v) + \boldsymbol{C},$$

where

$$\boldsymbol{P}_1(u, v) = \left(-\frac{u^3}{3} + uv^2 + u, -u^2 v + \frac{v^3}{3} - v, u^2 - v^2\right)^T,$$

$$\boldsymbol{P}_2(u, v) = \left(u^2 v - \frac{v^3}{3} - v, -\frac{u^3}{3} + uv^2 - u, -2uv\right)^T,$$

$$\boldsymbol{P}_3(u, v) = \left(u - \frac{1}{3}u\left(u^2 + v^2\right), v - \frac{1}{3}v\left(u^2 + v^2\right), u^2 + v^2\right)^T,$$

and $c_i \in \mathbb{R}$, $i = 1, 2, 3$, $\boldsymbol{C} \in \mathbb{R}^3$, are free constants. Let us examine how the generators $\boldsymbol{P}_i$, $i = 1, 2, 3$, can be constructed based on our approach. Observe first that the unit normals $\boldsymbol{N}$ of all three surfaces are equal to

$$\frac{1}{1 + u^2 + v^2}\left(2u, 2v, u^2 + v^2 - 1\right)^T.$$

Therefrom we obtain that a quaternion polynomial $\mathcal{A}$ must be chosen as $\mathcal{A}(u, v) = \left(-v, (1, 0, u)^T\right)$. Moreover, it can easily be checked that $\boldsymbol{P}_1$ follows by choosing

$$\mathcal{U}_1 = \left(0, (1, 0, 0)^T\right), \quad \mathcal{U}_2 = \left(0, (0, 1, 0)^T\right),$$

and to obtain $\boldsymbol{P}_2$ we must take

$$\mathcal{U}_1 = \left(0, (0, 1, 0)^T\right), \quad \mathcal{U}_2 = \left(0, (-1, 0, 0)^T\right).$$

But it turns out that $\boldsymbol{P}_3$ can not be derived from $\mathcal{A}$ by any constant quaternion $\mathcal{U}_1$ and $\mathcal{U}_2$. In this case we observe that a combination

$$\frac{v}{1 + u^2 + v^2}\boldsymbol{h}_1(u, v) + \frac{u}{1 + u^2 + v^2}\boldsymbol{h}_2(u, v) = (v, -u, 0) =: \widehat{\boldsymbol{h}}(u, v)$$

gives a linear mapping. Moreover, on can check that $\boldsymbol{P}_3$ follows from (3) by choosing

$$\boldsymbol{g}_1 = -\frac{4}{3}\widehat{\boldsymbol{h}} + \boldsymbol{h}_1, \quad \boldsymbol{g}_2 = \frac{4}{3}\widehat{\boldsymbol{h}} - \boldsymbol{h}_2$$

or equivalently

$$\mathcal{U}_1(u, v) = \left(0, \left(\frac{3 + 3u^2 - v^2}{3(1 + u^2 + v^2)}, -\frac{4uv}{3(1 + u^2 + v^2)}, 0\right)^T\right),$$

$$\mathcal{U}_2(u, v) = \left(0, \left(\frac{4uv}{3(1 + u^2 + v^2)}, -\frac{3 + 3v^2 - u^2}{3(1 + u^2 + v^2)}, 0\right)^T\right).$$

This example indicates that cubic PN surfaces can be obtained from linear quaternion polynomial also by choosing φ_i, $i = 1, 2, 3, 4$, as particular quadratic rational functions. Let us examine such possibilities for a general linear quaternion. The next lemma reveals linear combinations of $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ that result in a linear vector field orthogonal to $\boldsymbol{h}_3$.

Lemma 4. *Let $\mathcal{A}$ be a linear quaternion defined in (1) for $n = 1$ and let*

$$p_1(u,v) = \boldsymbol{w} \cdot \operatorname{vec}\left(\mathcal{A}_{0,0}^{-1}\,\mathcal{A}\,\mathbf{j}\right), \quad p_2(u,v) = -\boldsymbol{w} \cdot \operatorname{vec}\left(\mathcal{A}_{0,0}^{-1}\mathcal{A}\,\mathbf{i}\right),$$

where $\boldsymbol{w} = \operatorname{vec}\left(\mathcal{A}_{0,0}^{-1}\mathcal{A}_{1,0}\right) \times \operatorname{vec}\left(\mathcal{A}_{0,0}^{-1}\mathcal{A}_{0,1}\right)$. *Then*

$$\widehat{\boldsymbol{h}} = \frac{\|\mathcal{A}_{0,0}\|^4}{\|\mathcal{A}\|^2}\left(p_1\boldsymbol{h}_1 + p_2\boldsymbol{h}_2\right) \tag{20}$$

is a linear polynomial mapping.

Proof. By using rotations we can without loosing generality assume that

$$\mathcal{A}_{0,0} = \left(1, (0,0,0)^T\right).$$

Let us denote the components of the remained control quaternions as

$$\mathcal{A}_{1,0} = \left(a_0, (a_1, a_2, a_3)^T\right), \quad \mathcal{A}_{0,1} = \left(b_0, (b_1, b_2, b_3)^T\right).$$

A straightforward computation yields

$$p_1(u,v) = \boldsymbol{w} \cdot \left((0,1,0)^T + (-a_3, a_0, a_1)^T\, u + (-b_3, b_0, b_1)^T\, v\right),$$
$$p_2(u,v) = \boldsymbol{w} \cdot \left((-1,0,0)^T + (-a_0, -a_3, a_2)^T\, u + (-b_0, -b_3, b_2)^T\, v\right).$$

Furthermore, one can check that

$$p_1\boldsymbol{h}_1 + p_2\boldsymbol{h}_2 = \|\mathcal{A}\|^2 \begin{pmatrix} \boldsymbol{w} \cdot \left((0,1,0)^T + (a_3, a_0, a_1)^T\, u + (b_3, b_0, b_1)^T\, v\right) \\ \boldsymbol{w} \cdot \left((-1,0,0)^T + (-a_0, a_3, a_2)^T\, u + (-b_0, b_3, b_2)^T\, v\right) \\ \boldsymbol{w} \cdot \left((0,0,2a_3)^T\, u + (0,0,2b_3)^T\, v\right) \end{pmatrix},$$

which completes the proof. □

By using the result of Lemma 4 the additional cubic PN surfaces can be constructed as follows. Choose $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ as a combination

$$\begin{aligned} \boldsymbol{g}_1(u,v) &= \widehat{\varphi}_1(u,v)\widehat{\boldsymbol{h}}(u,v) + \alpha_1\boldsymbol{h}_1(u,v) + \alpha_2\boldsymbol{h}_2(u,v), \\ \boldsymbol{g}_2(u,v) &= \widehat{\varphi}_2(u,v)\widehat{\boldsymbol{h}}(u,v) + \alpha_3\boldsymbol{h}_1(u,v) + \alpha_4\boldsymbol{h}_2(u,v) \end{aligned} \tag{21}$$

for some linear functions $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$. The condition (4) results in nine scalar linear equations for six coefficients of $\widehat{\varphi}_1$, $\widehat{\varphi}_2$ and four parameters α_i. We can choose to fix one of α_i and all the other coefficients can then be expressed only with the coefficients of quaternions $\mathcal{A}_{1,0}$ and $\mathcal{A}_{0,1}$. The solution is not inserted in the paper since the expressions are quite involved. Instead, let us demonstrate a construction on a particular example. For

$$\mathcal{A}(u,v) = \left(2, (-16,2,0)^T\right) + \left(-2, (2,2,1)^T\right)u + \left(-4, (0,1,2)^T\right)v$$

expression (20) simplifies to

$$\widehat{\boldsymbol{h}}(u,v) = \left(2(183u + 50v - 880), 124(u + v + 9), 8(u + 2v + 28)\right)^T,$$

and by fixing $\alpha_4 = 342$, we compute

$$\widehat{\varphi}_1(u,v) = 95 + 5v,\ \widehat{\varphi}_2(u,v) = 50 - 5u,\ \alpha_1 = 78,\ \alpha_2 = 156,\ \alpha_3 = 171.$$

The PN surface derived from (21) is then equal to

$$\boldsymbol{S}(u,v) = \begin{pmatrix} 1404u^3 + 5574u^2v + 19170u^2 + 7708uv^2 - 14200uv - 314432u + 4902v^3 - 12100v^2 - 132224v \\ 520u^3 + 2180u^2v + 17708u^2 + 2800uv^2 + 27232uv + 124680u + 1140v^3 - 2008v^2 - 79920v \\ 208u^3 + 1288u^2v + 10120u^2 + 2656uv^2 + 39600uv + 21344u + 1824v^3 + 38420v^2 - 24112v \end{pmatrix} \tag{22}$$

and is presented in Fig. 2 together with its offset. Note that different values of α_4 imply different scalings of the obtained surface.

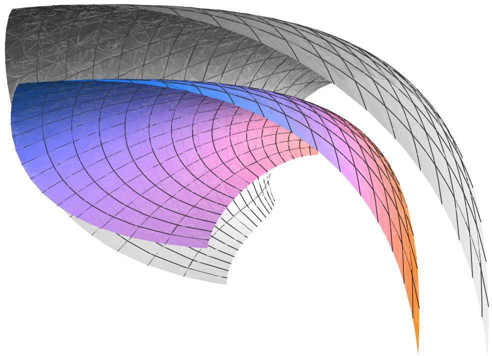

Fig. 2. A cubic PN surface (22) and its offset (gray) for $(u, v) \in [-4, 4] \times [-4, 4]$.

4.2. Quintic polynomial PN surfaces

A quintic polynomial PN surface can be constructed from a quadratic quaternion polynomial $\mathcal{A}$, defined in (1) for $n = 2$, with control quaternions being equal to (17) at linear terms, and to

$$\mathcal{A}_{1,1} = \mathcal{Q}_2, \quad \mathcal{A}_{2,0} = \frac{1}{2}\mathcal{Q}_2\mathcal{U}_1\mathcal{U}_2^{-1}, \quad \mathcal{A}_{0,2} = \frac{1}{2}\mathcal{Q}_2\mathcal{U}_2\mathcal{U}_1^{-1} \tag{23}$$

at additional quadratic ones. Recall that $\mathcal{Q}_1$ in (17) and $\mathcal{Q}_2$ in (23) are free quaternions that represent some of the degrees of freedom in a PN surface construction. Then $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are of the form (8) with $m = 4$,

$$\begin{aligned}
\boldsymbol{B}_{0,0}^{[\ell]} &= \mathcal{A}_{0,0}\mathcal{U}_\ell\overline{\mathcal{A}}_{0,0},\\
\boldsymbol{B}_{1-k,k}^{[\ell]} &= 2\mathrm{vec}\left(\mathcal{A}_{0,0}\mathcal{U}_\ell\overline{\mathcal{A}}_{1-k,k}\right),\\
\boldsymbol{B}_{1,1}^{[\ell]} &= 2\mathrm{vec}\left(\mathcal{A}_{0,1}\mathcal{U}_\ell\overline{\mathcal{A}}_{1,0} + \mathcal{A}_{0,0}\mathcal{U}_\ell\overline{\mathcal{A}}_{1,1}\right),\\
\boldsymbol{B}_{2-2k,2k}^{[\ell]} &= \mathcal{A}_{1-k,k}\mathcal{U}_\ell\overline{\mathcal{A}}_{1-k,k} + 2\mathrm{vec}\left(\mathcal{A}_{0,0}\mathcal{U}_\ell\overline{\mathcal{A}}_{2-2k,2k}\right),\\
\boldsymbol{B}_{3-3k,3k}^{[\ell]} &= 2\mathrm{vec}\left(\mathcal{A}_{1-k,k}\mathcal{U}_\ell\overline{\mathcal{A}}_{2-2k,2k}\right),\\
\boldsymbol{B}_{2-k,1+k}^{[\ell]} &= 2\mathrm{vec}\left(\mathcal{A}_{1-k,k}\mathcal{U}_\ell\overline{\mathcal{A}}_{1,1} + \mathcal{A}_{k,1-k}\mathcal{U}_\ell\overline{\mathcal{A}}_{2-2k,2k}\right),\\
\boldsymbol{B}_{2,2}^{[\ell]} &= \mathcal{A}_{1,1}\mathcal{U}_\ell\overline{\mathcal{A}}_{1,1} + 2\mathrm{vec}\left(\mathcal{A}_{0,2}\mathcal{U}_\ell\overline{\mathcal{A}}_{2,0}\right),\\
\boldsymbol{B}_{4-4k,4k}^{[\ell]} &= \mathcal{A}_{2-2k,2k}\mathcal{U}_\ell\overline{\mathcal{A}}_{2-2k,2k},\\
\boldsymbol{B}_{3-2k,1+2k}^{[\ell]} &= 2\mathrm{vec}\left(\mathcal{A}_{1,1}\mathcal{U}_\ell\overline{\mathcal{A}}_{2-2k,2k}\right),
\end{aligned} \tag{24}$$

for $k = 0, 1$, $\ell = 1, 2$, and a quintic PN surface is given in Lemma 2. The construction is summarized in Algorithm 2:

Algorithm 2 Construction of a quintic PN surface.

Input: Quaternions $\mathcal{A}_{0,0}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$, parameters $(\alpha_i)_{i=1}^4$, and $\boldsymbol{P} \in \mathbb{R}^3$.
Output: A quintic PN surface $\boldsymbol{S}$.
1: Set $\mathcal{U}_1 = \left(0, (\alpha_1, \alpha_2, 0)^T\right)$, $\mathcal{U}_2 = \left(0, (\alpha_3, \alpha_4, 0)^T\right)$;
2: Compute $\mathcal{A}_{1,0}$ and $\mathcal{A}_{0,1}$ by (17);
3: Compute $\mathcal{A}_{2,0}$, $\mathcal{A}_{1,1}$ and $\mathcal{A}_{0,2}$ by (23);
4: Compute $B_{i,j}^{[\ell]}$ for $\ell = 1, 2$ and $0 \le i + j \le 4$ by (24);
5: Set $\boldsymbol{S}_{0,0} = \boldsymbol{P}$, $m = 4$;
6: Compute $\boldsymbol{S}_{i,j}$ for $1 \le i + j \le 5$ by (11);
7: Compute $\boldsymbol{S}$ by (10).

To demonstrate the results let us choose

$$\begin{aligned}
&\mathcal{A}_{0,0} = \left(1, (0, 0, 0)^T\right), \mathcal{Q}_1 = \left(-\frac{1}{2}, \left(1, -\frac{1}{4}, \frac{2}{3}\right)^T\right), \mathcal{Q}_2 = \left(0, \left(-\frac{1}{8}, \frac{1}{3}, \frac{1}{10}\right)^T\right),\\
&\alpha_1 = \frac{1}{4}, \quad \alpha_2 = -\frac{3}{4}, \quad \alpha_3 = -\frac{1}{2}, \quad \alpha_4 = -\frac{1}{2}, \quad \boldsymbol{P} = (0, 0, 0)^T.
\end{aligned} \tag{25}$$

The surface is shown in Fig. 3 together with its triangular Bézier patch.

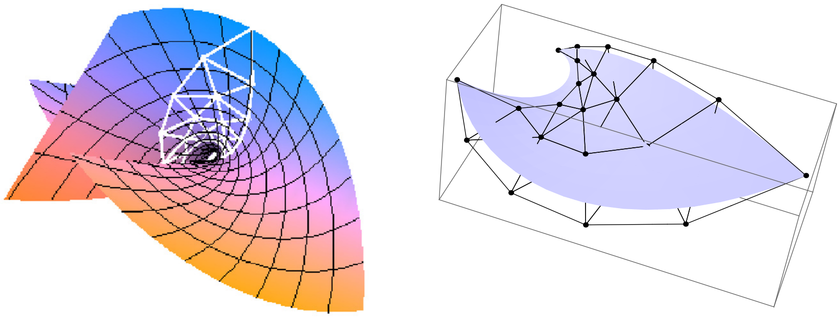

Fig. 3. A quintic PN surface, determined by (25), for $(u,v) \in [-1,1] \times [-1,1]$ (left) and its triangular Bézier patch (right).

5. Construction of polynomial PN surfaces of even degree

Let a quaternion polynomial $\mathcal{A}$ be of the form (1). To obtain a PN surface of an even degree, we must choose φ_i in (3) to be functions of an odd degree. In particular, let us examine the case when

$$\varphi_i(u,v) = \alpha_i + \beta_i u + \gamma_i v, \quad \alpha_i, \beta_i, \gamma_i \in \mathbb{R}, \quad i = 1,2,3,4, \tag{26}$$

are linear functions. Moreover, $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are given by (13) with $\mathcal{U}_1, \mathcal{U}_2$ defined by (12). The condition (4) is now fulfilled iff

$$\operatorname{vec}\left((2\mathcal{A}_v\mathcal{U}_1 - 2\mathcal{A}_u\mathcal{U}_2 + \mathcal{A}\left((\mathcal{U}_1)_v - (\mathcal{U}_2)_u\right))\overline{\mathcal{A}}\right) = \mathbf{0},$$

or equivalently, it must hold that

$$2\mathcal{A}_v\mathcal{U}_1 - 2\mathcal{A}_u\mathcal{U}_2 + \mathcal{A}\left((\mathcal{U}_1)_v - (\mathcal{U}_2)_u\right) = \psi\mathcal{A} \tag{27}$$

for some scalar function ψ. Since the left hand side of the expression in (27) is a polynomial of degree n in variables (u,v), we choose $\psi = \psi_0$ as a constant. Then (27) represents $4\binom{n+2}{2}$ equations that connect $4\binom{n+2}{2}$ coefficients of $\mathcal{A}$ and 12 coefficients that appear in $\mathcal{U}_1, \mathcal{U}_2$. Let us denote

$$\mathcal{U}_\ell(u,v) = \mathcal{U}_{0,0}^{[\ell]} + \mathcal{U}_{1,0}^{[\ell]}u + \mathcal{U}_{0,1}^{[\ell]}v, \quad \ell = 1,2,$$

where

$$\begin{aligned} &\mathcal{U}_{0,0}^{[\ell]} = \left(0, (\alpha_{2\ell-1}, \alpha_{2\ell}, 0)^T\right), \quad \mathcal{U}_{1,0}^{[\ell]} = \left(0, (\beta_{2\ell-1}, \beta_{2\ell}, 0)^T\right), \\ &\mathcal{U}_{0,1}^{[\ell]} = \left(0, (\gamma_{2\ell-1}, \gamma_{2\ell}, 0)^T\right). \end{aligned} \tag{28}$$

First, the case when $n = 1$ is analyzed which leads to quartic polynomial PN surfaces.

5.1. Quartic polynomial PN surfaces

For $n = 1$ the relation (27) represents three quaternion equations which can be, by evaluating the value for $(u,v) = (0,0)$ and by applying $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, written as

$$\begin{aligned} &2\mathcal{A}_{0,1}\mathcal{U}_{0,0}^{[1]} - 2\mathcal{A}_{1,0}\mathcal{U}_{0,0}^{[2]} + \mathcal{A}_{0,0}\left(\mathcal{U}_{0,1}^{[1]} - \mathcal{U}_{1,0}^{[2]} - \psi_0\mathbf{1}\right) = 0, \\ &2\mathcal{A}_{0,1}\mathcal{U}_{i,1-i}^{[1]} - 2\mathcal{A}_{1,0}\mathcal{U}_{i,1-i}^{[2]} + \mathcal{A}_{i,1-i}\left(\mathcal{U}_{0,1}^{[1]} - \mathcal{U}_{1,0}^{[2]} - \psi_0\mathbf{1}\right) = 0, \quad i = 0,1. \end{aligned} \tag{29}$$

The last two equations simplify to

$$\begin{aligned} &\mathcal{U}_{1,0}^{[1]} = -\frac{1}{2}\mathcal{C}\left(\mathcal{U}_{0,1}^{[1]} - 3\mathcal{U}_{1,0}^{[2]} - \psi_0\mathbf{1}\right) = 0, \\ &\mathcal{U}_{0,1}^{[2]} = \frac{1}{2}\mathcal{C}^{-1}\left(3\mathcal{U}_{0,1}^{[1]} - \mathcal{U}_{1,0}^{[2]} - \psi_0\mathbf{1}\right) = 0, \end{aligned} \tag{30}$$

where $\mathcal{C} = \left(c_0, (c_1, c_2, c_3)^T\right) := \mathcal{A}_{0,1}^{-1}\mathcal{A}_{1,0}$. Using (28) one can compute from (30) that

$$\beta_1 = \frac{c_1 \, \|\mathcal{C}\|^2 \, \psi_0}{2\,(c_1^2+c_2^2)}, \quad \beta_2 = \frac{c_2 \, \|\mathcal{C}\|^2 \, \psi_0}{2\,(c_1^2+c_2^2)}, \quad \beta_3 = \frac{(2c_0c_1 + c_2c_3)\,\psi_0}{4\,(c_1^2+c_2^2)},$$
$$\beta_4 = -\frac{(c_1c_3 - 2c_0c_2)\,\psi_0}{4\,(c_1^2+c_2^2)}, \quad \gamma_1 = \frac{(2c_0c_1 - c_2c_3)\,\psi_0}{4\,(c_1^2+c_2^2)}, \quad \gamma_2 = \frac{(2c_0c_2 + c_1c_3)\,\psi_0}{4\,(c_1^2+c_2^2)}, \tag{31}$$
$$\gamma_3 = \frac{c_1\psi_0}{2\,(c_1^2+c_2^2)}, \quad \gamma_4 = \frac{c_2\psi_0}{2\,(c_1^2+c_2^2)},$$

provided $c_1^2 + c_2^2 \neq 0$. The first equation in (29) is then equal to

$$\left(0, (\alpha_1, \alpha_2, 0)^T\right) - \mathcal{C}\left(0, (\alpha_3, \alpha_4, 0)^T\right) + \psi_0 \mathcal{D} = 0,$$

where

$$\mathcal{D} = \left(d_0, (d_1, d_2, d_3)^T\right) := \frac{1}{2}\mathcal{A}_{0,1}^{-1}\mathcal{A}_{0,0}\left(-1, \frac{c_3}{2\,(c_1^2+c_2^2)}(-c_2, c_1, 0)^T\right),$$

and it is straightforward to compute

$$\alpha_1 = c_0\alpha_3 - c_3\alpha_4 - d_1\psi_0, \quad \alpha_2 = c_3\alpha_3 + c_0\alpha_4 - d_2\psi_0,$$
$$\alpha_3 = -\psi_0 \frac{c_1d_0 + c_2d_3}{c_1^2+c_2^2}, \quad \alpha_4 = \psi_0 \frac{c_1d_3 - c_2d_0}{c_1^2+c_2^2}. \tag{32}$$

Note that the parameter ψ_0 does not bring any additional degrees of freedom and can be set to one. Namely, ψ_0 affects only the magnitude of $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, which can be changed also by multiplying a quaternion polynomial $\mathcal{A}$ by some constant factor. Note also that $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are by (26), (31) and (32) expressed with 12 free parameters.

Let us consider now a particular case where $c_1^2 + c_2^2 = 0$. From (30) it follows that $\psi_0 = 0$. Furthermore, it is easy to see that the equations (29) are satisfied iff

$$\mathcal{A}_{1,0} = \mathcal{A}_{0,1}\mathcal{C}, \quad \mathcal{C} = \left(c_0, (0, 0, c_3)^T\right),$$

and

$$\mathcal{U}_{1,0}^{[2]} = \mathcal{C}\mathcal{U}_{0,1}^{[2]}, \quad \mathcal{U}_1 = \mathcal{C}\mathcal{U}_2, \tag{33}$$

or equivalently

$$\mathcal{U}_{0,0}^{[2]} = \left(0, (\alpha_3, \alpha_4, 0)^T\right), \; \mathcal{U}_{0,1}^{[2]} = \left(0, (\gamma_3, \gamma_4, 0)^T\right), \; \mathcal{U}_{1,0}^{[2]} = \mathcal{C}\left(0, (\gamma_3, \gamma_4, 0)^T\right),$$
$$\mathcal{U}_{0,0}^{[1]} = \mathcal{C}\mathcal{U}_{0,0}^{[2]}, \quad \mathcal{U}_{0,1}^{[1]} = \mathcal{C}\mathcal{U}_{0,1}^{[2]}, \quad \mathcal{U}_{1,0}^{[1]} = \mathcal{C}\mathcal{U}_{1,0}^{[2]}. \tag{34}$$

This shows that a particular case where $c_1^2 + c_2^2 = 0$ provides much simpler solution with more degrees of freedom in comparison to a solution given by (31) and (32). The results are summarized in a following lemma.

Lemma 5. *Let a quaternion polynomial be of the form*

$$\mathcal{A}(u, v) = \mathcal{A}_{0,0} + \mathcal{A}_{0,1}\mathcal{C}u + \mathcal{A}_{0,1}v, \quad \mathcal{C} = \left(c_0, (c_1, c_2, c_3)^T\right) \in \mathbb{H}.$$

Suppose that $\mathcal{U}_1, \mathcal{U}_2$ are given by (12), (26), (31) and (32) in the case when $c_1^2 + c_2^2 \neq 0$, and suppose that for $c_1 = c_2 = 0$

$$\mathcal{U}_2(u, v) = \left(0, (\alpha_3, \alpha_4, 0)^T\right) + \mathcal{C}\left(0, (\gamma_3, \gamma_4, 0)^T\right)u + \left(0, (\gamma_3, \gamma_4, 0)^T\right)v,$$
$$\mathcal{U}_1(u, v) = \mathcal{C}\mathcal{U}_2(u, v)$$

for arbitrary $\alpha_3, \alpha_4, \gamma_3, \gamma_4 \in \mathbb{R}$. Then a parametric surface $\boldsymbol{S}$, defined by (5) and (13) is a PN surface of degree 4 with $\boldsymbol{N} = \boldsymbol{e}_3$.

If the suppositions of Lemma 5 hold true, then $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are of the form (8) with $m=3$, where

$$\begin{aligned}
\boldsymbol{B}_{0,0}^{[\ell]} &= \mathcal{A}_{0,0}\mathcal{U}_{0,0}^{[\ell]}\overline{\mathcal{A}}_{0,0},\\
\boldsymbol{B}_{1-k,k}^{[\ell]} &= \operatorname{vec}\left(\left(\mathcal{A}_{0,0}\mathcal{U}_{1-k,k}^{[\ell]} + 2\mathcal{A}_{1-k,k}\mathcal{U}_{0,0}^{[\ell]}\right)\overline{\mathcal{A}}_{0,0}\right),\\
\boldsymbol{B}_{2-2k,2k}^{[\ell]} &= \operatorname{vec}\left(\left(\mathcal{A}_{1-k,k}\mathcal{U}_{0,0}^{[\ell]} + 2\mathcal{A}_{0,0}\mathcal{U}_{1-k,k}^{[\ell]}\right)\overline{\mathcal{A}}_{1-k,k}\right),\\
\boldsymbol{B}_{1,1}^{[\ell]} &= 2\operatorname{vec}\left(\mathcal{A}_{0,1}\mathcal{U}_{0,0}^{[\ell]}\overline{\mathcal{A}}_{1,0} + \mathcal{A}_{1,0}\mathcal{U}_{0,1}^{[\ell]}\overline{\mathcal{A}}_{0,0} + \mathcal{A}_{0,1}\mathcal{U}_{1,0}^{[\ell]}\overline{\mathcal{A}}_{0,0}\right),\\
\boldsymbol{B}_{3-3k,3k}^{[\ell]} &= \mathcal{A}_{1-k,k}\mathcal{U}_{1-k,k}^{[\ell]}\overline{\mathcal{A}}_{1-k,k},\\
\boldsymbol{B}_{2-k,1+k}^{[\ell]} &= \operatorname{vec}\left(\left(\mathcal{A}_{1-k,k}\mathcal{U}_{k,1-k}^{[\ell]} + 2\mathcal{A}_{k,1-k}\mathcal{U}_{1-k,k}^{[\ell]}\right)\overline{\mathcal{A}}_{1-k,k}\right),
\end{aligned} \tag{35}$$

for $k=0,1$, $\ell=1,2$. The construction of a quartic PN surface is given in Algorithm 3:

Algorithm 3 Construction of a quartic PN surface.

Input: Quaternions $\mathcal{A}_{0,0}$, $\mathcal{A}_{0,1}$, parameters $c_0, c_1, c_2, c_3 \in \mathbb{R}$, and $\boldsymbol{P} \in \mathbb{R}^3$.
Output: A quartic PN surface $\boldsymbol{S}$.

Set $\mathcal{C} = \left(c_0, (c_1, c_2, c_3)^T\right)$;
Set $\mathcal{A}_{1,0} = \mathcal{A}_{0,1}\mathcal{C}$;
if $c_1^2 + c_2^2 \neq 0$ **then**
    Set $\psi_0 = 1$;
    Compute $(\beta_i)_{i=1}^4$ and $(\gamma_i)_{i=1}^4$ by (31);
    Compute $(\alpha_i)_{i=1}^4$ by (32);
    Compute $\mathcal{U}_{0,0}^{[\ell]}, \mathcal{U}_{1,0}^{[\ell]}, \mathcal{U}_{0,1}^{[\ell]}$ for $\ell = 1, 2$ by (28);
else
    Choose $\alpha_3, \alpha_4, \gamma_3, \gamma_4$;
    Compute $\mathcal{U}_{0,0}^{[\ell]}, \mathcal{U}_{1,0}^{[\ell]}, \mathcal{U}_{0,1}^{[\ell]}$ for $\ell = 1, 2$ by (34);
Compute $B_{i,j}^{[\ell]}$ for $\ell = 1, 2$ and $0 \le i + j \le 3$ by (35);
Set $\boldsymbol{S}_{0,0} = \boldsymbol{P}$, $m = 3$;
Compute $\boldsymbol{S}_{i,j}$ for $1 \le i + j \le 4$ by (11);
Compute $\boldsymbol{S}$ by (10).

As an example let us choose

$$\mathcal{A}_{0,0} = \left(1, (0,0,0)^T\right), \quad \mathcal{A}_{0,1} = \left(2, (1,3,4)^T\right), \quad \boldsymbol{P} = (0,0,0)^T, \tag{36}$$

and

$$\mathcal{C} = \frac{1}{12}\left(-3, (-4,2,3)^T\right). \tag{37}$$

The surface is shown in Fig. 4 (left). By choosing (36) and

$$\mathcal{C} = \frac{1}{4}\left(-1, (0,0,1)^T\right), \quad \alpha_3 = \frac{1}{2}, \quad \alpha_4 = -\frac{1}{3}, \quad \gamma_3 = \frac{1}{4}, \quad \gamma_4 = \frac{1}{3} \tag{38}$$

we obtain a surface presented in Fig. 4 (right).

5.2. Particular even degree polynomial PN surfaces

Suppose now that $\mathcal{A}$ is of degree n and let us try to find some simple relations between the coefficients $\mathcal{A}_{i,j}$, $i = 0,1,\ldots,n$, $j = 0,1,\ldots,n-i$, that guarantee (27) to hold true. Following a particular case for $n=1$ let us assume that $\psi_0 = 0$ and that linear quaternion polynomials $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfy (33) for some quaternion $\mathcal{C} = \left(c_0, (0,0,c_3)^T\right) \neq 0$. Then also $(\mathcal{U}_1)_v = (\mathcal{U}_2)_u$ and (27) simplifies to

$$\mathcal{A}_v\mathcal{U}_1 = \mathcal{A}_u\mathcal{U}_2,$$

which holds true if

$$\mathcal{A}_v\mathcal{C} = \mathcal{A}_u.$$

The next result follows by some straightforward computation.

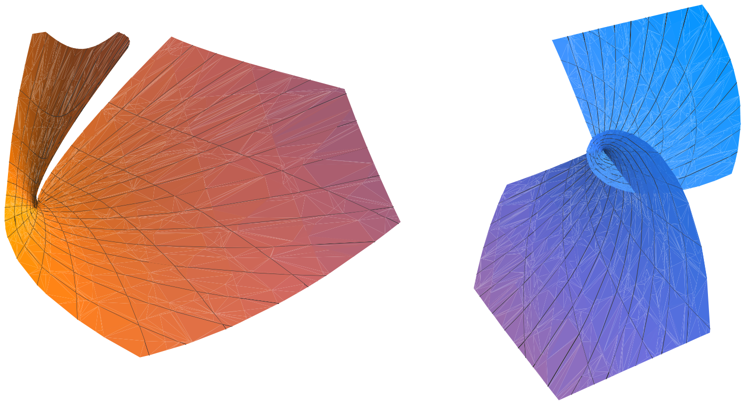

Fig. 4. A quartic PN surface determined by (36) and (37) for $(u, v) \in [-1, 1] \times [-1, 1]$ (left), and a quartic PN surface determined by (36) and (38) for $(u, v) \in [-1, 1] \times [-1, 1]$ (right).

Theorem 2. *Suppose that the coefficients in a quaternion polynomial (1) satisfy*

$$(j+1)\mathcal{A}_{i,j+1}\mathcal{C} = (i+1)\mathcal{A}_{i+1,j}, \quad i = 0, 1, \ldots, n-1, \ j = 0, 1, \ldots, n-1-i,$$

for some nonzero quaternion $\mathcal{C} = \left(c_0, (0, 0, c_3)^T\right)$, *and let*

$$\mathcal{U}_2(u, v) = \left(0, (\alpha_3, \alpha_4, 0)^T\right) + \mathcal{C}\left(0, (\gamma_3, \gamma_4, 0)^T\right) u + \left(0, (\gamma_3, \gamma_4, 0)^T\right) v,$$
$$\mathcal{U}_1(u, v) = \mathcal{C}\mathcal{U}_2(u, v).$$

Then a parametric surface $\boldsymbol{S}$, *defined by (5) and (13) is a PN surface of degree* $2n+2$ *with* $\boldsymbol{N} = \boldsymbol{e}_3$.

6. Mean curvature of the derived PN surfaces

The coefficients of the first and the second fundamental form of PN surfaces derived from (3) are equal to

$$E = \|\mathcal{U}_1\|^2 \|\mathcal{A}\|^4, \quad F = \mathcal{U}_1 \cdot \mathcal{U}_2 \|\mathcal{A}\|^4, \quad G = \|\mathcal{U}_2\|^2 \|\mathcal{A}\|^4,$$
$$L = 2\mathrm{vec}\left(\left(\mathcal{A}_u\mathcal{U}_1 + \mathcal{A}(\mathcal{U}_1)_u\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N},$$
$$M = 2\mathrm{vec}\left(\left(\mathcal{A}_v\mathcal{U}_1 + \mathcal{A}(\mathcal{U}_1)_v\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N},$$
$$N = 2\mathrm{vec}\left(\left(\mathcal{A}_v\mathcal{U}_2 + \mathcal{A}(\mathcal{U}_2)_v\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N},$$

where $\boldsymbol{N}$ is the unit normal. The next lemma reveals the values of the mean curvature for particular PN surfaces given in Theorem 1 and in Theorem 2.

Lemma 6. *Let a PN surface be given by Theorem 1 for an odd degree or by Theorem 2 for an even degree surface. Then its mean curvature is identically zero.*

Proof. It is easy to see that the mean curvature is identically zero iff

$$EN + GL - 2FM \equiv 0. \tag{39}$$

Note that $\mathcal{A}_v\mathcal{U}_1 = \mathcal{A}_u\mathcal{U}_2$ for both types of the PN surfaces. Then

$$EN = 2\|\mathcal{U}_1\|^2 \|\mathcal{A}\|^4 \mathrm{vec}\left(\left(\mathcal{A}_v\mathcal{U}_1(\mathcal{U}_1)^{-1}\mathcal{U}_2 + \mathcal{A}(\mathcal{U}_2)_v\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N}$$
$$= 2\|\mathcal{A}\|^4 \mathrm{vec}\left(\left(\mathcal{A}_v\mathcal{U}_1\overline{\mathcal{U}}_1\mathcal{U}_2 + \mathcal{A}(\mathcal{U}_2)_v\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N},$$
$$GL = 2\|\mathcal{U}_2\|^2 \|\mathcal{A}\|^4 \mathrm{vec}\left(\left(\mathcal{A}_u\mathcal{U}_2(\mathcal{U}_2)^{-1}\mathcal{U}_1 + \mathcal{A}(\mathcal{U}_1)_u\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N}$$
$$= 2\|\mathcal{A}\|^4 \mathrm{vec}\left(\left(\mathcal{A}_u\mathcal{U}_2\overline{\mathcal{U}}_2\mathcal{U}_1 + \mathcal{A}(\mathcal{U}_1)_u\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N}$$
$$= 2\|\mathcal{A}\|^4 \mathrm{vec}\left(\left(\mathcal{A}_v\mathcal{U}_1\overline{\mathcal{U}}_2\mathcal{U}_1 + \mathcal{A}(\mathcal{U}_1)_u\right)\overline{\mathcal{A}}\right) \cdot \boldsymbol{N},$$

and thus

$$
\begin{aligned}
&EN+GL-2FM=\\
&\quad 2\,\|\mathcal{A}\|^4 \operatorname{vec}\left(\mathcal{A}_v\mathcal{U}_1\left(\overline{\mathcal{U}}_1\mathcal{U}_2+\overline{\mathcal{U}}_2\mathcal{U}_1-2\left(\mathcal{U}_1\cdot\mathcal{U}_2,\mathbf{0}^T\right)\right)\overline{\mathcal{A}}\right)\cdot\boldsymbol{N}+\\
&\quad +2\,\|\mathcal{A}\|^4 \operatorname{vec}\left(\mathcal{A}\left(\|\mathcal{U}_1\|^2(\mathcal{U}_2)_v+\|\mathcal{U}_2\|^2(\mathcal{U}_1)_u-2\mathcal{U}_1\cdot\mathcal{U}_2\,(\mathcal{U}_1)_v\right)\overline{\mathcal{A}}\right)\cdot\boldsymbol{N}.
\end{aligned}
$$

It is straightforward to see that

$$
\overline{\mathcal{U}}_1\mathcal{U}_2+\overline{\mathcal{U}}_2\mathcal{U}_1-2\left(\mathcal{U}_1\cdot\mathcal{U}_2,\mathbf{0}^T\right)=0.
$$

For odd degree surfaces the quaternions $\mathcal{U}_1$ and $\mathcal{U}_2$ are constant and (39) holds true. For even degree surfaces relations $(\mathcal{U}_1)_v=(\mathcal{U}_2)_u$ and $\mathcal{U}_1=\mathcal{C}\mathcal{U}_2$ imply

$$
\begin{aligned}
&\|\mathcal{U}_1\|^2(\mathcal{U}_2)_v+\|\mathcal{U}_2\|^2(\mathcal{U}_1)_u-2\mathcal{U}_1\cdot\mathcal{U}_2\,(\mathcal{U}_1)_v=\\
&\qquad\left(\|\mathcal{U}_2\|^2\left(\mathcal{C}+\overline{\mathcal{C}}\right)-2\left(\mathcal{U}_1\cdot\mathcal{U}_2,\mathbf{0}^T\right)\right)(\mathcal{U}_2)_u=0
\end{aligned}
$$

which completes the proof for even degrees too. □

By Lemma 6 cubic PN surface (19), quintic PN surface determined by (25) and degree four PN surface determined by (36)–(37) belong to a class of minimal surfaces. Note that there exist also minimal PN surfaces which are not defined by Theorem 1 or by Theorem 2. Such surfaces can be for example constructed by choosing functions φ_i of higher degrees. Examples of PN surfaces that are not minimal are a $\boldsymbol{P}_3$ surface, surface (22) and a quartic PN surface given by (36) and (38).

7. Interpolation example

To show the practical value of the derived surfaces an interpolation scheme with PN surfaces of degree four and five is proposed. Suppose that three points $\boldsymbol{P}_i$, $i=0,1,2$, and three normal directions $\boldsymbol{N}_i$, $\|\boldsymbol{N}_i\|=1$, $i=0,1,2$, are given. The task is to find the PN surface $\boldsymbol{S}$, such that

$$
\boldsymbol{S}(u_i,v_i)=\boldsymbol{P}_i,\quad \boldsymbol{N}(u_i,v_i)=\boldsymbol{N}_i,\quad i=0,1,2,
$$

where $(u_0,v_0)=(0,0)$, $(u_1,v_1)=(1,0)$, $(u_2,v_2)=(0,1)$. Suppose first that $\boldsymbol{S}$ is of degree five, obtained by Theorem 1. From the interpolation of normal directions one obtains

$$
\mathcal{A}(u_i,v_i)=\lambda_i\mathcal{X}(\boldsymbol{N}_i,\phi_i),\quad \phi_i\in[-\pi,\pi),\quad i=0,1,2,\tag{40}
$$

where

$$
\mathcal{X}(\boldsymbol{N}_i,\phi_i)=\begin{cases}\sqrt{\|\boldsymbol{N}_i\|}\,\dfrac{\frac{\boldsymbol{N}_i}{\|\boldsymbol{N}_i\|}+\mathbf{k}}{\left\|\frac{\boldsymbol{N}_i}{\|\boldsymbol{N}_i\|}+\mathbf{k}\right\|}\left(\cos\phi_i+\mathbf{k}\sin\phi_i\right), & \frac{\boldsymbol{N}_i}{\|\boldsymbol{N}_i\|}\neq-\mathbf{k}\\[2ex] \sqrt{\|\boldsymbol{N}_i\|}\,\mathbf{i}\left(\cos\phi_i+\mathbf{k}\sin\phi_i\right), & \frac{\boldsymbol{N}_i}{\|\boldsymbol{N}_i\|}=-\mathbf{k}\end{cases}
$$

is a solution of a well known ‘star-equation’ (Farouki et al., 2002). Therefrom, $\mathcal{A}_{0,0}$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are expressed by four parameters α_i and six new free parameters, i.e., normal lengths λ_i and the angles ϕ_i, $i=0,1,2$. Interpolation of a point $\boldsymbol{P}_0$ is obtained by setting $\boldsymbol{S}(0,0)=\boldsymbol{P}_0$. The remaining two points imply six nonlinear scalar equations.

As an example let us choose

$$
\begin{aligned}
&\boldsymbol{P}_0=(0,0,0)^T,\quad \boldsymbol{P}_1=(-24,24,-18)^T,\quad \boldsymbol{P}_2=(-27,-42,10)^T,\\
&\boldsymbol{N}_0=\frac{1}{\sqrt{437}}(-6,1,20)^T,\,\boldsymbol{N}_1=\frac{1}{\sqrt{113}}(-10,-3,2)^T,\,\boldsymbol{N}_2=\frac{1}{3\sqrt{11}}(-7,7,1)^T.
\end{aligned}
$$

We can fix four parameters and compute the remaining by a Newton method. Namely, by choosing $\phi_0=0$, $\lambda_i=7$, $i=0,1,2$, one of the solutions is

$$
\begin{aligned}
&\alpha_1=0.5783,\quad \alpha_2=-0.608535,\quad \alpha_3=0.181048,\\
&\alpha_4=1.09587,\quad \phi_1=0.218641,\quad \phi_2=-0.300473,
\end{aligned}
$$

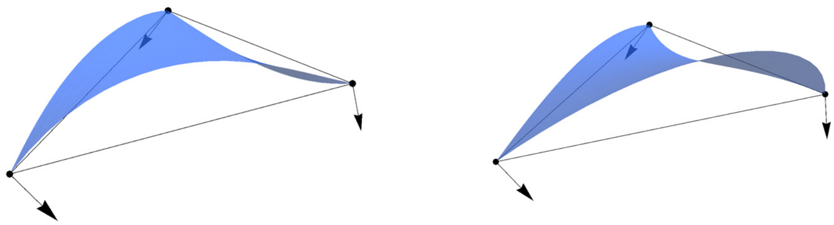

Fig. 5. Interpolating PN surface of degree five (left) and four (right).

which yields

$$\mathcal{A}_{0,0} = \left(0, (-1.01561, 0.169269, 6.92386)^T\right),$$
$$\mathcal{Q}_1 = \left(3.01248, (0.932408, -2.24853, -2.30311)^T\right),$$
$$\mathcal{Q}_2 = \left(-0.0481767, (5.40034, -0.948083, 2.2529)^T\right).$$

The corresponding quintic PN surface is shown in Fig. 5, left.

The above interpolation problem can be solved also by quartic PN surfaces, obtained by Theorem 2. In this case two of the parameters, coming from (40), can be fixed. By choosing $\phi_0 = 0$, $\lambda_0 = 5$, one of the solutions computed by a Newton method is

$$\alpha_1 = 0.158866,\ \alpha_2 = 1.46099,\ \alpha_3 = 2.03928,\ \alpha_4 = 0.234132,\ \lambda_1 = 7.1611,$$
$$\lambda_2 = 5.4437,\ \phi_1 = 0.258336,\ \phi_2 = 0.179764,\ c_0 = 0.434869,\ c_3 = -1.12697,$$
$$\mathcal{A}_{0,0} = \left(0, (-0.725438, 0.120906, 4.94562)^T\right),$$
$$\mathcal{Q}_1 = \left(-0.721999, (-1.35286, 2.88051, -0.972605)^T\right)$$

and the corresponding quartic PN surface is shown in Fig. 5, right.

Since nonlinear equations are involved the analysis of both interpolation schemes is beyond the scope of this paper and might be an interesting topic for future research. Note also that cubic PN surfaces unfortunately do not offer enough degrees of freedom to tackle this interpolation problem.

8. Conclusion

Although polynomial surfaces with rational field of unit normal vectors are important in practical applications, not much results about these surfaces is known. The present paper introduces a new approach for a construction based on bivariate quaternion polynomials. Particular relations between the quaternion coefficients are derived that allow us to construct polynomial PN surfaces of degrees $2n+1$ and $2n+2$ from degree n quaternion polynomial. As in the curve case such a representation could be particularly useful if the interpolation with PN surfaces is considered. The analysis of the interpolation scheme with quartic and quintic PN surfaces, proposed in Section 7, is left for a future research. Also the extension to interpolating PN G^1 splines over triangulations might be a useful topic to study. It would also be interesting for the future work to examine whether polynomials $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, defined in (3), that satisfy (4), could be constructed using φ_i being rational functions or polynomials of higher degrees. This may lead to find a full description of all polynomial PN parameterizations at least for some low degrees, which is still an open problem.

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