ARTICLE IN PRESS
COMAID:1577

Computer Aided Geometric Design ••• (••••) •••–•••

Contents lists available at ScienceDirect

Computer Aided Geometric Design

www.elsevier.com/locate/cagd

Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space

Georg Nawratil

Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10/104, 1040 Vienna, Austria

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Kinematic mapping
Equiform motion
Quaternion
Instantaneous screw
Line-element

ABSTRACT

We extend the quaternionic kinematic mapping of Euclidean displacements of Euclidean 4-space E^4 to the group of equiform transformations S(4). As a consequence the equiform motions of basic elements (points, oriented lines, oriented planes, oriented hyperplanes) of E^4 can be written compactly in terms of 2×2 quaternionic matrices. This representation is extended to oriented line-elements of E^4 and to instantaneous screws of S(4), for which a classification (incl. corresponding normal forms) is given. Based on this preparatory work we study the relation between instantaneous equiform motions and the geometry of line-elements (path normal-elements, path tangent-elements) in E^4. Finally, we show that the line-elements of projective 3-space can be mapped bijectively on the Segre variety $\Sigma_{3,2}$.

1. Introduction

This paper can be seen as a logical sequel to the author's work (Nawratil, 2016), which we will reference a few times in order to avoid unnecessary replications.

The composition of Euclidean displacements (= orientation preserving congruence transformations) and uniform scalings with a scale-factor α yields the group S(n) of equiform transformations of n-dimensional Euclidean space E^n (Bottema and Roth, 1979, §3 of Chapter 12).

The geometric object of a line combined with a point on it is called a line-element. It is a partial flag and also known as pointed line in the literature.

The elegance of the quaternion based analytical treatment of kinematics in Euclidean spaces of dimension 2 and 3 was pointed out and used by various authors (e.g. Blaschke, 1960; Müller, 1962; Ströher, 1973). Recently Nawratil (2016) extended this quaternionic kinematic to E^4. Based on these results we give a quaternionic kinematic mapping for equiform motions in E^4 and E^3 and study the corresponding point models (cf. Section 2). As a consequence the equiform transformations of points, oriented lines, oriented planes and oriented hyperplanes of E^4 can be written compactly in terms of 2×2 quaternionic matrices. Moreover we show that the transformation of oriented line-elements of E^4 can also be represented in this way. In Section 3 we proceed with a detailed study of instantaneous equiform motions of E^4, which results in a quaternionic definition of instantaneous screws of S(4) and their classification (incl. normal forms). Moreover the equiform transformation of these screws can also be embedded into the algebra of 2×2 quaternionic matrices. Inspired by the publication of Odehnal et al. (2006) we investigate the relation between equiform kinematics and the geometry of line-elements of E^4 in the Sections 4 and 5. In detail we define a linear complex of line-elements and show that it equals the set of path normal-elements of an instantaneous screw and vice versa. On this basis we extend some known results

E-mail address: nawratil@geometrie.tuwien.ac.at.

http://dx.doi.org/10.1016/j.cagd.2016.06.003

on path normal-elements from spatial equiform kinematics to 4 dimensions (see Section 5). Moreover we study the set of path tangent-elements of an instantaneous screw in Section 5.1. Finally in Section 6 the representation of finite lines of E^4 via so-called minimal coordinates is used to show that the line-elements of the projective closure P^3 of E^3 can be mapped bijectively on the Segre variety $\Sigma_{3,2}$.

But before we can plunge in medias res we have to provide a literature review on spatial equiform kinematics, as no compact survey of this topic, which can be referred to, is known to the author.

1.1. Literature review

Bottema and Roth (1979) noted that "not much attention has been given in the literature to spatial equiform kinematics" and therefore they reported some basic facts on pages 455–458. Before that time, the author's literature research yields following results:

- To the best knowledge of the author the first explicit result on spatial equiform kinematics was given by Burmester (1878, 1902) and reads as follows: All points, which have instantaneously the same velocity under an equiform motion, are located on an ellipsoid of rotation, whose center is the velocity pole and its symmetry axis equals the instantaneously fixed line. All other given results of Burmester belong to the more general field of spatial affine kinematics.[1]
- Seiliger (1892) proved that each spatial equiform motion can be composed of a central similarity and a rotation about an axis through its center. Further results on this topic were obtained by Pascal (1932) and Di Noi (1934), inter alia that each line has a point with a velocity vector orthogonal to it (Pascal, 1932) and that the tangent complex is a quadratic one (Di Noi, 1934).
- Moreover the following special equiform motions were studied: Kowalewski (1930) investigated point paths generated by an equiform rolling motion of two cylindro-conical spiral curves.[2] Wunderlich (1962) studied spatial equiform motions, where a line slides through a fixed point and a second line (skew to the first one) slides along itself.

Since 1979 the following results of spatial equiform kinematics were published to the best knowledge of the author:

- A lot of work was done on equiform motions with only planar/spherical trajectories, which are so-called equiform Darboux/Bricard motions. All equiform Darboux motions were determined by Karger (1981) and Röschel (1991). In contrast a complete listing of equiform Bricard motions is still missing, but a subclassification was obtained by Gfrerer (2008), which includes the above mentioned motion of Wunderlich (1962). Moreover equiform bundle motions with spherical point paths were discussed by Gfrerer and Lang (1998).
- Pottmann (1984) studied the envelopes of planes under equiform Schönflies motions. Moreover he discussed global kinematic properties of spatial equiform motions in Pottmann (1986, 1987). This work was furthered by Hager (1991) to n dimensions.
- From the application point of view, equiform motions were studied in context of 3D shape recognition and reconstruction (e.g. Hofer et al., 2005 and Odehnal et al., 2006). Moreover equiform motions were used by Röschel (2000, 2014) to construct overconstrained mechanisms and to define a performance index for Stewart–Gough platforms (Lang et al., 2001). In this context also the article of Hamdoon and Abdel-All (2004) on octahedral Stewart–Gough platforms should be noted, where spatial equiform motions were studied with three points moving on three circles possessing coplanar axes.

No explicit results on S(4) are known to the author, but there are the following on S(n): Somer (1979a) investigated closed equiform motion where some points trace one and the same trajectory. Moreover Somer (1979b) studied linear spaces, which are instantaneous invariant under S(n). A more detailed study of first order properties of S(n) was done by Spallek (1992) by generalizing his approach of *glide–glide-kinematics* to angle-preserving transformations.

2. Quaternionic formulation and kinematic mapping of S(4) and S(3)

We write quaternions, which are printed in bold letters, just side by side for multiplication instead of introducing an extra multiplication sign. q_0 is the so-called scalar part of the quaternion $\mathbf{Q} := q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ and $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ its pure part. By denoting the pure part by the small letter $\mathbf{q}$, the quaternion can also be written as $\mathbf{Q} := q_0 + \mathbf{q}$. Finally it should be noted that the conjugated quaternion to $\mathbf{Q}$ is given by $\widetilde{\mathbf{Q}} := q_0 - \mathbf{q}$. For more basics on quaternions we refer to Nawratil (2016, Section 1.1), where the same notation is used.

[1] Please see Pottmann (1994, page 149) for references to this topic.
[2] These spatial curves can be generated as trajectories of spiral motions, which are the most general uniform equiform motions in E^3.

2.1. Fundamentals

First of all we want to sharpen the definition of equiform motions given in the second paragraph of Section 1. The given definition yields for even dimensional spaces a double cover of orientation-preserving similarity transformations if $\alpha \in \mathbb{R} \setminus \{0\}$ holds.[3] This choice of α implies for odd dimensions no double cover but we get in addition the set of orientation-reversing similarity transformations. Note that different conventions/definitions can be found in the literature; especially in E^3 the equiform motion group is often identified with the group of orientation-preserving similarity transformations (e.g. Odehnal et al., 2006), which is denoted by $\mathrm{S}^+(3)$ within our notation.

Assume that $(x_0, x_1, x_2, x_3)^T$ is the Cartesian coordinate vector of a point X in E^4, then its image under an equiform mapping η can be written in terms of linear algebra as:

$$\eta : (x_0, x_1, x_2, x_3)^T \mapsto (x'_0, x'_1, x'_2, x'_3)^T = \alpha A (x_0, x_1, x_2, x_3)^T + (a_0, a_1, a_2, a_3)^T$$

with the scaling factor $\alpha \in \mathbb{R} \setminus \{0\}$ and the rotation matrix $A \in \mathrm{SO}(4)$ (Bottema and Roth, 1979, §3 of Chapter 12).

If we embed the point X of E^4 into the set of quaternions by the mapping:

$$\iota_4 : \mathbb{R}^4 \to \mathbb{H} \quad \text{with} \quad (x_0, x_1, x_2, x_3) \mapsto \mathbf{X} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

then $\eta \in \mathrm{S}(4)$ can be written in terms of quaternions as follows according to Nawratil (2016, Theorem 2.6):

Theorem 1. *The mapping of points $\mathsf{X} \in E^4$ to $\mathsf{X}' \in E^4$ induced by any element of $S(4)$ can be written as:*

$$\mathbf{X} \mapsto \mathbf{X}' \quad \textit{with} \quad \mathbf{X}' := \mathbf{E}\mathbf{X}\widetilde{\mathbf{F}} - 2\mathbf{E}\widetilde{\mathbf{T}} \tag{1}$$

with the unit-quaternion $\mathbf{E}$ *and* $\|\mathbf{F}\| \neq 0$. *Moreover the mapping of Eq. (1) is an element of* $S(4)$ *for any triplet of quaternions* $\mathbf{E}, \mathbf{F}, \mathbf{T}$*, where* $\mathbf{E}$ *is a unit-quaternions and* $\|\mathbf{F}\| \neq 0$.

As both triplets of quaternions $\pm(\mathbf{E}, \mathbf{F}, \mathbf{T})$, where $\mathbf{E}$ is a unit-quaternion, correspond to the same equiform motion of E^4 we consider the homogeneous 12-tuple $(e_0 : \ldots : e_3 : f_0 : \ldots : f_3 : t_0 : \ldots : t_3)$, which can be written abstractly in a quaternionic representation as $(\mathbf{E} : \mathbf{F} : \mathbf{T})$. These 12 homogeneous motion parameters for E^4 can be interpreted as a point of a projective 11-dimensional space P^{11}. Therefore there is a bijection – the so-called kinematic mapping – between $\mathrm{S}(4)$ and the set $\mathcal{S}_4$ of real points of P^{11}, which are not located in one of the two 7-dimensional spaces $e_0 = e_1 = e_2 = e_3 = 0$ and $f_0 = f_1 = f_2 = f_3 = 0$, respectively. These spaces need to be removed as $\mathbf{E}$ cannot be normalized and $\|\mathbf{F}\| = 0$ holds, respectively.

If we identify E^3 with the hyperplane $x_0 = 0$, the set $\mathcal{S}_3$ of all points of $\mathcal{S}_4$ fulfilling

$$f_0 : e_0 = f_1 : e_1 = f_2 : e_2 = f_3 : e_3 \tag{2}$$

and the condition that no translation is done in direction of x_0 ($\Leftrightarrow a_0 = 0$), map the hyperplane $x_0 = 0$ onto itself. Note that $a_0 = 0$ equals the so-called Study condition

$$e_0 t_0 + e_1 t_1 + e_2 t_2 + e_3 t_3 = 0. \tag{3}$$

Therefore there is a bijection – the so-called kinematic mapping – between $\mathrm{S}(3)$ and the points of $\mathcal{S}_3$, which is studied in more detail next. The condition given in Eq. (2) can be formulated algebraically by the set of equations:

$$e_i f_j - e_i f_j = 0 \quad \text{for} \quad i < j \quad \text{and} \quad i, j \in \{0, 1, 2, 3\}.$$

The Hilbert-polynomial of the ideal $\mathcal{I}$ generated by these 6 equations together with the Study equation (3) can be computed as

$$p(t) = \tfrac{1}{630}t^7 + \tfrac{11}{360}t^6 + \tfrac{11}{45}t^5 + \tfrac{19}{18}t^4 + \tfrac{479}{180}t^3 + \tfrac{1409}{360}t^2 + \tfrac{433}{140}t + 1.$$

Therefore $\mathcal{S}_3$ is an algebraic variety $V(\mathcal{I})$ of dimension 7 and degree[4] 8 in P^{11}, which is sliced along the two 7-dimensional spaces $e_0 = e_1 = e_2 = e_3 = 0$ and $f_0 = f_1 = f_2 = f_3 = 0$. The first space is completely contained in $V(\mathcal{I})$ in contrast to the second one, which intersects $V(\mathcal{I})$ in a quadric given by Eq. (3).

Remark 1. In comparison to $\mathcal{S}_4$ the point-model[5] $\mathcal{S}_3$ for $\mathrm{S}(3)$ is much more complicated, but until now no simpler one is known to the author. In this context it should be noted that another interesting approach for the description of $\mathrm{S}(3)$ was done in Combebiac (1902, pages 23–28) based on so-called tri-quaternions. Within this framework Combebiac also defined a so-called *linear element*, which can be viewed as an extension of a line-element as it[6] "depends upon a point, a segment

[3] The case $\alpha = 0$ is excluded as it yields a singular transformation.

[4] For $p(t) = c_n t^n + o(t^n)$ the degree of the variety is computed by $n! c_n$ according to Shafarevich (1988).

[5] Bijective mapping between $\mathrm{S}(3)$ and points within a projective space.

[6] The following is cited from Combebiac (1902, page 15).

Please cite this article in press as: Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Des. (2016), http://dx.doi.org/10.1016/j.cagd.2016.06.003

(direction and length), and a coefficient (a mass or tensor)". Therefore this work of Combebiac (1902) seems worth to be revisited in future studies on this topic. $\diamond$

As $\mathcal{S}_3$ is contained in $\mathcal{S}_4$ we can restrict to the discussion of S(4) in the following section.

2.2. Composition and representation of equiform motions

Analogously to Nawratil (2016, Section 2.4) the composition $\eta = \eta_2 \circ \eta_1$ of two equiform motions η_1, η_2 corresponds to the multiplication of lower triangular 2×2 quaternionic matrices; i.e.:

$$\begin{pmatrix} \mathbf{E} & \mathbf{O} \\ \mathbf{T} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_2 & \mathbf{O} \\ \mathbf{T}_2 & \mathbf{F}_2 \end{pmatrix} \begin{pmatrix} \mathbf{E}_1 & \mathbf{O} \\ \mathbf{T}_1 & \mathbf{F}_1 \end{pmatrix}.$$

Moreover in Nawratil (2016, Section 4) it was also shown that the displacements of the basic geometric elements in E^4 can be formulated in a unified way using 2×2 quaternionic matrices. In the following we extend this representation to S(4). We embed a point $\mathsf{X} \in E^4$ and its image $\mathsf{X}' \in E^4$ into the set of 2×2 quaternionic matrices by

$$\underline{\mathbf{X}} = \begin{pmatrix} -1 & \mathbf{X} \\ \mathbf{O} & 1 \end{pmatrix}, \qquad \underline{\mathbf{X}}' = \begin{pmatrix} -1 & \mathbf{X}' \\ \mathbf{O} & 1 \end{pmatrix}.$$

As for an equiform motion $(\mathbf{E}, \mathbf{F}, \mathbf{T})$ the scaling factor α is given by $\|\mathbf{F}\| > 0$ we can define the following quaternionic matrices:

$$\underline{\mathbf{D}} = \frac{1}{\sqrt{\alpha}} \begin{pmatrix} \mathbf{E} & \mathbf{O} \\ \mathbf{T} & \mathbf{F} \end{pmatrix}, \quad \widetilde{\underline{\mathbf{D}}}^T = \frac{1}{\sqrt{\alpha}} \begin{pmatrix} \widetilde{\mathbf{E}} & \widetilde{\mathbf{T}} \\ \mathbf{O} & \widetilde{\mathbf{F}} \end{pmatrix}, \quad \widetilde{\underline{\mathbf{D}}}^{-T} = \frac{1}{\alpha\sqrt{\alpha}} \begin{pmatrix} \alpha^2 \mathbf{E} & -\mathbf{E}\widetilde{\mathbf{T}}\mathbf{F} \\ \mathbf{O} & \mathbf{F} \end{pmatrix}, \tag{4}$$

where $\widetilde{\underline{\mathbf{D}}}^{-T}$ denotes the (left and right) multiplicative inverse of $\widetilde{\underline{\mathbf{D}}}^T$. Based on this notation we can formulate the corresponding theorem to Nawratil (2016, Theorem 4.1):

Theorem 2. *The mapping of points $\mathsf{X} \in E^4$ to $\mathsf{X}' \in E^4$ induced by any element of S(4) can be written as follows:*

$$\underline{\mathbf{X}} \mapsto \underline{\mathbf{X}}' \quad \textit{with} \quad \underline{\mathbf{X}}' := \widetilde{\underline{\mathbf{D}}}^{-T} \underline{\mathbf{X}} \widetilde{\underline{\mathbf{D}}}^T.$$

The proof is just done by a multiplication of the quaternionic matrices. Moreover it can be shown by analogous considerations as given in Nawratil (2016, Section 4) that also the equiform motion of oriented lines, planes and hyperplanes can be written in a similar fashion. The corresponding theorems[7] given in Nawratil (2016, Section 4) also hold for S(4) with respect to the matrices given in Eq. (4). In the following we only repeat the results for oriented lines in more detail, as they are needed for the study of line-elements later on.

Geometrically we characterize an oriented line $\vec{\mathsf{y}}$ by its pedal point C (given by the quaternion $\mathbf{C}$) with respect to the origin and by its direction, which can be written as a unit-quaternion $\mathbf{Y}$. Then we can represent $\vec{\mathsf{y}}$ by the pair $(\mathbf{Y}, \widetilde{\mathbf{Y}}\mathbf{C})$, which is the 4-dimensional analogue of the spear coordinates (oriented line coordinates) of E^3. Therefore we call the pure quaternion $\widetilde{\mathbf{Y}}\mathbf{C}$ the moment quaternion $\mathbf{m} := \widetilde{\mathbf{Y}}\mathbf{C}$. Moreover the expression $\mathbf{m}$ can be computed from any point $\mathsf{X} \in \vec{\mathsf{y}}$ with $\mathbf{X} = \mathbf{C} + \xi\mathbf{Y}$ as follows:

$$\tfrac{1}{2}\left(\widetilde{\mathbf{Y}}\mathbf{X} - \widetilde{\mathbf{X}}\mathbf{Y}\right) = \tfrac{1}{2}\left(\widetilde{\mathbf{Y}}\mathbf{C} + \xi - \widetilde{\mathbf{C}}\mathbf{Y} - \xi\right) = \widetilde{\mathbf{Y}}\mathbf{C}. \tag{5}$$

Now we introduce the following notation:

$$\underline{\mathbf{Y}} = \begin{pmatrix} \mathbf{O} & \mathbf{Y} \\ -\widetilde{\mathbf{Y}} & \mathbf{m} \end{pmatrix}, \qquad \underline{\mathbf{Y}}' = \begin{pmatrix} \mathbf{O} & \mathbf{Y}' \\ -\widetilde{\mathbf{Y}}' & \mathbf{m}' \end{pmatrix},$$

where $\mathbf{Y}'$ denotes the unit-quaternion in direction of the transformed spear $\vec{\mathsf{y}}'$ and $\mathbf{m}'$ its moment quaternion. Then the above mentioned analogous result to Nawratil (2016, Theorem 4.3) reads as:

Theorem 3. *The mapping of oriented lines $\vec{\mathsf{y}}$ of E^4 to oriented lines $\vec{\mathsf{y}}'$ of E^4 induced by any element of S(4) can be written as follows:*

$$\underline{\mathbf{Y}} \mapsto \underline{\mathbf{Y}}' \quad \textit{with} \quad \underline{\mathbf{Y}}' := \underline{\mathbf{D}}\,\underline{\mathbf{Y}}\,\widetilde{\underline{\mathbf{D}}}^T.$$

[7] For oriented lines see Nawratil (2016, Theorem 4.3), for oriented planes see Nawratil (2016, Theorem 4.4) and for oriented hyperplanes see Nawratil (2016, Theorem 4.2).

Please cite this article in press as: Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Des. (2016), http://dx.doi.org/10.1016/j.cagd.2016.06.003

Now we extend the oriented line coordinates $(\mathbf{Y}, \mathbf{m})$ of $\overrightarrow{\mathsf{y}}$ in E^4 to those of an oriented line-element $(\overrightarrow{\mathsf{y}}, \mathsf{X})$. As the scalar part of the moment quaternion $\mathbf{m}$ is zero we want to insert the information about the point $\mathsf{X} \in \overrightarrow{\mathsf{y}}$ at this position. Analogously to the method of Odehnal et al. (2006) we use the distance ξ of X to C in direction of $\overrightarrow{\mathsf{y}}$. Under consideration that $\xi = \langle \mathbf{X}, \mathbf{Y} \rangle$ holds we can define the oriented line-element coordinates by $(\mathbf{Y}, \xi + \mathbf{m})$.

Based on the notation:

$$\underline{\mathbf{Z}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -2\widetilde{\mathbf{Y}} & \xi + \mathbf{m} \end{pmatrix}, \qquad \underline{\mathbf{Z}}' = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -2\widetilde{\mathbf{Y}}' & \xi' + \mathbf{m}' \end{pmatrix},$$

with $\xi' = \langle \mathbf{X}', \mathbf{Y}' \rangle$ a straightforward computation shows the following result:

Theorem 4. *The mapping of oriented line-elements $(\overrightarrow{\mathsf{y}}, \mathsf{X})$ of E^4 to oriented line-elements $(\overrightarrow{\mathsf{y}}', \mathsf{X}')$ of E^4 induced by any element of $S(4)$ can be written as follows:*

$$\underline{\mathbf{Z}} \mapsto \underline{\mathbf{Z}}' \quad \textit{with} \quad \underline{\mathbf{Z}}' := \underline{\mathbf{D}}\,\underline{\mathbf{Z}}\,\widetilde{\underline{\mathbf{D}}}^T.$$

3. Instantaneous equiform motions in Euclidean 4-space

Now $\mathbf{X}$ contains the coordinates of X with respect to the moving coordinate frame $\mathcal{C}$ and $\mathbf{X}_\tau^{\oplus}$ denotes the coordinates of X with respect to the fixed frame $\mathcal{C}^{\oplus}$ in dependency of the time τ of the constrained motion. According to Eq. (1) the following relation holds:

$$\mathbf{X}_\tau^{\oplus} = \mathbf{E}_\tau \mathbf{X} \widetilde{\mathbf{F}}_\tau - 2\mathbf{E}_\tau \widetilde{\mathbf{T}}_\tau, \tag{6}$$

where $\mathbf{E}_\tau$, $\mathbf{F}_\tau$ and $\mathbf{T}_\tau$ are functions of the time τ. Eq. (6) can be rewritten in terms of 2×2 quaternionic matrices (cf. Theorem 2) as follows:

$$\underline{\mathbf{X}}_\tau^{\oplus} = \widetilde{\underline{\mathbf{D}}}_\tau^{-T} \underline{\mathbf{X}}\, \widetilde{\underline{\mathbf{D}}}_\tau^{T}.$$

Without loss of generality (w.l.o.g.) we can change the fixed frame from the old $\mathcal{C}^{\oplus}$ into the new one $\mathcal{C}^{\otimes}$ by an Euclidean motion in a way that at time $\tau = *$ the moving frame $\mathcal{C}$ and $\mathcal{C}^{\otimes}$ coincide. This is achieved by the Euclidean transformation (Nawratil, 2016):

$$\underline{\mathbf{X}}_\tau^{\otimes} = \widetilde{\underline{\mathbf{C}}}_*^{T} \underline{\mathbf{X}}_\tau^{\oplus} \widetilde{\underline{\mathbf{C}}}_*^{-T}$$

with

$$\widetilde{\underline{\mathbf{C}}}_*^{T} = \begin{pmatrix} \widetilde{\mathbf{E}}_* & \widetilde{\mathbf{T}}_* \\ \mathbf{0} & \frac{1}{\alpha_*}\widetilde{\mathbf{F}}_* \end{pmatrix}, \qquad \widetilde{\underline{\mathbf{C}}}_*^{-T} = \begin{pmatrix} \mathbf{E}_* & -\frac{1}{\alpha_*}\mathbf{E}_* \widetilde{\mathbf{T}}_* \mathbf{F}_* \\ \mathbf{0} & \frac{1}{\alpha_*}\mathbf{F}_* \end{pmatrix}.$$

By introducing the notation $\widetilde{\underline{\mathbf{B}}}_\tau^{-T} = \widetilde{\underline{\mathbf{C}}}_*^{T} \widetilde{\underline{\mathbf{D}}}_\tau^{-T}$ and $\widetilde{\underline{\mathbf{B}}}_\tau^{T} = \widetilde{\underline{\mathbf{D}}}_\tau^{T} \widetilde{\underline{\mathbf{C}}}_*^{-T}$ the constrained motion with respect to the system $\mathcal{C}^{\otimes}$ is written as:

$$\underline{\mathbf{X}}_\tau^{\otimes} = \widetilde{\underline{\mathbf{B}}}_\tau^{-T} \underline{\mathbf{X}}\, \widetilde{\underline{\mathbf{B}}}_\tau^{T} \quad \Longleftrightarrow \quad \mathbf{X}_\tau^{\otimes} = \mathbf{G}_\tau \mathbf{X} \widetilde{\mathbf{H}}_\tau - 2\mathbf{G}_\tau \widetilde{\mathbf{U}}_\tau \tag{7}$$

with

$$\mathbf{G}_\tau = \widetilde{\mathbf{E}}_* \mathbf{E}_\tau, \quad \mathbf{H}_\tau = \tfrac{1}{\alpha_*} \widetilde{\mathbf{F}}_* \mathbf{F}_\tau, \quad \mathbf{U}_\tau = \tfrac{1}{\alpha_*} \left(\widetilde{\mathbf{F}}_* \mathbf{T}_\tau - \widetilde{\mathbf{F}}_* \mathbf{T}_* \widetilde{\mathbf{E}}_* \mathbf{E}_\tau \right).$$

The advantage of this coordinate transformation is that the geometric properties can be studied in a more compact way.

3.1. Velocity quaternion and instantaneous screw

According to the calculation rules for the differentiation of quaternions, the time derivative of the normalizing condition $\mathbf{G}_\tau \widetilde{\mathbf{G}}_\tau = 1$ with respect to τ yields $\dot{\mathbf{G}}_\tau \widetilde{\mathbf{G}}_\tau + \mathbf{G}_\tau \dot{\widetilde{\mathbf{G}}}_\tau = 0$, where the superior dot denotes the time derivative. Evaluation of this formula at $\tau = *$ implies $\dot{g}_0(*) = 0$; i.e. $\dot{\mathbf{G}}_* = \dot{\mathbf{g}}_*$. Moreover by differentiation of Eq. (7) we get:

$$\dot{\underline{\mathbf{X}}}_\tau^{\otimes} = \dot{\widetilde{\underline{\mathbf{B}}}}_\tau^{-T} \underline{\mathbf{X}}\, \widetilde{\underline{\mathbf{B}}}_\tau^{T} + \widetilde{\underline{\mathbf{B}}}_\tau^{-T} \underline{\mathbf{X}}\, \dot{\widetilde{\underline{\mathbf{B}}}}_\tau^{T} \quad \Longleftrightarrow \dot{\mathbf{X}}_\tau^{\otimes} = \dot{\mathbf{G}}_\tau \mathbf{X} \widetilde{\mathbf{H}}_\tau + \mathbf{G}_\tau \mathbf{X} \dot{\widetilde{\mathbf{H}}}_\tau - 2\dot{\mathbf{G}}_\tau \widetilde{\mathbf{U}}_\tau - 2\mathbf{G}_\tau \dot{\widetilde{\mathbf{U}}}_\tau.$$

Its evaluation at time $\tau = *$ yields:

$$\dot{\mathbf{X}}_*^{\otimes} = \alpha_* \dot{\mathbf{g}}_* \mathbf{X} + \mathbf{X} \dot{\widetilde{\mathbf{H}}}_* - 2\dot{\widetilde{\mathbf{U}}}_*, \tag{8}$$

which we call the velocity quaternion of X implied by the equiform motion at time $\tau = *$ with respect to the fixed coordinate system $\mathcal{C}^{\otimes}$. Its norm gives the corresponding velocity.

Please cite this article in press as: Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Des. (2016), http://dx.doi.org/10.1016/j.cagd.2016.06.003

Table 1
Table of normal forms of instantaneous equiform motions $(\alpha_*\dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*)$ in E^4.

Instantaneous Equiform Motion	$\alpha_*\dot{\mathbf{g}}_*$	$\dot{\mathbf{H}}_*$	$\dot{\mathbf{U}}_*$	with
2-plane spiraling (rotation for $s=0$)	$w\mathbf{i}$	$s+(1-w)\mathbf{i}$	$\mathbf{O}$	$w\in(0,1)$
isoclinic left spiraling (rotation for $s=0$)	$\mathbf{i}$	s	$\mathbf{O}$	
isoclinic right spiraling (rotation for $s=0$)	$\mathbf{o}$	$s+\mathbf{i}$	$\mathbf{O}$	
1-plane spiraling (rotation for $s=0$)	$\frac{1}{2}\mathbf{i}$	$s+\frac{1}{2}\mathbf{i}$	$\mathbf{O}$	
central scaling	$\mathbf{o}$	1	$\mathbf{O}$	
translation	$\mathbf{o}$	$\mathbf{O}$	$-\frac{1}{2}$	
1-plane rotation + translation parallel to rotation plane	$\frac{1}{2}\mathbf{i}$	$\frac{1}{2}\mathbf{i}$	$-\frac{p}{2}$	$p\neq 0$

The matrix representation of $\underline{\dot{\mathbf{X}}}_*^{\otimes}$ reads as:

$$\begin{pmatrix} \dot{h}_0 & -\alpha\dot{g}_1+\dot{h}_1 & -\alpha\dot{g}_2+\dot{h}_2 & -\alpha\dot{g}_3+\dot{h}_3 \\ \alpha\dot{g}_1-\dot{h}_1 & \dot{h}_0 & -\alpha\dot{g}_3-\dot{h}_3 & \alpha\dot{g}_2+\dot{h}_2 \\ \alpha\dot{g}_2-\dot{h}_2 & \alpha\dot{g}_3+\dot{h}_3 & \dot{h}_0 & -\alpha\dot{g}_1-\dot{h}_1 \\ \alpha\dot{g}_3-\dot{h}_3 & -\alpha\dot{g}_2-\dot{h}_2 & \alpha\dot{g}_1+\dot{h}_1 & \dot{h}_0 \end{pmatrix}_* \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} + 2\begin{pmatrix} -\dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix}_* . \tag{9}$$

This affine mapping is singular over $\mathbb{R}$ if and only if $\dot{h}_0(*)=0$ and $\alpha_*^2\dot{\mathbf{g}}_*\dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\dot{\mathbf{h}}_*=0$ hold, which can easily be seen on closer inspection of the determinant of the above given 4×4 matrix. Note that $\dot{h}_0(*)=0$ implies an instantaneous Euclidean motion, as the differentiation of $\mathbf{H}_\tau\widetilde{\mathbf{H}}_\tau=\alpha_\tau^2$ with respect to τ and its evaluation at time $\tau=*$ yields $\alpha_*\dot{h}_0(*)=\dot{\alpha}_*$.

Definition 1. The triplet $(\alpha_*\dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*)$ is called the instantaneous screw $\$_*^{\otimes}$ of the equiform motion $(\mathbf{G}_\tau, \mathbf{H}_\tau, \mathbf{U}_\tau)$ in E^4 with $\|\mathbf{H}_\tau\|=\alpha_\tau$ at time $\tau=*$ with respect to the fixed coordinate system $\mathcal{C}^{\otimes}$. $\$_*^{\otimes}$ is called singular if $\dot{h}_0(*)=0$ and $\alpha_*^2\dot{\mathbf{g}}_*\dot{\mathbf{g}}_*-\dot{\mathbf{h}}_*\dot{\mathbf{h}}_*=0$ holds; otherwise regular.

It can be shown by analogous considerations as given in Nawratil (2016, Section 5) that equiform transformation of instantaneous screws can also be embedded into the algebra of 2×2 quaternionic matrices, as Nawratil (2016, Theorem 5.4) also holds for S(4) with respect to the matrices given in Eq. (4).

Remark 2. Note that the linear space of instantaneous screws is nothing but the Lie algebra of the group of similarity transformations. ◇

3.2. Classification of instantaneous equiform motions in Euclidean 4-space

For $\dot{h}_0(*)=0$ we are in the Euclidean case thus we can refer to Nawratil (2016, Theorem 5.7) (see also Table 1). For $\dot{h}_0(*)\neq 0$ there exists always a unique velocity pole P (= point with zero velocity) due to Section 3.1. W.l.o.g. we can assume that the origin of the moving frame equals P ($\Rightarrow \dot{\mathbf{U}}_*=\mathbf{O}$). Then the velocity quaternion simplifies to:

$$\dot{\mathbf{X}}_*^{\otimes}=\alpha_*\dot{\mathbf{g}}_*\mathbf{X}+\mathbf{X}\widetilde{\dot{\mathbf{H}}}_*=\left(\alpha_*\dot{\mathbf{g}}_*\mathbf{X}-\mathbf{X}\dot{\mathbf{h}}_*\right)+\mathbf{X}\dot{h}_0(*). \tag{10}$$

The second term is implied by an instantaneous scaling with center P and scaling velocity $\dot{h}_0(*)$ and the first term by an instantaneous spherical motion with center P. According to Nawratil (2016) the latter can be an instantaneous

(a) rotation about two total-orthogonal planes ($\Leftrightarrow$ $\dot{\mathbf{g}}_*\neq\mathbf{o}\neq\dot{\mathbf{h}}_*$ and $\alpha_*^2\dot{\mathbf{g}}_*\dot{\mathbf{g}}_*-\dot{\mathbf{h}}_*\dot{\mathbf{h}}_*\neq 0$),
(b) rotation about one plane ($\Leftrightarrow$ $\dot{\mathbf{g}}_*\neq\mathbf{o}\neq\dot{\mathbf{h}}_*$ and $\alpha_*^2\dot{\mathbf{g}}_*\dot{\mathbf{g}}_*-\dot{\mathbf{h}}_*\dot{\mathbf{h}}_*=0$),
(c) isoclinic left resp. right rotation ($\Leftrightarrow$ $\dot{\mathbf{g}}_*\neq\mathbf{o}=\dot{\mathbf{h}}_*$ resp. $\dot{\mathbf{g}}_*=\mathbf{o}\neq\dot{\mathbf{h}}_*$),
(d) standstill ($\Leftrightarrow$ $\dot{\mathbf{g}}_*=\mathbf{o}=\dot{\mathbf{h}}_*$).

For the computation of the rotation planes and corresponding angular velocities we refer to Nawratil (2016, Section 5.2).[8] We denote the resulting infinitesimal equiform motions an instantaneous (a) 2-plane spiraling, (b) 1-plane spiraling, (c) isoclinic left resp. right spiraling and (d) central scaling, respectively.

Based on equiform transformations of instantaneous screws it can easily be verified that they have the "normal forms" given in Table 1. For a complete listing of instantaneous equiform motions in Euclidean 3-space see Odehnal et al. (2006, Theorem 2).

[8] Alternatively the rotation planes can be computed by the approach of Somer (1979b).

4. Relation between instantaneous equiform motions and line-elements in Euclidean 4-space

In Section 2.2 we introduced coordinates of oriented line-elements based on the oriented line coordinates $(\mathbf{Y},\mathbf{m})$. In the next section we discuss a further representation of lines in E^4, namely the Grassmann coordinates.

4.1. Grassmann coordinates

In 3-space the Grassmann coordinates (also known as Plücker coordinates) of a finite line are just the homogenized spear coordinates, but in 4-space the situation is different. The Grassmann coordinates of a line in E^4 can be written in an abstract quaternionic representation as $(\overline{\mathbf{y}}:\widehat{\mathbf{y}}:\mathbf{Y})$ with:

$$\overline{\mathbf{y}} := \tfrac{1}{2}\left(\mathbf{Y}\widetilde{\mathbf{C}} + \widetilde{\mathbf{C}}\mathbf{Y}\right) \quad \text{and} \quad \widehat{\mathbf{y}} := \tfrac{1}{2}\left(\mathbf{Y}\widetilde{\mathbf{C}} - \widetilde{\mathbf{C}}\mathbf{Y}\right). \tag{11}$$

This can be proven by introducing projective point coordinates for $\mathbf{C}$ and $\mathbf{Y}$; i.e.

$$\begin{aligned}(\hat{c}_0:\hat{c}_1:\hat{c}_2:\hat{c}_3:\hat{c}_4) &:= (c_0:c_1:c_2:c_3:1),\\ (\hat{y}_0:\hat{y}_1:\hat{y}_2:\hat{y}_3:\hat{y}_4) &:= (y_0:y_1:y_2:y_3:0).\end{aligned}$$

Then the Grassmann coordinates of lines are defined by $l_{ij} := \hat{c}_i\hat{y}_j - \hat{c}_j\hat{y}_i$ (e.g. Pottmann and Wallner, 2001, Section 2.2 or Joswig and Theobald, 2008, Section 12). Now computation shows that following holds:

$$\begin{aligned}\mathbf{Y} &= l_{40} + l_{41}\mathbf{i} + l_{42}\mathbf{j} + l_{43}\mathbf{k},\\ \widetilde{\mathbf{C}}\mathbf{Y} + \mathbf{Y}\widetilde{\mathbf{C}} &= 2\left(l_{01}\mathbf{i} + l_{02}\mathbf{j} + l_{03}\mathbf{k}\right),\\ \widetilde{\mathbf{C}}\mathbf{Y} - \mathbf{Y}\widetilde{\mathbf{C}} &= 2\left(l_{23}\mathbf{i} + l_{31}\mathbf{j} + l_{12}\mathbf{k}\right).\end{aligned}$$

Therefore the triplet $(\overline{\mathbf{y}}:\widehat{\mathbf{y}}:\mathbf{Y})$ contains all 10 Grassmann coordinates. By the usage of these homogeneous coordinates one loses the information on the line's orientation. This can be avoided by using normalized Grassmann coordinates, where the normalization is done with respect to the direction vector of the line, which is represented by the quaternion $\mathbf{Y}$. As $\mathbf{Y}$ already denotes a unit-quaternion the normalized Grassmann coordinates of Eq. (11) can be written as $(\overline{\mathbf{y}},\widehat{\mathbf{y}},\mathbf{Y})$. Instead of this triplet we can use the following one:

Definition 2. The modified Grassmann coordinates of an oriented line $\overrightarrow{\mathrm{y}}$ in E^3, which are called oriented mG line coordinates for short, are defined as $(\mathbf{y}_+,\mathbf{y}_-,\mathbf{Y})$ with

$$\mathbf{y}_+ := \overline{\mathbf{y}} + \widehat{\mathbf{y}} = \mathbf{Y}\widetilde{\mathbf{C}}, \qquad \mathbf{y}_- := \overline{\mathbf{y}} - \widehat{\mathbf{y}} = \widetilde{\mathbf{C}}\mathbf{Y}. \tag{12}$$

If the orientation of the line y is not of importance one can also use homogeneous mG line coordinates $(\mathbf{y}_+:\mathbf{y}_-:\mathbf{Y})$.

Remark 3. Now the question arises, which homogeneous 10-tuples $(\mathbf{z}_+:\mathbf{z}_-:\mathbf{Z}) \neq (\mathbf{o},\mathbf{o},\mathbf{O})$ represent mG line coordinates. According to Pottmann and Wallner (2001, Theorem 2.2.4) and Joswig and Theobald (2008, Corollary 12.22), respectively, five conditions have to hold, which read as follows in the quaternionic formulation:

$$\langle \mathbf{z}_+,\mathbf{z}_+\rangle - \langle \mathbf{z}_-,\mathbf{z}_-\rangle = 0, \qquad \mathbf{z}_+\mathbf{Z} - \mathbf{Z}\mathbf{z}_- = \mathbf{O}. \tag{13}$$

By computing the Hilbert-polynomial of these 5 equations it can easily be seen (cf. footnote 4) that the set of lines is a 6-dimensional variety of degree 5 within the projective 9-dimensional space P^9. If we slice the variety along the 5-dimensional space $\mathbf{Z} = \mathbf{O}$ we get the set $\mathcal{L}$, which corresponds to lines of E^4. ⋄

Now we can clarify the relation between the oriented mG coordinates $(\mathbf{y}_+,\mathbf{y}_-,\mathbf{Y})$ of lines in E^4 and the 4-dimensional analogue of the spear coordinates $(\mathbf{Y},\mathbf{m})$ given in Section 2.2. The latter are just obtained by the following projection:

$$(\mathbf{y}_+,\mathbf{y}_-,\mathbf{Y}) \mapsto (\mathbf{Y},\mathbf{m}) := (\mathbf{Y},\widetilde{\mathbf{y}}_-) \tag{14}$$

due to Eq. (12). These projected coordinates have only the trivial side condition $\|\mathbf{Y}\| = 1$ in contrast to the oriented mG line coordinates, which have those of Eq. (13) in addition. This gives rise to the following nomenclature:

Definition 3. The minimal coordinates of an oriented line $\overrightarrow{\mathrm{y}} \in E^4$ are given by $(\mathbf{Y},\mathbf{m})$ and $(\mathbf{Y}:\mathbf{m})$ are the homogeneous minimal coordinates of an unoriented line $\mathrm{y} \in E^4$, respectively.

Consequently this implies the following notation:

Definition 4. The minimal coordinates of an oriented line-element $(\overrightarrow{\mathrm{y}},\mathrm{X}) \in E^4$ are defined as $(\mathbf{Y},\xi + \mathbf{m})$ and $(\mathbf{Y}:\xi + \mathbf{m})$ are the homogeneous minimal coordinates of an unoriented line-element $(\mathrm{y},\mathrm{X}) \in E^4$.

Please cite this article in press as: Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Des. (2016), http://dx.doi.org/10.1016/j.cagd.2016.06.003

As $\mathbf{y}_+$ and $\mathbf{y}_-$ of the oriented mG line coordinates are pure quaternions, we have now two possibilities to insert the information $\xi = \langle \mathbf{X}, \mathbf{Y}\rangle$ of the point $\mathsf{X} \in \overrightarrow{\mathsf{y}}$. The connection given in Eq. (14) implies the following definition:

Definition 5. The oriented mG line-element coordinates of an oriented line-element $(\overrightarrow{\mathsf{y}}, \mathsf{X}) \in E^4$ are defined as $(\mathbf{y}_+, \xi + \mathbf{y}_-, \mathbf{Y})$. Consequently $(\mathbf{y}_+ : \xi + \mathbf{y}_- : \mathbf{Y})$ are the homogeneous mG line-element coordinates of an unoriented line-element $(\mathsf{y}, \mathsf{X}) \in E^4$.

Remark 4. Any homogeneous 11-tuple $(\mathbf{z}_+ : \zeta + \mathbf{z}_- : \mathbf{Z}) \neq (\mathbf{o}, \mathbf{O}, \mathbf{O})$ defines a line-element of E^4 if and only if Eq. (13) and $\mathbf{Z} \neq \mathbf{O}$ hold. Therefore the set of line-elements of E^4 corresponds to a 7-dimensional variety $\mathcal{E}$ of degree 5, which is sliced along the 6-dimensional space $\mathbf{Z} = \mathbf{O}$, within the projective 10-dimensional space P^{10}. ⋄

4.2. Linear complexes of lines and line-elements in Euclidean 4-space

Let us start this section with the following three definitions:

Definition 6. A linear complex of lines of E^4 is the intersection of $\mathcal{L}$ (cf. Remark 3) with a hyperplane.

This definition is in accordance with the definition of a linear line complex given in Dolgachev (Section 10.2). Now we can extend this definition to line-elements of E^4 as follows:

Definition 7. A linear complex of line-elements of E^4 is the intersection of $\mathcal{E}$ (cf. Remark 4) with a hyperplane.

This definition is in accordance with Odehnal et al. (2006, Definition 3). For the formulation of the next theorem, a further definition is needed.

Definition 8. A path normal-element (y, X) of an instantaneous screw consists of a point $\mathsf{X} \in E^4$ and a path normal y of this point with respect to the instantaneous screw.

Theorem 5. *The set of path normal-elements of an instantaneous screw $\$_*^{\otimes} = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*) \neq (\mathbf{o}, \mathbf{O}, \mathbf{O})$ of E^4 is a linear complex of line-elements of E^4 and vice versa.*

Proof. We prove this by showing that the mG line-element coordinates $(\mathbf{y}_+, \xi + \mathbf{y}_-, \mathbf{Y})$ fulfill the linear equation:

$$\alpha_* \langle \mathbf{y}_+, \dot{\mathbf{g}}_* \rangle + \langle \xi + \mathbf{y}_-, \dot{\widetilde{\mathbf{H}}}_* \rangle - 2\langle \mathbf{Y}, \dot{\widetilde{\mathbf{U}}}_* \rangle = 0. \tag{15}$$

Therefore we investigate the condition that a unit-quaternion $\mathbf{Y}$ is orthogonal to the velocity quaternion $\dot{\mathbf{X}}_*^{\otimes}$ of Eq. (8). The corresponding condition $\langle \mathbf{Y}, \dot{\mathbf{X}}_*^{\otimes} \rangle = 0$ can be rewritten as follows:

$$\alpha_* \langle \mathbf{Y}\widetilde{\mathbf{X}}, \dot{\mathbf{g}}_* \rangle - \langle \widetilde{\mathbf{X}}\mathbf{Y}, \dot{\mathbf{h}}_* \rangle + \xi \dot{h}_0(*) - 2\langle \mathbf{Y}, \dot{\widetilde{\mathbf{U}}}_* \rangle = 0, \tag{16}$$

under consideration of $\dot{\mathbf{H}}_* = \dot{h}_0(*) + \dot{\mathbf{h}}_*$ and $\dot{\widetilde{\mathbf{H}}}_* = \dot{h}_0(*) - \dot{\mathbf{h}}_*$. Due to Eq. (5) the expressions of $\mathbf{y}_+$ and $\mathbf{y}_-$ given in Eq. (12) can be computed as:

$$\mathbf{y}_+ = \tfrac{1}{2}\left(\mathbf{Y}\widetilde{\mathbf{X}} - \mathbf{X}\widetilde{\mathbf{Y}}\right), \qquad \mathbf{y}_- = \tfrac{1}{2}\left(\widetilde{\mathbf{X}}\mathbf{Y} - \widetilde{\mathbf{Y}}\mathbf{X}\right). \tag{17}$$

These relations already imply the validity of the following equalities:

$$\langle \mathbf{Y}\widetilde{\mathbf{X}}, \dot{\mathbf{g}}_* \rangle = \langle \mathbf{Y}\widetilde{\mathbf{C}}, \dot{\mathbf{g}}_* \rangle = \langle \mathbf{y}_+, \dot{\mathbf{g}}_* \rangle, \quad \langle \widetilde{\mathbf{X}}\mathbf{Y}, \dot{\mathbf{h}}_* \rangle = \langle \widetilde{\mathbf{C}}\mathbf{Y}, \dot{\mathbf{h}}_* \rangle = \langle \mathbf{y}_-, \dot{\mathbf{h}}_* \rangle, \tag{18}$$

thus summed up we get our required result given in Eq. (15).

The converse statement is also true as the homogeneous coordinates of any linear line-element complex of E^4 can be identified with an instantaneous screw $\$_*^{\otimes}$ up to a real multiple different from zero. □

By setting $\dot{h}_0(*) = 0$ we get for the Euclidean case the following result:

Corollary 1. *The set of path normals of an instantaneous Euclidean motion $\$_*^{\otimes} = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*) \neq (\mathbf{o}, \mathbf{o}, \mathbf{O})$ of E^4 is a linear complex of lines of E^4 and vice versa.*

5. Extending known results to 4 dimensions

The 4-dimensional version of Odehnal et al. (2006, Theorem 3) reads as follows:

Theorem 6. *The elements of a linear complex $\mathcal{K}$ of line-elements in E^4, which are contained in a hyperplane are the path normal-elements of an instantaneous spatial equiform motion.*

Proof. W.l.o.g. we can consider the hyperplane given by $x_0 = 0$. As a consequence the mG coordinates of line-elements within this hyperplane are given by $(\widehat{\mathbf{y}}, \langle \mathbf{x}, \mathbf{y}\rangle - \widehat{\mathbf{y}}, \mathbf{y})$. Plugging these coordinates into Eq. (15) yields:

$$\langle \widehat{\mathbf{y}}, \alpha_* \dot{\mathbf{g}}_* + \dot{\mathbf{h}}_*\rangle + \dot{h}_0(*)\langle \mathbf{x}, \mathbf{y}\rangle - 2\langle \mathbf{y}, \dot{\widetilde{\mathbf{U}}}_*\rangle = 0. \tag{19}$$

Form Eq. (9) it can easily be seen that the velocity vectors are also contained in the hyperplane $x_0 = 0$ if and only if the instantaneous screw is of the form $(\dot{\mathbf{v}}_*, \dot{v}_0(*) + \dot{\mathbf{v}}_*, \dot{\mathbf{w}}_*)$. By setting $\dot{\mathbf{v}}_* = \frac{1}{2}\left(\alpha_* \dot{\mathbf{g}}_* + \dot{\mathbf{h}}_*\right)$, $\dot{v}_0(*) = \dot{h}_0(*)$ and $\dot{\mathbf{w}}_* = \dot{\mathbf{u}}_*$ we get the instantaneous spatial equiform motion, where the line-elements of $\mathcal{K}$ in $x_0 = 0$ are path normal-elements as

$$2\langle \widehat{\mathbf{y}}, \dot{\mathbf{v}}_*\rangle + \dot{v}_0(*)\langle \mathbf{x}, \mathbf{y}\rangle + 2\langle \mathbf{y}, \dot{\mathbf{w}}_*\rangle = 0$$

holds due to Eq. (19). □

As a consequence of Theorem 6 and Odehnal et al. (2006, Theorem 3) we get:

Corollary 2. *The elements of a linear complex $\mathcal{K}$ of line-elements in E^4, which are contained in a plane are the path normal-elements of an instantaneous planar equiform motion.*

The 4-dimensional version of the result of Pascal (1932) (see also Odehnal et al., 2006, Lemma 4.1), which was cited in the introduction, reads as follows:

Theorem 7. *Under an instantaneous equiform motion $\$_*^{\otimes} = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*)$ of E^4 with $\dot{h}_0(*) \neq 0$ each line $\mathsf{y} \in E^4$ has a unique point $\mathsf{X} \in E^4$ with a velocity vector orthogonal to it. Thus the line-element $(\mathsf{y}, \mathsf{X}) \in E^4$ is in the path normal-element complex of $\$_*^{\otimes}$.*

Under an instantaneous Euclidean motion $\$_^{\otimes} = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*) \neq (\mathbf{o}, \mathbf{o}, \mathbf{O})$ of E^4 the path normal-element complex of $\$_*^{\otimes}$ equals the set of line-elements in E^4, where the lines belong to the path normal complex of $\$_*^{\otimes}$.*

Proof. If $\dot{h}_0(*) \neq 0$ holds then Eq. (16) can uniquely be solved for $\xi = \langle \mathbf{X}, \mathbf{Y}\rangle$. For the case $\dot{h}_0(*) = 0$ this equation (16) is fulfilled for any point of a path normal (cf. Corollary 1). □

As a consequence of Theorem 6, Corollary 2 and Theorem 7 we get:

Corollary 3. *The path normals of an instantaneous Euclidean motion $\$_*^{\otimes} = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{h}}_*, \dot{\mathbf{U}}_*) \neq (\mathbf{o}, \mathbf{o}, \mathbf{O})$ of E^4, which are contained in a hyperplane/plane are the path normals of an instantaneous spatial/planar Euclidean motion.*

We proceed by extending the results (Odehnal et al., 2006, Lemma 4.2 and Theorem 4) to 4 dimensions. For their formulation we need the following definition of a line-element bundle according to Odehnal (2006, Section 4.6):

Definition 9. The 4-dimensional set $\mathcal{B}$ of all line-elements, those lines belong to a bundle of lines in E^4, constitute a line-element bundle. If the bundle vertex B is an ideal point it is called parallel line-element bundle.

Theorem 8. *We consider a bundle $\mathcal{B}$ of line-elements in E^4 with vertex B. The set of points X such that $(\mathsf{y}, \mathsf{X}) \in \mathcal{B}$ is a path normal-element of the instantaneous screw $\$_*^{\otimes} = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*)$ of E^4 with $\dot{h}_0(*) \neq 0$ is a*

1. *hyperplane if B is an ideal point,*
2. *hypersphere if B is a finite point. Its diameter is given by $\mathsf{B}\overline{\mathsf{B}}$ with $\overline{\mathsf{B}}$ according to Eq. (20).*

Proof. We have to distinguish two cases:

1. B is an ideal point: W.l.o.g. we can assume that B is in direction of the identity quaternion 1. Then the line-element (y, X) of $\mathcal{B}$ has mG line-element coordinates $(-\mathbf{x}, \widetilde{\mathbf{X}}, 1)$ according to Eq. (17). Now Eq. (15) of the path normal-element complex simplifies to

   $$\langle \mathbf{x}, \alpha_* \dot{\mathbf{g}}_* - \dot{\mathbf{h}}_*\rangle - x_0 \dot{h}_0(*) + 2\dot{u}_0(*) = 0.$$

   As $\dot{h}_0(*) \neq 0$ holds this linear equation depends on x_0 for sure.

2. B is a finite point: W.l.o.g. we can assume that B is the origin of our reference frame; i.e. $\mathbf{B} = \mathbf{O}$. Then the line-element (y, X) of $\mathcal{B}$ has mG line-element coordinates $(\mathbf{o}, \|\mathbf{X}\|, \mathbf{Y})$, where $\mathbf{Y}$ is any unit-quaternion. Now Eq. (15) implies:

$$\|\mathbf{X}\| = \frac{2}{h_0(*)} \langle \mathbf{Y}, \dot{\mathbf{U}}_* \rangle.$$

Therefore X is the pedal point on y of the point $\overline{\mathrm{B}}$ with

$$\overline{\mathbf{B}} := \mathbf{B} - \frac{1}{h_0(*)} \dot{\mathbf{B}}^{\otimes}_*, \tag{20}$$

where $\dot{\mathbf{B}}^{\otimes}_*$ is the velocity quaternion of the point B with respect to $\$^{\otimes}_*$. □

5.1. The set of path tangent-elements

Within this section we extend the result of Di Noi (1934), which is listed in the introduction (cf. Section 1.1), with respect to line-elements and the 4th dimension. Therefore we need the following definition:

Definition 10. A path tangent-element (y, X) of an instantaneous screw consists of a point $\mathrm{X} \in E^4$, which has a velocity different from zero with respect to the instantaneous screw, and the resulting path tangent y.

Theorem 9. *The 4-dimensional set of path tangent-elements of a regular instantaneous screw $\$^{\otimes}_* = (\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*)$ of E^4 has a quadratic rational parametrization in terms of minimal coordinates of line-elements of E^4.*

Proof. Due to the regularity assumption there exists a unique velocity pole P. W.l.o.g. we can assume that P is located in the origin, thus we have $\dot{\mathbf{U}}_* = \mathbf{O}$. Therefore the moment quaternion $\mathbf{m}$ of a tangent in direction of the velocity quaternion $\dot{\mathbf{X}}^{\otimes}_*$ can be computed according to Eq. (5) as

$$\mathbf{m} = \|\mathbf{X}\|^2 \dot{\mathbf{h}}_* - \alpha_* \widetilde{\mathbf{X}} \dot{\mathbf{g}}_* \mathbf{X}.$$

Moreover we can solve Eq. (10) with respect to $\mathbf{X}$ in a quaternionic way according to Jack (2008, Section "General solution to linear problems in quaternion variables"). Plugging the obtained expression for $\mathbf{X}$, which is linear in $\dot{\mathbf{X}}^{\otimes}_*$, into $\langle \mathbf{X}, \dot{\mathbf{X}}^{\otimes}_* \rangle + \mathbf{m}$ shows that we have obtained a quadratic rational parametrization of the set of path tangent-elements in terms of the first 4 entries of the homogeneous 8-tuple of minimal coordinates of line-elements of E^4. □

Remark 5. Note that in the case of an instantaneous central scaling the path tangent-elements form a line-element bundle, where the velocity pole P is its vertex. ◇

The cases excluded from Theorem 9 are treated next:

Theorem 10. *The set of path tangent-elements of an instantaneous rotation about one plane in E^4 has a quadratic rational parametrization in terms of minimal coordinates of line-elements of E^4. If this instantaneous motion is additionally composed with an instantaneous translation parallel to the rotation plane, then the corresponding rational parametrization is cubic. In the case of a pure instantaneous translation the set of path tangent-elements form a parallel line-element bundle, which is a 4-dimensional linear subspace of the projective space of minimal coordinates of line-elements in E^4.*

Proof. We can restrict the proof to the corresponding normal forms given in Table 1.

1. $(\alpha_* \dot{\mathbf{g}}_*, \dot{\mathbf{H}}_*, \dot{\mathbf{U}}_*) = \left(\frac{1}{2}\mathbf{i}, \frac{1}{2}\mathbf{i}, -\frac{p}{2}\right)$: In this case the velocity quaternion equals $p + 0\mathbf{i} - x_3\mathbf{j} + x_2\mathbf{k}$, the resulting moment quaternion $\mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}$ reads as:

$$(px_1 + x_2^2 + x_3^2)\mathbf{i} + (x_0x_3 - x_1x_2 + px_2)\mathbf{j} + (px_3 - x_0x_2 - x_1x_3)\mathbf{k}$$

and ξ equals px_0. Now we distinguish two cases:
(a) $p \neq 0$: In this case we can set:

$$x_0 = \frac{\xi}{p}, \quad x_1 = \frac{m_1 - x_2^2 + x_3^2}{p}.$$

Then the following two equations in the minimal coordinates $(p : 0 : -x_3 : x_2 : \xi : m_1 : m_2 : m_3)$ of line-elements of E^4 remain:

$$x_2^3 + x_2x_3^2 - x_2m_1 + x_3\xi + p^2x_2 - m_2p = 0,$$
$$p^2x_3 + x_3x_2^2 + x_3^3 - x_3m_1 - x_2\xi - m_3p = 0.$$

Solving these two equations with respect to m_2 and m_3 yields a rational cubic parametrization in m_1, x_2, x_3, ξ for a given parameter p.

(b) $p=0$: In this case the minimal coordinates $(0:0:-x_3:x_2:0:m_1:m_2:m_3)$ of line-elements of E^4 have to fulfill the quadratic equation $x_2^2+x_3^2-m_1=0$. Thus we have the following rational quadratic parametrization $(0:0:-x_3:x_2:0:x_2^2+x_3^2:m_2:m_3)$ in x_2,x_3,m_2,m_3.

2. $(\alpha_*\dot{\mathbf{g}}_*,\dot{\mathbf{H}}_*,\dot{\mathbf{U}}_*)=\left(\mathbf{o},\mathbf{O},-\frac{1}{2}\right)$: In the case of the instantaneous translation, the resulting minimal coordinates of line-elements of E^4 read as $(1:0:0:0:x_0:x_1:x_2:x_3)$. □

Remark 6. Note that based on the given parametrization of the set of path normal-elements the degree of the corresponding variety in the space of mG or minimal coordinates of line-elements of E^4 can e.g. be determined via the Hilbert-polynomial (cf. footnote 4), where one can restrict to the normal forms given in Table 1. This straightforward computation is left to the interested reader. ◇

6. Point-models for line-elements in projective 3-space

We start this section with a review of the known two models for line-elements of the projective 3-space P^3, which were both given by Odehnal (2006):

(i) Odehnal's point-model equals the intersection of the Segre variety $\Sigma_{3,5}$, which is a 7-dimensional variety in a 23-dimensional projective space P^{23}, and four hyperplanes (Odehnal, 2006, Section 2.2).

(ii) Odehnal's second model is based on the idea to combine the homogeneous Plücker coordinates $(l_{01}:l_{02}:l_{03}:l_{23}:l_{31}:l_{12})$ of the line $\in P^3$, which fulfill the Plücker condition

$$l_{01}l_{23}+l_{02}l_{31}+l_{03}l_{12}=0, \tag{21}$$

and the homogeneous point coordinates $(l_0:l_1:l_2:l_3)$ to the 10-tuple

$$(l_{01}:l_{02}:l_{03}:l_{23}:l_{31}:l_{12}:l_0:l_1:l_2:l_3).$$

In this case any line-element is mapped to a 1-parametric set of 10-tuples

$$(\mu l_{01}:\mu l_{02}:\mu l_{03}:\mu l_{23}:\mu l_{31}:\mu l_{12}:\lambda l_0:\lambda l_1:\lambda l_2:\lambda l_3) \tag{22}$$

with $\lambda,\mu\in\mathbb{R}\setminus\{0\}$, which corresponds to a line[9] in the projective 9-dimensional space P^9, where the two points $(\lambda:\mu)=(1:0)$ and $(\lambda:\mu)=(0,1)$ are removed. Therefore all points of this P^9, which correspond to a line-element of P^3, form a 6-dimensional[10] variety V of degree 5, which is sliced along the two spaces $l_{01}=l_{02}=\ldots=l_{12}=0$ and $l_0=\ldots=l_3=0$. The 3-dimensional space is completely contained in V in contrast to the 5-dimensional space, which intersects V in the quadric given by Eq. (21).

Odehnal noted in (2006, Section 6) that a point-model of line-elements is of interest for the (differential) geometric study of so-called ruled surface strips, which are ruled surfaces together with a curve on it. In the following such an improved point-model is presented.

6.1. Improved point-model

If we rewrite Odehnal's 10-tuple in the abstract quaternionic representation $(\bar{\mathbf{l}}:\widehat{\mathbf{l}}:\mathbf{L})$ with

$$\bar{\mathbf{l}}:=l_{01}\mathbf{i}+l_{02}\mathbf{j}+l_{03}\mathbf{k},\quad \widehat{\mathbf{l}}:=l_{23}\mathbf{i}+l_{31}\mathbf{j}+l_{12}\mathbf{k},\quad \mathbf{L}:=l_0+l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k},$$

then it can easily be seen that these are the Grassmann coordinates of a line in E^4, as the five conditions given in Eq. (13) are equal to those given in Odehnal (2006, Eqs. (2) and (7)). Therefore we can also use the minimal coordinates $(\mathbf{L}:\mathbf{m})$ of a line in E^4.

Due to the free parameter in Eq. (22) we can claim additionally that the pedal point C of the line in E^4 with respect to the origin has distance 1; i.e. the line is tangent to the unit-hypersphere. Under consideration that $\mathbf{m}=\widetilde{\mathbf{L}}\mathbf{C}$ holds we can compute $\mathbf{C}$ as $\mathbf{L}\mathbf{m}\|\mathbf{L}\|^{-2}$. Then the tangential condition $\widetilde{\mathbf{C}}\mathbf{C}=1$ to the unit-hypersphere simplifies to

$$\Phi:\quad \|\mathbf{L}\|^2-\|\mathbf{m}\|^2=0. \tag{23}$$

All real points $(\mathbf{L}:\mathbf{m})$ in the projective 6-dimensional space P^6 of minimal coordinates of lines in E^4, which are located on the quadric Φ of Eq. (23), correspond to a line-element of P^3. This is the case as only points with $\mathbf{Y}=\mathbf{O}$ or $\mathbf{m}=\mathbf{o}$ do not correspond with line-elements of P^3, but these two linear spaces have with the quadric Φ no real points in common.

[9] Therefore it is somehow misleading to speak in this context of a point-model as done in Odehnal (2006, Section 3).
[10] And not a 5-dimensional variety as outlined in Odehnal (2006, Section 3).

Please cite this article in press as: Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Des. (2016), http://dx.doi.org/10.1016/j.cagd.2016.06.003

The disadvantage of this point-model is that with $(\mathbf{L}:\mathbf{m})\in\Phi$ also $(\mathbf{L}:-\mathbf{m})\in\Phi$ holds, thus we get a double cover of the set of line-elements of P^3. We can get rid of this double cover by considering the Grassmann variety of lines spanned by $(\mathbf{L}:\mathbf{m})\in\Phi$ and $(\mathbf{L}:-\mathbf{m})\in\Phi$. This is trivially isomorphic to the Segre variety $\Sigma_{3,2}$ defined by the Segre embedding σ: $P^3\times P^2\to P^{11}$ with

$$(l_0:l_1:l_2:l_3)\times(m_1:m_2:m_3)\mapsto(l_0m_1:l_0m_2:l_0m_3:l_1m_1:\ldots:l_3m_3).$$

It is well known that the Segre variety $\Sigma_{m,n}$ has degree $\frac{(m+n)!}{m!n!}$. As a consequence $\Sigma_{3,2}$ is of degree 10 in contrast to the above noted point-model (i) based on $\Sigma_{3,5}$, which is of degree 56.

We can even give a rational parametrization of the Segre variety $\Sigma_{3,2}$ by parametrizing the tangents to the unit-hypersphere of E^4, which can be done as follows. By stereographic projection we obtain the set of pedal points in dependency of the variables u, v, w as:

$$\mathbf{C}=\frac{u^2+v^2+w^2-1+2u\mathbf{i}+2v\mathbf{j}+2w\mathbf{k}}{D}\quad\text{with}\quad D:=u^2+v^2+w^2+1.$$

As $\langle\frac{\partial\mathbf{C}}{\partial i},\frac{\partial\mathbf{C}}{\partial j}\rangle=0$ holds for $i\neq j\in\{u,v,w\}$ and $\|\frac{\partial\mathbf{C}}{\partial u}\|=\|\frac{\partial\mathbf{C}}{\partial v}\|=\|\frac{\partial\mathbf{C}}{\partial w}\|=2D$ we can parametrize the directions of the tangents e.g. by:

$$\mathbf{L}=\tfrac{1}{2D}\left(\sin\delta_1\cos\delta_2\frac{\partial\mathbf{C}}{\partial u}+\sin\delta_1\sin\delta_2\frac{\partial\mathbf{C}}{\partial v}+\cos\delta_1\frac{\partial\mathbf{C}}{\partial w}\right),$$

where the sin and cos functions can be replaced by the half-angle substitution.

Remark 7. In order to catch also the center of the stereographic projection, which is the point $(1,0,0,0)$ of the hyper-sphere, we can use the following homogenized parametrization:

$$\mathbf{C}=\frac{u^2+v^2+w^2-h^2+2uh\mathbf{i}+2vh\mathbf{j}+2wh\mathbf{k}}{D}\quad\text{with}\quad D:=u^2+v^2+w^2+h^2,$$

where h is the homogenizing variable. Then for $h=0$ the point $\mathbf{C}=1$ is obtained. Moreover for a complete covering of $\Sigma_{3,2}$ we also have to replace the half-angle substitution by its homogenized version. $\diamond$

7. Conclusion and outlook

In Section 1 we provided a detailed literature review on equiform kinematics in Euclidean spaces of dimension 3 and higher. In Section 2 a quaternion based kinematic mapping for equiform transformations of Euclidean spaces of dimension 3 and 4 was introduced. Based on this mapping and a study on instantaneous equiform motions of E^4 in Section 3, we discussed the set of path normal-elements and path tangent-elements in the Sections 4 and 5. Moreover in Section 6 we presented an improved point-model for unoriented line-elements of the projective 3-space.

Point-models for the set of oriented line-elements of E^3 are of practical interest for the motion representation of robots with an axial symmetric end-tool (e.g. a milling cutter). Therefore a forthcoming publication of the author is devoted to the study and comparison of such models; e.g. the quadric Φ of Eq. (23) sliced along the hyperplane $l_0=0$ represents one possible point-model.

Acknowledgements

The research is supported by Grant No. P 24927-N25 of the Austrian Science Fund FWF.

References

Blaschke, W., 1960. Kinematik und Quaternionen. VEB Deutscher Verlag der Wissenschaften, Berlin.
Bottema, O., Roth, B., 1979. Theoretical Kinematics. Applied Mathematics and Mechanics. North-Holland Publishing Company, Amsterdam.
Burmester, L., 1878. Kinematisch-geometrische Theorie der Bewegung der affin-veränderlichen, ähnlich-veränderlichen und starren räumlichen oder ebenen Systeme I. Z. Math. Phys. 23, 108–131.
Burmester, L., 1902. Kinematisch-geometrische Theorie der Bewegung der affin-veränderlichen, ähnlich-veränderlichen und starren räumlichen oder ebenen Systeme II. Z. Math. Phys. 47, 128–156.
Combebiac, G., 1902. Calculus of tri-quaternions. Phd thesis (translated by Delphenich, D.H.). Science Faculty of Paris.
Di Noi, S., 1934. Considerazioni geometriche sul moto di un corpo deformabile che si mantiene simile a se stesso. Rend. Napoli 3 (4), 176–181.
Dolgachev, I. Classical algebraic geometry: a modern view. http://www.math.lsa.umich.edu/~idolga/CAG.pdf.
Gfrerrer, A., 2008. Axial equiform Bricard motions. Grazer Math. Ber. 352, 17–40.
Gfrerrer, A., Lang, J., 1998. Equiform bundle motions in E_3 with spherical trajectories I and II. Beitr. Algebra Geom. 39 (2), 307–328.
Hager, G., 1991. Globale Eigenschaften von Zwangläufen in euklidischen und elliptischen Räumen höherer Dimension. Dissertation. Vienna University of Technology.
Hamdoon, F.M., Abdel-All, N.H., 2004. An investigation of an octahedral platform using equiform motions. J. Geom. Graph. 8 (1), 33–39.
Hofer, M., Odehnal, B., Pottmann, H., Steiner, T., Wallner, J., 2005. 3D shape recognition and reconstruction based on line element geometry. Proc. 10th IEEE Int. Conf. Comput. Vision 2, 1532–1538.

Jack, P., 2008. Three Papers on Quaternions 2006–2007. Timefoot Books. https://archive.org/details/ThreePapersOnQuaternions2006-2007.
Joswig, M., Theobald, T., 2008. Algorithmische Geometrie. Springer.
Karger, A., 1981. Similarity motions in E_3 with plane trajectories. Apl. Mat. 26 (3), 194–201.
Kowalewski, G., 1930. Rollkurven im Raume. Math. Z. 31 (1), 306–318.
Lang, J., Mick, S., Röschel, O., 2001. The rigidity rate of positions of Stewart–Gough platforms. J. Geom. Graph. 5 (2), 121–132.
Müller, H., 1962. Sphärische Kinematik. VEB Deutscher Verlag der Wissenschaften, Berlin.
Nawratil, G., 2016. Fundamentals of quaternionic kinematics in Euclidean 4-space. Adv. Appl. Clifford Algebras 26 (2), 693–717.
Odehnal, B., 2006. Die Linienelemente des P^3. Österr. Akad. Wiss. Math.-Nat. Kl. S-B II 215 (), 155–171.
Odehnal, B., Pottmann, H., Wallner, J., 2006. Equiform kinematics and the geometry of line elements. Beitr. Algebra Geom. 47 (2), 567–582.
Pascal, M., 1932. Sul moto di un corpo deformabile che si mantiene simile a se stesso. Ann. del Regio Istituto Super. Navale Napoli 1, 224–242.
Pottmann, H., 1984. Spezielle äquiforme Zwangläufe. Apl. Mat. 29 (3), 225–232.
Pottmann, H., 1986. Die Öffnungsstrecken der Bahnregelflächen geschlossener räumlicher äquiformer Zwangläufe. Monatshefte Math. 101, 317–326.
Pottmann, H., 1987. Zur globalen Raumkinematik. Monatshefte Math. 103, 289–302.
Pottmann, H., 1994. Kinematische Geometrie. In: Giering, O., Hoschek, J. (Eds.), Geometrie und ihre Anwendungen. Hanser, pp. 141–175.
Pottmann, H., Wallner, J., 2001. Computational Line Geometry. Springer.
Röschel, O., 1991. Darboux-Zwangläufe und äquiforme Kinematik. Appl. Math. 36 (3), 233–240.
Röschel, O., 2000. Möbius mechanisms. In: Lenarcic, J., Stanisic, M. (Eds.), Advances in Robot Kinematics. Kluwer, pp. 375–382.
Röschel, O., 2014. Overconstrained mechanisms based on planar four-bar-mechanisms. Comput. Aided Geom. Des. 31 (7–8), 595–601.
Seiliger, D., 1892. Theorie der Bewegung eines ähnlich veränderlichen Körpers. Kasan Ber. (2) II. No. 1.
Shafarevich, I., 1988. Algebraic Geometry. Springer.
Somer, J., 1979a. Geschlossene äquiforme Bewegungen der Räume endlicher Dimension. Apl. Mat. 24 (4), 304–314.
Somer, J., 1979b. Lineare invariante Gebilde in äquiformen Bewegungen der Räume endlicher Dimension. Apl. Mat. 24 (1), 56–66.
Spallek, K., 1992. W-Gleitgleitkinematik. Manuscr. Math. 77, 293–319.
Ströher, W., 1973. Sphärische und Räumliche Kinematik. Unpublished manuscript.
Wunderlich, W., 1962. Über eine spezielle Bewegung eines ähnlich-veränderlichen Raumsystems. Anz. Math.-Nat. Kl. Österr. Akad. Wiss. 12, 213–219.

Please cite this article in press as: Nawratil, G. Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Des. (2016), http://dx.doi.org/10.1016/j.cagd.2016.06.003