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# Complexity of hierarchical refinement for a class of admissible mesh configurations

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ABSTRACT

An adaptive isogeometric method based on $d$-variate hierarchical spline constructions can be derived by considering a refine module that preserves a certain class of admissibility between two consecutive steps of the adaptive loop (Buffa and Giannelli, 2016). In this paper we provide a complexity estimate, i.e., an estimate on how the number of mesh elements grows with respect to the number of elements that are marked for refinement by the adaptive strategy. Our estimate is in the line of the similar ones proved in the context of adaptive finite element methods.



## 1. Introduction

Throughout the last years, isogeometric methods have gained widespread interest and are a very active field of research (Cottrell et al., 2009; Beirão da Veiga et al., 2014) investigating a wide range of applications and theoretical questions. Due to the tensor-product structure of splines, there exist very stable procedures to perform mesh refinement and degree raising which are known in the literature as $h$-refinement, $p$-refinement, $k$-refinement (Cottrell et al., 2009). While these algorithms are very efficient, the preservation of the tensor-product structure at least locally on each patch, produces a dramatic increase of degrees of freedom together with elongated elements. Mainly for this reason, several approaches have been proposed to alleviate these constraints and they all need the definition of B-splines over non-tensor-product meshes. Indeed, there are several strategies and we mention here T-splines (Bazilevs et al., 2010), hierarchical B-splines (Forsey and Bartels, 1988; Kraft, 1997; Kuru et al., 2014) and THB-splines (Giannelli et al., 2012), but also LR splines (Dokken et al., 2013; Bressan, 2013), hierarchical T-splines (Evans et al., 2015), modified T-splines (Kang et al., 2013), PHT-splines (Deng et al., 2008; Wang et al., 2011) amongst others.

Clearly, the development of adaptive strategies exploiting the potential of non-tensor-product splines is an interesting and important step which has been approached in a number of papers, at least from the practical point of view. In fact, despite their performance in experiments (Bazilevs et al., 2010; Dörfel et al., 2010; Beirão da Veiga et al., 2014; Kuru et al., 2014; Evans et al., 2015), the advantages of mesh-adaptive isogeometric methods have not been assessed in theory until today. Partial results on approximation, efficient and reliable error estimates and convergence of the adaptive procedure, have been proven in preliminary work (Buffa and Giannelli, 2016) in the context of (truncated) hierarchical splines. We aim to

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continue this study and to provide further ingredients that are needed for a proof of optimal convergence of the proposed adaptive approach in the spirit of adaptive finite element methods (Binev et al., 2004; Stevenson, 2007; Cascón et al., 2008; Carstensen et al., 2014).

In particular, in this paper we address the complexity of the mesh refinement procedure proposed in (Buffa and Giannelli, 2016). The relation between the set of marked elements and the overall number of refined elements introduced by the refine module is not straightforward: additional elements may be refined to create only (strictly) *admissible* meshes. Admissibility is a restriction to suitably graded meshes that allows for the adaptivity analysis of hierarchical isogeometric methods. Consequently, in order to define an automatic strategy to steer the adaptive method, the refinement is recursively propagated over a *suitable neighborhood* of any marked element. By starting from an initial mesh configuration $\mathcal{Q}_0$, let $\left\{\mathcal{Q}_j, \mathcal{M}_j\right\}_{j\geq 0}$ be the sequence of meshes $\mathcal{Q}_j$ and marked elements $\mathcal{M}_j$ computed by the adaptive scheme. At step $j$ of the refinement loop, the adaptive algorithm refines the marked subset of elements $\mathcal{M}_{j-1} \subseteq \mathcal{Q}_{j-1}$, together with some additional ones, to obtain the refined mesh $\mathcal{Q}_j$ with the same properties as $\mathcal{Q}_{j-1}$. A complexity estimate of the form

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \leq \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j \,, \tag{1}$$

with some positive constant $\Lambda$, provides a bound for the ratio of the newly inserted elements $\#\mathcal{Q}_J - \#\mathcal{Q}_0$ introduced up to step $J$ and the cumulative number $\sum_{j=0}^{J-1} \#\mathcal{M}_j$ of elements marked for refinement at each intermediate step in the subdivision process that leads from the initial to the final mesh. This allows to control the propagation of the refinement beyond the set of elements initially selected by the marking strategy. Our main result provides a complexity estimate (1) for an adaptive isogeometric method based on $d$-variate hierarchical spline constructions of any degree. An analogous complexity analysis is currently available for bivariate and trivariate T-splines (Morgenstern and Peterseim, 2015; Morgenstern, submitted for publication).

This paper is organized as follows. In Section 2, we recall notation and basic results from (Buffa and Giannelli, 2016). Section 3 is devoted to the announced complexity estimate. Conclusions and an outlook to future work are given in Section 4.

## 2. Hierarchical refinement

In this section, we recall some notation and basic results from (Buffa and Giannelli, 2016). Since the complexity analysis of the REFINE module can be performed directly in the parametric setting, we avoid to introduce the two different notations for parametric/physical domains.

### 2.1. The truncated hierarchical basis

Let $V^0 \subset V^1 \subset \cdots \subset V^{N-1}$ be a nested sequence of tensor-product $d$-variate spline spaces of fixed degree $\mathbf{p} = (p_1, \ldots, p_d)$ defined on a closed hypercube $D$ in $\mathbb{R}^d$. For each level $\ell$, with $\ell = 0, 1, \ldots, N-1$, we denote by $\mathcal{B}^\ell$ the normalized tensor-product B-spline basis of the spline space $V^\ell$ defined on $d$ knot sequences $T_1^\ell, \ldots, T_d^\ell$, for $\ell = 0, \ldots, N-1$, containing the different knot values in any coordinate direction. Let $\mu(T_i, t)$ be the multiplicity of $t$ in $T_i$, where $0 \leq \mu(T_i, t) \leq p_i + 1$ and $\mu(T_i, t) = 0$ if $t$ is not a knot in $T_i$. In order to define nested spaces, the knot sequences are also assumed to be nested, namely $\mu(T_i^{\ell+1}, t) \geq \mu(T_i^\ell, t)$.

Each space $V^\ell$ has an associated grid $G^\ell$ consisting of axis-aligned boxes such that the restriction of a function that belongs to $V^\ell$ to any of these cells is a tensor-product polynomial of degree $\mathbf{p}$, and $G^\ell$ is the coarsest grid with that property. We assume that $G^0$ consists of open hypercubes with side length 1. The Cartesian product of $d$ open intervals between adjacent (and non-coincident) grid values defines a quadrilateral element $Q$ of $G^\ell$. For all $Q \in G^k$ we denote by $h_Q := 2^{-k}$ the length of its side, and by $\ell(Q)$ its level, i.e., $\ell(Q) = k$.

**Remark 1.** The analysis could be generalized to the more general case of a non-uniform initial knot configuration (by suitably taking into account the corresponding maximum local mesh size).

In order to define hierarchical spline spaces, we consider a nested sequence of closed subdomains $\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^{N-1}$ of $D$. Any $\Omega^\ell$ is the union of the closure of elements that belong to the tensor-product grid of the previous level. The *hierarchical mesh* $\mathcal{Q}$ is defined as

$$\mathcal{Q} := \left\{ Q \in \mathcal{G}^\ell,\ \ell = 0, \ldots, N-1 \right\}, \tag{2}$$

where

$$\mathcal{G}^\ell := \left\{ Q \in G^\ell : Q \subset \Omega^\ell \wedge Q \not\subset \Omega^{\ell+1} \right\} \tag{3}$$

is the set of active elements of level $\ell$. Fig. 1 shows two hierarchical meshes related to the case $d = 1$ and $d = 2$, respectively.

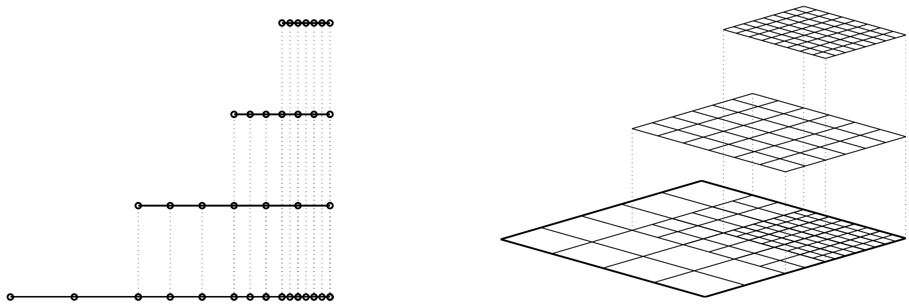


**Fig. 1.** Two examples of hierarchical meshes for the univariate (left) and bivariate (right) cases. The corresponding subdomain hierarchies are also shown.

We say that $\mathcal{Q}^*$ is a refinement of $\mathcal{Q}$, and denote $\mathcal{Q}^* \succeq \mathcal{Q}$, if $\mathcal{Q}^*$ is obtained from $\mathcal{Q}$ by splitting some of its elements via "$q$-adic" refinement. Although the hierarchical approach allows us to consider any integer $q \geq 2$, we will focus on the case of standard dyadic refinement with $q = 2$. Hierarchical B-splines are constructed according to a selection of active basis functions at different levels of detail, see also (Kraft, 1997; Vuong et al., 2011).

**Definition 2.** The hierarchical B-spline (HB-spline) basis $\mathcal{H}$ with respect to the mesh $\mathcal{Q}$ is defined as

$$\mathcal{H}(\mathcal{Q}) := \left\{ \beta \in \mathcal{B}^\ell : \operatorname{supp} \beta \subseteq \Omega^\ell \wedge \operatorname{supp} \beta \nsubseteq \Omega^{\ell+1},\ \ell = 0, \ldots, N-1 \right\},$$

where $\operatorname{supp} \beta$ denotes the intersection of the support of $\beta$ with $\Omega^0$.

Among the different adaptive spline structures currently available, the hierarchical B-spline basis is an effective solution defined as a straightforward extension of the tensor-product model that provides the possibility of local refinement within a multilevel setting. The spline hierarchy identifies different levels of refinement that can be exploited in order to obtain suitable adaptive solutions. The simple selection mechanism for the HB-spline basis construction introduced by Definition 2 guarantees not only the linear independence and non-negativity of the basis functions, but also a nested nature of the corresponding hierarchical spline spaces. Although these fundamental properties are directly preserved by construction, the B-spline refinement rules suggest an alternative basis construction for the same adaptive spline space. The following definition introduces the *truncation* mechanism, the key concept used to define the truncated basis for hierarchical splines (Giannelli et al., 2012).

**Definition 3.** Let

$$s = \sum_{\beta \in \mathcal{B}^{\ell+1}} c_\beta^{\ell+1}(s)\beta,$$

be the representation of $s \in V^\ell \subset V^{\ell+1}$ with respect to the finer basis $\mathcal{B}^{\ell+1}$. The truncation of $s$ with respect to $\mathcal{B}^{\ell+1}$ is defined as

$$\operatorname{trunc}^{\ell+1} s := \sum_{\substack{\beta \in \mathcal{B}^{\ell+1} \\ \operatorname{supp} \beta \nsubseteq \Omega^{\ell+1}}} c_\beta^{\ell+1}(s)\beta.$$

The truncation of a function that belongs to $V^\ell$ with respect to level $\ell + 1$ allows us to disregard the contribution of B-splines in $V^{\ell+1}$ whose support is contained in the refined domain $\Omega^{\ell+1}$. These refined B-splines are the ones who will be included in the HB-spline basis according to Definition 2. The iterative application of this truncation among the levels of the hierarchy leads to the following definition.

**Definition 4.** The truncated hierarchical B-spline (THB-spline) basis $\mathcal{T}$ with respect to the mesh $\mathcal{Q}$ is defined as

$$\mathcal{T}(\mathcal{Q}) := \left\{ \operatorname{Trunc}^{\ell+1} \beta : \beta \in \mathcal{B}^\ell \cap \mathcal{H}(\mathcal{Q}),\ \ell = 0, \ldots, N-1 \right\},$$

where $\operatorname{Trunc}^{\ell+1} \beta := \operatorname{trunc}^{N-1}(\operatorname{trunc}^{N-2}(\ldots(\operatorname{trunc}^{\ell+1}(\beta))\ldots))$, for any $\beta \in \mathcal{B}^\ell \cap \mathcal{H}(\mathcal{Q})$.

Let $\tau$ be a THB-spline and let $\beta$ be the hierarchical B-spline of a certain refinement level $\ell$ from which $\tau$ has been derived. We then say that $\tau$ belongs to the same level $\ell$ of $\beta$. HB-splines and THB-splines defined over the one-dimensional hierarchical mesh introduced in Fig. 1 (left) are illustrated in Fig. 2 for the cubic case.

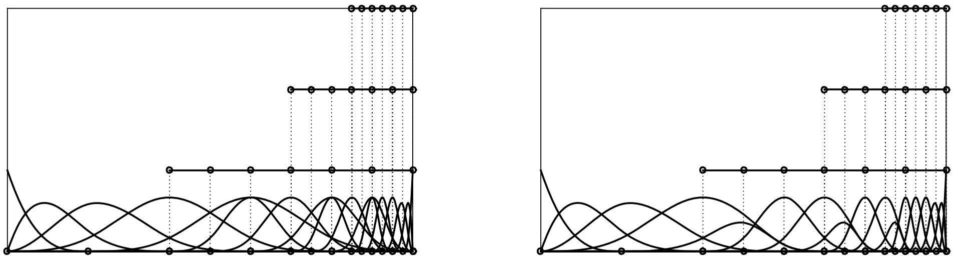
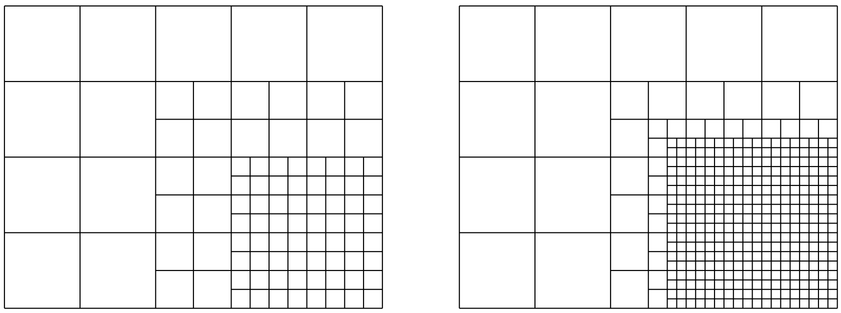


**Fig. 2.** HB-splines (left) and THB-splines (right) of degree 3 defined on the hierarchical mesh shown in Fig. 1 (left). Open uniform knot vectors are considered at any hierarchical level.



(a) (b)

**Fig. 3.** Two examples of admissible meshes of class $m$. The left mesh (a) is admissible of class 2 for $\mathbf{p} \le (2,2)$. It is *not* admissible of class 2 if $p_1 > 2$ or $p_2 > 2$, but it is admissible of class 3 for any $\mathbf{p}$. The mesh on the right-hand side (b) is admissible of class 2 only if $\mathbf{p} = (1,1)$, and admissible of class 4 for all other $\mathbf{p}$. Open uniform knot vectors are considered at any hierarchical level.

Note that the aforementioned properties of linear independence, non-negativity, and nested spaces are still valid. Moreover, let the B-spline $\beta$ related to the truncated basis function $\tau = \mathrm{Trunc}^{\ell+1}\, \beta$, $\ell = 0, \ldots, N-1$, be indicated as the *mother* B-spline of $\tau$. Being defined in terms of the truncation mechanism, each THB-spline is characterized by a *support that is either equal or smaller* than the one of its mother B-spline. This facilitates the identification of suitable graded meshes with certain local properties that do not depend on the overall number of hierarchical levels; see the discussion on the notion of *admissibility* in Subsection 2.2 below. In addition, the truncated basis satisfies the partition of unity property and improves the stability properties of the hierarchical construction. For details on the properties of the truncated basis, we refer to (Giannelli et al., 2012, 2014).

### 2.2. Admissible meshes and overlay

We restrict the adaptivity analysis of hierarchical isogeometric methods to a class of quasi-uniform mesh configurations, which we call *admissible* meshes. Considering the truncated basis, an admissible mesh guarantees the existence of an upper bound on the number of basis functions that take non-zero values on an arbitrary mesh element. This is a fundamental ingredient for the theoretical analysis of adaptive isogeometric methods, see (Buffa and Giannelli, 2016). In Fig. 3 we illustrate this concept through a couple of simple examples related to the case $d = 2$.

**Definition 5.** A mesh $\mathcal{Q}$ is admissible of class $m$ if the truncated basis functions in $\mathcal{T}(\mathcal{Q})$ which take non-zero values over any element $Q \in \mathcal{Q}$ belong to at most $m$ successive levels.

Since the case $m = 1$ refers to uniform meshes, we will from now on focus on the case $m \ge 2$. The relevance of admissible mesh configurations relies on two properties of THB-splines.

(P1) First, for each element $Q$ of an admissible mesh, the number of truncated basis functions of degree $\mathbf{p} = (p_1, \ldots, p_d)$ which are non-zero on $Q$ is less than $m \prod_{i=1}^{d} (p_i + 1)$.

(P2) Second, if $\mathcal{Q}$ is an admissible mesh of class $m$, then for all truncated basis functions $\tau \in \mathcal{T}(\mathcal{Q})$ and elements $Q \in \mathcal{Q}$ with $Q \cap \operatorname{supp} \tau \neq \emptyset$, we have $|Q| \lesssim |\operatorname{supp} \tau| \lesssim |Q|$, where the hidden constants in these inequalities depend on $m$ but not on $\tau$, $\mathcal{Q}$ and $N$.

In order to characterize a certain class of admissible meshes, we consider the generalization of the *support extension* usually considered in the tensor-product B-spline case to hierarchical configurations.

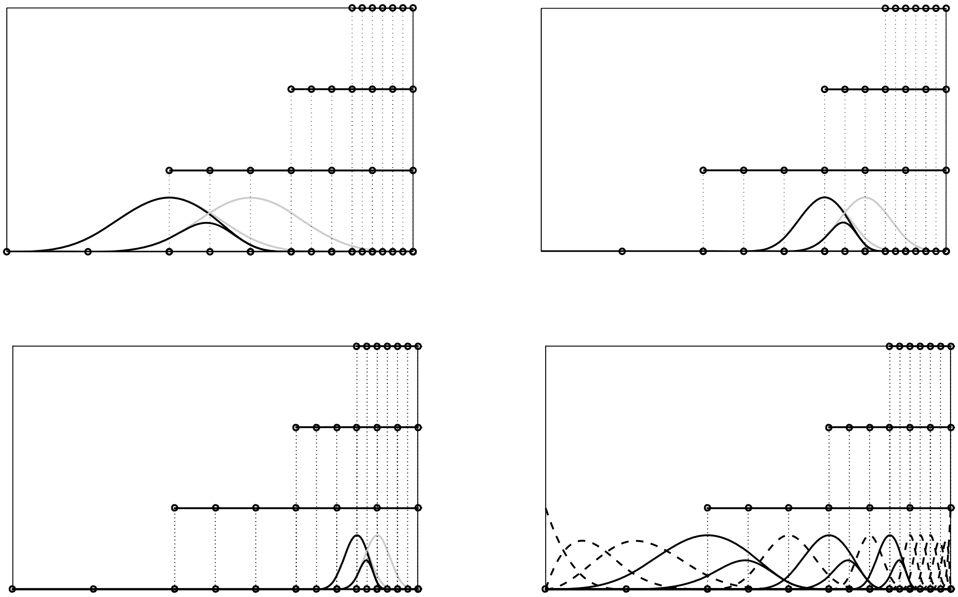


**Fig. 4.** We represent here the set of THB-splines (black) influenced by the truncation at level 0 (top left), 1 (top right), 2 (bottom left) together with their mother B-splines (gray). The underlying mesh is the one shown in Fig. 1 (left). In the bottom right plot, we show all levels together including (TH)B-splines not influenced by truncation (dashed). Open uniform knot vectors are considered at any hierarchical level.

**Definition 6.** The support extension $S(Q,k)$ of an element $Q \in G^\ell$ with respect to level $k$, with $0 \le k \le \ell$, is defined as

$$S(Q,k) := \left\{ Q' \in G^k : \exists\, \beta \in \mathcal{B}^k,\ \operatorname{supp}\beta \cap Q' \neq \emptyset \wedge \operatorname{supp}\beta \cap Q \neq \emptyset \right\}.$$

By a slight abuse of notation, we will also denote by $S(Q,k)$ the region occupied by the closure of elements in $S(Q,k)$. A relevant subset of admissible meshes can be defined according to (Buffa and Giannelli, 2016, Proposition 9).

**Definition 7.** Let $\mathcal{Q}$ be the mesh of active elements defined according to (2) and (3) with respect to the domain hierarchy $\Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^{N-1}$. Then $\mathcal{Q}$ is strictly admissible of class $m$ if

$$\Omega^\ell \subseteq \omega^{\ell-m+1} \tag{4}$$

where

$$\omega^{\ell-m+1} := \bigcup \left\{ \overline{Q} : Q \in G^{\ell-m+1} \wedge S(Q, \ell-m+1) \subseteq \Omega^{\ell-m+1} \right\},$$

for $\ell = m, m+1, \ldots, N-1$.

The smaller support that characterizes THB-splines motivates our interest in this *strict* version of admissibility. The properties (P1) and (P2) are fundamental ingredients for the adaptivity analysis of isogeometric methods, and they are valid for THB-splines considered on a strictly admissible mesh (thanks to the reduced support of truncated basis functions, see Fig. 4). However, (P1) and (P2) do not hold for the HB-spline basis defined over the same mesh, see e.g., Fig. 2.

The *overlay* $\mathcal{Q}_*$ of two meshes $\mathcal{Q}_1$, $\mathcal{Q}_2$ is the mesh obtained as the coarsest common refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, usually denoted by

$$\mathcal{Q}_* = \mathcal{Q}_1 \otimes \mathcal{Q}_2.$$

Let $\{\Omega_1^\ell\}_{\ell=0,\ldots,N_1-1}$ and $\{\Omega_2^\ell\}_{\ell=0,\ldots,N_2-1}$ with $\Omega_1^0 = \Omega_2^0$ be the nested sequence of domains that define the hierarchical meshes $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively. The domain hierarchy $\{\Omega_*^\ell\}_{\ell=0,\ldots,N_*-1}$, with $N_* = \max(N_1, N_2)$, associated to $\mathcal{Q}_*$ satisfies

$$\Omega_*^\ell = \Omega_1^\ell \cup \Omega_2^\ell \quad \text{and} \quad \omega_*^\ell \supseteq \omega_1^\ell \cup \omega_2^\ell$$

for $\ell = 1, \ldots, N_* - 1$, where $\Omega_i^\ell = \emptyset$ if $\ell \ge N_i$, for $i = 1, 2$. If $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are strictly admissible, then we have for any level $\ell$

$$\Omega_*^\ell \ = \ \Omega_1^\ell \cup \Omega_2^\ell \ \subseteq \ \omega_1^{\ell-m+1} \cup \omega_2^{\ell-m+1} \ \subseteq \ \omega_*^{\ell-m+1}.$$

```
𝒬* = REFINE(𝒬, ℳ, m)
  for all Q ∈ 𝒬 ∩ ℳ do
    𝒬 = REFINE_RECURSIVE(𝒬, Q, m)
  end for
  𝒬* = 𝒬
```

```
𝒬 = REFINE_RECURSIVE(𝒬, Q, m)
  for all Q' ∈ 𝒩(𝒬, Q, m) do
    𝒬 = REFINE_RECURSIVE(𝒬, Q', m)
  end for
  subdivide Q and
  update 𝒬 by replacing Q with its children
```

**Fig. 5.** The REFINE and REFINE_RECURSIVE modules.

Hence, the overlay $\mathcal{Q}_*$ of two strictly admissible meshes is a strictly admissible mesh. Note that the number of elements of the overlay mesh $\mathcal{Q}_*$ is bounded as follows,

$$\#\mathcal{Q}_* = \#(\mathcal{Q}_1 \otimes \mathcal{Q}_2) \leq \#\mathcal{Q}_1 + \#\mathcal{Q}_2 - \mathcal{Q}_0,$$

where $\mathcal{Q}_0$ is the initial mesh configuration, see e.g., (Bonito and Nochetto, 2010; Morgenstern and Peterseim, 2015). Analogously to the adaptive finite element setting, the above inequality may be used for discussing the rate optimality of the resulting adaptive isogeometric method (Buffa and Giannelli, in preparation).

### 2.3. *The* REFINE *module*

By exploiting the truncation mechanism, we can consider strictly admissible meshes according to Definition 7, in order to be able to design a refine module that preserves a certain class of admissibility between two consecutive steps of the adaptive loop. This refinement procedure recursively propagates the refinement in a certain *neighborhood* of any marked element so that the refined mesh produced by the algorithm is still strictly admissible. Consequently, the use of this kind of meshes facilitates the design of an automatic strategy to steer the adaptive method as it is common practice in adaptive finite element methods.

**Definition 8.** The neighborhood of $Q \in \mathcal{Q} \cap \mathcal{G}^\ell$ with respect to $m$ is defined as

$$\mathcal{N}(\mathcal{Q}, Q, m) := \left\{ Q' \in \mathcal{G}^{\ell-m+1} : \exists\, Q'' \in S(Q, \ell-m+2),\, Q'' \subseteq Q' \right\},$$

when $\ell - m + 1 \geq 0$, and $\mathcal{N}(\mathcal{Q}, Q, m) = \emptyset$ for $\ell - m + 1 < 0$.

A sequence of *strictly* admissible meshes can be recursively defined by suitably extending the refinement of coarser regions beyond the set of marked elements $\mathcal{M}$ through the algorithms presented in Fig. 5. Note that these algorithms follow the structure of informal high-level descriptions in the spirit of the analogous modules related to the adaptive finite element methods.

By exploiting key properties of the REFINE_RECURSIVE module, summarized in Lemma 9 and Proposition 10 below, Corollary 11 characterizes the output of the REFINE procedure (Buffa and Giannelli, 2016).

**Lemma 9** *(Recursive refinement). (See Buffa and Giannelli, 2016, Lemma 15.) Let $\mathcal{Q}$ be a strictly admissible mesh of class m and $Q \in \mathcal{Q}$. The call to $\mathcal{Q}^* =$ REFINE_RECURSIVE$(\mathcal{Q}, Q, m)$ terminates and returns a refined mesh $\mathcal{Q}^*$ with elements that either were already active in $\mathcal{Q}$ or are obtained by single refinement of an element of $\mathcal{Q}$.*

In addition, if $Q \in \mathcal{G}^\ell$, the level $\ell^*$ of all newly created elements $Q'$ generated by the call to $\mathcal{Q}^* =$ REFINE_RECURSIVE$(\mathcal{Q}, Q, m)$ satisfies

$$\ell^* \leq \ell + 1\,. \tag{5}$$

In order to verify this, we note that the recursion is applied to elements of level $< \ell$, and, in particular, of level $\leq \ell - m + 1$. If $Q'$ is a child of $Q$ then $\ell^* = \ell + 1$. Otherwise, $Q'$ is obtained by splitting some elements in the sequence of neighborhoods generated by the set of recursive calls and, consequently, $\ell^* \leq \ell - m + 2 < \ell + 1$ since $m \geq 2$.

**Proposition 10.** *(See Buffa and Giannelli, 2016, Proposition 16.) Let $\mathcal{Q}$ be a strictly admissible mesh of class $m \geq 2$ and let $Q \in \mathcal{G}^\ell$, for some $0 \leq \ell \leq N-1$. Then it follows that the call to $\mathcal{Q}^* =$ REFINE_RECURSIVE$(\mathcal{Q}, Q, m)$ returns a strictly admissible mesh $\mathcal{Q}^* \succeq \mathcal{Q}$ of class m.*

**Corollary 11.** *(See Buffa and Giannelli, 2016, Corollary 17.) Let $\mathcal{Q}$ be a strictly admissible mesh of class $m \geq 2$ and $\mathcal{M}$ the set of elements of $\mathcal{Q}$ marked for refinement. The call to $\mathcal{Q}^* =$ REFINE $(\mathcal{Q}, \mathcal{M}, m)$ terminates and returns a strictly admissible mesh $\mathcal{Q}^* \succeq \mathcal{Q}$ of class m.*

Note that in each computation of the neighborhood $\mathcal{N}(\mathcal{Q}, Q, m)$, the choice of level $\ell - m + 2$ for the support extension yields the smallest neighborhood that is necessary for preserving the class of admissibility of the mesh when subdividing the given element $Q$. Nevertheless, depending on the underlying hierarchical mesh configurations, the basis functions could also be truncated at different intermediate levels.

## 3. Linear complexity

This section is devoted to a complexity estimate in the spirit of Binev et al. (2004) and Stevenson (2007) in the context of adaptive finite element methods.

### 3.1. Auxiliary results

For every pair of mesh elements $(Q, Q')$, let $\mathrm{dist}(Q, Q')$ be the Euclidean distance of their midpoints. Given a $Q \in \mathcal{G}^\ell$, all $Q' \in \mathcal{N}(\mathcal{Q}, Q, m)$ satisfy

$$\mathrm{dist}(Q, Q') \le \frac{\sqrt{d}}{2}\,\mathrm{diam}(S(Q, \ell - m + 2))\,,$$

where $\ell = \ell(Q)$ and

$$\mathrm{diam}(S(Q, \ell - m + 2)) := 2^{-\ell+m-2}\,(2\,p + 1) = 2^{-\ell} C_s\,,$$

with $C_s = C_s(p, m) := 2^{m-2}(2\,p + 1)$, $p := \max_{i=1,\ldots,d} p_i$. Hence,

$$\mathrm{dist}(Q, Q') \le 2^{-\ell-1}\,C_d, \qquad C_d = C_d(d, p, m) := \sqrt{d}\,C_s\,. \tag{6}$$

**Lemma 12.** *Let $\mathcal{Q}$ be a strictly admissible mesh of class $m \ge 2$, $\mathcal{M}$ the set of elements of $\mathcal{Q}$ marked for refinement, and $Q' \in \mathcal{Q} \cap \mathcal{M}$. Any newly created $Q \in \mathcal{Q}^* \setminus \mathcal{Q}$ obtained by the call to $\mathcal{Q}^* = \mathrm{REFINE\_RECURSIVE}(\mathcal{Q}, Q', m)$ satisfies*

$$\mathrm{dist}(Q, Q') \le 2^{-\ell(Q)} C \quad \textit{with} \quad C := \sqrt{d}\,\tilde{C}, \quad \tilde{C} := \left(2^{-1} + \frac{2}{1 - 2^{1-m}} C_s\right), \tag{7}$$

*where then $C$ depends on $d$, $p$ and $m$.*

**Proof.** The existence of $Q \in \mathcal{Q}^* \setminus \mathcal{Q}$ means that REFINE_RECURSIVE is called over a sequence of elements $Q' = Q_J, Q_{J-1}, \ldots, Q_0$ and corresponding meshes $\mathcal{Q}_J, \ldots, \mathcal{Q}_0$ so that $Q_{j-1} \in \mathcal{N}(\mathcal{Q}_j, Q_j, m)$, with $Q' \in \mathcal{M}$ and $Q$ being a child of $Q_0$, namely $\ell(Q) = \ell(Q_0) + 1$. Since $\ell(Q_{j-1}) = \ell(Q_j) - m + 1$, it follows

$$\ell(Q_j) = \ell(Q_0) + j\,(m - 1). \tag{8}$$

We have

$$\mathrm{dist}(Q, Q') \le \mathrm{dist}(Q, Q_0) + \mathrm{dist}(Q_0, Q')$$

and

$$\mathrm{dist}(Q, Q_0) = 2^{-\ell(Q)} 2^{-1}\sqrt{d}\,, \qquad \mathrm{dist}(Q_0, Q') \le \sum_{j=1}^{J} \mathrm{dist}(Q_j, Q_{j-1})\,.$$

According to (6) and (8), we obtain

$$\begin{aligned}\sum_{j=1}^{J} \mathrm{dist}(Q_j, Q_{j-1}) &\le \sum_{j=1}^{J} 2^{-\ell(Q_j)-1}\,C_d = \sum_{j=1}^{J} 2^{-\ell(Q_0)-1-j(m-1)}\,C_d \\ &< 2^{-\ell(Q_0)} C_d \sum_{j=0}^{\infty} 2^{-j(m-1)} = \frac{2^{-\ell(Q_0)}}{1 - 2^{1-m}} C_d = \frac{2^{-\ell(Q)+1}}{1 - 2^{1-m}} C_d\,.\end{aligned}$$

Hence, $\mathrm{dist}(Q, Q') \le 2^{-\ell(Q)} C$, where $C$ is the constant defined in (7). □

*3.2. Main result*

The main result of this paper states the existence of a generic constant $\Lambda = \Lambda(d, p, m) < \infty$ that bounds the ratio between the number of new elements in the final mesh $\mathcal{Q}_J$ and the number of all marked elements encountered in the sequence of successive refinements from $\mathcal{Q}_0$ to $\mathcal{Q}_J$.

**Theorem 13** *(Complexity of* REFINE*). Let $\mathcal{M} := \bigcup_{j=0}^{J-1} \mathcal{M}_j$ be the set of marked elements used to generate the sequence of strictly admissible meshes $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_J$ starting from $\mathcal{Q}_0 = G^0$, namely*

$$\mathcal{Q}_j = \mathrm{REFINE}(\mathcal{Q}_{j-1}, \mathcal{M}_{j-1}, m), \quad \mathcal{M}_{j-1} \subseteq \mathcal{Q}_{j-1} \quad \textit{for } j \in \{1, \ldots, J\}.$$

*Then, there exists a positive constant $\Lambda = \Lambda(d, p, m) \le 4(4\tilde{C} + 1)^d$ so that*

$$\#\mathcal{Q}_J - \#\mathcal{Q}_0 \le \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j ,$$

*where $\tilde{C} = \tilde{C}(d, \mathbf{p}, m)$ is defined in (7).*

**Proof.** We denote by $\mathbb{G} := \bigcup_j G^j$ the set of the initial mesh elements and all elements that can be generated from their successive dyadic subdivision. Let $Q \in \mathbb{G}$, $Q' \in \mathcal{M}$, and

$$\lambda(Q, Q') := \begin{cases} 2^{\ell(Q)-\ell(Q')} & \text{if } \ell(Q) \le \ell(Q') + 1 \text{ and } \mathrm{dist}(Q, Q') < 2^{1-\ell(Q)} C, \\ 0 & \text{otherwise.} \end{cases}$$

The proof consists of two main steps devoted to identify

(i) a lower bound for the sum of the $\lambda$ function as $Q'$ varies in $\mathcal{M}$ so that each $Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0$ satisfies

$$\sum_{Q' \in \mathcal{M}} \lambda(Q, Q') \ \ge\ 1 ; \tag{9}$$

(ii) an upper bound for the sum of the $\lambda$ function as the refined element $Q$ varies in $\mathcal{Q}_J \setminus \mathcal{Q}_0$ so that, for any $j = 0, \ldots, J-1$, each $Q' \in \mathcal{M}_j$ satisfies

$$\sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \lambda(Q, Q') \ \le \Lambda . \tag{10}$$

If inequalities (9) and (10) hold for a certain constant $\Lambda$, we have

$$\begin{aligned} \#\mathcal{Q}_J - \#\mathcal{Q}_0 &= \sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} 1 \le \sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \sum_{Q' \in \mathcal{M}} \lambda(Q, Q') \\ &\le \sum_{Q' \in \mathcal{M}} \Lambda = \Lambda \sum_{j=0}^{J-1} \#\mathcal{M}_j , \end{aligned}$$

and the proof of the theorem is complete. We detail below the analysis of (i) and (ii).

(i) Let $Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0$ be an element generated in the refinement process from $\mathcal{Q}_0$ to $\mathcal{Q}_J$, and let $j_1 < J$ be the index so that $Q \in \mathcal{Q}_{j_1+1} \setminus \mathcal{Q}_{j_1}$. Lemma 12 together with (5) states the existence of $Q_1 \in \mathcal{M}_{j_1}$ with

$$\mathrm{dist}(Q, Q_1) \le 2^{-\ell(Q)} C \quad \text{and} \quad \ell(Q) \le \ell(Q_1) + 1 ,$$

and, consequently $\lambda(Q, Q_1) = 2^{\ell(Q)-\ell(Q_1)} > 0$. The repeated use of Lemma 12 yields a sequence $\{Q_2, Q_3, \ldots\}$ with $Q_{i-1} \in \mathcal{Q}_{j_i+1} \setminus \mathcal{Q}_{j_i}$, for $j_1 > j_2 > j_3 > \ldots$, and $Q_i \in \mathcal{M}_{j_i}$ such that

$$\mathrm{dist}(Q_{i-1}, Q_i) \le 2^{-\ell(Q_{i-1})} C \quad \text{and} \quad \ell(Q_{i-1}) \le \ell(Q_i) + 1. \tag{11}$$

We iteratively apply Lemma 12 as long as

$$\lambda(Q, Q_i) > 0 \quad \text{and} \quad \ell(Q_i) > 0 ,$$

until we reach the first index $L$ with $\lambda(Q, Q_L) = 0$ or $\ell(Q_L) = 0$. By considering the three possible cases below, inequality (9) may be derived as follows.

- If $\ell(Q_L) = 0$ and $\lambda(Q, Q_L) > 0$, then

$$\sum_{Q' \in \mathcal{M}} \lambda(Q, Q') \geq \lambda(Q, Q_L) = 2^{\ell(Q)-\ell(Q_L)} > 1\,,$$

  since $\ell(Q) > \ell(Q_L) = 0$.
- If $\lambda(Q, Q_L) = 0$ because $\ell(Q) > \ell(Q_L) + 1$, then (11) yields $\ell(Q_{L-1}) \leq \ell(Q_L) + 1 < \ell(Q)$ and hence

$$\sum_{Q' \in \mathcal{M}} \lambda(Q, Q') \geq \lambda(Q, Q_{L-1}) = 2^{\ell(Q)-\ell(Q_{L-1})} > 1.$$

- If $\lambda(Q, Q_L) = 0$ because $\operatorname{dist}(Q, Q_L) \geq 2^{1-\ell(Q)} C$, then a triangle inequality combined with Lemma 12 leads to

$$2^{1-\ell(Q)} C \leq \operatorname{dist}(Q, Q_1) + \sum_{i=1}^{L-1} \operatorname{dist}(Q_i, Q_{i+1}) \leq 2^{-\ell(Q)} C + \sum_{i=1}^{L-1} 2^{-\ell(Q_i)} C\,.$$

  Consequently, $2^{-\ell(Q)} \leq \sum_{i=1}^{L-1} 2^{-\ell(Q_i)}$, and we obtain

$$1 \;\leq\; \sum_{i=1}^{L-1} 2^{\ell(Q)-\ell(Q_i)} \;=\; \sum_{i=1}^{L-1} \lambda(Q, Q_i) \;\leq\; \sum_{Q' \in \mathcal{M}} \lambda(Q, Q').$$

(ii) Inequality (10) can be derived as follows. For any $0 \leq j \leq J-1$, we consider the set of elements of level $j$ whose distance from $Q'$ is less than $2^{1-j} C$ defined as

$$B(Q', j) := \left\{Q \in G^j : \operatorname{dist}(Q, Q') < 2^{1-j} C\right\}.$$

According to the definition of $\lambda$, the set $B(Q', j)$ collects the elements at level $j$ so that $\lambda(Q, Q') > 0$. We then have

$$\sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \lambda(Q, Q') \leq \sum_{Q \in \mathbb{G} \setminus \mathcal{Q}_0} \lambda(Q, Q') = \sum_{j=1}^{\ell(Q')+1} 2^{j-\ell(Q')} \# B(Q', j)\,. \tag{12}$$

Since the diagonal of an element $Q$ of level $j$ is $2^{-j}\sqrt{d}$, the diagonal of the hypercube composed by the union of the closure of all elements in $B(Q', j)$ is less or equal to

$$2 \cdot 2^{1-j} C + 2^{-j}\sqrt{d} = 2^{-j}\sqrt{d}\,(4\tilde{C} + 1)\,,$$

where $\tilde{C}$ is defined in (7). Hence,

$$\# B(Q', j) \leq (4\tilde{C} + 1)^d\,,$$

and the index substitution $k := 1 - j + \ell(Q')$ reduces (12) to

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_J \setminus \mathcal{Q}_0} \lambda(Q, Q') &\leq \sum_{j=1}^{\ell(Q')+1} 2^{j-\ell(Q')} \# B(Q', j) = \sum_{k=0}^{\ell(Q')} 2^{1-k} \# B(Q', j) \\ &< 2 \sum_{k=0}^{\infty} 2^{-k} \# B(Q', j) = 4\, \# B(Q', j) \leq \Lambda\,, \end{aligned}$$

with $\Lambda = \Lambda(d, p, m) = 4(4\tilde{C} + 1)^d$. □

## 4. Conclusions

We developed a complexity estimate which states that the ratio between the refined elements and the marked elements along the refinement history stays bounded if refinement is performed as proposed in (Buffa and Giannelli, 2016). In particular, this estimate guarantees that if the refinement routine is applied very often in the same location (e.g., for resolving a singularity), then it will asymptotically remain local. Note that for a single refinement step, a uniform (with constants independent on the level) estimate bounding the number of refined elements in terms of the marked ones is not possible (Nochetto and Veeser, 2012).

Our work paves the way to the analysis of optimal convergence of the adaptive strategy proposed in (Buffa and Giannelli, 2016) that will be addressed in further studies (Buffa and Giannelli, in preparation).

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