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Isoptic surfaces of polyhedra

Géza Csima *, Jenő Szirmai

Budapest University of Technology and Economics, Institute of Mathematics, Department of Geometry, Budapest, P.O. Box 91, H-1521, Hungary

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ABSTRACT

The theory of the isoptic curves is widely studied in the Euclidean plane $\mathbf{E}^2$ (see Cieślak et al., 1991 and Wieleitner, 1908 and the references given there). The analogous question was investigated by the authors in the hyperbolic $\mathbf{H}^2$ and elliptic $\mathcal{E}^2$ planes (see Csima and Szirmai, 2010, 2012, submitted for publication), but in the higher dimensional spaces there are only few results in this topic.

In Csima and Szirmai (2013) we gave a natural extension of the notion of the isoptic curves to the n-dimensional Euclidean space $\mathbf{E}^n$ ($n \geq 3$) which is called isoptic hypersurface. Now we develope an algorithm to determine the isoptic surface $\mathcal{H}_\mathcal{P}$ of a 3-dimensional polyhedron $\mathcal{P}$.

We will determine the isoptic surfaces for Platonic solids and for some semi-regular Archimedean polytopes and visualize them with *Wolfram Mathematica* (Wolfram Research, Inc., 2015).

1. Introduction

Let G be one of the constant curvature plane geometries, either the Euclidean $\mathbf{E}^2$ or the hyperbolic $\mathbf{H}^2$ or the elliptic $\mathcal{E}^2$. The isoptic curve of a given plane curve $\mathcal{C}$ is the locus of points $P \in G$ where $\mathcal{C}$ is seen under a given fixed angle α $(0 < \alpha < \pi)$. An isoptic curve formed by the locus of tangents meeting at right angles is called orthoptic curve. The name isoptic curve was suggested by Taylor (1884).

In Cieślak et al. (1991, 1996) the isoptic curves of the closed, strictly convex curves are studied, by use of their support function. The explicit formula for the isoptic curve of the triangle can be found in Michalska and Mozgawa (2015). The papers Wunderlich (1971a, 1971b) deal with curves having a circle or an ellipse by an isoptic curve. Further curves appearing as isoptic curves are well studied in the Euclidean plane geometry $\mathbf{E}^2$, see *e.g.* Loria (1911), Wieleitner (1908). Isoptic curves of conic sections have been studied in Holzmüller (1882) and Siebeck (1860). Isoptic curves of Bézier curves are considered in Kunkli et al. (2013). A lot of papers concentrate on the properties of the isoptics *e.g.* Miernowski and Mozgawa (1997), Michalska (2003), and the references given there. The papers Kurusa (2012) and Kurusa and Ódor (2015) deal with inverse problems.

In the hyperbolic and elliptic planes $\mathbf{H}^2$ and $\mathcal{E}^2$ the isoptic curves of segments and proper conic sections are determined by the authors Csima and Szirmai (2010, 2012, 2014). In Csima and Szirmai (submitted for publication) we extended the notion of the isoptic curves to the outer (non-proper) points of the hyperbolic plane and determined the isoptic curves of the generalized conic sections.

It is known, that the angle between two half-lines with the vertex A in the plane can be measured by the arc length on the unit circle around the point A. This statement can be generalized to the higher dimensional Euclidean spaces. The

* Corresponding author.
E-mail addresses: csgeza@math.bme.hu (G. Csima), szirmai@math.bme.hu (J. Szirmai).

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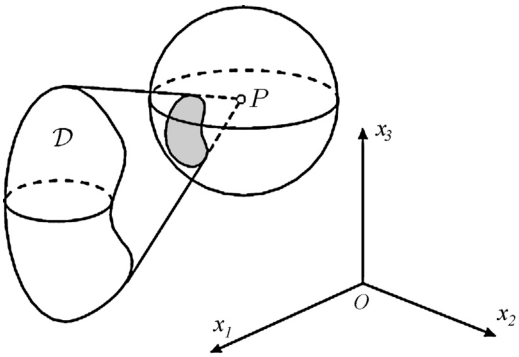

Fig. 1. Projection of a compact domain $\mathcal{D}$ onto a unit sphere in $\mathbf{E}^3$.

notion of the *solid angle* is well known and widely studied in the literature (see Gardner and Verghese, 1971). We recall this definition concerning the 3-dimensional Euclidean space $\mathbf{E}^3$.

Definition 1.1. The solid angle $\Omega_S(\mathbf{p})$ subtended by a surface S is defined as the surface area of the projection of S onto the unit sphere around $P(\mathbf{p})$, where $\mathbf{p}$ is the coordinates of P.

The solid angle is measured in *steradians*, e.g. the solid angle subtended by the whole Euclidean space $\mathbf{E}^3$ is equal to 4π steradians. Moreover, this notion has several important applications in physics (in particular in astrophysics, radiometry or photometry) (see Camp and Van Lehn, 1969), computational geometry (see Joe, 1991) and we will use it for the definition of the isoptic surfaces.

The isoptic hypersurface in the n-dimensional Euclidean space ($n \geq 3$) is defined in Csima and Szirmai (2013) and now, we recall some statements and specify them to $\mathbf{E}^3$.

Definition 1.2. The isoptic hypersurface $\mathcal{H}^{\alpha}_{\mathcal{D}}$ in $\mathbf{E}^3$ of an arbitrary 3-dimensional compact domain $\mathcal{D}$ is the locus of points P where the measure of the projection of $\mathcal{D}$ onto the unit sphere around P is equal to a given fixed value α, where $0 < \alpha < 2\pi$ (see Fig. 1).

In this paper we develope an algorithm and the corresponding computer program to determine the isoptic surface of an arbitrary convex polyhedron in the 3-dimensional Euclidean space. We apply this algorithm for the regular Platonic solids and some semi-regular Archimedean solids as well. We note here that this generalization of the isoptic curves to the 3-dimensional space provides possible research to extend the notion of isoptic surfaces to bounded polyhedral surfaces and, with triangulations, to 'smooth surfaces'.

2. The algorithm

In this section we discuss the algorithm developed to determine the isoptic surface of a given polyhedron.

1. We assume that an arbitrary polyhedron $\mathcal{P}$ is given by the usual data structure. This consists of the list of facets $\mathcal{F}_{\mathcal{P}}$ with the set of vertices V_i in clockwise order. Each facet can be embedded into a plane.
   It is well known, that if $\mathbf{a} \in \mathbf{R}^3 \setminus \{\mathbf{0}\}$ and $b \in \mathbf{R}$ then $\{\mathbf{x} \in \mathbf{R}^3 | \mathbf{a}^T\mathbf{x} = b\}$ is a plane and $\{\mathbf{x} \in \mathbf{R}^3 | \mathbf{a}^T\mathbf{x} \leq b\}$ defines a halfspace. Every polyhedron is the intersection of finitely many halfspaces. Therefore an arbitrary polyhedron can also be given by a system of inequalities $A\mathbf{x} \leq \mathbf{b}$ where $A \in \mathbf{R}^{m\times 3}$ $(4 \leq m \in \mathbb{N})$, $\mathbf{x} \in \mathbf{R}^3$ and $\mathbf{b} \in \mathbf{R}^m$.
2. For an arbitrary point $P(\mathbf{p}) \in \mathbf{E}^3$ we have to decide, that which facets of $\mathcal{P}$ 'can be seen' from it. Let us denote the ith facet of $\mathcal{P}$ by $\mathcal{F}^i_{\mathcal{P}}$ $(i = 1, \dots, m)$ and by $\mathbf{a}^i$ the vector derived by the ith row of the matrix A which characterize the facet $\mathcal{F}^i_{\mathcal{P}}$.
   Since the polyhedron $\mathcal{P}$ is given by the system of inequalities $A\mathbf{x} \leq \mathbf{b}$, where each inequality $\mathbf{a}^i\mathbf{x} \leq b_i$ $(i \in \{1, 2 \dots, m\})$ is assigned to a certain facet, therefore the facet $\mathcal{F}^i_{\mathcal{P}}$ is visible from P if an only if the inequality $\mathbf{a}^i\mathbf{p} > b_i$ holds.
   Now, we define the characteristic function $\mathbb{I}^i_{\mathcal{P}}(\mathbf{x})$ for each facet $\mathcal{F}^i_{\mathcal{P}}$:
   $$\mathbb{I}^i_{\mathcal{P}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{a}^i\mathbf{x} > b_i \\ 0 & \mathbf{a}^i\mathbf{x} \leq b_i. \end{cases}$$
3. Using the Definition 1.1, let $\Omega_i(\mathbf{p}) := \Omega_{\mathcal{F}^i_{\mathcal{P}}}(\mathbf{p})$ be the solid angle of the facet $\mathcal{F}^i_{\mathcal{P}}$ regarding the point $P(\mathbf{p})$.

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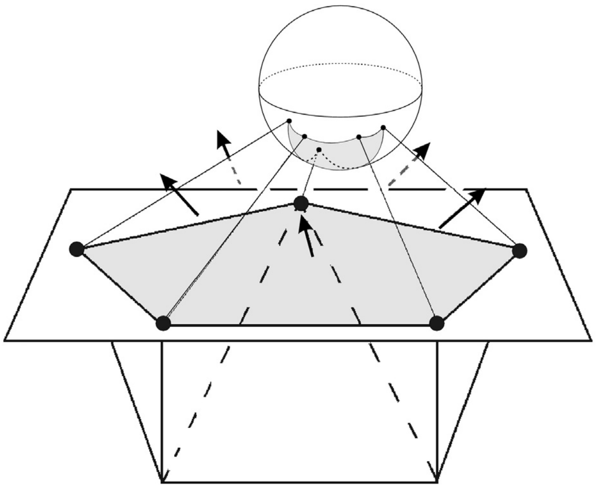

Fig. 2. Projection of a facet $\mathcal{F}^i_{\mathcal{P}}$ onto the unit sphere in $\mathbf{E}^3$.

To determine $\Omega_i(\mathbf{p})$, we use the methods of the spherical geometry. Let us suppose that $\mathcal{F}^i_{\mathcal{P}}$ contains n_i vertices, $V_{i_j}(\mathbf{x}_{i_j})$ $(j = 1, \dots, n_i)$ where the vertices are given in clockwise order. Projecting these vertices onto the unit sphere centered in $P(\mathbf{p})$ we get a spherical n_i-gon (see Fig. 2), whose area can be calculated by the usual formula

$$\Omega_i(\mathbf{p}) = \Theta - (n_i - 2)\pi\,.$$

Here Θ is the sum of the angles τ_j of the spherical projection of the polygon $\mathcal{F}^i_{\mathcal{P}}$ where the angles are measured in radians.

4. To obtain angles τ_j, we need to determine the angles between the two planes containing the neigbouring edges $\overline{PV}_{i_{j-1}}$, $\overline{PV}_{i_j}$ and $\overline{PV}_{i_j}$, $\overline{PV}_{i_{j+1}}$. Thus for $j = 1, 2, \dots, n$ $(i_0 := i_{n_i}$ and $i_{n_i+1} := i_1)$, we have:

$$\tau_j = \pi - \arccos\left(\frac{\left\langle \overrightarrow{PV}_{i_{j-1}} \times \overrightarrow{PV}_{i_j}, \overrightarrow{PV}_{i_j} \times \overrightarrow{PV}_{i_{j+1}} \right\rangle}{\left|\overrightarrow{PV}_{i_{j-1}} \times \overrightarrow{PV}_{i_j}\right| \left|\overrightarrow{PV}_{i_j} \times \overrightarrow{PV}_{i_{j+1}}\right|}\right).$$

Finally, we get the solid angle function $\Omega_i(\mathbf{x})$ of the facet $\mathcal{F}^i_{\mathcal{P}}$ for any $\mathbf{x} \in \mathbf{R}^3$:

$$\Omega_i(\mathbf{x}) = 2\pi - \sum_{j=1}^{n_i} \arccos\left(\frac{\left\langle \overrightarrow{XV}_{i_{j-1}} \times \overrightarrow{XV}_{i_j}, \overrightarrow{XV}_{i_j} \times \overrightarrow{XV}_{i_{j+1}} \right\rangle}{\left|\overrightarrow{XV}_{i_{j-1}} \times \overrightarrow{XV}_{i_j}\right| \left|\overrightarrow{XV}_{i_j} \times \overrightarrow{XV}_{i_{j+1}}\right|}\right).$$

5. We can summarize our results in the following

Theorem 2.1. *Let us consider a solid angle α $(0 < \alpha < 2\pi)$ and a convex polyhedron $\mathcal{P}$ given by its data structure and its set of inequality. Then the isoptic surface of $\mathcal{P}$ can be determined by the equation*

$$\alpha = \sum_{i=1}^{m} \mathbb{I}^i_{\mathcal{P}}(\mathbf{x})\Omega_i(\mathbf{x}).$$

The results and the computations will be demonstrated in the following subsection through the computation related to the regular tetrahedron and along with some figures.

Remark 2.2.

1. The algorithm can be easily extended for non-closed directed surfaces *e.g.* for subdivision surfaces.
2. If we have a **convex** polyhedron, then projecting its whole surface to the unit sphere, we obtain a double coverage (double solid angle) of the given polyhedron, therefore the algorithm can be changed *i.e.* it is not necessary to determine the visible facets. In this case the isoptic surfaces are determined by the following **implicit** equation:

$$\alpha = \frac{1}{2}\sum_{i=1}^{m} \Omega_i(\mathbf{x}).$$

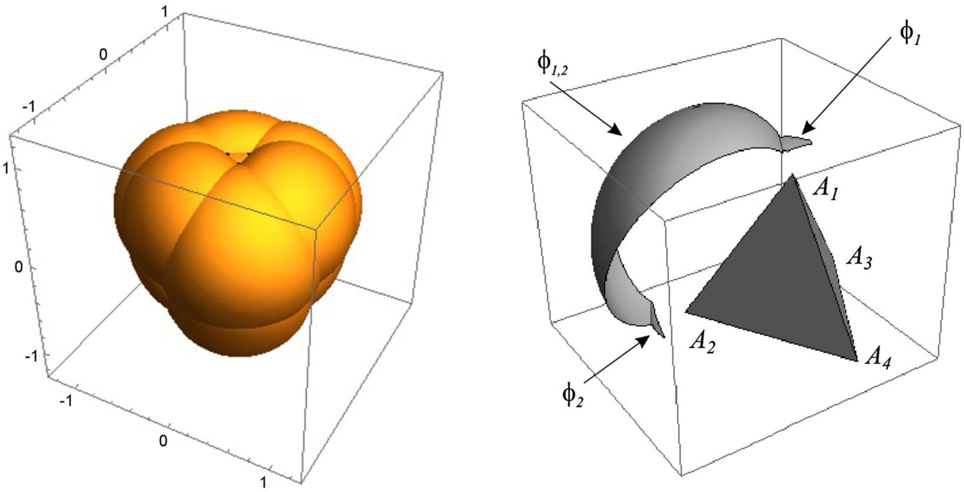

Fig. 3. Isoptic surface of the regular tetrahedron for $\alpha = \pi/8$ (left). A part of the tetrahedral isoptic surface (right). E.g. the surface $\phi_{1,2}$ is derived by the solid angles of the facets $\mathcal{F}^3_{\mathcal{T}}$ and $\mathcal{F}^4_{\mathcal{T}}$.

2.1. Computations of the isoptic surface of a given regular tetrahedron

Following the steps of the above described algorithm, we will calculate the compound isoptic surface of a given regular tetrahedron $\mathcal{T}$ whose data structure is determined by its vertices and facets where the facets $\mathcal{F}^i_{\mathcal{P}}$ are given by their clockwise ordered vertices:

$$A_1 = \left(0, 0, \sqrt{\frac{2}{3}} - \frac{1}{2\sqrt{6}}\right), A_2 = \left(-\frac{1}{2\sqrt{3}}, -\frac{1}{2}, -\frac{1}{2\sqrt{6}}\right),$$
$$A_3 = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2}, -\frac{1}{2\sqrt{6}}\right), A_4 = \left(\frac{1}{\sqrt{3}}, 0, -\frac{1}{2\sqrt{6}}\right),$$
$$\left\{\mathcal{F}^1_{\mathcal{T}}, \{A_2, A_3, A_4\}\right\}, \left\{\mathcal{F}^2_{\mathcal{T}}, \{A_3, A_2, A_1\}\right\},$$
$$\left\{\mathcal{F}^3_{\mathcal{T}}, \{A_4, A_1, A_2\}\right\}, \left\{\mathcal{F}^4_{\mathcal{T}}, \{A_1, A_4, A_3\}\right\}.$$

This tetrahedron can also be given by its system of inequalities:

$$\begin{pmatrix} 0 & 0 & -4\sqrt{3} \\ -8\sqrt{6} & 0 & 4\sqrt{3} \\ 4\sqrt{6} & -12\sqrt{3} & 4\sqrt{3} \\ 4\sqrt{6} & 12\sqrt{3} & 4\sqrt{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq \begin{pmatrix} \sqrt{2} \\ 3\sqrt{2} \\ 3\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}.$$

Since the final formula of the isoptic surface is too long to appear in print even for this simple example (the right side of the equation consists of twelve arccos function), we choose $i = 1$ and derive only $\mathbb{I}^1_{\mathcal{P}}(\mathbf{x})$.

$$\mathbb{I}^1_{\mathcal{P}}(x, y, z) = \begin{cases} 1 & -4\sqrt{3}z > \sqrt{2} \\ 0 & -4\sqrt{3}z \leq \sqrt{2} \end{cases}.$$

For the obtained surface, see Fig. 3 (left), and Fig. 3 (right) shows a part of the tetrahedral isoptic surface for the solid angle $\alpha = \pi/8$. The surface $\phi_{1,2}$ is derived by the solid angles of the facets $\mathcal{F}^3_{\mathcal{T}}$ and $\mathcal{F}^4_{\mathcal{T}}$. The isoptic surface ϕ_1 is derived by the solid angles of the facets $\mathcal{F}^2_{\mathcal{T}}$, $\mathcal{F}^3_{\mathcal{T}}$ and $\mathcal{F}^4_{\mathcal{T}}$ and the isoptic surface ϕ_2 is derived by the solid angles of the facets $\mathcal{F}^1_{\mathcal{T}}$, $\mathcal{F}^3_{\mathcal{T}}$ and $\mathcal{F}^4_{\mathcal{T}}$.

2.2. Isoptic surfaces to regular polyhedra and some Archimedean polyhedra

In the following we apply our algorithm to some polyhedra. We note here that the algorithm provides the implicit equation of the compound isoptic surface related to the given polyhedron (see Figs. 4 and 5).

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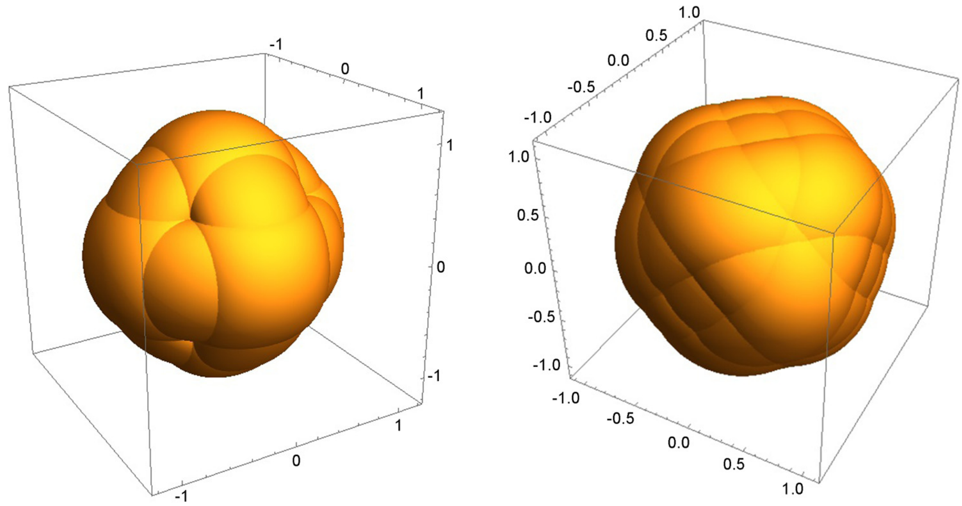

Fig. 4. Isoptic surface of the cube for $\alpha = \pi/2$ (left). Isoptic surface of the regular octahedron for $\alpha = \pi/7$ (right).

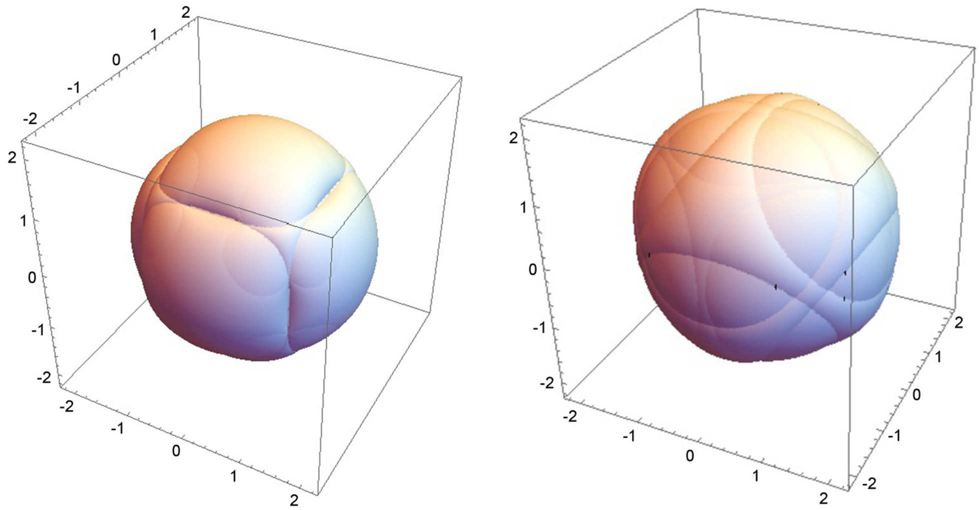

Fig. 5. Isoptic surface of the truncated cube for $\alpha = \pi$ (left). Isoptic surface of the truncated octahedron $\alpha = 2\pi/3$ (right).

Remark 2.3. Despite the equation can be obtained in $\mathcal{O}(e)$ steps, where e is the number of edges, the rendering of these figures by *Wolfram Mathematica* (Wolfram Research, Inc., 2015) takes 20–40 minutes. The implicit equation of the isoptic surface is so complicated that it seems difficult to draw further consequences from it.

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