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Design with Quasi Extended Chebyshev piecewise spaces

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ABSTRACT

A Quasi Extended Chebyshev (QEC) space is a space of sufficiently differentiable functions in which any Hermite interpolation problem which is not a Taylor problem is unisolvent. On a given interval the class of all spaces which contains constants and for which the space obtained by differentiation is a QEC-space has been identified as the largest class of spaces (under ordinary differentiability assumptions) which can be used for design. As a first step towards determining the largest class of splines for design, we consider a sequence of QEC-spaces on adjacent intervals, all of the same dimension, we join them via connection matrices, so as to maintain both the dimension and the unisolvence. The resulting space is called a Quasi Extended Chebyshev Piecewise (QECP) space. We show that all QECP-spaces are inverse images of two-dimensional Chebyshev spaces under piecewise generalised derivatives associated with systems of piecewise weight functions. We show illustrations proving that QECP-spaces can produce interesting shape effects.

1. Introduction

Throughout this article we consider a fixed interval $[a,b]$, $a<b$, and a fixed positive integer n. An $(n+1)$-dimensional space $\mathbb{E}\subset C^n([a,b])$ is an *Extended Chebyshev-space (for short EC-space) on* $[a,b]$ if any non-zero element of $\mathbb{E}$ vanishes at most n times on $[a,b]$, counting multiplicities as far as possible for C^n functions, that is, up to $(n+1)$. Equivalently, $\mathbb{E}$ is an EC-space on $[a,b]$ if any Hermite interpolation problem in $(n+1)$ data on $[a,b]$ has a unique solution in $\mathbb{E}$. There exists a classical procedure to build $(n+1)$-dimensional EC-spaces from systems of weight functions on $[a,b]$, that is, from sequences $(w_0,\dots,w_n)$ such that each w_i is C^{n-i} and positive on $[a,b]$. With such a system one can associate generalised derivatives $L_0,\dots,L_n$ defined in a recursive way as follows:

$$L_0F := \frac{F}{w_0}\,,\quad L_iF := \frac{1}{w_i}DL_{i-1}F,\quad 1\leqslant i\leqslant n, \tag{1}$$

when D stands for the ordinary differentiation. Then, the set $EC(w_0,\dots,w_n)$ composed of all functions $F\in C^n([a,b])$ for which L_nF is constant is an EC-space on $[a,b]$. This is due to Rolle's theorem (Karlin and Studden, 1966; Schumaker, 1981). Because we are working on a closed bounded interval we can even state that (Pottmann, 1993; Mazure, 2007, 2011b)

Theorem 1.1. *The class of all $(n+1)$-dimensional EC-spaces on $[a,b]$ coincides with the class of all spaces of the form $EC(w_0,\dots,w_n)$, where $(w_0,\dots,w_n)$ is any system of weight functions on $[a,b]$.*

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It is well known that EC-spaces can advantageously serve as substitutes for polynomials in many issues requiring shape/tension parameters (Schweikert, 1966). To gain the maximal potential from the great variety of parameters provided by this class of spaces, it is interesting to mix different $(n+1)$-dimensional EC-spaces on adjacent intervals so as to obtain an $(n+1)$-dimensional space possessing similar properties. With this in view, take:

– a fixed sequence $\mathbb{T} = (t_1, \ldots, t_q)$ of knots interior to $[a,b]$, with

$$t_0 := a < t_1 < \cdots < t_q < t_{q+1} := b;$$

– an associated sequence $M_1, \ldots, M_q$ of connection matrices of order $(n+1)$, each of them being lower triangular with positive diagonal entries;
– a sequence $\mathbb{E}_k$, $0 \leqslant k \leqslant q$, of section-spaces: for each k, $\mathbb{E}_k \subset C^n([t_k, t_{k+1}])$ is an $(n+1)$-dimensional EC-space on $[t_k, t_{k+1}]$.

These ingredients provide us with an $(n+1)$-dimensional *Piecewise Extended Chebyshev space* (for short, PEC-space) *on* $([a,b];\mathbb{T})$, defined as the set $\mathbb{E}$ composed of all *piecewise functions* F such that:

1) for $k = 0, \ldots, q$, there exists a function $F_k \in \mathbb{E}_k$ such that F coincides with F_k on $[t_k^+, t_{k+1}^-]$;
2) for $k = 1, \ldots, q$, the following connection condition is fulfilled:

$$\left(F(t_k^+), F'(t_k^+), \ldots, F^{(n)}(t_k^+)\right)^T = M_k \left(F(t_k^-), F'(t_k^-), \ldots, F^{(n)}(t_k^-)\right)^T. \tag{2}$$

Not all such PEC-spaces on $([a,b];\mathbb{T})$ are of interest. Nevertheless, note that the assumptions on the connection matrices make it possible to count the total number of zeroes of any element of $\mathbb{E}$, including multiplicities up to $(n+1)$. We denote it by $Z_{n+1}(F)$. By analogy with the non-piecewise case, the $(n+1)$-dimensional PEC-space $\mathbb{E}$ is said to be *an Extended Chebyshev Piecewise space* (for short, ECP-space) *on* $([a,b];\mathbb{T})$ when $Z_{n+1}(F) \leqslant n$ for any non-zero $F \in \mathbb{E}$.

An extensive study of ECP-spaces was developed in a number of earlier papers Mazure (1999, 2005b, 2006, 2007, 2011a), Mazure and Laurent (1999). In particular, the class of all ECP-spaces was clearly identified in Mazure (2007), see also Mazure (2011a). With a view to describe it, consider a system $(w_0, \ldots, w_n)$ of *piecewise weight functions on* $([a,b];\mathbb{T})$, with the meaning that each w_i is C^{n-i} and positive separately on each $[t_k^+, t_{k+1}^-]$. With such a system we can associate linear piecewise generalised derivatives $L_0, \ldots, L_n$ via the procedure already recalled in (1). By $ECP(w_0, \ldots, w_n)$ we denote the set of all piecewise functions which are piecewise C^n on $([a,b];\mathbb{T})$ and such that $L_n F$ is constant on $[t_k^+, t_{k+1}^-]$ for $k = 0, \ldots q$, with the additional requirement that

$$L_i F(t_k^+) = L_i F(t_k^-) \text{ for } i = 0, \ldots, n, \text{ and for } k = 1, \ldots, q.$$

A piecewise version of Rolle's theorem Mazure (2006) proves that the space $ECP(w_0, \ldots, w_n)$ is an $(n+1)$-dimensional ECP-space on $([a,b];\mathbb{T})$. It was proved that (Mazure, 2007)

Theorem 1.2. *The class of all $(n+1)$-dimensional ECP-spaces on $([a,b];\mathbb{T})$ coincides with the class of all spaces of the form $ECP(w_0, \ldots, w_n)$, where $(w_0, \ldots, w_n)$ is any system of piecewise weight functions on $([a,b];\mathbb{T})$.*

For any real numbers $p, q > n-1$, the space $\mathbb{E}_{p,q}$ spanned on $[a,b] = [0,1]$ by the $(n+1)$ functions $1, x, \ldots, x^{n-2}, x^p, (1-x)^q$, is not an EC-space on $[0,1]$ — except for $p = q = n$ where it is the degree n polynomial space on $[0,1]$. Nevertheless, this $(n+1)$-dimensional space behaves similarly to EC-spaces, and the parameters p, q act as useful tension parameters for spline interpolation as well as for design (Costantini, 1986, 2000; Kaklis and Pandelis, 1990; Kaklis and Sapidis, 1995; Goodman and Mazure, 2001; Costantini et al., 2005; Mazure, 2011d; Bosner and Rogina, 2013). These spaces belong to a larger class of spaces, named *Quasi-Extended Chebyshev-spaces* (for short QEC-spaces), see Mazure (2008b). An $(n+1)$-dimensional space $\mathbb{E} \subset C^{n-1}([a,b])$ is a QEC-space on $[a,b]$ if any non-zero element of $\mathbb{E}$ vanishes at most n times on $[a,b]$, counting multiplicities as far as possible for C^{n-1} functions, that is, up to n, or, equivalently, if any Hermite interpolation problem in $(n+1)$ data $[a,b]$ which is not a Taylor interpolation problem has a unique solution in $\mathbb{E}$. An $(n+1)$-dimensional space $\mathbb{E} \subset C^0([a,b])$ is a *Chebyshev space* (C-space) *on* $[a,b]$ if any non-zero $F \in \mathbb{E}$ vanishes at most once in $[a,b]$, not counting multiplicities. In dimension two, being a QEC-space on $[a,b]$ is therefore the same as being a QEC-space on $[a,b]$. Select any two-dimensional C-space $\mathbb{C}$ on $[a,b]$, containing constants, and any system $(w_0, \ldots, w_{n-1})$ of weight functions on $[a,b]$, with associated generalised derivatives $L_0, \ldots, L_{n-1}$. Denote by $QEC(w_0, \ldots, w_{n-1}; \mathbb{C})$ the set composed of all functions $F \in C^{n-1}([a,b])$ for which $L_{n-1}F \in \mathbb{C}$. The class of all QEC-spaces on $[a,b]$ was clearly identified in Mazure (2011c) where Theorem 1.1 was extended to QEC-spaces as follows:

Theorem 1.3. *The class of all $(n+1)$-dimensional QEC-spaces on $[a,b]$ coincides with the class of all spaces of the form $QEC(w_0, \ldots, w_{n-1}; \mathbb{C})$, where $(w_0, \ldots, w_{n-1})$ is any system of weight functions on $[a,b]$ and where $\mathbb{C}$ is any two-dimensional C-space on $[a,b]$ containing constants.*

If only for the sake of theoretical completeness, the brief record above makes it natural to address the following question: is it possible to establish a result similar to Theorems 1.2 and 1.3 for $(n+1)$-dimensional *Quasi Extended Piecewise spaces* (for short, QECP-spaces) obtained by connecting in an appropriate way $(n+1)$-dimensional QEC-spaces on adjacent intervals. This article gives a positive answer to this question. We will indeed prove that

Theorem 1.4. *The class of all $(n+1)$-dimensional QECP-spaces on $([a,b];\mathbb{T})$ coincides with the class of all spaces of the form $QECP(w_0,\dots,w_{n-1};\mathbb{C})$, where $(w_0,\dots,w_{n-1})$ is any system of piecewise weight functions on $([a,b];\mathbb{T})$ and where $\mathbb{C}$ is any two-dimensional C-space on $[a,b]$ containing constants.*

The motivation for this work is also on the practical side. Indeed, in several previous situations (see, for instance, Laurent et al., 1997; Mazure, 2011d) we have observed that allowing zero multiplicities in spline spaces can produce powerful shape effects. This will be permitted by Theorem 1.4 for splines based on QECP-spaces, which is a priori not obvious at first with QEC-section-spaces.

In Section 3 the reader will learn how to build and how to handle $(n+1)$-dimensional *Piecewise QEC-spaces (PQEC-spaces)* obtained by connecting different $(n+1)$-dimensional QEC-section-spaces. Among all such PQEC-spaces, in Section 4 we distinguish those which meet the same requirement on the number of zeroes as QEC-spaces and we call them QECP-spaces. We show that QECP-spaces really behave as QEC-spaces, and that one can build such spaces with the help of systems of piecewise weight functions. Section 5 is devoted to QECP-spaces *good for design*, in which *ready-to-blossom bases* exist and make the existence of *blossoms* easily perceived. As usual, the crucial and tricky point is to prove that blossoms are *pseudoaffine*. As was already the case for Theorems 1.1, 1.2, and 1.3, pseudoaffinity is a key-point for Theorem 1.4. The proof of this theorem is given in Section 6, and some consequences are developed. The results are illustrated in Section 7 with QECP-spaces built from sections of the form $\mathbb{E}_{p,q}$ (up to translations).

The present work was made possible by many earlier results by the same author, in particular on QEC-spaces and on ECP-spaces. Out of necessity, this article cannot be self-contained. However in order to facilitate the reading a brief commented background is given is Section 2 below.

2. Vocabulary and technical comments

From the next section we will be in a piecewise context. In order to facilitate the reading we have gathered here some of the most important definitions we will deal with, introducing them in the non-piecewise context, and possibly commenting on the extensions to the piecewise framework.

2.1. (Piecewise) weight functions

A system $(w_0,\dots,w_n)$ of weight functions on $[a,b]$ (see Section 1) generates a nested sequence of EC-spaces on $[a,b]$

$$EC(w_0)\subset EC(w_0,w_1)\subset\cdots\subset EC(w_0,w_1,\dots,w_n).$$

Take any non-zero $U_0\in EC(w_0)$, and, for $i=1,\dots,n$, any $U_i\in EC(w_0,\dots,w_i)\setminus EC(w_0,\dots,w_{i-1})$. Then, the functions $U_0,U_1,\dots,U_n$ form a *Complete W-sequence* (for short, CW-sequence) *on* $[a,b]$ in the sense that their successive Wronskians never vanish on $[a,b]$, *i.e.*,

$$W(U_0,\dots,U_i)(x):=\det\left(U_k{}^{(j)}(x)\right)_{0\leqslant k,j\leqslant i}\neq 0 \text{ for all } x\in[a,b],\quad i=0,\dots,n. \tag{3}$$

Conversely, let $(n+1)$ functions $U_0,U_1,\dots,U_n\in C^n([a,b])$ form a CW-sequence on $[a,b]$, in the sense of (3). Then, for any $F\in C^n(I)$ we can set

$$L_iF=\frac{W(U_0,\dots,U_{i-1},F)}{W(U_0,\dots,U_{i-1},U_i)},\quad 0\leqslant i\leqslant n. \tag{4}$$

Clearly, each L_i is a linear differential operator of order i. By differentiation, we obtain (Mazure, 2006)

$$DL_iF=\frac{W(U_0,\dots,U_{i-1})\,W(U_0,\dots,U_{i-1},U_i,F)}{W(U_0,\dots,U_i)^2}. \tag{5}$$

Accordingly, the operators $L_0,L_1,\dots,L_n$ satisfy (1) provided that we define the functions $w_0,\dots,w_n$ by

$$w_i:=\frac{W(U_0,\dots,U_{i-2})\,W(U_0,\dots,U_i)}{W(U_0,\dots,U_{i-1})^2},\quad 0\leqslant i\leqslant n, \tag{6}$$

with the convention that $W(\emptyset)=\mathbb{1}$. The sequence $(w_0,\dots,w_n)$ will be a system of weight functions on $[a,b]$ if all Wonskians (3) are positive. If it is not the case, we simply have to replace U_i by $-U_i$ whenever needed. What the reader should retain is that generalised derivatives can be associated with systems of weight functions or with CW-sequences as well.

In the presence of a sequence $\mathbb{T}$ of interior knots, we will make use of *piecewise functions on* $([a,b];\mathbb{T})$. Stricto sensu, a piecewise function F on $([a,b];\mathbb{T})$ should be defined as a sequence $F=(F_0,\dots,F_q)$, where for $k=0,\dots,q$, F_k is a function on $[t_k,t_{k+1}]$. It will be convenient to proceed similarly to the non-piecewise case, saying that F is defined separately on each $[t_k^+,t_{k+1}^-]$, the equality $F=G$ between two piecewise functions F,G on $([a,b];\mathbb{T})$ meaning that $F(x)=G(x)$ for all $x\in[a,b]\setminus\{t_1,\dots,t_n\}$, and that both $F(t_k^-)=G(t_k^-)$ and $F(t_k^+)=G(t_k^+)$, for $k=1,\dots,q$. This will be summarised by saying that $F(x^\varepsilon)=G(x^\varepsilon)$ for all $x\in[a,b]$, ε having the meaning of both $-$, $+$. One can similarly consider positive piecewise functions on $([a,b];\mathbb{T})$, and so forth. In particular we will denote by $\mathcal{PC}^j([a,b];\mathbb{T})$ the set of all piecewise functions on $([a,b];\mathbb{T})$ which are C^j on each $[t_k^+,t_{k+1}^-]$.

As mentioned in Section 1, we can introduce systems of piecewise weight functions on $([a,b];\mathbb{T})$. One can also define *piecewise CW-sequences* (for short PCW-sequences) *on* $([a,b];\mathbb{T})$, with the meaning of a sequence $(U_0,\dots,U_n)$ in $\mathcal{PC}^n([a,b];\mathbb{T})$ which satisfy all relations (3) in which we replace x by x^ε. When the piecewise functions $U_0,\dots,U_n\in\mathcal{PC}^n([a,b];\mathbb{T})$ satisfy a common connection relation of the form (2), if they form a PCW-system on $([a,b];\mathbb{T})$, then for each $i=0,\dots,n$, the piecewise function $W(U_0,\dots,U_i)$ keeps the same strict sign on all $[t_k^+,t_{k+1}^-]$. After possible multiplications by (-1) they will similarly give birth to a system of piecewise weight functions on $([a,b];\mathbb{T})$ via (6), and therefore to piecewise generalised derivatives. In the piecewise context, the reader should keep in mind that (4) and all similar equalities are then piecewise equalities.

2.2. QEC-spaces for design

Definition 2.1. Given $(n+1)$ functions $B_0,\dots,B_n\in C^{n-1}([a,b])$ we say that $(B_0,\dots,B_n)$ is a *quasi-Bernstein-like basis relative to* (a,b) if these functions satisfy the following properties:

(1) $B_0(a)\neq 0$ and B_0 vanishes n times at b;
(2) for each $i=1,\dots,n-1$, B_i vanishes exactly i times at a and exactly $(n-i)$ times at b;
(3) B_n vanishes n times at a and $B_n(b)\neq 0$.

We say that it is *a positive quasi-Bernstein-like basis relative to* (a,b) when in addition to the previous properties they satisfy

(4) for each $i=0,\dots,n$, B_i is positive on $]a,b[$.

Clearly, a quasi-Bernstein-like basis relative to (a,b) is a basis of the space it spans. Given an $(n+1)$-dimensional space $\mathbb{E}\subset C^{n-1}([a,b])$, the presence of quasi-Bernstein-like bases relative to all couples $(c,d)\in[a,b]^2$, $c<d$, is one of the many possible ways to characterise that this space $\mathbb{E}$ is a QEC-space on $[a,b]$. For other characterisations, readers are mainly referred to Mazure (2008b). If we know that is a QEC-space on $[a,b]$, if needed quasi-Bernstein-like bases can be assumed to be positive (on the concerned interval).

Subsequently, we consider a given $(n+1)$-dimensional space $\mathbb{E}\subset C^{n-1}([a,b])$ containing constants. By $\mathbb{1}$ we mean the constant function $\mathbb{1}(x):=1$ for all $x\in[a,b]$. We will keep the same notation for its restriction to any sub-interval.

Definition 2.2. A basis $(B_0,\dots,B_n)$ in $\mathbb{E}$ is said to be *normalised* if $\sum_{i=0}^n B_i=\mathbb{1}$. A *quasi-Bernstein basis relative to* (a,b) is a positive quasi-Bernstein-like basis relative to (a,b) which is normalised.

The presence in $\mathbb{E}$ of quasi-Bernstein bases relative to all couples $(c,d)\in[a,b]^2$, $c<d$, is necessary and sufficient for the n-dimensional space $D\mathbb{E}$ (obtained from $\mathbb{E}$ via differentiation) to be a QEC-space on $[a,b]$, and it implies that $\mathbb{E}$ itself is a QEC-space on $[a,b]$.

A *mother-function* in $\mathbb{E}$ is a function $\Phi:=(\Phi_1,\dots,\Phi_n)\in\mathbb{E}^n$ such that $(\mathbb{1},\Phi_1,\dots,\Phi_n)$ is a basis of $\mathbb{E}$. In that case, for any $d\geqslant 1$, any $F\in\mathbb{E}^d$ can be uniquely obtained as the image $F=h\circ\Phi$ of Φ under an affine map $h:\mathbb{R}^n\to\mathbb{R}^d$. Given any non-negative integer $i\leqslant n-1$, and any $x\in[a,b]$, the *ith order osculating flat of* Φ *at* x is the affine flat through $\Phi(x)$ with direction spanned by the i derivatives $\Phi^{(j)}(x)$, $j=1,\dots,i$. We denote it by $\mathrm{Osc}_i\,\Phi(x)$. Note that $\mathrm{Osc}_0\,\Phi(x)=\{\Phi(x)\}$.

Definition 2.3. An $(n+1)$-dimensional space $\mathbb{E}\subset C^{n-1}([a,b])$, with $n\geqslant 2$, is said to be *a QEC-space good for design on* $[a,b]$ when $\mathbb{E}$ contains constants and when the n-dimensional space $D\mathbb{E}$ is a QEC-space on $[a,b]$.

Assume $\mathbb{E}$ to be a QEC-space good for design on $[a,b]$. Given any $a_1<\cdots<a_r$ in $[a,b]$ and any positive integers $\mu_1,\dots,\mu_r$ such that $\sum_{i=1}^r\mu_i=n$, one can prove that the r osculating flats $\mathrm{Osc}_{n-\mu_i}\,\Phi(a_i)$, $i=1,\dots,r$, have in common a single point. The *blossom* φ of Φ is the function $\varphi=\big(\varphi_1,\dots,\varphi_n\big):[a,b]^n\to\mathbb{R}^n$ defined by setting

$$\{\varphi(x_1,\dots,x_n)\}:=\bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i}\,\Phi(a_i) \tag{7}$$

whenever $(x_1, \ldots, x_n) = (a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]})$ up to permutation, where $x^{[k]}$ stands for x repeated k times. The function φ is thus symmetric and it gives Φ by restriction to the diagonal of $[a, b]^n$. For any affine map $h : \mathbb{R}^n \to \mathbb{R}^d$, the blossom f of $F := h \circ \Phi$ is then defined as $f := h \circ \varphi$. It is important to mention that it does not depend on the selected mother-function. In particular, *the Bézier points of F relative to* (a, b) are the points $f\left(a^{[n-i]}, b^{[i]}\right)$, $i = 0, \ldots, n$.

It is due to the presence of blossoms and to their properties (see Mazure, 2008b) that the class of all QEC-spaces good for design has been identified as the largest class of spaces (with ordinary differentiation assumptions) which can be used for design. Existence and properties of blossoms will be extended to the piecewise QEC context in Section 5.

3. Piecewise Quasi Extended Chebyshev spaces

In this section we connect different $(n + 1)$-dimensional QEC-spaces on adjacent intervals so as to form an $(n + 1)$-dimensional space on the global interval. The connection relations cannot be of the form (2) since, at a given point, either the nth derivatives do not exist or, if they exist, they may be linearly dependent of the derivatives of lesser orders.

3.1. Connecting QEC-spaces on adjacent intervals

For a while we assume that $\mathbb{E} \subset C^{n-1}([a, b])$ is an $(n + 1)$-dimensional QEC-space on $[a, b]$ and that $a < c < b$. Consider the linear functionals $\ell_0, \ldots, \ell_n, \overline{\ell}_n : \mathbb{E} \to \mathbb{R}$ defined by

$$\ell_i(F) := F^{(i)}(c) \text{ for } i = 0, \ldots, n-1, \quad \ell_n(F) := F(a), \quad \overline{\ell}_n(F) := F(b).$$

Due to $\mathbb{E}$ being a QEC-space on $[a, b]$, the $(n + 1)$ linear functionals $\ell_0, \ldots, \ell_n$ are linearly independent. It results that the linear functional $\overline{\ell}_n$ can be expanded in a unique way as a linear combination of $\ell_0, \ldots, \ell_n$. In other words, there exists a unique $(A_0, \ldots, A_n) \in \mathbb{R}^n$ such that

$$F(b) = \sum_{i=0}^{n-1} A_i F^{(i)}(c) + A_n F(a) \quad \text{for all } F \in \mathbb{E}. \tag{8}$$

Given a basis $(U_0, \ldots, U_n)$ in $\mathbb{E}$, let us set $\mathbf{U} := (U_0, \ldots, U_n)^T$. Due to $\mathbb{E}$ being a QEC-space on $[a, b]$, we know that the determinant

$$\det\left(\mathbf{U}(a_1), \mathbf{U}'(a_1), \ldots, \mathbf{U}^{(\mu_1-1)}(a_1),\ \ldots\ldots\ , \mathbf{U}(a_r), \ldots, \mathbf{U}^{(\mu_r-1)}(a_r)\right)$$

keeps the same strict sign independently of the integer $r \geqslant 2$, of the positive integers $\mu_1, \ldots, \mu_r$, such that $\sum_{i=1}^r \mu_i = n+1$, and of $a \leqslant a_1 < a_2 < \cdots < a_r \leqslant b$ (see Lemma 2.14 in Mazure, 2008b). For any given $x^* \in [a, b]$, the non-zero function $\Psi_n^{x^*}$ defined by

$$\Psi_n^{x^*}(y) := \det\left(\mathbf{U}(x^*), \mathbf{U}'(x^*), \ldots, \mathbf{U}^{(n-1)}(x^*), \mathbf{U}(y)\right), \quad y \in [a, b], \tag{9}$$

belongs to $\mathbb{E}$, it vanishes n times at x^*, and it vanishes nowhere else. As a straightforward consequence of the result recalled above we can say that, if $x^* \in]a, b[$, the (strict) sign of the function $\Psi_n^{x^*}$ changes n times at x^*, *i.e.*,

$$(-1)^n \Psi_n^{x^*}(y_1)\Psi_n^{x^*}(y_2) > 0 \quad \text{for all } y_1, y_2 \text{ such that } a \leqslant y_1 < x^* < y_2 \leqslant b. \tag{10}$$

Accordingly, applying (8) with $F := \Psi_n^c$ proves that $(-1)^n A_n > 0$.

After this preliminary observation, from now on, we work with a fixed integer $n \geqslant 1$ and the following data:

(1) a sequence of interior knots $\mathbb{T} = (t_1, \ldots, t_q)$ with $t_0 := a < t_1 < \cdots < t_q < t_{q+1} := b$;
(2) a corresponding sequence $(M_1, \ldots, M_q)$ of connection matrices: for each $k = 1, \ldots, q$, $M_k = \left(m_{i,j}^k\right)_{0\leqslant i,j\leqslant n}$ is a lower triangular matrix of order $(n + 1)$, of which the diagonal entries $m_{j,j}^k$, $j = 0, \ldots, n$, satisfy

$$m_{j,j}^k > 0 \text{ for } j = 0, \ldots, n-1, \quad (-1)^n m_{n,n}^k > 0; \tag{11}$$

(3) a sequence $(\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_q)$ of section-spaces: for $k = 0, \ldots, q$, $\mathbb{E}_k \subset C^{n-1}([t_k, t_{k+1}])$ is an $(n+1)$-dimensional QEC-space on $[t_k, t_{k+1}]$.

Consider the space $\mathbb{E} \subset \mathcal{P}C^{n-1}([a, b]; \mathbb{T})$ composed of all piecewise functions $F \in \mathcal{P}C^{n-1}([a, b]; \mathbb{T})$ such that:

(i) for $k = 0, \ldots, q$, the restriction of F to $[t_k^+, t_{k+1}^-]$ belongs to $\mathbb{E}_k$;

(ii) for each $k = 1, \dots, q$, F satisfies the connection condition

$$\Big(F(t_k^+), F'(t_k^+), \dots, F^{(n-1)}(t_k^+), F(t_{k+1}^-)\Big)^T = M_k\Big(F(t_k^-), F'(t_k^-), \dots, F^{(n-1)}(t_k^-), F(t_{k-1}^+)\Big)^T. \tag{12}$$

Definition 3.1. The $(n+1)$-dimensional space $\mathbb{E}$ defined by the two conditions (i) and (ii) above is said to be a *piecewise Quasi Extended Chebyshev space* (for short, PQEC-space) *on* $([a,b]; \mathbb{T})$.

The requirement that $(-1)^n m_{n,n}^k > 0$ for $k = 1, \dots, q$ is a demanding one, aimed at eliminating uninteresting cases. It is justified by our preliminary observation which enables to state:

Proposition 3.2. *Any QEC-space on* $[a,b]$ *is a PQEC-space on* $([a,b]; \mathbb{T})$.

Remark 3.3. One can also check that any PEC-space on $([a,b]; \mathbb{T})$ is a PQEC-space on $([a,b]; \mathbb{T})$. Finally, our preliminary observation makes it clear that, by restriction to any non-trivial subinterval, a PQEC-space on $([a,b]; \mathbb{T})$ remains a PQEC-space.

3.2. Operating on PQEC-spaces

In this section we gather the main technical points about PQEC-spaces which will be involved in the next sections. They concern in particular the connection matrices, and how they are transformed via standard operations.

3.2.1. Constants

The special case where $\mathbb{E}$ contains constants is crucial for design.

Lemma 3.4. *The PQEC-space* $\mathbb{E}$ *contains constants if and only if the following two properties are satisfied*

(1) *for* $k = 0, \dots, q$, *the section-space* $\mathbb{E}_k$ *contains constants;*
(2) *for* $k = 1, \dots, q$, *the first column of* M_k *is equal to* $(1, 0, \dots, 0, 1 - m_{n,n}^k)^T$.

Assuming that the PQEC-space $\mathbb{E}$ *contains constants, then* $\mathbb{E} \subset C^0([a,b])$ *and for* $k = 1, \dots, q$, *the connection condition (12) can be replaced by*

$$\Big(F(t_k), F'(t_k^+), \dots, F^{(n-1)}(t_k^+), F(t_{k+1}) - F(t_k)\Big)^T = N_k\Big(F(t_k), F'(t_k^-), \dots, F^{(n-1)}(t_k^-), F(t_k) - F(t_{k-1})\Big)^T, \tag{13}$$

where the n first rows of N_k *coincide with those of* M_k, *the last diagonal entry of* N_k *is equal to* $-m_{n,n}^k$ *(hence it has the same strict sign as* $(-1)^{n-1}$*), and its first column to* $(1, 0, \dots, 0)$.

Proof. The new connection condition (13) is obtained by applying (12) to constants. Details are left to the reader. □

Below we state an important consequence of the previous lemma.

Proposition 3.5. *A two-dimensional PQEC-space on* $([a,b]; \mathbb{T})$ *which contains constants is a C-space on* $[a,b]$.

Proof. Let $\mathbb{E}$ be a two-dimensional PQEC-space on $([a,b]; \mathbb{T})$ containing constants. We thus know that $\mathbb{E} \subset C^0([a,b])$ and from (13) we can state the existence of positive $m_1, \dots, m_q$ such that any element $F \in \mathbb{E}$ satisfies

$$F(t_{k+1}) - F(t_k) = m_k\big(F(t_k) - F(t_{k-1})\big), \quad k = 1, \dots, q. \tag{14}$$

For each $k = 0, \dots, q$, the section-space $\mathbb{E}_k$ is a two-dimensional C-space on $[t_k, t_{k+1}]$ which contains constants. Accordingly, for each $k = 0, \dots, q$, the restriction $F_k \in \mathbb{E}_k$ of F to $[t_k, t_{k+1}]$ is either constant or strictly monotone on $[t_k, t_{k+1}]$. Therefore, due to (14), the function F is constant (resp. strictly increasing, strictly decreasing) on $[a,b]$ if and only if it is constant (resp. strictly increasing, strictly decreasing) on one of the intervals $[t_k, t_{k+1}]$, which proves the claimed result. □

3.2.2. Differentiation

Lemma 3.6. *Assume that $n \geq 2$, that the $(n+1)$-dimensional PQEC-space $\mathbb{E}$ contains constants and that, for $k = 0, \ldots, q$, the section-space $\mathbb{E}_k$ is good for design on $[t_k, t_{k+1}]$. Then the n-dimensional space $D\mathbb{E}$ is a PQEC-space on $([a,b]; \mathbb{T})$.*

Proof. Two things are to be checked: firstly, that the section-spaces in $D\mathbb{E}$ are QEC-spaces; secondly that they are connected by convenient matrices. The first point is guaranteed by the assumption that, for $k = 0, \ldots, q$, the section-space $\mathbb{E}_k$ is good for design on $[t_k, t_{k+1}]$. Take $F \in \mathbb{E}$. The latter assumption also guarantees the existence of blossoms in each $\mathbb{E}_k$ (see the reminder in Section 2.2). For $k = 0, \ldots, q$, we can therefore consider the Bézier points relative to (t_k, t_{k+1}) of the restriction $F_k \in \mathbb{E}_k$ of F to $[t_k, t_{k+1}]$. Denote them by $P_0^k, \ldots, P_n^k$.

We know that the function F satisfies the connection conditions (13), in which we can replace $F(t_{k+1}) - F(t_k)$ by $P_n^k - P_0^k$ and $F(t_k) - F(t_{k-1})$ by $P_n^{k-1} - P_0^{k-1}$. The last line in the connection condition (13) can be written as

$$\left[P_n^k - P_{n-1}^k\right] + \left[P_{n-1}^k - P_0^k\right] = \sum_{i=1}^{n-1} m_{n,i}^k F^{(i)}(t_k^-) - m_{n,n}^k\left(\left[P_n^{k-1} - P_1^{k-1}\right] + \left[P_1^{k-1} - P_0^{k-1}\right]\right),$$

with $(-1)^n m_{n,n}^k > 0$. Denoting by $f_k : [t_k, t_{k+1}]^n \to \mathbb{R}$ the blossom of the function $F_k \in \mathbb{E}_k$, we know that

$$P_{n-1}^k = f_k\left(t_k, {t_{k+1}}^{[n-1]}\right), \quad P_1^{k-1} = f_{k-1}\left({t_{k-1}}^{[n-1]}, t_k\right).$$

The value at $\left({t_{k-1}}^{[n-1]}, t_k\right)$ of the blossom of a mother function in $\mathbb{E}_{k-1}$ is the unique point belonging both to its osculating flat of order $(n-1)$ at t_k and to its osculating flat of order 1 at t_{k-1}. The value at $\left(t_k, {t_{k+1}}^{[n-1]}\right)$ of the blossom of a mother function in $\mathbb{E}_k$ is the unique point belonging both to its osculating flat of order $(n-1)$ at t_k and to its osculating flat of order 1 at t_{k+1}. This ensures the existence of real numbers λ_{k+1}^-, λ_{k-1}^+, $\mu_{k,1}^-, \ldots, \mu_{k,n-1}^-$, $\mu_{k,1}^+, \ldots, \mu_{k,n-1}^+$, independent of F, such that

$$\begin{aligned} &\left[P_n^k - P_{n-1}^k\right] = \lambda_{k+1}^- F'(t_{k+1}^-), \quad \left[P_1^{k-1} - P_0^{k-1}\right] = \lambda_{k-1}^+ F'(t_{k-1}^+); \\ &\left[P_{n-1}^k - P_0^k\right] = \sum_{i=1}^{n-1} \mu_{k,i}^+ F^{(i)}(t_k^+), \quad \left[P_n^{k-1} - P_1^{k-1}\right] = \sum_{i=1}^{n-1} \mu_{k,i}^- F^{(n-1)}(t_k^-). \end{aligned} \tag{15}$$

One can prove that the two real numbers λ_{k+1}^- and λ_{k-1}^+ are positive (see for instance Lemma 3.21 in Mazure, 2008b). Combining the previous relations with the first n lines of (13) leads to

$$F'(t_{k+1}^-) = \sum_{i=1}^{(n-1)} \overline{m}_{n-1,i}^k F^{(i)}(t_k^-) + \overline{m}_{n-1,n-1}^k F'(t_{k-1}^+), \quad \text{with } \overline{m}_{n-1,n-1}^k = -m_{n,n}^k \frac{\lambda_{k-1}^+}{\lambda_{k+1}^-} > 0, \tag{16}$$

where all coefficients $m_{n,i}^k$ are independent of F. The connection matrix at t_k in the space $D\mathbb{E}$ is obtained by deleting the first row and the first column in N_k and then replacing the last line by (16). Since $(-1)^{n-1}\overline{m}_{n-1,n-1}^k > 0$, it satisfies the appropriate requirements for the space $D\mathbb{E}$ to be a PQEC-space on $([a,b]; \mathbb{T})$. $\square$

3.2.3. Integration

Lemma 3.7. *Given an $(n+1)$-dimensional PQEC-space $\mathbb{E}$ on $([a,b]; \mathbb{T})$, let $\widehat{\mathbb{E}}$ denote the space of all continuous functions $\widehat{F}$ on $[a,b]$ which are piecewise differentiable on $([a,b]; \mathbb{T})$ and such that $D\widehat{F} \in \mathbb{E}$. This is an $(n+2)$-dimensional PQEC-space on $([a,b]; \mathbb{T})$ containing constants.*

Proof. For each $k = 1, \ldots, q$, the connection condition (12) in $\mathbb{E}$ can now rewritten as a connection on the (left/right) derivatives in $\widehat{\mathbb{E}}$:

$$\left(\widehat{F}'(t_k^+), \ldots, \widehat{F}^{(n)}(t_k^+), \widehat{F}'(t_{k+1}^-)\right)^T = M_k\left(\widehat{F}'(t_k^-), \ldots, \widehat{F}^{(n)}(t_k^-), \widehat{F}'(t_{k-1}^+)\right)^T. \tag{17}$$

Clearly, the section-spaces in $\widehat{\mathbb{E}}$ are QEC-spaces good for design on their intervals, and $\widehat{\mathbb{E}}$ contains constants. Accordingly, we can proceed exactly the reverse way as we did in the previous subsection, going this time from derivatives to Bézier points in each section, *e.g.*, via relations of the form

$$\widehat{F}'(t_{k+1}^-) = \frac{1}{\widehat{\lambda}_{k+1}^-}\left[\widehat{P}_n^k - \widehat{P}_{n-1}^k\right], \quad \widehat{F}'(t_{k-1}^+) = \frac{1}{\widehat{\lambda}_{k-1}^+}\left[\widehat{P}_1^{k-1} - \widehat{P}_0^{k-1}\right],$$

with positive real numbers $\widehat{\lambda}_{k+1}^-$, $\widehat{\lambda}_{k-1}^+$. Given that $\widehat{P}_n^k = \widehat{F}(t_{k+1})$, $\widehat{P}_0^{k-1} = \widehat{F}(t_{k-1})$ and $\widehat{P}_0^k = \widehat{P}_n^{k-1} = F(t_k)$, these reverse arguments will finally provide us with a connection condition of the form

$$\widehat{F}(t_{k+1}) = \sum_{i=0}^{n} \widehat{m}_{n+1,i}\widehat{F}^{(i)}(t_k^-) + \widehat{m}^k_{n+1,n+1}\widehat{F}(t_{k-1}),$$

with $(-1)^{n+1}\widehat{m}^k_{n+1,n+1} > 0$. We can then transform (17) into a convenient connection condition at t_k with a lower triangular matrix $\widehat{M}_k$ of order $(n+2)$ obtained from M_k by adding $(1, 0, \dots, 0)$ and $(1, 0, \dots, \widehat{m}_{n+1,0})^T$ as its first row and column, respectively. □

3.2.4. Strict signed piecewise multiplication

Lemma 3.8. *Assume that $\mathbb{E}$ is an $(n+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$. For each piecewise function $\omega \in \mathcal{PC}^{n-1}([a,b];\mathbb{T})$ which has the same strict sign on all $[t_k^+, t_{k+1}^-]$, the space $\omega\mathbb{E}$ obtained by piecewise multiplication of all elements of $\mathbb{E}$ by ω is an $(n+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$.*

Proof. The section spaces in $\omega\mathbb{E}$ are QEC-spaces on their intervals. Therefore, for each $k = 1, \dots, q$, there is indeed a lower triangular connection matrix $\widetilde{M}_k$ of order $(n+1)$ in $\omega\mathbb{E}$ such that, for each $F \in \mathbb{E}$

$$\begin{aligned}&\Big((\omega F)(t_k^+), (\omega F)'(t_k^+), \dots, (\omega F)^{(n-1)}(t_k^+), (\omega F)(t_{k+1}^-)\Big)^T\\ &\quad= \widetilde{M}_k\Big((\omega F)(t_k^-), (\omega F)'(t_k^-), \dots, (\omega F)^{(n-1)}(t_k^-), (\omega F)(t_{k-1}^+)\Big)^T.\end{aligned} \tag{18}$$

Let $M_k^\sharp$ and $\widetilde{M}_k^\sharp$ be the lower triangular matrices obtained by deleting the last row and column in M_k and $\widetilde{M}_k$, respectively. Applying the Leibniz formula yields

$$\widetilde{M}_k^\sharp := \mathcal{C}_{n-1}(\omega, t_k^+)\, M_k^\sharp\, \mathcal{C}_{n-1}(\omega, t_k^-)^{-1}, \quad k = 1, \dots, q, \tag{19}$$

where, for $x \in [a,b]$ and $\varepsilon \in \{-,+\}$, $\mathcal{C}_{n-1}(\omega, x^\varepsilon) = \big(\mathcal{C}_{n-1}(\omega, x^\varepsilon)_{p,q}\big)_{0\leqslant p,q\leqslant n-1}$ stands for the lower triangular square matrix of order n defined by (see Lemma 40 in Mazure, 2006)

$$\mathcal{C}_{n-1}(\omega, x^\varepsilon)_{p,q} := \tbinom{p}{q}\, \omega^{(p-q)}(x^\varepsilon), \quad \text{for } 0 \leqslant q \leqslant p \leqslant n-1.$$

Taking additionally account of the last line in (18), one can see that the diagonal entries of the matrix $\widetilde{M}_k$ are given by

$$\widetilde{m}^k_{j,j} = \frac{\omega(t_k^+)}{\omega(t_k^-)}\, m^k_{j,j} \text{ for } j = 0, \dots, n-1, \quad \widetilde{m}^k_{n,n} = \frac{\omega(t_{k+1}^-)}{\omega(t_{k-1}^+)}\, m^k_{n,n}.$$

Since ω keeps the same strict sign on all $[t_k^+, t_{k+1}^-]$, the signs on the diagonal of $\widetilde{M}_k$ are the same as on the diagonal of M_k. The proof is thus complete. □

3.2.5. Generalised differentiation

Combining the last two operations, we can describe the behaviour of PEQC-spaces under suitable generalised differentiation.

Lemma 3.9. *Assume that $n \geq 2$ and that $\mathbb{E}$ is an $(n+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$. Given $w \in \mathbb{E}$, we assume that, for $k = 0, \dots, q$, its restriction w_k to $[t_k^+, t_{k+1}^-]$ has positive coordinates in any positive quasi-Bernstein-like basis of $\mathbb{E}_k$ relative to (t_k, t_{k+1}). Then, the piecewise function w is positive on each $[t_k^+, t_{k+1}^-]$, and if L_0 denotes the piecewise division by w, the set $DL_0\mathbb{E}$ is an n-dimensional PQEC-space on $([a,b];\mathbb{T})$.*

Proof. Take any $k = 0, \dots, q$. That w_k has positive coordinates in any positive quasi-Bernstein-like basis of $\mathbb{E}_k$ relative to (t_k, t_{k+1}) is necessary and sufficient to ensure both that w_k is positive on $[t_k^+, t_{k+1}^-]$, and that the space $L_0^k\mathbb{E}_k$ obtained by dividing all elements of $\mathbb{E}_k$ by w_k is a QEC-space good for design on $[t_k, t_{k+1}]$ (see Mazure, 2011c).

From Lemma 3.8 we know that $L_0\mathbb{E}$ is a PEQC-space on $([a,b];\mathbb{T})$. Moreover, $L_0\mathbb{E}$ clearly contains constants because it is obtained by piecewise division by w which belongs to $\mathbb{E}$. Since each $L_0^k\mathbb{E}_k$ is a QEC-space good for design on $[t_k, t_{k+1}]$, it follows from Lemma 3.6 that the n-dimensional space $DL_0\mathbb{E}$ is PQEC-space on $([a,b];\mathbb{T})$. □

4. Quasi Extended Chebyshev Piecewise spaces

In this section, we work with a given $(n+1)$-dimensional PQEC-space $\mathbb{E}$ on $([a,b];\mathbb{T})$, with $n \geqslant 1$, defined by the sequence $\mathbb{T} = (t_1, \dots, t_q)$ of interior knots in $[a,b]$, connection matrices $M_1, \dots, M_q$ satisfying (11), and QEC section-spaces $\mathbb{E}_0, \dots, \mathbb{E}_q$. The results developed here will generalise those proved in earlier articles both on QEC-spaces and on ECP-spaces, in particular Mazure (2005b, 2006, 2008b).

4.1. Definition and first characterisations

Select any non-zero piecewise function $F \in \mathbb{E}$. For $k = 1, \dots, q$, and for $0 \leqslant p \leqslant n$, F vanishes p times at t_k^- (*i.e.*, $F(t_k^-) = F'(t_k^-) = \cdots = F^{(p-1)}(t_k^-) = 0$) if and only if it vanishes p times at t_k^+ (*i.e.*, $F(t_k^+) = F'(t_k^+) = \cdots = F^{(p-1)}(t_k^+) = 0$). This is due to the connection matrix M_k being lower triangular and regular. In that case we will simply say that F vanishes p times at t_k. If $p \leqslant n-1$, F vanishes exactly p times at t_k^- (*i.e.*, it vanishes p times, but not $(p+1)$ times at t_k^-) if and only if it vanishes exactly p times at t_k^+. These remarks make it possible to count the total number of zeroes of F on $[a,b]$, including multiplicities up to n, which we denote as $Z_n^{[a,b]}(F)$. Given $k = 0, \dots, q$, since the section-space $\mathbb{E}_k$ is a QEC-space on $[t_k, t_{k+1}]$, we know that $Z_n^{[t_k,t_{k+1}]}(F) \leqslant n$.

Definition 4.1. An $(n+1)$-dimensional PQEC-space $\mathbb{E}$ on $([a,b]; \mathbb{T})$ is said to be a *Quasi Extended Chebyshev Piecewise space* (for short, QECP-space) *on* $([a,b]; \mathbb{T})$ when, for any non-zero element of $F \in \mathbb{E}$, the total number $Z_n^{[a,b]}(F)$ of zeroes of F on $[a,b]$, multiplicities included up to n, is bounded above by n.

Remark 4.2. Given a basis $(U_0, \dots, U_n)$ in the PQEC-space $\mathbb{E}$ on $([a,b]; \mathbb{T})$, set $\mathbf{U} := (U_0, \dots, U_n)^T$. Then, obviously, $\mathbb{E}$ is a QECP-space on $([a,b]; \mathbb{T})$ when any of the following equivalent properties below are satisfied

(1) any Hermite interpolation problem

$$\text{find } F \in \mathbb{E} \text{ such that } F^{(j)}(\tau_i^{\varepsilon_i}) = \alpha_{i,j}\,, \quad 1 \leqslant i \leqslant r,\ 0 \leqslant j \leqslant \mu_i - 1\,, \tag{20}$$

– in which r is any integer greater than or equal to 2, $\mu_1, \dots, \mu_r$ are positive integers such that $\sum_{i=1}^r \mu_i = n+1$, $\tau_1, \dots, \tau_r \in [a,b]$ are pairwise distinct, $\alpha_{i,j}$, $1 \leqslant i \leqslant r$, $0 \leqslant j \leqslant \mu_i - 1$ are given real numbers, and where $\varepsilon_1, \dots, \varepsilon_r \in \{-,+\}$ – has a unique solution.

(2) with any data as in (1), the $(n+1)$ vectors $\mathbf{U}(\tau_i^{\varepsilon_i}), \mathbf{U}'(\tau_i^{\varepsilon_i}), \dots, \mathbf{U}^{(\mu_i-1)}(\tau_i^{\varepsilon_i})$, $i = 1, \dots, r$, are linearly independent.

Remark 4.3. In the $(n+1)$-dimensional PQEC-space $\mathbb{E}$, for any $x^* \in [a,b]$ and any j, $0 \leqslant j \leqslant n-1$, it is possible to select a piecewise function which vanishes exactly j times at x^*. We systematically denote such a function as $\Psi_j^{x^*}$. Similarly, the notation $\Psi_n^{x^*}$ will stand for a non-zero element of $\mathbb{E}$ which vanishes n times at x^*. Such an element is unique up to multiplication by a non-zero constant.

In spite of the piecewise nature of the elements of the PQEC-space $\mathbb{E}$, the fact that we can count the zeroes makes it possible to keep the same definition for quasi-Bernstein-like bases in $\mathbb{E}$ as in QEC-spaces (see Definition 2.1).

Theorem 4.4. *Given $n \geqslant 1$ and given an $(n+1)$-dimensional PQEC-space $\mathbb{E}$ on $([a,b]; \mathbb{T})$, the following six properties are equivalent:*

(i) *$\mathbb{E}$ is a QECP-space on $([a,b]; \mathbb{T})$;*
(ii) *any Hermite interpolation problem (20) based on* exactly two points *of $[a,b]$ has a unique solution in $\mathbb{E}$;*
(iii) *given any $x_1, x_2 \in [a,b]$, $x_1 \neq x_2$, any non-zero element F of $\mathbb{E}$ vanishes at most n times on $\{x_1, x_2\}$, counting multiplicities up to order n, i.e., $Z_n^{\{x_1,x_2\}}(F) \leqslant n$;*
(iv) *for any positive integers i_1, i_2 such that $i_1 + i_2 = n+1$, any $x_1, x_2 \in [a,b]$, $x_1 \neq x_2$, and any $\varepsilon_1, \varepsilon_2 \in \{+,-\}$, the $(n+1)$ vectors $\mathbf{U}(x_1^{\varepsilon_1}), \dots, \mathbf{U}^{(i_1-1)}(x_1^{\varepsilon_1}), \mathbf{U}(x_2^{\varepsilon_2}), \dots, \mathbf{U}^{(i_2-1)}(x_2^{\varepsilon_2})$ are linearly independent;*
(v) *for any $(c,d) \in [a,b]^2$, $c < d$, $\mathbb{E}$ possesses a quasi-Bernstein-like basis relative to (c,d);*
(vi) *for any $c \in [a,b]$, and any integer p, $1 \leqslant p \leqslant n$,*

$$W(\Psi_n^c, \dots, \Psi_p^c)(x^\varepsilon) \neq 0 \quad \textit{for any } x \in [a,b] \setminus \{c\}. \tag{21}$$

Proof. This is a strong result which says that, in the PQEC-space $\mathbb{E}$, the unisolvence of any Hermite interpolation problem (20) is obtained as soon as it is obtained for any Hermite interpolation problem based on exactly two points in $[a,b]$; equivalently the bound on the number of zeroes is obtained by bounding zeroes at any pair of points. There is hardly any difference in the proofs of the equivalences stated above between the present piecewise situation and the non-piecewise one addressed in Mazure (2008b) to which we refer the reader to. □

4.2. QECP-spaces and piecewise weight functions

In Goodman and Mazure (2001) we showed how to build QEC-spaces from systems of weight functions. The theorem below is the piecewise version of this construction.

Theorem 4.5. *Let $(w_0, \dots, w_{n-1})$ be a system of piecewise weight functions on $([a,b]; \mathbb{T})$, with associated piecewise generalised derivatives denoted by $L_0, \dots, L_{n-1}$, and let $\mathbb{C}$ be a two-dimensional C-space on $[a,b]$ containing constants. Let $QECP(w_0, \dots, w_{n-1}; \mathbb{C}) \subset \mathcal{P}C^{n-1}([a,b]; \mathbb{T})$ be composed of all piecewise functions $F \in \mathcal{P}C^{n-1}([a,b]; \mathbb{T})$ meeting the following two requirements:*

$$\begin{array}{ll}
\textbf{step 2n – 1:} & QECP(w_0, w_1, \dots, w_{n-1}; \mathbb{C}) = (n+1)\text{-dim. QECP-space on } ([a,b];\mathbb{T}) \\
& \quad \textit{multiplication} \uparrow \textit{by } w_0 \\
\textbf{step 2n – 2:} & QECP(\mathbb{1}, w_1, \dots, w_{n-1}; \mathbb{C}) = (n+1)\text{-dim. QECP-space on } ([a,b];\mathbb{T}) \text{ containing constants} \\
& \quad \textit{continuous} \uparrow \textit{integration} \\
& \quad \vdots \\
\textbf{step 3:} & QECP(w_{n-2}, w_{n-1}; \mathbb{C}) = 3\text{-dim. QECP-space on } ([a,b];\mathbb{T}) \\
& \quad \textit{multiplication} \uparrow \textit{by } w_{n-2} \\
\textbf{step 2:} & QECP(\mathbb{1}, w_{n-1}; \mathbb{C}) = 3\text{-dim. QECP-space on } ([a,b];\mathbb{T}) \text{ containing constants} \\
& \quad \textit{continuous} \uparrow \textit{integration} \\
\textbf{step 1:} & QECP(w_{n-1}; \mathbb{C}) = 2\text{-dim. QECP-space on } ([a,b];\mathbb{T}) \\
& \quad \textit{multiplication} \uparrow \textit{by } w_{n-1} \\
\textbf{step 0:} & \mathbb{C} = 2\text{-dim. C-space on } [a,b] \text{ containing constants}
\end{array}$$

Fig. 1. Constructing QECP-spaces on $([a,b];\mathbb{T})$ with the help of piecewise weight functions.

(i) $L_{n-1}F \in \mathbb{C}$;
(ii) *for* $i = 0, \dots, n-2$, $L_i F$ *is continuous on* $[a,b]$.

Then, the set $QECP(w_0, \dots, w_{n-1}; \mathbb{C})$ *is an* $(n+1)$*-dimensional QECP-space on* $([a,b];\mathbb{T})$.

Proof. For $i = 0, \dots, n$, we can also consider the following sets:

$$\mathbb{E}^{<i>} := QECP(w_{n-1-i}, \dots, w_{n-1}; \mathbb{C}), \quad i = n, \dots, 0, \quad \widehat{\mathbb{E}}^{<i>} = QECP(\mathbb{1}, w_{n-i}, \dots, w_{n-1}; \mathbb{C})$$

so that

$$\widehat{\mathbb{E}}^{<0>} = \mathbb{C}, \quad \mathbb{E}^{<n-1>} = QECP(w_0, \dots, w_{n-1}; \mathbb{C}).$$

As indicated in Fig. 1, the recursive definition of the generalised derivatives implies that

(1) for each i, $0 < i \leqslant n-1$, we go from $\widehat{\mathbb{E}}^{<i>}$ to $\mathbb{E}^{<i>}$ by piecewise multiplication by w_{n-1-i};
(2) for each i, $0 < i \leqslant n-2$, we go from $\mathbb{E}^{<i>}$ to $\widehat{\mathbb{E}}^{<i+1>}$ by continuous integration.

First of all, since $\widehat{\mathbb{E}}^{<0>}$ is a C-space on $[a,b]$, iterated applications of Lemmas 3.8 and 3.7 guarantee that all spaces in Fig. 1 are PQEC-spaces on $([a,b],\mathbb{T})$. Moreover, for $i = 0, \dots, n$, the passage from $\widehat{\mathbb{E}}^{<i>}$ to $\mathbb{E}^{<i>}$ keeps the dimension and the count of zeroes unchanged. For $i = 1, \dots, n-1$, a given function $\widehat{F} \in \widehat{\mathbb{E}}^{<i>}$ is obtained by continuous integration of a piecewise function in $\mathbb{E}^{<i-1>}$. On account of the assumption on the connection matrices we can apply a piecewise version of Rolle's theorem guarantee that (see Lemma 38 in Mazure, 2006)

$$Z_{i+1}^{[a,b]}(\widehat{F}) \leqslant Z_i^{[a,b]}(D\widehat{F}) + 1 \text{ for each non-zero } \widehat{F} \in \widehat{\mathbb{E}}^{<i>}.$$

Starting from $Z_0^{[a,b]}(F) \leqslant 1$ for each non-zero $F \in \mathbb{C}$, by induction one can prove that $Z_n^{[a,b]}(F) \leqslant n$ for any non-zero $F \in \mathbb{E}^{<n-1>}$. The proof is complete. $\square$

The following two results will be crucial in subsequent sections. In particular Theorem 4.6 below is a piecewise version of Theorem 3.10 in Mazure (2008b).

Theorem 4.6. *An* $(n+1)$*-dimensional PQEC-space* $\mathbb{E}$ *on* $([a,b];\mathbb{T})$ *being given, we assume the existence of a system* $(w_0, \dots, w_{n-1})$ *of piecewise weight functions on* $([a,b];\mathbb{T})$ *such that*

$$ECP(w_0, \dots, w_{n-1}) \subset \mathbb{E}, \tag{22}$$

and we denote by $L_0, \dots, L_{n-1}$ *the associated piecewise generalised derivatives. Then,* $\mathbb{C} := L_{n-1}\mathbb{E}$ *is a two-dimensional C-space on* $[a,b]$ *and* $\mathbb{E} = QECP(w_0, \dots, w_{n-1}; \mathbb{C})$.

Proof. Clearly, the only thing that we have to prove is that $\mathbb{C} := L_{n-1}\mathbb{E}$ is a C-space on $[a,b]$. On account of Proposition 3.5, it is sufficient to prove by induction on $i = 0, \dots, n-1$, that $L_i\mathbb{E}$ is an $(n-i+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$ which contains constants.

Select any element U_n in $\mathbb{E} \setminus ECP(w_0, \dots, w_{n-1})$. For $k = 0, \dots, q$, denote by $w_0^k, \dots, w_{n-1}^k, U_n^k$ the restrictions of $w_0, \dots, w_{n-1}, U_n$ to $[t_k^+, t_{k+1}^-]$ and by $L_0^k, \dots, L_{n-1}^k$ the generalised derivatives associated with $(w_0^k, \dots, w_{n-1}^k)$. From (22) we can derive that

$$EC(w_0^k, \dots, w_{n-1}^k) \subset \mathbb{E}_k, \quad k = 0, \dots, q.$$

Since $\mathbb{E}_k$ is a QEC-space on $[t_k, t_{k+1}]$, Theorem 3.10 in Mazure (2008b) implies that $\mathbb{E}_k = QEC(w_0^k, \dots, w_{n-1}^k; \mathbb{C}_k)$, where $\mathbb{C}_k$ is the two-dimensional C-space on $[t_k^+, t_{k+1}^-]$ spanned by $(\mathbb{1}, L_{n-1}^k U_n^k)$. Moreover, we also have

$$L_i^k \mathbb{E}_k = QEC(\mathbb{1}, w_{i+1}^k, \dots, w_{n-1}^k; \mathbb{C}_k) \text{ for } i = 0 \leqslant n-1, \quad k = 0, \dots, q.$$

From Theorem 3.2 of Mazure (2011c) we know that w_{i+1}^k has positive coordinates in any positive quasi-Bernstein-like basis of $DL_i^k\mathbb{E}_k$. The expected result follows by iterated application of Lemma 3.9. □

Proposition 4.7. *Select any $\overline{a} < a$, and denote by $\overline{\mathbb{T}}$ the sequence $(a, \mathbb{T})$. Then, any $(n+1)$-dimensional QECP-space $\mathbb{E}$ on $([a, b]; \mathbb{T})$ which is of the form $\mathbb{E} := QECP(w_0, \dots, w_{n-1}; \mathbb{C})$ can be extended into an $(n+1)$-dimensional QEC-space on $([\overline{a}, b]; \overline{\mathbb{T}})$.*

Proof. Select any two-dimensional C-space $\widetilde{\mathbb{C}} \subset C^0([\overline{a}, a])$ containing constants. The set $\overline{\mathbb{C}}$ composed of all continuous functions on $[\overline{a}, b]$ which belong to $\widetilde{\mathbb{C}}$ and $\mathbb{C}$ by restriction to $[\overline{a}, a]$ and to $[a, b]$, respectively, and which additionally satisfy

$$F(a) - F(\overline{a}) = c\big[F(b) - F(a)\big] \quad \text{for some given positive } c,$$

is a two-dimensional C-space on $[\overline{a}, b]$. Considering any system $(\overline{w}_0, \dots, \overline{w}_{n-1})$ of piecewise weight functions on $[\overline{a}, b]$ which coincides with $(w_0, \dots, w_{n-1})$ on $([a, b]; \mathbb{T})$, the $(n+1)$-dimensional QECP-space $QECP(\overline{w}_0, \dots, \overline{w}_{n-1}; \overline{\mathbb{C}})$ on $\big([\overline{a}, b]; (a, \mathbb{T})\big)$ gives $\mathbb{E}$ by restriction to $([a, b]; \mathbb{T})$. □

4.3. Towards blossoms

In Theorem 4.4 we have given different ways to characterise QECP-spaces on $([a, b]; \mathbb{T})$ among all PQEC-spaces on $([a, b]; \mathbb{T})$. Below, we give an additional one, which will be crucial for blossoms in next section.

Theorem 4.8. *Let $\mathbb{E}$ be an $(n+1)$-dimensional PQEC-space on $([a, b]; \mathbb{T})$. Then, it is a QECP-space on $([a, b]; \mathbb{T})$ if and only if it meets the following requirement:*

(vii) *given any integer $r \geqslant 1$, any pairwise distinct $a_1, \dots, a_r \in [a, b]$, and any positive integers $\mu_1, \dots, \mu_r$,*
  (a) *when $\sum_{i=1}^r \mu_i = n+1$, any sequence of the form $(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \ \dots\dots\ , \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r})$ is a basis of $\mathbb{E}$;*
  (b) *when $\sum_{i=1}^r \mu_i \leqslant n$, then for each $x \in [a, b] \setminus \{a_1, \dots, a_r\}$,*

$$W(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \ \dots\dots\ , \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r})(x^\varepsilon) \neq 0. \tag{23}$$

Proof. The statement in (b) of (vii) with $r = 1$ corresponds to property (vi) of Theorem 4.4. Accordingly, we only have to assume that $\mathbb{E}$ is QECP-space on $([a, b]; \mathbb{T})$, and to prove that (vii) is satisfied. Moreover, given that (vii) is satisfied for $r = 1$, we can additionally assume that the result holds with some integer $r - 1 \geqslant 1$ in any QECP-space. Then, we just have to prove that $\mathbb{E}$ satisfies (vii) with the integer r.

Without loss of generality we can assume that $a_1 < a_2 < \cdots < a_r$. Let $\mathbb{F} \subset \mathbb{E}$ be spanned by the n piecewise functions $\Psi_n^{a_1}, \dots, \Psi_1^{a_1}$. Let the interval $[a^*, b^*]$ be chosen so that $a_1 < a^* < a_2$, $b^* = b$, let $\mathbb{E}^*$, $\mathbb{F}^*$ denote the restrictions of $\mathbb{E}$, $\mathbb{F}$ to $[a^*, b^*]$, respectively, and let $\mathbb{T}^*$ be the sequence of knots which are interior to $[a^*, b^*]$. From (vi) of Theorem 4.4 we know that the sequence $\big(\Psi_n^{a_1}, \dots, \Psi_1^{a_1}\big)$ is a PCW-sequence on $([a^*, b^*]; \mathbb{T}^*)$ (see Section 2.1). Let $\big(w_0^*, \dots, w_{n-1}^*\big)$ be the associated system of piecewise weight functions on $([a^*, b^*]; \mathbb{T}^*)$, given by (6) (up to signs), and let $L_0^*, \dots, L_{n-1}^*$ denote the corresponding piecewise generalised derivatives. We thus have

$$\mathbb{F}^* = ECP\big(w_0^*, \dots, w_{n-1}^*\big) \subset \mathbb{E}^*.$$

Given that $\mathbb{E}^*$ is an $(n+1)$-dimensional QECP-space on $([a^*, b^*]; \mathbb{T}^*)$, by Theorem 4.6 this implies that

$$\mathbb{E}^* = QECP\big(w_0^*, \dots, w_{n-1}^*; \mathbb{C}^*\big) \quad \text{for some convenient C-space } \mathbb{C}^* \text{ on } [a^*, b^*]. \tag{24}$$

From (24) we can derive that

$$\widetilde{\mathbb{E}^*} := DL_{\mu_1-1}^* \mathbb{E}^* = QECP\left(w_{\mu_1}^*, \dots, w_{n-1}^*; \mathbb{C}^*\right).$$

Setting $\widetilde{n} := n - \mu_1$, the space $\widetilde{\mathbb{E}^*}$ is thus an $(\widetilde{n}+1)$-dimensional QECP-space on $([a^*, b^*]; \mathbb{T}^*)$ and the positive integers $\mu_2, \dots, \mu_r$ satisfy $\sum_{i=2}^r \mu_i \leqslant \widetilde{n} + 1$. Therefore, the inductive assumption applies to $\widetilde{\mathbb{E}^*}$. For any $x \in [a^*, b^*]$, the notation $\widetilde{\Psi^*}_k^x$ will stand for any element in $\widetilde{\mathbb{E}^*}$ which vanishes exactly k times at x if $k < \widetilde{n}$, and for any non-zero element in $\widetilde{\mathbb{E}^*}$ vanishing $\widetilde{n}$ times at x if $k = \widetilde{n}$. This enables us to state the following two properties:

(1) if $\sum_{i=1}^r \mu_i = n+1$, any sequence of the form $\left(\widetilde{\Psi^*}_{\widetilde{n}}^{a_2}, \dots, \widetilde{\Psi^*}_{\widetilde{n}-\mu_2+1}^{a_2}, \ \dots\dots\ , \widetilde{\Psi^*}_{\widetilde{n}}^{a_r}, \dots, \widetilde{\Psi^*}_{\widetilde{n}-\mu_r+1}^{a_r}\right)$ is a basis of $\widetilde{\mathbb{E}^*}$;

(2) if $\sum_{i=1}^{r}\mu_i \leqslant n$, then, for all $x \in [a^*, b^*] \setminus \{a_2, \ldots, a_r\}$,

$$W\left(\widetilde{\Psi^*}^{a_2}_{\widetilde{n}}, \ldots, \widetilde{\Psi^*}^{a_2}_{\widetilde{n}-\mu_2+1}, \ \ldots\ldots\ , \widetilde{\Psi^*}^{a_r}_{\widetilde{n}}, \ldots, \widetilde{\Psi^*}^{a_r}_{\widetilde{n}-\mu_r+1}\right)(x^{\varepsilon}) \neq 0. \tag{25}$$

From the piecewise version of (5) we know that $\widetilde{\mathbb{E}}^*$ is composed of all piecewise functions $\widetilde{F^*} \in \mathcal{PC}^{\widetilde{n}-1}([a^*, b^*]; \mathbb{T}^*)$ which are of the form

$$\widetilde{F^*}(x^{\varepsilon}) = \frac{W(\Psi^{a_1}_n, \ldots, \Psi^{a_1}_{n-\mu_1+2})(x^{\varepsilon})W(\Psi^{a_1}_n, \ldots, \Psi^{a_1}_{n-\mu_1+2}, \Psi^{a_1}_{n-\mu_1+1}, F)(x^{\varepsilon})}{\left[W(\Psi^{a_1}_n, \ldots, \Psi^{a_1}_{n-\mu_1+2}, \Psi^{a_1}_{n-\mu_1+1})(x^{\varepsilon})\right]^2}, \quad x \in [a^*, b^*], \tag{26}$$

obtained when F ranges over $\mathbb{E}$. For any $x^* \in [a^*, b^*]$ and any $0 \leqslant k \leqslant \widetilde{n}$, take $F = \Psi^{x^*}_{k+\mu_1}$. The corresponding piecewise function $\widetilde{F^*}$ vanishes exactly k times at x^* if $k < \widetilde{n}$ and when $k = \widetilde{n}$ it is non-zero and it vanishes $\widetilde{n}$ times at x^* (see Lemma 24 in Mazure, 2006). We can thus take this function as $\widetilde{\Psi^*}^{x^*}_k$ in $\widetilde{\mathbb{E}}^*$. From (26), classical relations on Wronskians, and less classical ones (Lemma 20 in Mazure, 2006) it is easy to derive that

(1) if $\sum_{i=1}^{r}\mu_i = n+1$, restricted to $[a^*, b^*]$ the $(n+1)$ piecewise functions $\Psi^{a_i}_n, \ldots, \Psi^{a_i}_{n-\mu_i+1}$, $i = 1, \ldots, r$, form a basis of $\mathbb{E}^*$, and therefore these $(n+1)$ piecewise functions form a basis of $\mathbb{E}$;
(2) if $\sum_{i=1}^{r}\mu_i \leqslant n$, then (23) holds on $[a^*, b^*] \setminus \{a_2, \ldots, a_r\}$. This being true for any $a_1 < a^* < a_2$, it is true also on $]a_1, b] \setminus \{a_2, \ldots, a_r\}$.

Exchanging the roles of a, a_1 and b, a_r would similarly enable us to show that (23) holds on $[a, a_r[\setminus \{a_1, \ldots, a_{r-1}\}$, which completes the proof. □

5. QECP-spaces for design

Being interested in design, in this section we assume that, for each $k = 0, \ldots, q$, $\mathbb{E}_k$ is good for design on $[t_k, t_{k+1}]$, and that the PQEC-space $\mathbb{E}$ contains constants. In particular $\mathbb{E} \subset C^0([a, b])$.

5.1. Blossoms and ready-to-blossom bases

The results presented here extend those previously obtained for EC-spaces (*e.g.*, Mazure, 2005a, 2006), QEC-spaces (Mazure, 2008b), ECP-spaces (*e.g.*, Mazure, 2005b, 2006).

Given any basis of the form $(\mathbb{1}, \Phi_1, \ldots, \Phi_n)$ in $\mathbb{E}$, we consider the associated mother-function $\Phi := (\Phi_1, \ldots, \Phi_n)$. Due to the structure of the connection matrices $M_1, \ldots, M_q$, for any $k = 1, \ldots, q$, and any $i = 1, \ldots, n-1$, the linear space spanned by the right derivatives $\Phi'(t_k^+), \ldots, \Phi^{(i)}(t_k^+)$ coincides with the linear space spanned by the left ones $\Phi'(t_k^-), \ldots, \Phi^{(i)}(t_k^-)$. The left and right osculating flats of order i, that is, $\mathrm{Osc}_i\Phi(t_k^+)$ and $\mathrm{Osc}_i\Phi(t_k^-)$ thus coincide too. We will thus denote both of them by $\mathrm{Osc}_i\Phi(t_k)$. We can therefore define possible blossoms exactly as explained in Section 2.2 for QEC-spaces.

Definition 5.1. Let $\mathbb{E}$ be an $(n+1)$-dimensional PQEC-space on $\big([a, b]; \mathbb{T}\big)$ containing constants. Given any pairwise distinct $a_1, \ldots, a_r \in [a, b]$ and any positive integers $\mu_1, \ldots, \mu_r$ with $\sum_{i=1}^{r}\mu_i = n$, if a sequence $\Big(\mathbb{1}, \Psi^{a_1}_n, \ldots, \Psi^{a_1}_{n-\mu_1+1}, \ldots\ldots, \Psi^{a_r}_n, \ldots, \Psi^{a_r}_{n-\mu_r+1}\Big)$ forms a basis of $\mathbb{E}$ we say that this basis is a *ready-to-blossom basis relative to any n-tuple* $(x_1, \ldots, x_n)$ which is equal to $\big({a_1}^{[\mu_1]}, \ldots, {a_r}^{[\mu_r]}\big)$ up to permutation.

Theorem 5.2. *Let $\mathbb{E} \subset \mathcal{PC}^{n-1}([a, b]; \mathbb{T})$ be an $(n+1)$-dimensional PQEC-space containing constants, with $n \geqslant 2$, and let $\mathbb{U} \subset \mathcal{PC}^{n-2}([a, b]; \mathbb{T})$ be the n-dimensional PQEC-space on $([a, b]; \mathbb{T})$ obtained by differentiation of $\mathbb{E}$ (see Lemma 3.6). The following four properties are equivalent:*

(i) *$\mathbb{U}$ is a QECP-space on $([a, b]; \mathbb{T})$;*
(ii) *given any $c, d \in [a, b]$, with $c < d$, the space $\mathbb{E}$ possesses a normalised quasi-Bernstein-like basis relative to (c, d);*
(iii) *given any $(x_1, \ldots, x_n) \in [a, b]^n$, the space $\mathbb{E}$ possesses a ready-to-blossom basis relative to $(x_1, \ldots, x_n)$;*
(iv) *blossoms exist in the space $\mathbb{E}$.*

Moreover, when any of these properties is satisfied, $\mathbb{E}$ itself is a QECP-space on $([a, b]; \mathbb{T})$.

Proof. The fact that $\mathbb{E}$ is a QECP-space on $([a, b]; \mathbb{T})$ as soon as (i) holds results from the piecewise version of Rolle's theorem already mentioned.

• (i) ⇔ (iii) is obtained by applying Theorem 4.8 in the PQEC-space $\mathbb{U}$.

• (i) ⇒ (ii). Assume that (i) holds, and consider any $c, d \in [a, b]$, with $c < d$. We know that the QECP-space $\mathbb{U}$ possesses a quasi-Bernstein-like basis $(V_0, \ldots, V_{n-1})$ relative to (c, d). Each piecewise function V_i does not vanish on $[a, b] \setminus \{c, d\}$ and therefore it piecewisely keeps the same strict sign on the whole of $]c, d[$, thus ensuring that $\int_c^d V_i(t)dt \neq 0$. Setting

$$B_0(x) = 1 - \frac{\int_c^x V_0(t)\,\mathrm{d}t}{\int_c^d V_0(t)\,\mathrm{d}t},$$

$$B_i(x) = \frac{\int_c^x V_{i-1}(t)\,\mathrm{d}t}{\int_c^d V_{i-1}(t)\,\mathrm{d}t} - \frac{\int_c^x V_i(t)\,\mathrm{d}t}{\int_c^b V_i(t)\,\mathrm{d}t}, \quad 1 \leqslant i \leqslant n-1,$$

$$B_n(x) = \frac{\int_c^x V_{n-1}(t)\,\mathrm{d}t}{\int_c^d V_{n-1}(t)\,\mathrm{d}t}, \quad x \in [a, b], \tag{27}$$

yields a normalised quasi-Bernstein-like basis relative to (c, d) in $\mathbb{E}$.

• (ii) ⇒ (i). If $(B_0, \ldots, B_n)$ is a normalised quasi-Bernstein-like basis relative to some $(c, d) \in [a, b]^2$, $c < d$, then as is classical the piecewise functions $V_0, \ldots, V_{n-1} \in \mathbb{U}$ defined by

$$V_i := \sum_{j=i+1}^{n} DB_j = -\sum_{j=0}^{i} DB_j, \quad i = 0, \ldots, n-1, \tag{28}$$

form a quasi-Bernstein-like basis relative to (c, d) in $\mathbb{U}$ (see (4.1) and (4.2) in Mazure, 2005a). Therefore (v) ⇒ (i) of Theorem 4.4 guarantees that $\mathbb{U}$ is a QECP-space on $([a, b]; \mathbb{T})$.

• (iii) ⇒ (iv). Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be any mother-function in $\mathbb{E}$. Given an n-tuple $(x_1, \ldots, x_n) = \left(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}\right) \in [a, b]^n$, with pairwise distinct $a_1, \ldots, a_r$ and positive $\mu_1, \ldots, \mu_r$ we have to prove that the r osculating flats $\mathrm{Osc}_{n-\mu_i} \Phi(a_i)$, $i = 1, \ldots, r$, have in common a unique point. Since this can be done with any mother-function, let us select Φ so that $(\mathbb{1}, \Phi_1, \ldots, \Phi_n)$ is a ready-to-blossom basis relative to $(x_1, \ldots, x_n)$. With this choice it can be easily checked that

$$\bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \Phi(a_i) = \{(0, \ldots, 0)\}. \tag{29}$$

• (iv) ⇒ (i). Assume the existence of blossoms in $\mathbb{E}$. In particular, given any $x_1, x_2 \in [a, b]$, $x_1 < x_2$, and any integer i, $1 \leqslant i \leqslant n-1$, the Bézier point $\Pi_i := \varphi\left(x_1{}^{[n-i]}, x_2{}^{[i]}\right)$ of a mother-function Φ is the unique point obtained by intersecting the i-dimensional affine flat $\mathrm{Osc}_i \Phi(x_1)$ with the $(n-i)$-dimensional affine flat $\mathrm{Osc}_{n-i} \Phi(x_2)$. It implies that for $\varepsilon_1, \varepsilon_2 \in \{-, +\}$, the n vectors $\Phi'(x_1^{\varepsilon_1}), \ldots, \Phi^{(i)}(x_1^{\varepsilon_1})$, $\Phi'(x_2^{\varepsilon_2}), \ldots, \Phi^{(n-i)}(x_2^{\varepsilon_2})$, are linearly independent. Accordingly, the space $\mathbb{U}$ satisfies (iv) of Theorem 4.4: it is thus a QECP-space on $([a, b]; \mathbb{T})$. □

The last statement in Theorem 5.2 makes the following definition consistent.

Definition 5.3. Given an $(n+1)$-dimensional PQEC-space $\mathbb{E}$ on $([a, b]; \mathbb{T})$, with $n \geqslant 2$, we say that it is a *QECP-space good for design on* $([a, b]; \mathbb{T})$ when it meets the following two requirements:

(1) $\mathbb{E}$ contains constants;
(2) the n-dimensional piecewise space $\mathbb{U} := D\mathbb{E}$ is a QECP-space on $([a, b]; \mathbb{T})$.

Theorem 5.4. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QECP-space good for design on $([a, b]; \mathbb{T})$, with $n \geqslant 2$. Then, for any $F \in \mathbb{E}$, and any $(x_1, \ldots, x_n) \in [a, b]^n$, we have that:*

1– *the value* $f(x_1, \dots, x_n)$ *of the blossom* f *of* F *at* $(x_1, \dots, x_n)$ *is the first coordinate of* F *in any ready-to-blossom basis relative to* $(x_1, \dots, x_n)$*;*
2– *if* $(x_1, \dots, x_n) = \left(a_1{}^{[\mu_1]}, \dots, a_r{}^{[\mu_r]}\right) \in [a, b]^n$*, with pairwise distinct* $a_1, \dots, a_r \in [a, b]$ *and positive* $\mu_1, \dots, \mu_r$*,*

$$f(x_1, \dots, x_n) = \frac{W\left(\Psi_n^{a_2}, \dots, \Psi_{n-\mu_2+1}^{a_2}, \;\dots\dots\;, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r}, F\right)(a_1^{\varepsilon})}{W\left(\Psi_n^{a_2}, \dots, \Psi_{n-\mu_2+1}^{a_2}, \;\dots\dots\;, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r}, \mathbb{1}\right)(a_1^{\varepsilon})}, \tag{30}$$

all other expressions obtained by permutation of $a_1, \dots, a_r$ *being also valid.*

Proof. As in the proof of Theorem 5.2 assume that the mother function $\Phi = (\Phi_1, \dots, \Phi_n)$ is selected so that $(\mathbb{1}, \Phi_1, \dots, \Phi_n)$ is a ready-to-blossom basis relative to $(x_1, \dots, x_n)$. The first point is obtained from (29) via affine maps. The second one readily follows (see Corollary 3.19 in Mazure, 2008b). □

Remark 5.5. In order to be able to consider the space $\mathbb{U} := D\mathbb{E}$ we had to assume the integer n to be greater than or equal to 2, that is, the space $\mathbb{E}$ to be of dimension at least three. Nevertheless, apart from (i) of Theorem 5.2, all other properties/formulæ involved in this section are meaningful even when $n = 1$. What therefore should be the definition of a two-dimensional QECP-space good for design on $([a, b]; \mathbb{T})$? The answer is clear: it is a PQEC-space on $([a, b]; \mathbb{T})$ containing constants, that is, a C-space on $[a, b]$ containing constants.

5.2. Pseudoaffinity of blossoms

In this section we assume that $n \geqslant 2$ and that $\mathbb{E} \subset \mathcal{P}C^{n-1}\left([a, b]; \mathbb{T}\right)$ is an $(n+1)$-dimensional QECP-space good for design on $\left([a, b]; \mathbb{T}\right)$. As usual, the blossom f of any $F \in \mathbb{E}^d$, $d \geqslant 1$, satisfies two elementary properties which readily follow from the geometrical definition of the blossom φ of any mother-function Φ:

- **symmetry:** f is symmetric on $[a, b]^n$;
- **diagonal property:** by restriction to the diagonal of $[a, b]^n$, f gives F, that is, $f(x^{[n]}) = F(x)$ for all $x \in [a, b]$.

The interest of blossoms lies in that they also satisfy the crucial pseudoaffinity property described in Theorem 5.6 below. In all situations previously addressed, proving pseudoaffinity was a difficult challenge worthwhile resolving, for it is the property which guarantees that the spaces satisfy all expected properties for design. It was first proved for C^∞ EC-spaces in Mazure and Pottmann (1996), then for ECP-spaces with C^∞ sections in Mazure (1999), Mazure and Laurent (1999), for C^n EC-spaces and ECP-spaces with C^n sections in Mazure (2006), for QEC-spaces in Mazure (2008b). Each extension required the development of new tools or approaches. It will be so here too.

Theorem 5.6. *Let* $\mathbb{E}$ *be an* $(n+1)$*-dimensional QECP-space good for design on* $([a, b]; \mathbb{T})$*. Then, the blossom* $f : [a, b]^n \to \mathbb{R}^d$ *of any* $F \in \mathbb{E}^d$*, is* **pseudoaffine in each variable***, in the following sense: for any* $x_0, x_1, \dots, x_n \in [a, b]$*, with* $x_0 < x_n$*, there exists a strictly increasing continuous function* $\beta(x_1, \dots, x_{n-1}; x_0, x_n; .) : [a, b] \to \mathbb{R}$*, independent of* F*, such that*

$$\begin{aligned} f(x_1, \dots, x_{n-1}, x) = {} & [1 - \beta(x_1, \dots, x_{n-1}; x_0, x_n; x)] f(x_1, \dots, x_{n-1}, x_0) \\ & + \beta(x_1, \dots, x_{n-1}; x_0, x_n; x) f(x_1, \dots, x_{n-1}, x_n), \quad x \in [a, b]. \end{aligned} \tag{31}$$

Proof. It is sufficient to prove (31) replacing f by the blossom φ of any possible mother-function Φ in $\mathbb{E}$. In that case, proving (31) consists in proving that the point $\varphi(x_1, \dots, x_{n-1}, x)$ moves in a continuous strictly monotonic way along an affine line $\mathcal{L} \subset \mathbb{R}^n$ as x moves from a to b. Moreover, due to the symmetry of blossoms we can assume that

$$(x_1, \dots, x_{n-1}) = \left(a_1{}^{[\mu_1]}, \dots, a_r{}^{[\mu_r]}\right), \quad \text{with } a \leqslant a_1 < a_2 < \dots < a_r \leqslant b,$$

and with positive integers $\mu_1, \dots, \mu_r$, with $\sum_{i=1}^r \mu_i = n - 1$. The definition of blossoms makes it clear that

$$\text{for any } x \in [a, b], \text{ the point } \varphi(x_1, \dots, x_{n-1}, x) \text{ lies in } \mathcal{L} := \bigcap_{i=1}^r \mathrm{Osc}_{n-\mu_i} \Phi(a_i) \subset \mathbb{R}^n.$$

At each step of the proof, $[a^*, b^*]$ will denote a generic non-trivial interval strictly contained in $[a, b]$, and $\mathbb{T}^*$ will always denote the associated sequence composed of all the knots in $\mathbb{T}$ which are interior to $[a^*, b^*]$. In accordance to the result we want to prove we will select a specific n-dimensional space $\mathbb{F} \subset \mathbb{E}$ and we will consider the corresponding inclusion

$$\mathbb{F}^* \subset \mathbb{E}^*, \quad \text{where } \mathbb{E}^*, \mathbb{F}^* \text{ are the restrictions of } \mathbb{E}, \mathbb{F} \text{ to } [a^*, b^*]. \tag{32}$$

The choice of F and $[a^*, b^*]$ will be guided by the fact that we want to ensure the following property

$$\mathbb{F}^* \text{ is an } (n\text{-dimensional}) \text{ ECP-space good for design on } \big([a^*, b^*]; \mathbb{T}^*\big). \tag{33}$$

When needed, the choice of $\mathbb{F}$ can be connected with the choice of an appropriate mother-function $\Phi = (\Phi_1, \dots, \Phi_n)$: the first $(n-1)$ components of Φ can be chosen so as to form a mother-function in $\mathbb{F}$, and the last one, Φ_n, has then to be selected in $\mathbb{E} \setminus \mathbb{F}$. Moreover, in the situation described by (32) and (33) we will select a convenient basis $(U_0, \dots, U_{n-1})$ of $\mathbb{F}$ whose restriction $(U_0^*, \dots, U_{n-1}^*)$ to $[a^*, b^*]$ will form a PCW-sequence on $([a^*, b^*]; \mathbb{T}^*)$. Applying Theorem 4.6 to the inclusion (32) will then imply that

$$\mathbb{F}^* = ECP(w_0^*, w_1^*, \dots, w_{n-1}^*), \quad \mathbb{E}^* = QECP(w_0^*, w_1^*, \dots, w_{n-1}^*; \mathbb{C}^*), \tag{34}$$

where $(w_0^*, \dots, w_{n-1}^*)$ is the system of piecewise weight functions on $([a^*, b^*]; \mathbb{T}^*)$ deduced from $(U_0^*, \dots, U_{n-1}^*)$ (up to signs) as reminded in Section 2.1, and where $\mathbb{C}^* := L_{n-1}^*\mathbb{E}^*$ is a two-dimensional C-space on $[a^*, b^*]$, $L_0^*, \dots, L_{n-1}^*$ standing for the piecewise generalised derivatives on $\mathcal{PC}^{n-1}([a^*, b^*]; \mathbb{T}^*)$ associated with $(w_0^*, \dots, w_{n-1}^*)$. Depending on our purpose we will not necessarily have $w_0^* = \mathbb{1}$, that is $U_0 = \mathbb{1}$. The interest of such a situation (34) lies in that it will enable us to use the results developed in Section 4.2.

• **First step:** *We will first prove that $\mathcal{L}$ is an affine line along which the point $\varphi(x_1, \dots, x_{n-1}, x)$ moves in a continuous strictly monotonic way as x moves from a^* to b^*, where $[a^*, b^*]$ is any non-trivial interval selected so that*

$$[a^*, b^*] \subset [a, b] \setminus \{a_1, \dots, a_r\}. \tag{35}$$

In $\mathbb{E}$, take a mother-function $\Phi = (\Phi_1, \dots, \Phi_n)$ of the form

$$\Phi = \Big(\Psi_n^{a_1}, \dots, \Psi_{n-\mu_1+1}^{a_1}, \dots\dots, \Psi_n^{a_r}, \dots, \Psi_{n-\mu_r+1}^{a_r}, \Phi_n\Big). \tag{36}$$

It readily follows that $\mathcal{L}$ is the affine line composed of all the points in $\mathbb{R}^n$ of which the $(n-1)$ first components are equal to zero. We more precisely have

$$\varphi(x_1, \dots, x_{n-1}, x) = \big(0, \dots, 0, \varphi_n(x_1, \dots, x_{n-1}, x)\big) \quad \text{for each } x \in [a, b], \tag{37}$$

where φ_n is the blossom of Φ_n. We therefore have to prove that *the function $\varphi_n(x_1, \dots, x_{n-1}, .)$ is continuous and strictly monotone on $[a^*, b^*]$.*

Let $\mathbb{F} \subset \mathbb{E}$ be spanned by the n functions $\mathbb{1}, \Phi_1, \dots, \Phi_{n-1}$. Applying (23) in the space $\mathbb{U} = D\mathbb{E}$, we can see that $(\mathbb{1}, \Phi_1, \dots, \Phi_{n-1})$ is a PCW-sequence on $([a^*, b^*]; \mathbb{T}^*)$. Therefore in particular (33) is satisfied. Applying (23) in the space $\mathbb{E}$ itself enables us to state that the sequence $(U_0, \dots, U_{n-1}) := (\Phi_1, \dots, \Phi_{n-1}, \mathbb{1})$ is a PCW-sequence on $([a^*, b^*]; \mathbb{T}^*)$. We work within the corresponding situation (34). In particular, according to Theorem 4.6, we can therefore assert that the correspondence

$$x \in [a^*, b^*] \mapsto L_{n-1}^*\Phi_n(x) = \frac{W(\Phi_1, \dots, \Phi_{n-2}, \Phi_n)(x^\varepsilon)}{W(\Phi_1, \dots, \Phi_{n-2}, \mathbb{1})(x^\varepsilon)} \tag{38}$$

is a continuous strictly monotone function on $[a^*, b^*]$. As a matter of fact, due to (35), we know that the right-hand side of (38) is equal to $\varphi_n\left(a_1{}^{[\mu_1]}, \dots, a_r{}^{[\mu_r]}, x\right)$ for any $x \in [a^*, b^*]$ (see formula (30)). The claimed result is thus proved.

• **Second step:** *It remains to prove the continuity of the function $\varphi(x_1, \dots, x_{n-1}, .)$ at each point a_i along with the compatibility of the strict monotonicities on the left and right of a_i, for $i = 1, \dots, r$. Here we prove it with $r = 1$, and therefore with $\mu_1 = n - 1$.*

We first assume that $a \leqslant a_1 < b$. Selecting any c, with $a_1 < c < b$, our non-trivial interval $[a^*, b^*]$ is the interval $[a, c]$. Here the space $\mathbb{F}$ is spanned by $\mathbb{1}, \Psi_n^b, \Psi_{n-1}^b, \dots, \Psi_2^b$. Since the expected result can be proved using any mother-function Φ, here we will work with the mother-function $\overline{\Phi}$ defined by

$$\overline{\Phi} = (\Theta, \overline{\Phi}_n), \text{ with } \Theta := \Big(\Psi_n^b, \Psi_{n-1}^b, \dots, \Psi_2^b\Big) \text{ and, for instance } \overline{\Phi}_n := \Psi_1^b.$$

The function $\Theta : [a, b] \to \mathbb{R}^{n-1}$ is a mother-function in the n-dimensional space $\mathbb{F}$. For the sake of simplicity, the restriction of Θ to $[a^*, b^*]$ will still be denoted by Θ. Given that (33) holds for the restriction $\mathbb{F}^*$ of $\mathbb{F}$ to $[a^*, b^*]$, blossoms exist in $\mathbb{F}^*$, and we denote by

$$\vartheta : [a^*, b^*]^{n-1} \to \mathbb{R}^{n-1}$$

the blossom of Θ as a mother-function of $\mathbb{F}^*$, known to be continuous on $[a^*, b^*]^{n-1}$ and pseudoaffine in each variable, see Mazure (2006). Accordingly, denoting by $\mathcal{L}^*$ the segment in $\mathbb{R}^{n-1}$ with end-points $\vartheta(a_1{}^{[n-2]}, a)$ and $\vartheta(a_1{}^{[n-2]}, c)$, we know that

$$\mathcal{L}^* = \{\vartheta(a_1{}^{[n-2]}, x) \mid x \in [a^*, b^*]\}.$$

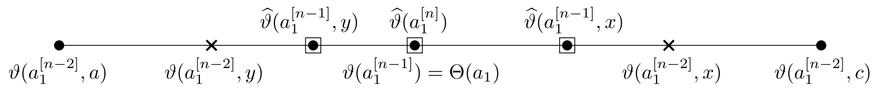

Fig. 2. Pseudoaffinity of blossoms in $\mathbb{F}^*$ and dimension elevation from $\mathbb{F}^*$ to $\mathbb{E}^*$.

Moreover we also know that

$$\vartheta({a_1}^{[n-2]}, x) \text{ moves in a continuous strictly monotonic way along } \mathcal{L}^* \text{ as } x \text{ goes from } a \text{ to } c. \tag{39}$$

Here the situation (34) corresponds to $(U_0, \ldots, U_{n-1}) = (\mathbb{1}, \Psi_n^b, \Psi_{n-1}^b, \ldots, \Psi_2^b)$, and therefore $w_0^* = \mathbb{1}$. From Proposition 4.7 we know that we can add an interval $[\overline{a}, a]$, with $\overline{a} < a$, on the left of a while preserving the equalities (34) on $\big([\overline{a}, b^*]; (a, \mathbb{T}^*)\big)$, and therefore the existence of blossoms in the extensions of $\mathbb{F}^*$, $\mathbb{E}^*$, thus defined on $[\overline{a}, b^*]^{n-1}$ and $[\overline{a}, b^*]^n$, respectively. If the claimed result is proved on the extended interval $[\overline{a}, b^*]$, we will obtain it in particular on $[a^*, b^*]$. Without loss of generality from now on we can therefore assume that $a < a_1$.

Viewed as an element of $\mathbb{E}^{*n-1}$, the function Θ possesses a blossom in n variables, say

$$\widehat{\vartheta} : [a^*, b^*]^n \to \mathbb{R}^{n-1}$$

the $(n-1)$ components of which are simply the restrictions to $[a^*, b^*]^{n-1}$ of the first $(n-1)$ components of the blossom $\overline{\varphi}$ of the mother-function $\overline{\Phi}$. Consider any x, y such that $a < y < a_1 < x < c$. Corresponding to the inclusion (32) a dimension elevation formula permits the calculation of the $(n+1)$ Bézier points of Θ in $\mathbb{E}^*$ from its n (affinely independent) Bézier points in $\mathbb{F}^*$, both relative to (y, a_1) and to (a_1, x). It can be proved that this works exactly as in QEC-spaces good for design, and we therefore refer the reader to Mazure (2008a) for more details. In addition to the equality $\vartheta({a_1}^{[n-1]}) = \Theta(a_1) = \widehat{\vartheta}({a_1}^{[n]})$, we know that

$$\widehat{\vartheta}({a_1}^{[n-1]}, x) \text{ is a strictly convex combination of } \Theta(a_1) \text{ and } \vartheta({a_1}^{[n-2]}, x);$$

$$\widehat{\vartheta}(y, {a_1}^{[n-1]}) \text{ is a strictly convex combination of } \Theta(a_1) \text{ and } \vartheta(y, {a_1}^{[n-2]}). \tag{40}$$

We are thus in the situation depicted in Fig. 2 where we represent the segment $\mathcal{L}^* \subset \mathbb{R}^{n-1}$. According to the first step, we know that the points $\widehat{\vartheta}({a_1}^{[n-1]}, x)$ and $\widehat{\vartheta}({a_1}^{[n-1]}, y)$ move in a continuous strictly monotonic way on $\mathcal{L}^*$ as x ranges over $]a_1, c]$ and y ranges over $[a, a_1[$, respectively. As clear from Fig. 2, (40) and (39) ensure the continuity and strict monotonicity of $\widehat{\vartheta}({a_1}^{[n-1]}, .)$ on the whole of $[a, c] = [a^*, b^*]$, whatever the point $c \in]a_1, b[$, and therefore on the whole of $[a, b[$. Continuity and strict monotonicity on $]a, b]$ can be proved symmetrically by exchanging the roles of a and b. The proof is thus complete in that case.

- **Third step:** *The objective is the same as in the second step, but now with any $r \geqslant 1$. The proof will be done by induction on r.*

Assume that $r \geqslant 2$ and the result holds for $(r-1)$ in any QECP-space good for design. We work with the n-dimensional space $\mathbb{F}$ spanned by $\mathbb{1}, \Psi_n^{a_r}, \ldots, \Psi_2^{a_r}$ and its restriction F^* to $[a^*, b^*] = [a, c]$, where now $a_{r-1} < c < a_r$. For any $F \in \mathbb{E}$, let us consider the following function

$$\widetilde{F}(x) := \varphi\left(x^{[n-\mu_r]}, {a_r}^{[\mu_r]}\right), \quad x \in [a, b]. \tag{41}$$

Let $\widetilde{\mathbb{E}}$ be the set of all such functions, and $\widetilde{\mathbb{E}}^*$ its restriction to $[a^*, b^*]$. From (30) we know that

$$\widetilde{F}(x) := \frac{W\left(\Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, F\right)(x^{\varepsilon})}{W\left(\Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \mathbb{1}\right)(x^{\varepsilon})}, \quad x \in [a, b] \setminus \{a_r\}. \tag{42}$$

Here, we take $(U_0, \ldots, U_{n-1}) = \left(\Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}, \mathbb{1}, \Psi_{n-\mu_r}^{a_r}, \ldots, \Psi_2^{a_r}\right)$. By application of (23) both in $\mathbb{E}$ and in $\mathbb{U} = D\mathbb{E}$, we know that this is indeed a PCW-sequence on $([a^*, b^*]; \mathbb{T}^*)$. We can therefore consider the corresponding situation (34). From (42) we can see that

$$\widetilde{\mathbb{E}}^* = L_{\mu_r-1}^* \mathbb{E}^* = QECP(\mathbb{1}, w_{\mu_1^*}, \ldots, w_{n-1}^*; \mathbb{C}^*).$$

Given any $y_1, \ldots, y_{n-\mu_r} \in [a, b] \setminus \{a_r\}$, let $(\mathbb{1}, \Theta_1, \ldots, \Theta_n)$ be a ready-to-blossom basis relative to $(y_1, \ldots, y_{n-\mu_r}, {a_r}^{[\mu_r]})$, with its last components equal to $\Psi_n^{a_r}, \ldots, \Psi_{n-\mu_r+1}^{a_r}$. Let us set

$$A := f\left(y_1, \ldots, y_{n-\mu_r}, {a_r}^{[\mu_r]}\right).$$

From Theorem 5.4 we know that

$$F = A\mathbb{1} + \sum_{i=1}^{n} \lambda_i \Theta_i, \quad \text{for some } \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$

Using (42) it readily follows that:

– restricted to $[a^*, b^*]$, $(\mathbb{1}, \widetilde{\Theta}_1, \ldots, \widetilde{\Theta}_{n-\mu_r})$ is a ready-to-blossom basis relative to $(y_1, \ldots, y_{n-\mu_r})$ in the space $\widetilde{\mathbb{E}}^*$;
– A is the first coordinate of the restriction of $\widetilde{F}$ in that basis.

Accordingly, in $\widetilde{\mathbb{E}}^*$, the blossom $\widetilde{f}^* : [a^*, b^*]^{n-\mu_r} \to \mathbb{R}$ of $\widetilde{F}^*$ is given by

$$\widetilde{f}^*(y_1, \ldots, y_{n-\mu_r}) = f(y_1, \ldots, y_{n-\mu_r}, a_r{}^{[\mu_r]}), \quad \text{for all } (y_1, \ldots, y_{n-\mu_r}) \in [a^*, b^*]^{n-\mu_r}.$$

Let us come back to the mother-function Φ defined in (36). The function $\widetilde{\Phi}^* := \left(\widetilde{\Phi}_1^*, \ldots, \widetilde{\Phi}_{n-1-\mu_r}^*, \widetilde{\Phi}_n^*\right)$ is a mother-function in $\widetilde{\mathbb{E}}^*$. Its blossom $\widetilde{\varphi}^*$ satisfies in particular

$$\begin{aligned}\widetilde{\varphi}^*\left(a_1{}^{[\mu_1]}, \ldots, a_{r-1}{}^{[\mu_{r-1}]}, x\right) &= \left(0, \ldots, 0, \widetilde{\varphi}_n^*(a_1{}^{[\mu_1]}, \ldots, a_{r-1}{}^{[\mu_{r-1}]}, x)\right)\\ &= \left(0, \ldots, 0, \varphi_n(a_1{}^{[\mu_1]}, \ldots, a_{r-1}{}^{[\mu_{r-1}]}, a_r{}^{[\mu_r]}, x)\right).\end{aligned} \tag{43}$$

Applying the inductive assumption in $\widetilde{\mathbb{E}}^*$, we can say that the function $\widetilde{\varphi}_n^*\left(a_1{}^{[\mu_1]}, \ldots, a_{r-1}{}^{[\mu_{r-1}]}, .\right)$ is continuous and strictly monotone on $[a^*, b^*]$. In other words, the function $\varphi_n\left(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}, .\right)$ is strictly monotone on any such interval $[a^*, b^*]$, hence on $[a, a_r[$.

Exchanging the roles of a_r and a_1, one can similarly prove that $\varphi_n\left(a_1{}^{[\mu_1]}, \ldots, a_r{}^{[\mu_r]}, .\right)$ is continuous and strictly monotone on $]a_1, b]$. It is therefore continuous and strictly monotone on the whole of $[a, b]$. □

5.3. Consequences

The pseudoaffinity of blossoms was not easy to achieve. It is a key-property, which is the reason why blossoms are so powerful tools. Indeed, together with symmetry and diagonal property, it is the property which makes it possible to describe in an elegant and efficient way the classical design algorithms (evaluation, subdivision) in the QECP-space $\mathbb{E}$ good for design on $([a, b]; \mathbb{T})$. Pseudoaffinity is the reason why we obtain *corner cutting* algorithms on the concerned intervals, and this is crucial for shape preservation.

As an instance, given a function $F \in \mathbb{E}^d$, $d \geqslant 1$, and its Bézier points relative to $(c, d) \in [a, b]^2$, $c < d$, $f(c^{[n-i]}, d^{[i]})$, $0 \leqslant i \leqslant n$, an n-step corner-cutting de Casteljau algorithm permits the evaluation of all values of the blossom f of F on $[a, b]^n$. Due to (31), at the nth step, we obtain $f(x_1, \ldots, x_n)$ as an affine combination of the Bézier points (a convex one when $x_1, \ldots, x \in [c, d]$, a strictly convex one when $x_1, \ldots, x \in]c, d[$), with coefficients independent of F. In other words, denoting by $b_i(x_1, \ldots, x_n)$, $0 \leqslant i \leqslant n$, the coefficients in question, we directly obtain

$$\begin{aligned}&f(x_1, \ldots, x_n) = \sum_{i=0}^{n} b_i(x_1, \ldots, x_n) f(c^{[n-i]}, d^{[i]}), \quad \sum_{i=0}^{n} b_i(x_1, \ldots, x_n) = 1, \quad x_1, \ldots, x_n \in [a, b],\\ &\quad \text{with } b_i(x_1, \ldots, x_n) > 0 \quad \text{for } c < x_1, \ldots, x_n < d, \quad 0 \leqslant i \leqslant n.\end{aligned} \tag{44}$$

As a special case, for $x_1 = \cdots = x_n = x$ we obtain

$$\begin{aligned}&F(x) = \sum_{i=0}^{n} B_i(x), \quad \sum_{i=0}^{n} B_i(x) = 1, \quad x \in [a, b],\\ &\quad \text{with } B_i(x) := b_i(x^{[n]}) > 0 \quad \text{for } c < x < d, \quad 0 \leqslant i \leqslant n.\end{aligned} \tag{45}$$

The functions $B_0, \ldots, B_n$ form a quasi-Bernstein basis relative to (c, d), their zeroes at c, d following from the geometrical definition of the Bézier points. Note that, for $i = 0, \ldots, n$, the function b_i involved in (44) is the blossom of B_i. That $(B_0, \ldots, B_n)$ is produced by the corner-cutting de Casteljau algorithm enables us to state the following result, the proof of which can be obtained as in Mazure (2004). For the importance of normalised totally positivity we refer the reader to Goodman (1989).

Theorem 5.7. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QECP-space good for design on $([a, b]; \mathbb{T})$. For any $(c, d) \in [a, b]$, $c < d$, the normalised quasi-Bernstein-like basis of $\mathbb{E}$ relative to (c, d) is a quasi-Bernstein basis relative to (c, d). It is the optimal normalised totally positive basis in the restriction of $\mathbb{E}$ to $[c, d]$.*

6. Building all QECP-spaces

The object of the present section is to show how to obtain all possible QECP-spaces on $([a, b]; \mathbb{T})$ by proving Theorem 1.4. On account of Theorem 4.5 what remains to prove is that any $(n+1)$-dimensional QECP-space on $([a, b]; \mathbb{T})$ is of the form $QECP(w_0, \ldots, w_{n-1}; \mathbb{C})$.

The results presented here extend similar results obtained for EC-spaces in Mazure (2011b), for ECP-spaces in Mazure (2011a), and for QEC-spaces in Mazure (2011c).

6.1. Theoretical characterisation

We will first work in QECP-spaces good for design where we now know that blossoms are pseudoaffine. As a matter of fact, the following lemma readily follows from the pseudoaffinity of blossoms exactly as in Mazure (2011b).

Lemma 6.1. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QECP-space good for design on $([a,b];\mathbb{T})$. Given a function $U \in \mathbb{E}$, the following three properties are equivalent:*

(i) *the Bézier points of U relative to (a,b) form a strictly increasing sequence;*
(ii) *for any $c,d \in [a,b]$, $c<d$, the Bézier points of U relative to (c,d) form a strictly increasing sequence;*
(iii) *the blossom u of U is strictly increasing in each variable on $[a,b]^n$.*

This lemma will be essential to establish Theorem 6.2 below.

Theorem 6.2. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QECP-space good for design on $([a,b];\mathbb{T})$, with $n \geqslant 2$, and let $(B_0,\dots,B_n)$ denote its quasi-Bernstein basis relative to (a,b), and let $\mathbb{U} = D\mathbb{E}$. Given any $U = \sum_{i=0}^n u_i B_i$, let $w_1 \in \mathcal{P}C^{n-2}([a,b];\mathbb{T})$ denote the piecewise function DU. The following properties are equivalent:*

(i) *the Bézier points $u_0,\dots,u_n$ of U form a strictly increasing sequence;*
(ii) *w_1 has positive coordinates in any positive quasi-Bernstein-like basis of $\mathbb{U}$ relative to (a,b);*
(iii) *w_1 is positive on $([a,b];\mathbb{T})$, and if L_1 is the piecewise generalised derivative on $\mathcal{P}C^{n-1}([a,b];\mathbb{T}))$ defined by $L_1F = DF/w_1$, $L_1\mathbb{E}$ is an n-dimensional QECP-space good for design on $([a,b];\mathbb{T})$.*

Proof. As a positive quasi-Bernstein-like basis in the space $D\mathbb{E}$ we can take the basis $(V_0,\dots,V_{n-1})$ defined in (28). In this basis, the piecewise function w_1 can be expanded as

$$w_1 = \sum_{i=0}^{n-1} \alpha_i V_i, \quad \alpha_i := u_{i+1} - u_i \text{ for } i = 0,\dots,n-1. \tag{46}$$

The equivalence between (i) and (ii) readily follows from (46).

Assume that (iii) holds. Let us denote by $\left(B_0^{\{1\}},\dots,B_{n-1}^{\{1\}}\right)$ the quasi-Bernstein basis relative to (a,b) in the n-dimensional space $L_1\mathbb{E}$ good for design on $([a,b];\mathbb{T})$. Clearly, after piecewise multiplication by w_1, the normalisation property $\mathbb{1} = \sum_{i=0}^{n-1} B_i^{\{1\}}$ implies that

$$w_1 = \sum_{i=0}^{n-1} w_1 B_i^{\{1\}}.$$

Now, $\left(w_1B_0^{\{1\}},\dots,w_1B_{n-1}^{\{1\}}\right)$ is a positive quasi-Bernstein-like basis in the QECP-space $\mathbb{U} = D\mathbb{E}$. Therefore (ii) is obviously satisfied.

Let us now assume that (i) holds and prove that (iii) is satisfied. From (46) we can see that the piecewise function $w_1 \in \mathbb{U} = D\mathbb{E}$ is positive on $([a,b];\mathbb{T})$. The n-dimensional space $L_1\mathbb{E}$ obtained by piecewise division of all elements of $\mathbb{U}$ by w_1 contains constants. Moreover it is a PEQC-space on $([a,b];\mathbb{T})$ according to Lemma 3.8. Piecewise division of the two sides of (46) by w_1 yields

$$\mathbb{1} = \sum_{i=0}^{n-1} B_i^{\{1\}}, \quad \text{with } B_i^{\{1\}} := \frac{\alpha_i V_i}{w_1} \text{for } i = 1,\dots,n-1.$$

Clearly, $\left(B_0^{\{1\}},\dots,B_{n-1}^{\{1\}}\right)$ is a quasi-Bernstein basis relative to (a,b) in $L_1\mathbb{E}$. Lemma 6.1 ensures that, for any $a^*,b^* \in [a,b]$, $a^* < b^*$, U has strictly increasing Bézier points relative to (a^*,b^*). We can thus similarly expand w_1 in the quasi-Bernstein-like basis $(V_0^*,\dots,V_{n-1}^*)$ of $\mathbb{U}$ relative to (a^*,b^*) as

$$w_1 = \sum_{i=0}^{n-1} \alpha_i^* V_i^*, \quad \text{with } \alpha_i^* > 0 \text{ for } i = 0,\dots,n-1.$$

As previously, after division by w_1 we obtain a quasi-Bernstein basis relative to (a^*,b^*) in $L_1\mathbb{E}$. The PQEC-space $L_1\mathbb{E}$ on $([a,b];\mathbb{T})$ thus possesses a quasi-Bernstein basis relative to any $(a^*,b^*) \in [a,b]^2$ with $a^* < b^*$. According to Theorem 5.2 this ensures that $DL_1\mathbb{E}$ is a QECP-space on $([a,b];\mathbb{T})$, *i.e.* (see Definition 5.3) $L_1\mathbb{E}$ is a QECP-space good for design on $([a,b];\mathbb{T})$. $\square$

Theorem 6.3. *Given an $(n+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$, containing constants, the following properties are equivalent:*

(i) *$\mathbb{E}$ is a QECP-space good for design on $([a,b];\mathbb{T})$;*
(ii) *there exists a system $(w_1,\ldots,w_{n-1})$ of piecewise weight functions on $([a,b];\mathbb{T})$ and a two-dimensional C-space $\mathbb{C}$ on $[a,b]$, containing constants, such that*

$$\mathbb{E} = QECP(\mathbb{1}, w_1, \ldots, w_{n-1}; \mathbb{C}). \tag{47}$$

Proof. This is obtained by iteration of Theorem 6.2, taking additionally into account of Proposition 3.5. □

Corollary 6.4. *Given an $(n+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$, containing constants, the following properties are equivalent*

(i) *$\mathbb{E}$ is a QECP-space on $([a,b];\mathbb{T})$;*
(ii) *there exists a system $(w_0,\ldots,w_{n-1})$ of piecewise weight functions on $([a,b];\mathbb{T})$ and a two-dimensional C-space $\mathbb{C}$ is on $[a,b]$ such that*

$$\mathbb{E} = QECP(w_0, \ldots, w_{n-1}; \mathbb{C}). \tag{48}$$

Proof. Let $\widehat{\mathbb{E}}$ be the $(n+2)$-dimensional PQEC-space on $([a,b];\mathbb{T})$ obtained from $\mathbb{E}$ by continuous integration. The claimed result is obtained by applying Theorem 6.3 to $\widehat{\mathbb{E}}$. □

Remark 6.5. Theorem 6.3 and therefore its corollary (which is exactly Theorem 1.4) have been obtained by iterated application of Theorem 6.2. It is worthwhile mentioning two most interesting consequences of Theorem 6.2.

1- Assuming that (i) of Theorem 6.2 is satisfied, we know that the piecewise function w_1 is positive on $([a,b];\mathbb{T})$. Accordingly the function U is strictly increasing on $[a,b]$. We can thus define a strictly increasing sequence $a = \xi_0 < \xi_1 < \cdots < \xi_n = b$ by setting

$$\xi_i := U^{-1}(\alpha_i), \quad i = 0, \ldots, n.$$

Let us define the operator $\mathbb{B} : C^0([a,b]) \to \mathbb{E}$ by

$$\mathbb{B}F := \sum_{i=0}^{n} F(\xi_i) B_i \quad \text{for all } F \in C^0([a,b]).$$

This is a Bernstein-type positive linear operator based on the QECP-space $\mathbb{E}$. It reproduces constants and the function U, and it is variation-diminishing due to the quasi-Bernstein basis being normalised and totally positive. Such operators can be handled as in the case of EC- or QEC-spaces Mazure (2009, 2011b, 2011c) to which we refer the reader (also see Aldaz et al., 2009 on this subject).

2- Given any positive $\omega_0,\ldots,\omega_n$, consider the function

$$\Omega := \sum_{i=0}^{n} \omega_i B_i \in \mathbb{E}.$$

As an application of (ii) ⇒ (iii) of Theorem 6.2, we can deduce that the space

$$\mathcal{R}(\mathbb{E};\Omega) := \left\{ \frac{F}{\Omega} \mid F \in \mathbb{E} \right\},$$

is a QECP-space good for design on $([a,b];\mathbb{T})$. It can also be described as

$$\mathcal{R}(\mathbb{E};\Omega) = \left\{ \frac{\sum_{i=0}^{n} \gamma_i \omega_i B_i}{\sum_{i=0}^{n} \omega_i B_i} \mid \gamma_0, \ldots, \gamma_n \in \mathbb{R} \right\}.$$

By analogy with the classical rational spaces, we call it *the rational QECP-space on $([a,b];\mathbb{T})$ based on $\mathbb{E}$ and Ω*. The properties and advantages of such rational QECP-spaces can be developed as we did for rational EC-spaces in Mazure (2013). We just mention two facts:

– for each $F \in \mathbb{E}$, the blossom of $\dfrac{F}{\Omega} \in \mathcal{R}(\mathbb{E};\Omega)$ is the quotient $\dfrac{f}{\omega}$ of the blossoms f, ω of F, Ω in $\mathbb{E}$;

– any QECP-space good for design is a rational QECP-space, since $\mathcal{R}\left(\mathcal{R}(\mathbb{E};\Omega); \dfrac{1}{\Omega}\right) = \mathbb{E}$.

6.2. Practical characterisation

We can actually write Theorem 6.3 as stated below, where, for the sake of simplicity we denote the same way an element of $\mathbb{E}$ and its restriction to each $[t_k^+, t_{k+1}^-]$:

Theorem 6.6. *Given an $(n+1)$-dimensional PQEC-space $\mathbb{E}$ on $([a,b];\mathbb{T})$, and given any $c \in [a,b]$, the following properties are equivalent:*

(i) *is a QECP-space on $([a,b];\mathbb{T})$;*
(ii) *for each $k = 0,\dots,q$, we can find a system $(w_0^k,\dots,w_{n-1}^k)$ of weight functions on $[t_k, t_{k+1}]$, with associated differential operators on $C^{n-1}([t_k,t_{k+1}]$ denoted by $L_0^k,\dots,L_{n-1}^k$, so that the following three requirements are fulfilled:*
   1) *the section spaces are given by*

$$\mathbb{E}_k = QEC(w_0^k,\dots,w_{n-1}^k;\mathbb{C}_k), \quad k = 0,\dots,q, \tag{49}$$

   where $\mathbb{C}_k$ is a two-dimensional C-space on $[t_k,t_{k+1}]$;
   2) *any $F \in \mathbb{E}$ satisfies the connection conditions*

$$L_i^k F(t_k^+) = L_i^{k-1} F(t_k^-) \quad \text{for } i = 0,\dots,n-1 \text{ and for } k = 1,\dots,q; \tag{50}$$

   3) *the continuous function whose restriction to $[t_k,t_{k+1}]$ is $L_{n-1}^k \Psi_n^c$, $k = 0,\dots,q$, is strictly monotone on $[a,b]$.*

Remark 6.7. Let $\mathbb{E}$ be an $(n+1)$-dimensional PQEC-space on $([a,b];\mathbb{T})$. The practical problem that we want to solve here is the following one: *how to know whether or not $\mathbb{E}$ is a QECP-space on $([a,b];\mathbb{T})$?* Equivalently, *how to establish necessary and sufficient conditions for $\mathbb{E}$ to be a QECP-space on $([a,b];\mathbb{T})$?* These conditions should involve all the parameters defining $\mathbb{E}$. What are therefore these parameters? The most obvious ones are the $q(n+1)(n+2)/2$ entries of all connection matrices, but the section-spaces themselves may depend on parameters, and even the knots $(t_0,\dots,t_{q+1})$ can also possibly serve as parameters. Answering these questions is not an easy task. Given that, for each $k = 0,\dots,q$, we can build all possible systems $(w_0^k,\dots,w_{n-1}^k)$ of weight functions on $[t_k,t_{k+1}]$ such that $\mathbb{E}_k = QEC(w_0^k,\dots,w_{n-1}^k;\mathbb{C}_k)$ by iteration of the non-piecewise version of Theorem 6.3, the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 6.6 provides us with a possible constructive approach to answer these questions. An alternative approach will consist in finding conditions to diminish the dimension within the class of all PECQ-spaces on $([a,b];\mathbb{T})$ via Proposition 6.8 below.

Proposition 6.8. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QECP-space on $([a,b];\mathbb{T})$ and let $(V_0,\dots,V_n)$ be a quasi-Bernstein-like basis relative to (a,b). Then, for each integer i, $0 \leqslant i \leqslant n$, all coordinates of V_i in a quasi-Bernstein-like basis $(W_0,\dots,W_n)$ relative to (c,d), with $a \leqslant c < d \leqslant b$, are positive — except, possibly, for those required to be 0 by the zeroes of V_i at a, b —. More precisely, if $V_i = \sum_{j=0}^n v_{i,j} W_j$*

- *if $a < c < d < b$, then all $v_{i,j}$ are positive;*
- *if $a = c < d < b$, then $v_{i,j} = 0$ for $j = 0,\dots,i-1$, and all other $v_{i,j}$ are positive;*
- *if $a < c < d = b$, then $v_{i,j} = 0$ for $j = i+1,\dots,n$, and all other $v_{i,j}$ are positive.*

Proof. Let $\widehat{\mathbb{E}}$ be the QECP-space good for design on $([a,b];\mathbb{T})$ obtained from $\mathbb{E}$ by continuous integration. Without loss of generality, we can assume that $(V_0,\dots,V_n)$ is obtained from the quasi-Bernstein basis $(\widehat{B}_0,\dots,\widehat{B}_n)$ relative to (a,b) in $\widehat{E}$ via formulæ (28). We can make a similar assumption for $(W_0,\dots,W_n)$ on $[c,d]$. The claimed result follows from the lemma below via (46). $\square$

Lemma 6.9. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QECP-space good for design on $([a,b];\mathbb{T})$ and let $(B_0,\dots,B_n)$ be its quasi-Bernstein basis relative to (a,b). Given $a < c < d < b$, and given an integer i, $0 \leqslant i \leqslant n-1$, we denote by $\beta_{i,j}$, $\gamma_{i,j}$, $\delta_{i,j}$, $j = 0,\dots,n$, the Bézier points of the function*

$$\mathcal{B}_i := \sum_{j=i+1}^{n} B_j,$$

relative to (a,c), (c,d), and (d,b), respectively. They satisfy

$$\begin{aligned}
&0 = \beta_{i,0} = \beta_{i,1} = \cdots = \beta_{i,i} < \beta_{i,i+1} < \beta_{i,i+2} < \cdots < \beta_{i,n} < 1,\\
&0 < \gamma_{i,0} < \gamma_{i,1} < \cdots < \gamma_{i,n-1} < \gamma_{i,n} < 1,\\
&0 < \delta_{i,0} < \delta_{i,1} < \cdots < \delta_{i,i} < \delta_{i,i+1} = \cdots = \delta_{i,n-1} = \delta_{i,n} = 1.
\end{aligned} \tag{51}$$

Proof. This is a straightforward consequence of the de Casteljau evaluation algorithm of $\mathcal{B}_i(c)$ from the Bézier points

$$(\underbrace{0, 0, \ldots, 0}_{(i+1)\text{ times}}, \underbrace{1, 1, \ldots, 1}_{(n-i)\text{ times}})$$

of $\mathcal{B}_i$ relative to (a, b) and then of the de Casteljau evaluation algorithm of $\mathcal{B}_i(d)$ from its Bézier points relative to (c, d). Therefore, it is a straightforward consequence of the properties of blossoms, and in particular of their pseudoaffinity. □

Remark 6.10. To continue the comments in Remark 6.7, the necessary and sufficient conditions for a PQEC-space $\mathbb{E}$ on $([a, b]; \mathbb{T})$ to be a QECP-space on $([a, b]; \mathbb{T})$ will be obtained step by step. From Proposition 6.8 we know that we should first find necessary and sufficient conditions for $\mathbb{E}$ to possesses a "convenient" quasi-Bernstein-like basis $(V_0, \ldots, V_n)$ relative to (a, b), in the sense that this basis should satisfy the positivity conditions stated in Proposition 6.8 in each interval $[t_k, t_{k+1}]$. Then, for any positive $\alpha_0, \ldots, \alpha_n$, the piecewise function $w_0 := \sum_{i=0}^{n} \alpha_i V_i \in \mathbb{E}$ will satisfy the assumptions of Lemma 3.9. Accordingly, the corresponding space $DL_0\mathbb{E}$ will be an n-dimensional PQEC-space on $([a, b]; \mathbb{T})$, possessing a quasi-Bernstein-like basis relative to (a, b) obtained from (28). The second step will consist in finding additional necessary and sufficient conditions for this basis to be a "convenient" one, so as to decrease the dimension again via Lemma 3.9, and so forth... This will be the method adopted in next section.

7. Illustrations

In this section we illustrate our results with the relatively simple case where $n = 2$ with a view to design in the four-dimensional space of continuous functions obtained by integration.

7.1. Building three-dimensional QECP-spaces

We start with only two sections spaces. We thus work with $t_0 = a < t_1 < t_2 = b$ and $\mathbb{T} = (t_1)$. In order to avoid too many indices, we are changing our notations compared to the previous sections. We denote by $\mathbb{E}$ and $\overline{\mathbb{E}}$ two three-dimensional QEC-spaces on $[t_0, t_1]$ and $[t_1, t_2]$, respectively. By (V_0, V_1, V_2) and $(\overline{V}_0, \overline{V}_1, \overline{V}_2)$ we denote a positive quasi-Bernstein-like basis relative to (t_0, t_1) and to (t_1, t_2) in $\mathbb{E}$ and $\overline{\mathbb{E}}$, respectively. Without loss of generality we can assume that

$$V_0(t_0) = V_2(t_1) = \overline{V}_0(t_1) = \overline{V}_2(t_2) = 1. \tag{52}$$

Since $n = 2$, the three-dimensional PQEC-space $\widehat{\mathbb{E}}$ on $([a, b]; \mathbb{T})$ we are working with is thus defined by the two section-spaces $\mathbb{E}$ and $\overline{\mathbb{E}}$ and a lower triangular connection matrix of order 3 with positive diagonal entries. Therefore, all piecewise functions $\widehat{F} \in \widehat{\mathbb{E}}$ satisfy the same connection condition

$$\begin{bmatrix} \widehat{F}(t_1^+) \\ \widehat{F}'(t_1^+) \\ \widehat{F}(t_2) \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} \widehat{F}(t_1^-) \\ \widehat{F}'(t_1^-) \\ \widehat{F}(t_0) \end{bmatrix}, \tag{53}$$

where $a, c, f > 0$.

Theorem 7.1. *The PQEC-space $\widehat{\mathbb{E}}$ is a QECP-space on $([t_0, t_2]; \mathbb{T})$ if and only if the following properties are satisfied*

$$e > 0, \quad d + eV_2'(t_1) > 0, \quad eb - cd - ea\overline{V}_0'(t_1) > 0. \tag{54}$$

Proof. The proof comprises several steps.

- **First step:** *Expliciting the connection conditions (53) via (V_0, V_1, V_2) and $(\overline{V}_0, \overline{V}_1, \overline{V}_2)$*. Consider two functions $F \in \mathbb{E}$ and $\overline{F} \in \overline{\mathbb{E}}$ expanded in the corresponding quasi-Bernstein-like bases as

$$F = \alpha_0 V_0 + \alpha_1 V_1 + \alpha_2 V_2, \quad \overline{F} = \overline{\alpha}_0 \overline{V}_0 + \overline{\alpha}_1 \overline{V}_1 + \overline{\alpha}_2 \overline{V}_2. \tag{55}$$

On account of our choice (52), we have

$$\begin{aligned} &F(t_1) = \alpha_2, \quad F'(t_1) = \alpha_1 V_1'(t_1) + \alpha_2 V_2'(t_1), \quad F(t_0) = \alpha_0, \\ &\overline{F}(t_1) = \overline{\alpha}_0, \quad \overline{F}'(t_1) = \overline{\alpha}_0 \overline{V}_0'(t_1) + \overline{\alpha}_1 \overline{V}_1'(t_1), \quad \overline{F}(t_2) = \overline{\alpha}_2. \end{aligned} \tag{56}$$

Accordingly, from (53) we can say that the two functions F and $\overline{F}$ are the restrictions to $[t_0, t_1]$ and $[t_1, t_2]$, of a piecewise function $\widehat{F} \in \widehat{\mathbb{E}}$ if and only if

$$\begin{cases} \overline{\alpha}_0 = a\alpha_2, \\ \overline{\alpha}_1 \overline{V}_1'(t_1) = \Delta\alpha_2 + cV_1'(t_1)\alpha_1, \\ \overline{\alpha}_2 = \mathrm{H}\alpha_2 + eV_1'(t_1)\alpha_1 + f\alpha_0, \end{cases} \tag{57}$$

with

$$\Delta := b + cV_2'(t_1) - a\overline{V}_0'(t_1), \quad \mathrm{H} := d + eV_2'(t_1). \tag{58}$$

• **Second step:** *Necessary and sufficient conditions ensuring the presence of a "convenient" quasi-Bernstein-like basis relative to* (t_0, t_2) *in* $\widehat{\mathbb{E}}$.

With notations consistent with the previous point, the restrictions B_0, $\overline{B}_0$, B_1, $\overline{B}_1$, B_2, $\overline{B}_2$, to $[t_0, t_1]$ and $[t_1, t_2]$, of a possible quasi-Bernstein-like basis $(\widehat{B}_0, \widehat{B}_1, \widehat{B}_2)$ in $\widehat{\mathbb{E}}$ can be written as

$$\begin{aligned}
&B_0 = \alpha_{0,0}V_0 + \alpha_{0,1}V_1 + \alpha_{0,2}V_2, \quad \overline{B}_0 = \overline{\alpha}_{0,0}\overline{V}_0,\\
&B_1 = \alpha_{1,1}V_1 + \alpha_{1,2}V_2, \quad \overline{B}_1 = \overline{\alpha}_{1,0}\overline{V}_0 + \overline{\alpha}_{1,1}\overline{V}_1,\\
&B_2 = \alpha_{2,2}V_2, \quad \overline{B}_2 = \overline{\alpha}_{2,0}\overline{V}_0 + \overline{\alpha}_{2,1}\overline{V}_1 + \overline{\alpha}_{2,2}\overline{V}_2.
\end{aligned} \tag{59}$$

Not only do we require the existence of such a quasi-Bernstein-like basis $(\widehat{B}_0, \widehat{B}_1, \widehat{B}_2)$ but we even require that all α'-s and all $\overline{\alpha}$'-s involved in (59) be positive. Since (V_0, V_1, V_2) and $(\overline{V}_0, \overline{V}_1, \overline{V}_2)$ are positive quasi-Bernstein-like bases in $\mathbb{E}$ and $\overline{\mathbb{E}}$, respectively, we know that

$$V_1'(t_1) < 0, \quad \overline{V}_1'(t_1) > 0. \tag{60}$$

1) B_2, $\overline{B}_2$ are obtained by solving (57) under the requirements: $\alpha_0 = \alpha_1 = 0$. We therefore have to solve

$$\overline{\alpha}_{2,1}\overline{V}_1'(t_1) = \Delta\alpha_{2,2}, \quad \overline{\alpha}_{2,2} = \mathrm{H}\alpha_{2,2}, \tag{61}$$

under the additional requirement $\alpha_{2,2}, \overline{\alpha}_{2,1}, \overline{\alpha}_{2,2} > 0$. Due to (60), a necessary condition for this to be possible is that

$$\Delta > 0 \quad \text{and} \quad \mathrm{H} > 0. \tag{62}$$

Conversely, if (62) holds, any choice of a positive $\alpha_{2,2}$ generates positive $\overline{\alpha}_{2,1}$, $\overline{\alpha}_{2,2}$.

2) Let us assume that (62) holds. Then, B_0, $\overline{B}_0$ are obtained by solving (57) under the requirements $\overline{\alpha}_1 = \overline{\alpha}_2 = 0$. We have to solve

$$-eV_1'(t_1)\alpha_{0,1} = \mathrm{H}\alpha_{0,2} + f\alpha_{0,0}, \quad -cV_1'(t_1)\alpha_{0,1} = \Delta\alpha_{0,2}, \tag{63}$$

with $\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2} > 0$. Substituting in it the value of $\alpha_{0,1}$ obtained from the second equation in (63), the first equation can equivalently be replaced by

$$\Gamma\alpha_{0,2} = cf\alpha_{0,0}, \quad \text{with } \Gamma := e\Delta - c\mathrm{H} = eb - cd - ea\overline{V}_0'(t_1). \tag{64}$$

On account of (62) we can therefore obtain positive $\alpha_{0,0}, \alpha_{0,1}, \alpha_{0,2} > 0$ satisfying (63) if and only if we additionally require that $\Gamma > 0$.

3) Let us assume that (62) holds and that $\Gamma > 0$. Then, B_1, $\overline{B}_1$ are obtained by solving (57) under the requirements $\alpha_0 = \overline{\alpha}_2 = 0$. We thus have to find positive $\alpha_{1,1}$, $\alpha_{1,2}$, $\overline{\alpha}_{1,1}$ such that

$$\overline{\alpha}_{1,1}\overline{V}_1'(t_1) = \Delta\alpha_{1,2} + cV_1'(t_1)\alpha_{1,1}, \quad \mathrm{H}\alpha_{1,2} + eV_1'(t_1)\alpha_{1,1} = 0. \tag{65}$$

After elimination of $\alpha_{1,1}$, the left equation can equivalently be replaced by

$$e\overline{\alpha}_{1,1}\overline{V}_1'(t_1) = \Gamma\alpha_{1,2}. \tag{66}$$

It readily follows that positive $\alpha_{1,1}$, $\alpha_{1,2}$, $\overline{\alpha}_{1,1}$ can satisfy (65) if and only if we additionally require that $e > 0$.

All in all, existence of a Bernstein-like basis $(\widehat{B}_0, \widehat{B}_1, \widehat{B}_2)$ with positive α'-s and $\overline{\alpha}$'-s in (59) is obtained if and only if

$$\Delta > 0, \quad \mathrm{H} > 0, \quad e > 0, \quad \Gamma > 0.$$

However, since $e\Delta = c\mathrm{H} + \Gamma$, these four conditions reduce to the three conditions $\mathrm{H} > 0$, $e > 0$, $\Gamma > 0$, which is exactly (54).

• **Third step:** *Assume that (54) holds. Let us prove that* $\widehat{\mathbb{E}}$ *is a QECP-space on* $([t_0, t_2]; \mathbb{T})$. Given any positive real numbers $\widehat{A}_0$, $\widehat{A}_1$, $\widehat{A}_2$, consider

$$\widehat{W}_0 := \widehat{A}_0\widehat{B}_0 + \widehat{A}_1\widehat{B}_1 + \widehat{A}_2\widehat{B}_2 \in \widehat{\mathbb{E}}. \tag{67}$$

Denote by W_0 and $\overline{W}_0$ the restrictions of $\widehat{W}_0$ to $[t_0, t_1]$ and $[t_1, t_2]$, respectively, and by L_0 and $\overline{L}_0$ the divisions by W_0 and $\overline{W}_0$. Let us expand these functions as

$$W_0 = \alpha_0V_0 + \alpha_1V_1 + \alpha_2V_2, \quad \overline{W}_0 = \overline{\alpha}_0\overline{V}_0 + \overline{\alpha}_1\overline{V}_1 + \overline{\alpha}_2\overline{V}_2. \tag{68}$$

On account of (67) and of our second step, we can say that α_0, α_1, α_n and $\overline{\alpha}_0$, $\overline{\alpha}_1$, $\overline{\alpha}_2$ are all positive. From Lemma 3.9 we can thus assert that $\widehat{W}_0$ is positive on $([a, b]; \mathbb{T})$ and that, if $\widehat{L}_0\widehat{\mathbb{E}}$ denotes the piecewise division by $\widehat{W}_0$, then $D\widehat{L}_0\widehat{\mathbb{E}}$ is a two-dimensional PQEC-space on $([t_0, t_2]; \mathbb{T})$. Moreover, we also know that $D\widehat{L}_0\widehat{\mathbb{E}}$ possesses a quasi-Bernstein-like basis relative to (t_0, t_2), say $\big(\widehat{\beta}_0, \widehat{\beta}_1\big)$. Indeed such a basis can be derived, for instance, by the systematic procedure (28) from the quasi-Bernstein basis relative to (t_0, t_2) in $\widehat{L}_0\widehat{\mathbb{E}}$ obtained by division of the sides of (67) by $\widehat{W}_0$, *i.e.*,

$$\left(\frac{\widehat{A}_0\widehat{B}_0}{\widehat{W}_0}, \frac{\widehat{A}_1\widehat{B}_1}{\widehat{W}_0}, \frac{\widehat{A}_2\widehat{B}_2}{\widehat{W}_0}\right).$$

The piecewise functions $\widehat{\beta}_0, \widehat{\beta}_1 \in D\widehat{L}_0\widehat{\mathbb{E}}$ can thus be taken as

$$\widehat{\beta}_0 := -\widehat{A}_0\left(\frac{\widehat{B}_0}{\widehat{W}_0}\right)' = \widehat{A}_0\frac{\widehat{W}_0'\widehat{B}_0 - \widehat{W}_0\widehat{B}_0'}{\widehat{W}_0^2}, \quad \widehat{\beta}_1 := \widehat{A}_2\left(\frac{\widehat{B}_2}{\widehat{W}_0}\right)' = \widehat{A}_2\frac{\widehat{W}_0\widehat{B}_2' - \widehat{W}_0'\widehat{B}_2}{\widehat{W}_0^2}.$$

In $D\widehat{L}_0\widehat{\mathbb{E}}$ the section-spaces $DL_0\mathbb{E}$ and $D\overline{L}_0\overline{\mathbb{E}}$ are two-dimensional C-spaces on $[t_0, t_1]$ and $[t_1, t_2]$, respectively. Moreover, we know that the first diagonal entry of the connection matrix at t_1 in $D\widehat{L}_0\widehat{\mathbb{E}}$ is positive. Accordingly, all components of $\widehat{\beta}_0, \widehat{\beta}_1$ in quasi-Bernstein-like bases of $DL_0\mathbb{E}$ and $D\overline{L}_0\overline{\mathbb{E}}$ — other than those which are equal to zero due to $\big(\widehat{\beta}_0, \widehat{\beta}_1\big)$ being a quasi-Bernstein-like basis relative to (t_0, t_2) — are positive if and only if we have

$$\widehat{\beta}_0(t_1^+) > 0 \quad \text{and} \quad \widehat{\beta}_1(t_1^-) > 0,$$

that is, if and only if

$$\overline{W}_0'(t_1)\overline{B}_0(t_1) - \overline{W}_0(t_1)\overline{B}_0'(t_1) > 0, \quad W_0(t_1)B_2'(t_1) - W_0'(t_1)B_2(t_1) > 0.$$

From (68), (52) and (59), we can derive that

$$\begin{aligned} W_0(t_1)B_2'(t_1) - W_0'(t_1)B_2(t_1) &= \alpha_2\alpha_{2,2}V_2'(t_1) - \big(\alpha_1 V_1'(t_1) + \alpha_2 V_2'(t_1)\big)\alpha_{2,2} \\ &= -\alpha_1\alpha_{2,2}V_1'(t_1) > 0, \end{aligned} \tag{69}$$

the positivity being guaranteed by (60). One can similarly check that

$$\overline{W}_0'(t_1)\overline{B}_0(t_1) - \overline{W}_0(t_1)\overline{B}_0'(t_1) = \overline{\alpha}_1\overline{\alpha}_{0,0}\overline{V}_1'(t_1) > 0.$$

Accordingly, for any positive numbers $\widehat{\gamma}_0, \widehat{\gamma}_1$, the piecewise function

$$\widehat{W}_1 := \widehat{\gamma}_0\widehat{\beta}_0 + \widehat{\gamma}_1\widehat{\beta}_1 \in D\widehat{L}_0\widehat{\mathbb{E}}$$

is positive on $([t_0, t_2]; \mathbb{T})$. Piecewise division by $\widehat{W}_1$ transforms the PQEC-space $D\widehat{L}_0\widehat{\mathbb{E}}$ on $([t_0, t_2]; \mathbb{T})$ into a two-dimensional PQEC-space $\widehat{L}_1\widehat{\mathbb{E}}$ on $([t_0, t_2]; \mathbb{T})$ containing constants, that is, into a two-dimensional C-space $\mathbb{C}$ on $[t_0, t_2]$ (see Proposition 3.5). It readily follows that $\widehat{\mathbb{E}} = QECP(\widehat{W}_0, \widehat{W}_1; \mathbb{C})$.

We thus have proved that the conditions (54) are indeed necessary and sufficient for $\widehat{\mathbb{E}}$ to be a QECP-space on $([t_0, t_2]; \mathbb{T})$. $\square$

Besides, from (61) and (63)–(66) we can derive the following result:

Proposition 7.2. *Assume that (54) holds. As a Bernstein-like basis relative to (t_0, t_2) in the QECP-space $\widehat{\mathbb{E}}$ we can take the basis $(\widehat{B}_0, \widehat{B}_1, \widehat{B}_2)$ whose restrictions to $[t_0, t_1]$ and $[t_1, t_2]$ are given by:*

$$\begin{aligned} &B_0 = V_0 + \frac{f\Delta}{-V_1'(t_1)\Gamma}V_1 + \frac{cf}{\Gamma}V_2, \quad \overline{B}_0 = \frac{acf}{\Gamma}\overline{V}_0, \\ &B_1 = \frac{\mathrm{H}}{-eV_1'(t_1)}V_1 + V_2, \quad \overline{B}_1 = a\overline{V}_0 + \frac{\Gamma}{e\overline{V}_1'(t_1)}\overline{V}_1, \\ &B_2 = \frac{1}{\mathrm{H}}V_2, \quad \overline{B}_2 = \frac{a}{\mathrm{H}}\overline{V}_0 + \frac{\Delta}{\overline{V}_1'(t_1)\mathrm{H}}\overline{V}_1 + \overline{V}_2, \end{aligned} \tag{70}$$

where Δ, H are defined in (58), and Γ in (64).

Note that the latter formulæ are deliberately written so as to point out the positivity of all coefficients (see (60)).

Remark 7.3. We have only addressed the case where we are connecting two three-dimensional QEC section-spaces. In practice, for the construction of splines, the resulting QECP-spaces can usefully play the role of one section-space (see Laurent et al., 1997). Nevertheless, we would like to draw the reader's attention to the fact that it is possible to iterate the process. Indeed, if, for instance, we have a third QEC-section space $\overline{\overline{\mathbb{E}}}$ on $[t_2, t_3]$ with a corresponding connection matrix at t_2, the QECP-space $\widehat{E}$ can be considered as one section-space on $[t_0, t_2]$ that we are connecting with the second section-space $\overline{\overline{\mathbb{E}}}$. The same approach can be developed, provided that we do not work with the given connection matrix at t_2 but with a modified one, expressing the values at t_3 of piecewise functions $\widehat{\widehat{F}}$ in the resulting PQEC-space $\widehat{\widehat{\mathbb{E}}}$ on $\big([t_0, t_3]; (t_1, t_2)\big)$ as linear combinations of $\widehat{\widehat{F}}(t_2^-)$, $\widehat{\widehat{F}}'(t_2^-)$ and $\widehat{\widehat{F}}(t_0)$. With the help of the quasi-Bernstein-like basis (70) and of a positive quasi-Bernstein-like basis $(\overline{\overline{V}}_0, \overline{\overline{V}}_1, \overline{\overline{V}}_2)$ relative to (t_2, t_3) in $\overline{\overline{\mathbb{E}}}$ we simply have to find necessary and sufficient conditions ensuring the existence of a "convenient" quasi-Bernstein-like basis relative to (t_0, t_3) in the $\widehat{\widehat{\mathbb{E}}}$. Such conditions are similar to (54).

7.2. Illustrations

Here we assume that $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, that $\mathbb{E} = \mathbb{E}_{p,q}$ and that $\overline{\mathbb{E}}$ is obtained from $\mathbb{E}_{\overline{p},\overline{q}}$ by translation from $[0, 1]$ to $[1, 2]$, where p, q, $\overline{p}$, $\overline{q}$ are any real numbers greater than 1.

As a positive quasi-Bernstein-like basis in the section-space $\mathbb{E}$, we can take the quasi-Bernstein basis relative to $(0, 1)$, that is,

$$V_0(x) = (1-x)^p, \quad V_2(x) := x^q, \quad V_1(x) := 1 - V_0(x) - V_2(x), \quad x \in [0,1],$$

and the similar quasi-Bernstein basis $(\overline{V}_0, \overline{V}_1, \overline{V}_2)$ relative to $(1, 2)$ in $\overline{E}$. We therefore have

$$V_1'(t_0) = p = -V_0'(t_0), \quad V_2'(t_1) = -V_1'(t_1) = q, \quad \overline{V}_1'(t_1) = \overline{p} = -\overline{V}_0'(t_1), \quad \overline{V}_2'(t_2) = -\overline{V}_1'(t_2) = \overline{q}.$$

In this example, the quantities Δ, H, Γ introduced in (58) and (64) are thus given by

$$\Delta = b + cq + a\overline{p}, \quad \mathrm{H} = d + eq, \quad \Gamma = eb - cd + ea\overline{p}. \tag{71}$$

In particular, Theorem 7.1 yields the following result.

Corollary 7.4. *Under the data above, the PQEC-space $\widehat{\mathbb{E}}$ is a QECP-space on $\big([0, 2]; (1)\big)$ if and only if*

$$e > 0, \quad d + eq > 0, \quad eb - cd + ea\overline{p} > 0. \tag{72}$$

Under the conditions (72) the space $\mathcal{F}$ obtained from $\widehat{\mathbb{E}}$ by continuous integration is a four-dimensional QECP-space good for design on $\big([0, 2]; (1)\big)$. It is in that space $\mathcal{F}$ that we want to illustrate the shape effects permitted by the context. Nevertheless, we have to handle a total amount of ten parameters: namely the six entries a, b, c, d, e, f of the connection matrix, and the two pairs of real numbers (p, q) and $(\overline{p}, \overline{q})$ defining the two section-spaces. This is not a reasonable amount to give illustrations in an article, and this is why we significantly reduce the number of free parameters by requiring the space $\mathcal{F}$ to preserve symmetry, in the sense that symmetric control polygons should provide symmetric parametric curves.

Proposition 7.5. *The space $\mathcal{F}$ preserves symmetry if and only if*

$$\overline{p} = q, \quad \overline{q} = p, \quad a = c = f = 1, \quad d = \frac{eb}{2}. \tag{73}$$

When (73) holds, the four free parameters e, b, p, q are assigned to satisfy

$$e > 0, \quad b + 2q > 0. \tag{74}$$

Proof. Since the Bernstein-like basis (70) is chosen so that $\widehat{B}_0(0) = \widehat{B}_0(2) = 1$, symmetry is preserved if and only if

$$B_0(x) = \overline{B}_2(2-x), \quad B_2(x) = \overline{B}_0(x), \quad B_1(x) = \overline{B}_1(2-x), \quad x \in [0,1].$$

From (70) it readily follows that these conditions are satisfied if and only if $\overline{p} = q$, $\overline{q} = p$, $a = c = f = 1$ and $\Gamma = \mathrm{H}$. This yields (73). In that case (72) becomes (74). □

Under the requirements (73) and (74), the functions in the four-dimensional space $\mathcal{F}$ are C^1 and F^2 (or G^2 as well): continuity of the Frenet frames of order two and of the first curvature at t_1. A few illustrations are provided in Figs. 3 and 4, with special insistence on the limit cases $b = -2q^+$, $b \to +\infty$ and $e = 0^+$, $e \to +\infty$.

Example 7.6. The technique used for connecting two three-dimensional QEC-spaces so as to form a QECP-space on $([t_0, t_2]; (t_1))$ can of course be used to connect two three-dimensional EC-spaces so as to form an ECP-space on $([t_0, t_2]; (t_1))$. We illustrate this with a simple example within the symmetry conditions (73). Here we assume that $p = q = 2$. We are thus connecting two cubic polynomials F, $\overline{F}$ on $[t_0, t_1] = [0, 1]$ and $[t_1, t_2] = [1, 2]$ via $F(t_1) = \overline{F}(t_1)$ and

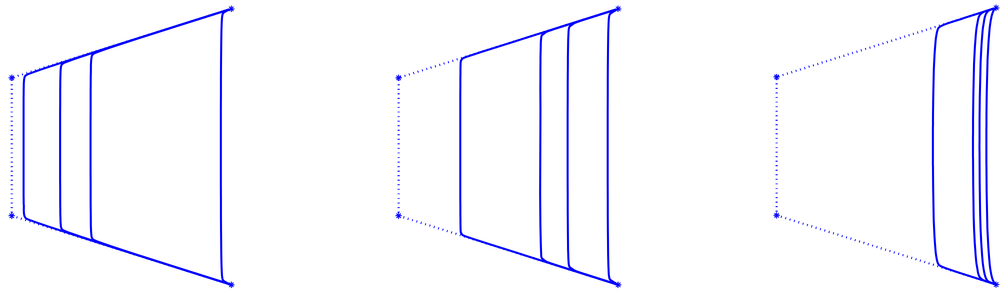

Fig. 3. Parametric curves in the symmetric QECP-space $\mathcal{F}$ on $([0, 2]; (1))$ described in Subsection 7.2 with, on each picture, from right to left $e = 0.1$; 10; 20; 100. Left: $p = 2$, $q = 20$, $b = -39.9$. Middle: $p = q = 20$, $b = -39.9$. Right: $p = 20$, $q = 2$, $b = -3.9$.

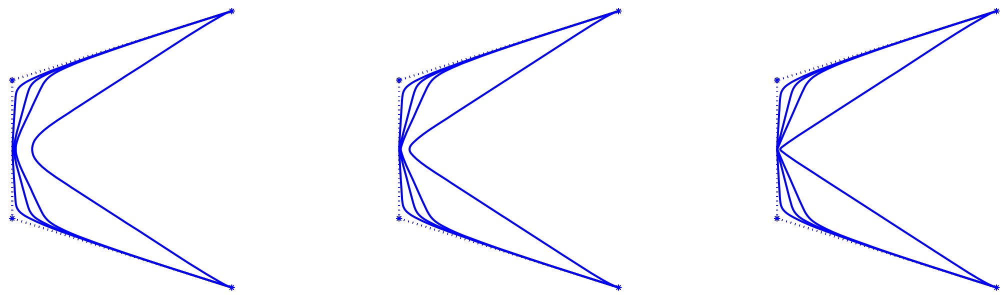

Fig. 4. The same with $p = q = 20$ everywhere. Left: $b = -20$. Middle: $b = 0$. Right: $b = 100$.

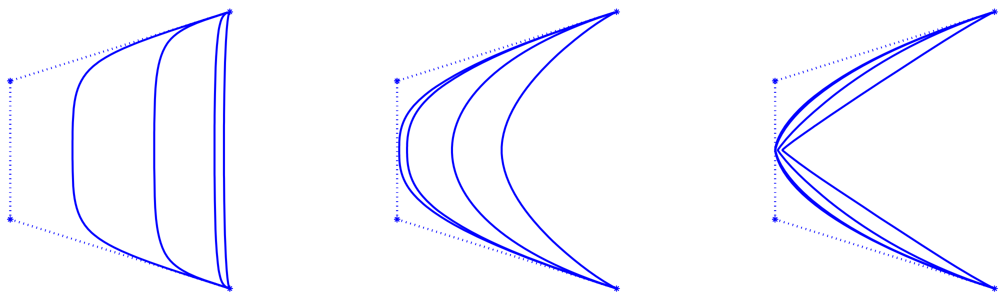

Fig. 5. Parametric curves in the symmetric ECP-space good for design on $([0, 2]; (1))$ composed of C^1 and F^3 piecewise cubic functions satisfying (76). In each figure, from right to left $\eta = -3.8$; 0; 36; 196 (that is, $e = 0.1$; 2; 20; 100). Left: $b = -3.9$. Middle: $b = 0$ (ordinary cubics for $\eta = 0$). Right: $b = 100$.

$$\begin{bmatrix} \overline{F}'(t_1) \\ \overline{F}''(t_1) \\ \overline{F}'(t_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ \frac{eb}{2} & e & 1 \end{bmatrix} \begin{bmatrix} F'(t_1) \\ F''(t_1) \\ F'(t_0) \end{bmatrix}. \tag{75}$$

Via Taylor expansions of $F'(t_0)$ and $\overline{F}'(t_2)$ at t_1, (75) can equivalently be written in the more usual way

$$\begin{bmatrix} \overline{F}'(t_1) \\ \overline{F}''(t_1) \\ \overline{F}'''(t_1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ \frac{\eta b}{2} & \eta & 1 \end{bmatrix} \begin{bmatrix} F'(t_1) \\ F''(t_1) \\ F'''(t_1) \end{bmatrix}, \quad \text{with } \eta := 2e - 4. \tag{76}$$

The functions in $\mathcal{F}$ are piecewise cubic on $\big([0, 2]; (1)\big)$, they are C^1 and they are F^3 in the usual sense of continuity of the Frenet frames of order three and of the first two curvatures at $t_1 = 1$. The symmetric space $\mathcal{F}$ is an ECP-space good for design on $\big([0, 2]; (1)\big)$ if and only if

$$b + 4 > 0 \quad \text{and} \quad \eta + 4 > 0.$$

This is illustrated in Fig. 5, for various values of the parameters η and b.

8. Conclusion

In this article we have identified the class of all QECP-spaces. This is interesting from a theoretical side, but the illustrations provided show that their construction also permits interesting shape effects.

In a QECP-space $\mathbb{E}$ assumed to be good for design, the presence of blossoms and their pseudoaffinity makes it possible to develop results for splines based on $\mathbb{E}$, with any multiplicities at the interior knots, exactly as we did for splines based on one single QEC-space in Mazure (2011d), or earlier for splines based on a given ECP-space in Mazure (1999). As usual, the forcefulness of blossoms is that their pseudoaffinity guarantees that such spline spaces satisfy all expected properties for design (development of all the classical CAGD algorithms for splines; presence of quasi-B-spline bases which are the optimal normalised totally positive bases). This is thus one step towards the identification of the largest possible class of splines (with ordinary differentiability assumptions on the sections) which can be used not only for Design, but also for Approximation, Interpolation, and even Isogeometric Analysis. The complete identification of this class, announced in Mazure (2015), will be the object of a further article.

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