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Greville abscissae of totally positive bases ☆

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ABSTRACT

For a given totally positive space of continuous functions, we analyze the construction of totally positive bases of the space of antiderivatives. If the functions of the totally positive space have continuous derivatives, normalization properties can be used to describe totally positive bases of the space of derivatives and relate them with properties of the Greville abscissae.

1. Introduction

Integral recurrence formulae for B-splines have been often used in the past. The definition of B-spline as a divided difference of a truncated power function and the Hermite–Gennochi formula lead to integral recursions. One of the first papers where it is observed that the sequence of B-spline bases can be obtained by successive integration in the general context of Chebyshevian splines is Bister and Prautzsch (1997). Bernstein polynomials as well as many other examples of totally positive bases in extended Chebyshev spaces are included in this setting.

Totally positive bases (TP) are bases whose collocation matrices have nonnegative minors. This kind of bases are commonly used in computer-aided design due to their shape preserving properties (see Goodman, 1996). Among all normalized TP bases of a space, we can find normalized B-bases, which are the optimal shape preserving bases (cf. Carnicer and Peña, 1994).

Spaces containing algebraic polynomials and trigonometric or hyperbolic functions have attracted much interest in the field of computer-aided geometric design (Zhang, 1996; Mainar et al., 2001). In Chen and Wang (2003), integral constructions of Bernstein-like basis for cycloidal spaces

$$C_n = \langle \cos t, \sin t, 1, t, \ldots, t^{n-2} \rangle$$

have been provided. In Costantini et al. (2005), such constructions are discussed in a more general setting, showing that the integral constructions provide TP bases. In particular, the normalized B-basis is expressed using integrals of a B-basis of the space of derivatives.

Greville abscissae are the coefficients of the function t with respect to a given basis and play a fundamental role in the definition of Bernstein-like operators in spaces of exponential polynomials (cf. Aldaz et al., 2009).

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In this paper, the construction of TP bases for the space of antiderivatives of a given space of continuous functions with a TP basis is analyzed. We also describe TP bases of the space of derivatives, whose normalization is related with properties of the Greville abscissae.

In contrast to other approaches, we do not require that the spaces with TP bases are extended Chebyshev or piecewise extendend Chebyshev spaces. In Mazure (2009), similar problems on integral constructions, derivative spaces and Greville abscissae are analyzed with powerful techniques, under the hypothesis that the space of derivatives is extended Chebyshev. In Section 7 of Mazure (2009), the question of extending the results to a more general context is explored. In Mazure (2011), piecewise Chebyshev spaces are analyzed using knot insertion techniques. In our approach, we deal not only with the normalized B-basis and we show that the integral or derivative constructions can be applied to the more general class of TP bases.

In Section 2, we describe integral constructions of normalized TP bases and normalized B-bases. The construction of TP bases and B-bases of the space of derivatives is presented in Section 3. In Section 4, we consider shape preserving representations of curves and we obtain a derivative formula of the curve involving the Greville abscissae. This formula relates the normalized B-basis of a given space with that of the space of derivatives. We also include examples illustrating that the integral constructions are valid even when the starting space is not extended Chebyshev. In Section 5, we show the equivalence of the existence of a normalized TP basis in the space of derivatives with the fact that Greville abscissae of shape preserving representations with the endpoint interpolation property are increasing. In Section 6, we present some applications, including sufficient conditions for Bernstein-like operators to be convexity preserving. Finally, we obtain a generalization of Theorem 25 of Aldaz et al. (2009) for general cycloidal spaces, deriving conditions on the length of the interval domain to ensure that the Greville abscissae of the normalized B-basis (corresponding to the nodes of the associated Bernstein operator) are strictly increasing.

2. Integral constructions with totally positive bases

Let us denote by $D : f \in C^1[a,b] \mapsto f' \in C[a,b]$ the derivative operator. For a given space of functions $U \subset C[a,b]$, we introduce the space

$$D^{-1}U := \{v \in C^1[a,b] \mid v' \in U\}.$$

Observe that $\ker D$ is the one dimensional space of constant functions. Hence, if $\dim U = n$, then $D^{-1}U$ contains the constant functions and $\dim D^{-1}U = n+1$.

Definition 1. A matrix is *totally positive* (TP) if all its minors are nonnegative. A system of functions $(u_0,\dots,u_n)$ defined on the subset $I \subseteq \mathbf{R}$ is *totally positive* (TP) if all its collocation matrices

$$M\begin{pmatrix} u_0,\dots,u_n \\ t_0,\dots,t_n \end{pmatrix} := (u_j(t_i))_{i,j=0,\dots,n}, \quad t_0 < \cdots < t_n \text{ in } I$$

are TP. A TP system of functions on I is *normalized* (NTP) if $\sum_{i=0}^n u_i(t) = 1$, for all $t \in I$.

In the following result we show how to construct a TP system of functions in $D^{-1}U$, starting from a TP system of functions in U.

Proposition 2. *Let U be an n-dimensional subspace of $C[a,b]$. If $(u_0,\dots,u_{n-1})$ is a TP system of functions in U, then the system $(f_0,\dots,f_n)$ defined by*

$$f_0(t) := 1, \quad f_i(t) := \int_a^t u_{i-1}(x)dx, \quad i = 1,\dots,n, \quad t \in [a,b], \tag{1}$$

is TP. Moreover, if $(u_0,\dots,u_{n-1})$ is a TP basis of U, then $(f_0,\dots,f_n)$ is a TP basis of $D^{-1}U$.

Proof. In order to prove the total positivity of $(f_0,\dots,f_n)$, it is sufficient to show that

$$d := \det M\begin{pmatrix} f_{i_0},\dots,f_{i_k} \\ t_0,\dots,t_k \end{pmatrix} \geq 0,$$

for every $i_0 < \cdots < i_k$ in $\{0,\dots,n\}$ and all $t_0 < \cdots < t_k$ in I. First let us analyze the case $i_0 \neq 0$. Subtracting to each row of the matrix $M\begin{pmatrix} f_{i_0},\dots,f_{i_k} \\ t_0,\dots,t_k \end{pmatrix}$ the previous one, taking into account the multilinearity of the determinant and the total positivity of $(u_0,\dots,u_{n-1})$, we deduce that

$$d=\det\begin{pmatrix}\int_a^{t_0} u_{i_0-1}(x)dx & \cdots & \int_a^{t_0} u_{i_k-1}(x)dx\\ \int_{t_0}^{t_1} u_{i_0-1}(x)dx & \cdots & \int_{t_0}^{t_1} u_{i_k-1}(x)dx\\ \vdots & \ddots & \vdots\\ \int_{t_{k-1}}^{t_k} u_{i_0-1}(x)dx & \cdots & \int_{t_{k-1}}^{t_k} u_{i_k-1}(x)dx\end{pmatrix}$$
$$=\int_a^{t_0}\int_{t_0}^{t_1}\cdots\int_{t_{k-1}}^{t_k}\det M\binom{u_{i_0-1},\dots,u_{i_k-1}}{s_0,\dots,s_k}ds_k\cdots ds_1 ds_0\geq 0.$$

The case $i_0=0$ follows analogously

$$d=\det\begin{pmatrix}\int_{t_0}^{t_1} u_{i_1-1}(x)dx & \cdots & \int_{t_0}^{t_1} u_{i_k-1}(x)dx\\ \vdots & \ddots & \vdots\\ \int_{t_{k-1}}^{t_k} u_{i_1-1}(x)dx & \cdots & \int_{t_{k-1}}^{t_k} u_{i_k-1}(x)dx\end{pmatrix}$$
$$=\int_{t_0}^{t_1}\cdots\int_{t_{k-1}}^{t_k}\det M\binom{u_{i_1-1},\dots,u_{i_k-1}}{s_1,\dots,s_k}ds_k\cdots ds_1\geq 0.$$

Therefore $(f_0,\dots,f_n)$ is TP.

Now, assume that $(u_0,\dots,u_{n-1})$ is a basis of U. Let $c_0,\dots,c_n$ be such that $\sum_{i=0}^n c_i f_i(t)=0$. Differentiating, we have that $\sum_{i=1}^n c_i u_{i-1}(t)=0$. By the linear independence of $(u_0,\dots,u_{n-1})$, $c_1=\cdots=c_n=0$ and then $c_0=c_0 f_0=0$. Therefore $(f_0,\dots,f_n)$ are linearly independent. Taking into account that $\dim D^{-1}U=n+1$, it follows that $(f_0,\dots,f_n)$ is a TP basis of $D^{-1}U$. $\square$

Let T,S,R be subsets of the real line and μ be a Borel measure. If $K:T\times S\to\mathbf{R}$ and $L:S\times R\to\mathbf{R}$ are two kernels such that $K(t,s)L(s,r)$ are μ-integrable on $s\in S$ for each $t\in T$, $r\in R$, the composition $M:T\times R\to\mathbf{R}$ is defined by

$$M(t,r):=\int_S K(t,s)L(s,r)d\mu(s).$$

The basic composition formula (2.5) of Section 2 of Chapter 1 of Karlin (1968) can be written as

$$\det(M(t_i,r_k))_{i,k=1}^m=\int_{s_1<\cdots<s_m}\det(K(t_i,s_j))_{i,j=1}^m\det(L(s_j,r_k))_{j,k=1}^m d\mu(s_1)\cdots d\mu(s_m).$$

Let us observe that the expansions of the determinants as multiple integrals in the proof of Proposition 2 can be regarded as a particular case, taking $T=[a,b]$, $S=[a,b]$, $R=\{0,\dots,n-1\}$, $K(t,s)=(t-s)_+^0$ and $L(s,i)=u_i(s)$. Then the total positivity of the system $(f_0,\dots,f_n)$ in (1) follows from the total positivity of the Heaviside kernel $K(t,s)=(t-s)_+^0$.

The following results summarize some properties of totally positive systems that can be found in Lemma 2.1 and Lemma 2.2 (i) of Carnicer and Peña (1994).

Lemma 3. *Let $(u_0,\dots,u_n)$ be a TP system of functions defined on $I\subseteq\mathbf{R}$.*

(a) The function $u_j(t)/u_i(t)$, defined on $I_i:=\{t\in I\mid u_i(t)\neq 0\}$, is nondecreasing for any $j>i$.
(b) Let $t_0\in I$ be such that $u_i(t_0)=0$ for some $i\in\{0,\dots,n\}$. Then either $u_i(t)=0$ for all $t\leq t_0$ or $u_j(t_0)=0$ for all $j\geq i$.

In Carnicer and Peña (1994), special TP bases called B-bases were introduced.

Definition 4. A system $(b_0,\dots,b_n)$ of linearly independent functions defined on I is a *B-basis* if it is TP and

$$\inf\left\{\frac{b_i(t)}{b_j(t)}\mid t\in I, b_j(t)\neq 0\right\}=0,\quad \forall i\neq j \text{ in } \{0,\dots,n\}. \tag{2}$$

In Carnicer and Peña (1994), it was shown that each finite dimensional space with a TP basis has a B-basis. Let us state the following result on NTP bases corresponding to Theorem 4.2 of Carnicer and Peña (1994).

Theorem 5. *Let U be a vector space of functions defined on an interval with an NTP basis. Then*

(i) There exists a unique NTP B-basis $(b_0, \dots, b_n)$.
(ii) A basis $(u_0, \dots, u_n) = (b_0, \dots, b_n)K$ of U is NTP if and only if the matrix K of change of basis is TP and stochastic.

Relevant examples of normalized B-bases are the Bernstein basis and the B-spline basis.

In Proposition 2.6 of Carnicer and Peña (1994), two TP bases were related by means of a lower triangular matrix L of change of basis. This result was obtained by applying Proposition 2.5 of Carnicer and Peña (1994) to a reordered basis with an upper triangular matrix of change of basis. We restate Proposition 2.6 of Carnicer and Peña (1994) adding extra information on the lower triangular matrix L. This information can be obtained from the proof of Proposition 2.5 of Carnicer and Peña (1994), where the matrix of change of basis is the inverse of a bidiagonal matrix.

Proposition 6. *Let $(v_0, \dots, v_n)$ be a TP system of linearly independent functions on a subset $I \subseteq \mathbf{R}$. Then there exists a TP system $(b_0, \dots, b_n)$ such that*

$$(v_0, \dots, v_n) = (b_0, \dots, b_n)L,$$

where L is a lower triangular TP matrix with unit diagonal whose inverse L^{-1} is a bidiagonal matrix and

$$\inf\left\{\frac{b_i(t)}{b_n(t)} \mid t \in I, b_n(t) \neq 0\right\} = 0, \quad i = 0, \dots, n-1.$$

The following result is a key tool to relate a B-basis of U with the normalized B-basis of $D^{-1}U$.

Theorem 7. *Let $(u_0, \dots, u_{n-1})$ be a TP basis of a space U in $C[a, b]$ such that*

$$\int_a^b u_i(x)dx = 1, \quad i = 0, \dots, n-1. \tag{3}$$

Let us define

$$\begin{aligned} u_0^1(t) &:= 1 - \int_a^t u_0(x)dx, \\ u_k^1(t) &:= \int_a^t (u_{k-1}(x) - u_k(x))dx, \quad k = 1, \dots, n-1, \\ u_n^1(t) &:= \int_a^t u_{n-1}(x)dx. \end{aligned} \tag{4}$$

Then $(u_0^1, \dots, u_n^1)$ is an NTP basis of $D^{-1}U$. Moreover, if $(u_0, \dots, u_{n-1})$ is a B-basis of U, then $(u_0^1, \dots, u_n^1)$ is the normalized B-basis of $D^{-1}U$.

Proof. By Proposition 2, the system $(f_0, \dots, f_n)$ given by (1) is TP. Taking into account (3), we deduce that $f_k(b) = 1$, $k = 0, \dots, n$. By Proposition 6, there exists a lower triangular TP matrix with unit diagonal L whose inverse is bidiagonal such that

$$(b_0, \dots, b_n) = (f_0, \dots, f_n)L^{-1}$$

is a TP basis of $D^{-1}U$ satisfying

$$\inf\{b_i(t)/b_n(t) \mid t \in [a, b], b_n(t) \neq 0\} = 0, \quad i = 0, \dots, n-1.$$

Let us observe that $b_n(t) = \int_a^t u_{n-1}(x)dx$ is a nonnegative nondecreasing continuous function with $b_n(b) = \int_a^b u_{n-1}(x)dx = 1$. Therefore

$$\{t \in [a, b] \mid b_n(t) \neq 0\} = (\alpha_n, b]$$

for some $\alpha_n \in (a, b]$. Using Lemma 3 (a), we deduce from the total positivity of $(b_0, \dots, b_n)$ that the functions $b_i(t)/b_n(t)$ are nonincreasing on $t \in (\alpha_n, b]$, $i = 0, \dots, n-1$. Therefore

$$0=\inf_{t\in(\alpha_n,b]}\frac{b_i(t)}{b_n(t)}=\frac{b_i(b)}{b_n(b)},$$

and $b_i(b)=0$ for all $i=0,\dots,n-1$. Since the matrix $L^{-1}=(\hat{l}_{ij})_{0\le i,j\le n}$ is a lower triangular and bidiagonal matrix with unit diagonal, we have

$$b_{k-1}(t)=f_{k-1}(t)+\hat{l}_{k,k-1}f_k(t),\quad k=1,\dots,n.$$

Evaluating at $t=b$, we deduce that

$$0=b_{k-1}(b)=f_{k-1}(b)+\hat{l}_{k,k-1}f_k(b)=\hat{l}_{k,k-1}+1,\quad k=1,\dots,n,$$

and $\hat{l}_{k,k-1}=-1$, $k=1,\dots,n$. Therefore the system $(u_0^1,\dots,u_n^1)$ defined in (4) coincides with the TP basis $(b_0,\dots,b_n)$ of the space $D^{-1}U$. We observe that

$$\sum_{k=0}^{n}u_k^1(t)=1+\int_a^t\Big[-u_0(t)+\sum_{k=1}^{n-1}(u_{k-1}(t)-u_k(t))+u_{n-1}(t)\Big]dt=1$$

and the system is a NTP basis of $D^{-1}U$.

Let us assume that $(u_0,\dots,u_{n-1})$ is a B-basis. Let

$$I_k:=\{t\in[a,b]\mid u_k(t)\ne 0\},\quad k=0,\dots,n-1,$$

and

$$I_k^1:=\{t\in[a,b]\mid u_k^1(t)\ne 0\},\quad k=0,\dots,n.$$

From (2), it is sufficient to show that $\inf_{t\in I_j^1}u_i^1(t)/u_j^1(t)=0$ for all $i\ne j$. Let us first assume that $j=0$. By Lemma 3 (a), u_i^1/u_0^1 is nondecreasing. Since $u_0^1(a)=1$, we have $a\in I_0^1$. Hence

$$\inf_{t\in I_0^1}\frac{u_i^1(t)}{u_0^1(t)}=\frac{u_i^1(a)}{u_0^1(a)}=0.$$

Now, let us consider the case, where $i>j>0$. By Lemma 3 (a), u_k/u_{j-1} are nondecreasing functions on I_{j-1} with $\inf_{t\in I_{j-1}}u_k(t)/u_{j-1}(t)=0$, for all $k\ge j$. Then, for any $\varepsilon>0$, there exists $\tau\in I_{j-1}$ such that

$$\frac{u_k(t)}{u_{j-1}(t)}\le\varepsilon\quad\text{for all } t\le\tau \text{ in } I_{j-1},\quad k\ge j.$$

Using Lemma 3 (b), we have that $u_k(t)=0$ if $u_{j-1}(t)=0$, $t\le\tau$, and we deduce that

$$0\le u_k(t)\le\varepsilon u_{j-1}(t),\quad u_{j-1}(t)-u_k(t)\ge(1-\varepsilon)u_{j-1}(t),\quad t\le\tau,\quad k\ge j.$$

Taking $k=j$, we have that

$$u_j^1(\tau)=\int_a^\tau(u_{j-1}(x)-u_j(x))dx\ge(1-\varepsilon)\int_a^\tau u_{j-1}(x)dx.$$

The integral $\int_a^\tau u_{j-1}(x)dx>0$ because $u_{j-1}\ge 0$ is continuous and $u_{j-1}(\tau)>0$. So $u_j^1(\tau)>0$, which implies that $\tau\in I_j^1$. We also deduce that

$$u_i^1(\tau)\le\varepsilon\int_a^\tau u_{j-1}(x)dx.$$

Thus $u_i^1(\tau)/u_j^1(\tau)\le\varepsilon/(1-\varepsilon)$. Taking ε arbitrarily small, we deduce that $\inf_{t\in I_j^1}u_i^1(t)/u_j^1(t)=0$. If $j=n$, then u_i^1/u_n^1 is nonincreasing by Lemma 3 (a). By (3), $u_n^1(b)=\int_a^b u_{n-1}(x)dx=1$ and $b\in I_n^1$. So, we have that

$$\inf_{t\in I_n^1}\frac{u_i^1(t)}{u_n^1(t)}=\frac{u_i^1(b)}{u_n^1(b)}=0.$$

Finally, we deal with the remaining case $i<j<n$. By Lemma 3 (a), u_k/u_j are nonincreasing functions on I_j with $\inf_{t\in I_j}u_k(t)/u_j(t)=0$, for all $k\le j-1$. Then, there exists $\sigma\in I_j$ such that

$$\frac{u_k(t)}{u_j(t)} \le \varepsilon \quad \text{for all } t \ge \sigma \text{ in } I_j, \quad k \le j-1.$$

Using Lemma 3 (b), we have that $u_k(t) = 0$ if $u_j(t) = 0$, $t \ge \sigma$, and we deduce that

$$0 \le u_k(t) \le \varepsilon u_j(t), \quad u_j(t) - u_k(t) \ge (1-\varepsilon) u_j(t), \quad t \ge \sigma, \quad k \le j-1.$$

Therefore we have that

$$u_j^1(\sigma) = \int_\sigma^b (u_j(x) - u_{j-1}(x))dx \ge (1-\varepsilon) \int_\sigma^b u_j(x)dx.$$

The integral $\int_\sigma^b u_j(x)dx > 0$ because $u_j \ge 0$ is continuous and $u_j(\sigma) > 0$. So $u_j^1(\sigma) > 0$, which implies that $\sigma \in I_j^1$. We also deduce the following inequality

$$u_i^1(\sigma) \le \varepsilon \int_\sigma^b u_j(x)dx.$$

Thus $u_i^1(\sigma)/u_j^1(\sigma) \le \varepsilon/(1-\varepsilon)$. Taking ε arbitrarily small, we deduce that $\inf_{t \in I_j^1} u_i^1(t)/u_j^1(t) = 0$. □

In Theorem 7 we have used a basis with the normalization property (3). Let us extend the above result for arbitrary TP bases.

Corollary 8. *Let $(u_0^0, \dots, u_{n-1}^0)$ be a TP basis of a space U in $C[a,b]$. Then $\int_a^b u_i^0(x)dx > 0$, for $i = 0, \dots, n-1$. Let us define*

$$\begin{aligned} u_0^1(t) &:= 1 - w_0 \int_a^t u_0^0(x)dx, \\ u_k^1(t) &:= w_{k-1} \int_a^t u_{k-1}^0(x)dx - w_k \int_a^t u_k^0(x)dx, \quad k = 1, \dots, n-1, \\ u_n^1(t) &:= w_{n-1} \int_a^t u_{n-1}^0(x)dx, \end{aligned} \tag{5}$$

where

$$w_i := \frac{1}{\int_a^b u_i^0(x)dx}, \quad i = 0, \dots, n-1. \tag{6}$$

Then $(u_0^1, \dots, u_n^1)$ is an NTP basis of $D^{-1}U$. Moreover, if $(u_0^0, \dots, u_{n-1}^0)$ is a B-basis of U then $(u_0^1, \dots, u_n^1)$ is the normalized B-basis of $D^{-1}U$.

Proof. Since $u_i^0 \ge 0$ are nonzero continuous functions, $\int_a^b u_i^0(x)dx > 0$, $i = 0, \dots, n-1$. So, the constants $w_0, \dots, w_{n-1}$ defined by (6) are positive. Let $u_i := w_i u_i^0$, $i = 0, \dots, n-1$. Then $(u_0, \dots, u_{n-1})$ is a TP basis of U satisfying (3). By Theorem 7, $(u_0^1, \dots, u_n^1)$ is an NTP basis of $D^{-1}U$.

If $(u_0^0, \dots, u_{n-1}^0)$ is a B-basis, so is $(w_0 u_0^0, \dots, w_{n-1} u_{n-1}^0)$ and hence, by Theorem 7, $(u_0^1, \dots, u_n^1)$ is the normalized B-basis. □

3. B-bases of the space of derivatives

In the following result, we show how to obtain a TP basis of the space of derivatives DU, by combining the derivatives of an NTP basis of U.

Theorem 9. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \ge 2$, with an NTP basis $(u_0, \dots, u_n)$. Then*

$$v_i(t) := \sum_{j=i}^n u_j(t), \quad t \in [a,b], \quad i = 0, \dots, n,$$

is a TP basis of U and $(v_1', \dots, v_n')$ is a TP basis of DU. Moreover, if $(u_0, \dots, u_n)$ is the normalized B-basis of U, then $(v_1', \dots, v_n')$ is a B-basis of DU.

Proof. Let E_{n+1} be the lower triangular $(n+1) \times (n+1)$ TP matrix

$$E_{n+1} := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

By the Cauchy–Binet formula (see page 1 of Karlin, 1968),

$$(v_0, \dots, v_n) = (u_0, \dots, u_n) E_{n+1},$$

is a TP system. Since E_{n+1} is nonsingular, $(v_0, \dots, v_n)$ is a TP basis of U. Hence $v_0', \dots, v_n'$ generate DU. Since $v_0' = 0$, we have that $(v_1', \dots, v_n')$ is a basis of DU. Given $a \le t_1 < \cdots < t_n < b$, we consider the TP matrix

$$M\binom{v_0, \dots, v_n}{t_1, t_1+h, t_2, t_2+h, \dots, t_n, t_n+h}, \quad 0 < h < b - t_n.$$

By subtracting to each row of the above matrix the previous row, we obtain a matrix of the form

$$\begin{pmatrix} 1 & * \\ 0 & M \end{pmatrix},$$

where M is a $(2n-1) \times n$ matrix. From Proposition 3.2 of Carnicer and Peña (1993), we have that M is a TP matrix. We extract the rows with odd index and deduce that the $n \times n$ matrix

$$h^{-1} \begin{pmatrix} v_1(t_1+h) - v_1(t_1) & \cdots & v_n(t_1+h) - v_n(t_1) \\ v_1(t_2+h) - v_1(t_2) & \cdots & v_n(t_2+h) - v_n(t_2) \\ \vdots & \ddots & \vdots \\ v_1(t_n+h) - v_1(t_n) & \cdots & v_n(t_n+h) - v_n(t_n) \end{pmatrix}$$

is TP. Taking limits as $h \to 0^+$, we obtain the TP matrix

$$M\binom{v_1', \dots, v_n'}{t_1, t_2, \dots, t_n}$$

and deduce that $(v_1', \dots, v_n')$ is TP on $[a, b)$. Since the functions $v_1', \dots, v_n'$ are continuous, we can take limits as $t_n \to b^-$ in the matrices above to obtain that $(v_1', \dots, v_n')$ is a TP system on $[a, b]$.

Let us now assume that $(u_0, \dots, u_n)$ is the normalized B-basis of U. Let $(g_0, \dots, g_{n-1})$ be a TP basis of DU. Let us construct $(g_0^1, \dots, g_n^1)$ from the basis $(g_0, \dots, g_{n-1})$ as in (5). By Corollary 8, $(g_0^1, \dots, g_n^1)$ is an NTP basis of U. By Theorem 5 (ii), there exists a stochastic TP matrix K such that

$$(g_0^1, \dots, g_n^1) = (u_0, \dots, u_n) K.$$

Then we have that

$$(g_0^1, \dots, g_n^1) E_{n+1} = (u_0, \dots, u_n) K E_{n+1} = (v_0, \dots, v_n) E_{n+1}^{-1} K E_{n+1}. \tag{7}$$

Observe that, by the Cauchy–Binet formula (cf. page 1 of Karlin, 1968), $K E_{n+1}$ is a TP matrix. The first column of $K E_{n+1}$ is $(1, \dots, 1)^T$ because K is stochastic. Subtracting to each row the previous one, we obtain the matrix

$$E_{n+1}^{-1} K E_{n+1} = \begin{pmatrix} 1 & * \\ 0 & K_1 \end{pmatrix},$$

which is TP by Proposition 3.2 of Carnicer and Peña (1993). Differentiating in (7), we deduce that

$$(0, g_0, \dots, g_{n-1}) = (0, v_1', \dots, v_n') E_{n+1}^{-1} K E_{n+1},$$

or equivalently $(g_0, \dots, g_{n-1}) = (v_1', \dots, v_n') K_1$. So, we have shown that each TP basis of DU can be expressed as $(v_1', \dots, v_n') K_1$, where K_1 is TP. By Proposition 3.11 of Carnicer and Peña (1994), $(v_1', \dots, v_n')$ is a B-basis. □

The following result extends Theorem 9 to obtain different normalizations of TP bases of the space of derivatives.

Corollary 10. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \geq 2$ with an NTP basis $(u_0^1, \dots, u_n^1)$. Let $w_0, \dots, w_{n-1}$ be a sequence of positive numbers. Then the system of functions $(u_0^0, \dots, u_{n-1}^0)$ defined by*

$$u_i^0(t) := \frac{1}{w_i} \sum_{j=i+1}^{n} (u_j^1)'(t), \quad t \in [a,b], \quad i = 0, \dots, n-1, \tag{8}$$

is a TP basis of DU. Moreover, if $(u_0^1, \dots, u_n^1)$ is the normalized B-basis of U, then $(u_0^0, \dots, u_{n-1}^0)$ is a B-basis of DU.

Proof. Let $u_i(t) := \sum_{j=i+1}^{n} (u_j^1)'(t)$, $i = 0, \dots, n-1$. By Theorem 9, $(u_0, \dots, u_{n-1})$ is a TP basis. Therefore the basis $(u_0^0, \dots, u_{n-1}^0)$ defined in (8) is also TP. Again by Theorem 9, if $(u_0^1, \dots, u_n^1)$ is the normalized B-basis of U, then $(u_0^0, \dots, u_{n-1}^0)$ is a B-basis of DU. $\square$

As a consequence of Corollary 8 and Corollary 10, we deduce the following generalization of Theorem 4.1 of Carnicer et al. (2004).

Theorem 11. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \geq 2$, such that $1 \in U$. Then U has an NTP basis if and only if DU has a TP basis.*

4. Derivatives of curve representations

A parametric curve can be represented in terms of a basis of nonnegative functions $(u_0, \dots, u_n)$ in the form

$$\gamma(t) = \sum_{i=0}^{n} P_i u_i(t), \quad t \in [a,b],$$

where $P_0, \dots, P_n \in \mathbb{R}^s$. The polygon $P_0 \cdots P_n$ is called the control polygon of γ. It is usually required the normalization property $\sum_{i=0}^{n} u_i(t) = 1$. In this case the representation is affine invariant and has the convex hull property.

The *endpoint interpolation property* of a curve representation means that $\gamma(a) = P_0$ and $\gamma(b) = P_n$. It is easy to verify that the endpoint interpolation property is equivalent to the fact that

$$u_i(a) = \delta_{i,0}, \quad u_i(b) = \delta_{i,n}, \quad i = 0, \dots, n, \tag{9}$$

where $\delta_{i,j}$ is the usual Kronecker symbol.

A representation of a curve is shape preserving if the shape properties of the curve γ are inherited from corresponding shape properties of its control polygon. Shape preserving representations are associated with NTP bases (cf. Goodman, 1996; Carnicer and Peña, 1993).

The analysis of geometric features of the curves demands the computation of the derivative of a curve. Many shape preserving representations corresponding to smooth bases have a formula for the derivative in terms of the sides $\Delta P_i = P_{i+1} - P_i$, $i = 0, \dots, n-1$. For obtaining such a formula, the following identity

$$\sum_{i=0}^{n} c_i u_i = c_0 \sum_{i=0}^{n} u_i + \sum_{i=0}^{n-1} (c_{i+1} - c_i) \sum_{j=i+1}^{n} u_j, \text{ for any } c_0, \dots, c_n, u_0, \dots, u_n \in \mathbf{R},$$

known as *Abel's Lemma*, can be used. Our purpose is to show that this kind of formula arises in each shape preserving representation.

Proposition 12. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \geq 2$ with an NTP basis $(u_0^1, \dots, u_n^1)$. Let $w_0, \dots, w_{n-1}$ be a sequence of positive numbers and let $(u_0^0, \dots, u_{n-1}^0)$ be the TP basis defined by (8). Then the derivative of the curve $\gamma(t) = \sum_{i=0}^{n} P_i u_i^1(t)$, $t \in [a,b]$, can be expressed by the formula*

$$\gamma'(t) = \sum_{i=0}^{n-1} w_i \Delta P_i u_i^0(t), \quad t \in [a,b].$$

Proof. By Abel's Lemma and the fact that $\sum_{j=0}^{n} u_j^1(t) = 1$, it follows that

$$\gamma(t) = P_0 \sum_{j=0}^{n} u_j^1(t) + \sum_{i=0}^{n-1} \Delta P_i \sum_{j=i+1}^{n} u_j^1(t) = P_0 + \sum_{i=0}^{n-1} \Delta P_i \sum_{j=i+1}^{n} u_j^1(t).$$

Taking into account that $w_i u_i^0(t)$ is the derivative of $\sum_{j=i+1}^{n} u_j^1(t)$, we may differentiate and obtain the desired formula. $\square$

We now introduce Greville abscissae and show how they can be used to find a normalized basis of the space of derivatives and relate the control polygons of a curve and its derivative with respect to two normalized bases.

Definition 13. Let U be a space of continuous functions on $[a,b]$ and let $(u_0,\dots,u_n)$ be a basis of nonnegative functions of U such that $\sum_{i=0}^n u_i(t)=1$, $t\in[a,b]$. If the function t belongs to U, the *Greville abscissae* with respect to the basis $(u_0,\dots,u_n)$ are defined as the unique coefficients $t_0,\dots,t_n$ in the expansion $t=\sum_{i=0}^n t_i u_i(t)$.

The Greville abscissae can be used to represent geometrically a control polygon for the graph of a function $u=\sum_{i=0}^n c_i u_i(t)$. From the formula

$$\begin{pmatrix} t \\ u(t) \end{pmatrix} = \sum_{i=0}^n \begin{pmatrix} t_i \\ c_i \end{pmatrix} u_i(t),$$

we see that the control points of the graph are $(t_i,c_i)^T$, $i=0,\dots,n$.

Observe that, if U and DU are spaces with NTP bases on $[a,b]$, then U contains all polynomials of degree less than or equal to 1, that is, $1,t\in U$.

Among all shape preserving representations there exists an optimal one, where the curve best imitates the shape of its control polygon. The normalized B-basis provides this optimal shape preserving representation, as shown in Carnicer and Peña (1994). Let us show that a derivative formula involving the Greville abscissae relates the normalized B-bases of both spaces.

Theorem 14. *Let U be a subspace of $C^1[a,b]$, $\dim U=n+1\geq 2$, with an NTP basis $(u_0^1,\dots,u_n^1)$. Assume that $t\in U$ and that the Greville abscissae with respect to the basis $(u_0^1,\dots,u_n^1)$ form a strictly increasing sequence*

$$t_0<t_1<\cdots<t_n. \tag{10}$$

Then the derivative of $\gamma(t)=\sum_{i=0}^n P_i u_i^1(t)$ is given by

$$\gamma'(t)=\sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\Delta P_i u_i^0(t), \tag{11}$$

where

$$u_i^0(t):=(t_{i+1}-t_i)\sum_{j=i+1}^n (u_j^1)'(t), \quad t\in[a,b], \quad i=0,\dots,n-1, \tag{12}$$

is an NTP basis of DU. Furthermore, if $(u_0^1,\dots,u_n^1)$ is the normalized B-basis, then $(u_0^0,\dots,u_{n-1}^0)$ is the normalized B-basis of DU.

Proof. If we apply Proposition 12 to the 1-dimensional curve whose control polygon is $t_0\cdots t_n$

$$t=\sum_{i=0}^n t_i u_i^1(t),$$

we get

$$1=\sum_{i=0}^{n-1}(t_{i+1}-t_i)w_i u_i^0(t)$$

and $(u_0^0,\dots,u_{n-1}^0)$ is a TP basis. So we see that the necessary and sufficient condition for $(u_0^0,\dots,u_{n-1}^0)$ defined in (8) to be an NTP basis of DU is that

$$w_i=\frac{1}{t_{i+1}-t_i}, \quad i=0,\dots,n-1. \tag{13}$$

Furthermore if $(u_0^1,\dots,u_n^1)$ is the normalized B-basis, then $(u_0^0,\dots,u_{n-1}^0)$ must be the normalized B-basis of DU. □

Formula (11) is a generalization of the well known formulae for the derivative of Bézier and B-spline curves.

In formula (13) we have suggested a relation between $w_0,\dots,w_n$ and the Greville abscissae. Let us restate Corollary 8 in the case that U and DU have normalized TP bases, expressing $w_0,\dots,w_n$ of formula (6) in terms of the Greville abscissae. We also deduce that the Greville abscissae must be strictly increasing.

Theorem 15. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \geq 2$, such that $1, t \in U$. If DU has an NTP basis $(u_0^0, \dots, u_{n-1}^0)$, then there exists an NTP basis of U given by*

$$u_0^1(t) := 1 - \frac{1}{t_1 - t_0} \int_a^t u_0^0(x)dx,$$

$$u_k^1(t) := \frac{1}{t_k - t_{k-1}} \int_a^t u_{k-1}^0(x)dx - \frac{1}{t_{k+1} - t_k} \int_a^t u_k^0(x)dx, \quad k = 1, \dots, n-1,$$

$$u_n^1(t) := \frac{1}{t_n - t_{n-1}} \int_a^t u_{n-1}^0(x)dx, \tag{14}$$

such that the endpoint interpolation property and (12) hold. The Greville abscissae of $(u_0^1, \dots, u_n^1)$ are

$$t_i = a + \sum_{j=0}^{i-1} \int_a^b u_j^0(x)dx, \quad i = 0, \dots, n, \tag{15}$$

and then they are strictly increasing. Moreover, if $(u_0^0, \dots, u_{n-1}^0)$ is the normalized B-basis of DU, then $(u_0^1, \dots, u_n^1)$ is the normalized B-basis of U.

Proof. Since u_k^0 is a nonnegative continuous and nonzero function, we have that $\int_a^b u_k^0(x)dx > 0$, $k = 0, \dots, n-1$. Let t_i be defined by (15). Clearly

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b.$$

Let us define $w_k := 1/\int_a^b u_k^0(x)dx > 0$, $k = 0, \dots, n-1$ as in (6). Then we have that

$$t_i = a + \sum_{j=0}^{i-1} w_i^{-1}, \quad i = 0, \dots n,$$

and

$$\frac{1}{t_{k+1} - t_k} = w_k, \quad k = 0, \dots, n-1.$$

By Corollary 8, $(u_0^1, \dots, u_n^1)$ is an NTP basis. Moreover, if $(u_0^0, \dots, u_{n-1}^0)$ is the normalized B-basis of DU, then $(u_0^1, \dots, u_n^1)$ is the normalized B-basis of U. Let us remark that $t_0 = a$. By Abel's Lemma, we have that

$$\sum_{i=0}^n t_i u_i^1(t) = a + \sum_{i=0}^n (t_i - a) u_i^1(t) = a + \sum_{i=0}^{n-1} (t_{i+1} - t_i) \sum_{j=i+1}^n u_j^1(t).$$

Using the normalization property of $(u_0^0, \dots, u_{n-1}^0)$ we have that

$$\sum_{i=0}^n t_i u_i^1(t) = a + \sum_{i=0}^{n-1} \int_a^t u_i^0(x)dx = a + \int_a^t dx = t,$$

which implies that $t_0, \dots, t_n$ are the Greville abscissae with respect to the basis $(u_0^1, \dots, u_n^1)$. From (14), it follows that

$$\sum_{j=i+1}^n u_j^1(t) = \frac{1}{t_{i+1} - t_i} \int_a^t u_i^0(x)dx. \tag{16}$$

Differentiating the above equation, we get (12). Evaluating (14) at a we have

$$u_0^1(a) = 1, \quad u_k^1(a) = 0, \quad k = 1, \dots, n.$$

Taking into account that

$$t_{k+1} - t_k = w_k^{-1} = \int_a^b u_k^0(x)dx, \quad k = 0, \dots, n-1,$$

we obtain evaluating (14) at b

$$u_k^1(b)=0, \quad k=0,\dots,n-1, \quad u_n^1(b)=1.$$

Therefore $(u_0^1,\dots,u_n^1)$ satisfies the endpoint interpolation property. □

Remark 16. The basis in (14) is the unique basis with the endpoint interpolation property such that (12) holds. Differentiating in (14) we obtain the relations

$$\begin{aligned}
(u_0^1)'(t) &= -\frac{u_0^0(t)}{t_1-t_0},\\
(u_k^1)'(t) &= \frac{u_{k-1}^0(t)}{t_k-t_{k-1}} - \frac{u_k^0(t)}{t_{k+1}-t_k}, \quad k=1,\dots,n-1,\\
(u_n^1)'(t) &= \frac{u_{n-1}^0(t)}{t_n-t_{n-1}},
\end{aligned}$$

which are equivalent to (12). The endpoint interpolation property determines the integration constant showing the uniqueness.

Theorem 15 can be successively applied to construct B-bases of spaces obtained by successive integration. If we start with a space $U_n \in C[a,b]$, $\dim U_n = n+1$, with a normalized B-basis, we can construct normalized B-bases of the spaces $U_m = D^{-(m-n)}U_n$, $m>n$. We remark that the functions in the space U_n need not be differentiable and hence U_n might not be an extended Chebyshev space.

Starting with the normalized B-basis $(b_0^n(x),\dots,b_n^n(x))$ of U_n, the normalized B-basis of U_m can be obtained by the recurrence

$$\begin{aligned}
b_0^m(t) &:= 1 - \frac{1}{t_1^m - t_0^m}\int_a^t b_0^{m-1}(x)dx,\\
b_j^m(t) &:= \frac{1}{t_j^m - t_{j-1}^m}\int_a^t b_{j-1}^{m-1}(x)dx - \frac{1}{t_{j+1}^m - t_j^m}\int_a^t b_j^{m-1}(x)dx, \; j=1,\dots,m-1,\\
b_m^m(t) &:= \frac{1}{t_m^m - t_{m-1}^m}\int_a^t b_{m-1}^{m-1}(x)dx,
\end{aligned}$$

where

$$t_i^m = a + \sum_{j=0}^{i-1}\int_a^b b_j^{m-1}(x)dx.$$

In Theorem 4.8 of Mazure (2009), it is shown that the Greville abscissae are increasing if the space of derivatives is extended Chebyshev. Theorem 15 is a similar result, but we require the weaker hypothesis that the space of derivatives has an NTP basis. In the following examples we show that Theorem 15 can be applied when differentiability conditions fail.

Example 17. A remarkable example of the above construction (already mentioned in Bister and Prautzsch, 1997), is the case where the starting space U_n is the space of linear splines on an interval $[a,b]$ with knots

$$a=\tau_0=\tau_1<\tau_2<\cdots<\tau_n<\tau_{n+1}=\tau_{n+2}=b.$$

The normalized B-basis of U_n is the B-spline basis $b_j^n(t)=N(t;\tau_j,\tau_{j+1},\tau_{j+2})$, $j=0,\dots,n$, consisting of "hat" functions. Successive integration gives rise to b_j^{n+k-1}, $j=0,\dots,n+k-1$, the B-spline basis of degree k of the space U_{n+k-1} of splines of degree k with the same interior knots. This important example shows that our construction is valid if the space is not extended Chebyshev. We remark that the space U_n is not an extended Chebyshev space if $n>1$ and that the functions are not differentiable everywhere.

Example 18. The construction of Theorem 15 allows us to deal with other situations where the space fails to be extended Chebyshev. In Costantini et al. (2005), the authors consider the spaces

$$U_3 := \langle 1, t, (1-t)^{m_0}, t^{m_1}\rangle, \quad t\in[0,1], \quad m_0, m_1 \ge 3$$

that provide interesting performances related with shape control. The space U_1, generated by $(1-t)^{m_0-2}, t^{m_1-2}$ on the interval $[0,1]$ is not an extended Chebyshev space, unless $m_0 = m_1 = 3$, because there are too many zeros at the ends of the interval.

Let us start with the B-basis $(m_0-1)(1-t)^{m_0-2}, (m_1-1)t^{m_1-2}$ of U_1 and apply Theorem 7 to obtain the normalized B-basis of the space $U_2 = D^{-1}U_1$

$$b_0^2(t) = (1-t)^{m_0-1}, \quad b_1^2(t) = 1-(1-t)^{m_0-1} - t^{m_1-1}, \quad b_2^2(t) = t^{m_1-1}.$$

Again, taking into account the multiplicity of the zeros at the ends of the interval, we have that U_2 is not an extended Chebyshev space, unless $m_0 = m_1 = 3$.

Using Theorem 15, we obtain the Greville abscissae

$$t_0^3 = 0, \quad t_1^3 = \frac{1}{m_0}, \quad t_2^3 = 1 - \frac{1}{m_1}, \quad t_2^3 = 1.$$

We define

$$\varphi(t) := \frac{1}{t_2^2 - t_1^2} \int_0^t b_1^2(x)dx = \frac{t - (1-(1-t)^{m_0})/m_0 - t^{m_1}/m_1}{1 - 1/m_0 - 1/m_1}$$

and using (14), we can express the normalized B-basis of the space U_3 in the following form

$$b_0^3(t) = 1 - m_0 \int_0^t b_0^2(x)dx = (1-t)^{m_0},$$

$$b_1^3(t) = m_0 \int_0^t b_0^2(x)dx - \varphi(t) = 1 - (1-t)^{m_0} - \varphi(t),$$

$$b_2^3(t) = \varphi(t) - m_1 \int_0^t b_2^2(x)dx = \varphi(t) - t^{m_1},$$

$$b_3^3(t) = m_1 \int_0^t b_2^2(x)dx = t^{m_1}.$$

5. Characterizations and properties of the Greville abscissae

Corollary 8 constructs an NTP basis of $D^{-1}U$ from a TP basis of U. Let us show that formulae (5) and (6) can provide a TP basis only when the initial basis of U is TP.

Theorem 19. *Let $(u_0, \ldots, u_{n-1})$ be a basis of a space U in $C[a,b]$ such that $\int_a^b u_i(x)dx \neq 0$, $i = 0, \ldots, n$. Let $w_0, \ldots, w_{n-1}$ be given by (6) and $(u_0^1, \ldots, u_n^1)$ be the basis defined by (5). Then $(u_0, \ldots, u_{n-1})$ is a TP basis of U if and only if $(u_0^1, \ldots, u_n^1)$ is an NTP basis of $D^{-1}U$ and $w_0, \ldots, w_n$ are positive. Moreover, $(u_0, \ldots, u_{n-1})$ is a B-basis of U if and only if $(u_0^1, \ldots, u_n^1)$ is the normalized B-basis of $D^{-1}U$.*

Proof. If $(u_0, \ldots, u_{n-1})$ is a TP basis, then by Corollary 8, $(u_0^1, \ldots, u_n^1)$ is an NTP basis. Moreover, if $(u_0, \ldots, u_{n-1})$ is a B-basis, then $(u_0^1, \ldots, u_n^1)$ is the normalized B-basis. Conversely, let us assume that $(u_0^1, \ldots, u_n^1)$ is an NTP basis and let

$$v_i(t) := \sum_{j=i}^n u_j^1(t), \quad t \in [a,b], \quad i = 0, \ldots, n.$$

Then by Theorem 9, $(v_1', \ldots, v_n')$ is a TP basis. Since

$$v_i'(t) = w_{n-1}u_{n-1}(t) + \sum_{j=i}^{n-1}(w_{j-1}u_{j-1}(t) - w_j u_j(t)) = w_{i-1}u_{i-1}(t),$$

we have that $(w_0u_0, \ldots, w_{n-1}u_{n-1})$ is a TP basis and, since $w_0, \ldots, w_{n-1}$ are positive, then $(u_0, \ldots, u_{n-1})$ is also a TP basis. Moreover if $(u_0^1, \ldots, u_n^1)$ is the normalized B-basis, then by Theorem 9, the system $(w_0u_0, \ldots, w_{n-1}u_{n-1})$ is a B-basis and so $(u_0, \ldots, u_{n-1})$ is also a B-basis. □

The following theorem relates the existence of an NTP basis of the space of derivatives with properties of the Greville abscissae of shape preserving representations with the endpoint interpolation property.

Theorem 20. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \geq 2$, such that $1, t \in U$, with an NTP basis. The following properties are equivalent:*

(a) There exists an NTP basis of DU.
(b) There exists an NTP basis of U with strictly increasing Greville abscissae.
(c) The Greville abscissae with respect to any NTP basis of U with the endpoint interpolation property are strictly increasing.

Furthermore, if $U \subset C^2[a,b]$, then the previous properties are also equivalent to the fact that D^2U has a TP basis.

Proof. Since U has an NTP basis, we deduce from Theorem 5 that U has an NTP B-basis. By Proposition 4.5 of Carnicer and Peña (1994), the NTP B-basis satisfies the endpoint interpolation property. Therefore there exists an NTP basis of U with the endpoint interpolation property.

(a) $\Longrightarrow$ (c). By Theorem 5, there exists a normalized B-basis of DU and each TP basis of DU can be normalized. Let $(u_0^1, \ldots, u_n^1)$ be an NTP basis of U with the endpoint interpolation property. By Proposition 10, the basis

$$u_i^0(t) := \frac{1}{w_i} \sum_{j=i+1}^{n} (u_j^1)'(t), \quad t \in [a,b], \quad i = 0, \ldots, n-1,$$

is TP. Since each TP basis of DU can be normalized, we can choose suitable positive constants $w_0, \ldots, w_n$ so that $(u_0^0, \ldots, u_{n-1}^0)$ is an NTP basis. Let $t_0 = a$, $t_i = a + \sum_{j=0}^{i-1} w_j^{-1}$, $i = 1, \ldots, n$. Therefore

$$w_i = \frac{1}{t_{i+1} - t_i}, \quad i = 0, \ldots, n-1,$$

and (12) holds. By Remark 16, $(u_0^1, \ldots, u_n^1)$ is the unique basis with the endpoint interpolation property satisfying relations (12). Integrating we deduce that (14) holds. Evaluating at $t = b$ in (11) and taking into account that $(u_0^1, \ldots, u_n^1)$ satisfies the endpoint interpolation property we obtain

$$\frac{\int_a^b u_0^0(x)dx}{t_1 - t_0} = 1, \quad \frac{\int_a^b u_{n-1}^0(x)dx}{t_n - t_{n-1}} = 1,$$
$$\frac{\int_a^b u_{k-1}^0(x)dx}{t_k - t_{k-1}} = \frac{\int_a^b u_k^0(x)dx}{t_{k+1} - t_k}, \quad k = 1, \ldots, n-1.$$

Therefore

$$w_k = \frac{1}{t_{k+1} - t_k} = \int_a^b u_k^0(x)dx, \quad k = 0, \ldots, n.$$

By Theorem 15, the strictly increasing sequence $t_0 < t_1 < \cdots < t_n$ are the Greville abscissae of $(u_0^1, \ldots, u_n^1)$.

(c) $\Longrightarrow$ (b) follows from the fact that U has NTP bases with the endpoint interpolation property. Finally, for (b) $\Longrightarrow$ (a) we use Theorem 14.

Let us finally assume that $U \subset C^2[a,b]$. By Theorem 11, D^2U has a TP basis if and only if DU has an NTP basis, which is equivalent to (a). □

Corollary 21. *Let U be a subspace of $C^1[a,b]$, $\dim U = n+1 \geq 2$, such that $1, t \in U$ with an NTP basis. Then the Greville abscissae with respect to the normalized B-basis of U are strictly increasing if and only if DU has a normalized B-basis.*

Proof. From Lemma 3 (a) and (2), it follows that the normalized B-basis of U always satisfies the endpoint interpolation property. By Theorem 5, the existence of an NTP basis of DU implies that DU has a normalized B-basis. The result readily follows applying Theorem 20. □

In Example 18 we describe a space $U_3 := \langle 1, t, (1-t)^{m_0}, t^{m_1} \rangle \subset C^2[0,1]$, where DU_3 has a normalized B-basis. Corollary 21 implies that the normalized B-basis of U_3 has strictly increasing Greville abscissae although the space DU_3 is not an extended Chebyshev space. Analogously, the Greville abscissae of the normalized B-basis of the quadratic spline space U_{n+1} considered in Example 17 are strictly increasing. However the space of derivatives U_n is the space of linear splines, which is not an extended Chebyshev space if $n > 1$.

6. Applications

Let us first apply the above results to the construction of Bernstein-like operators $B : C[a,b] \to U$ on spaces U with NTP bases. A Bernstein-like operator is

$$B : f \in C[a,b] \mapsto B[f] := \sum_{i=0}^{n} f(t_i) b_i,$$

where $(b_0, \dots, b_n)$ is an NTP basis of U. From its definition it follows that B preserves constants. However it is usually required that a strictly monotonic function is also preserved. With a change of variables, one can assume that this function is t. So we have

$$t = B[t] = \sum_{i=0}^{n} t_i b_i(t),$$

and $t_0, \dots, t_n$ are just the Greville abscissae. It is important that the Greville abscissae are distinct in order to have a surjective operator. For many questions concerning the analysis of Bernstein operators, the fact that the Greville abscissae are strictly increasing plays an important role.

The operator B is convexity preserving if $B[f]$ is convex for any convex function $f \in C[a,b]$. If the Greville abscissae are $t_0 < \cdots < t_n$, then Proposition 2.7 of Carnicer et al. (1995) shows that $B[f]$ is convexity preserving. The proof is based on the fact that

$$M\binom{1, t, B[f]}{x_0, x_1, x_2} = M\binom{b_0, \dots, b_n}{x_0, x_1, x_2} \begin{pmatrix} 1 & t_0 & f(t_0) \\ 1 & t_1 & f(t_1) \\ \vdots & \vdots & \vdots \\ 1 & t_n & f(t_n) \end{pmatrix}, \quad x_0 < x_1 < x_2.$$

All 3×3 minors of the first factor are nonnegative because $(b_0, \dots, b_n)$ is TP. Convexity of f implies that all 3×3 minors of the second factor are nonnegative. Using the Cauchy–Binet formula (see page 1 of Karlin, 1968), we find that the left hand side has nonnegative determinant for any $x_0 < x_1 < x_2$ and thus $B[f]$ is convex.

The results in this paper can be used in the problem of finding conditions for spaces of exponential polynomials to have strictly increasing Greville abscissae addressed in Aldaz et al. (2009). By Theorem 20, the search of NTP bases in the space of derivatives DU is a key tool to deal with this question.

A space of $U \subset C^n(I)$, $\dim U = n+1$, defined on an interval I is *extended Chebyshev* on a subinterval $J \subseteq I$ if the number of zeros in J counting multiplicities of any nonzero function in U is less than or equal to n. In Theorem 2.4 of Carnicer et al. (2004), it was shown that if U is extended Chebyshev on $[a,b]$, then there exists a TP basis of U. In Theorem 4.1 of Carnicer et al. (2004), it was shown that if DU is extended Chebyshev on $[a,b]$ and $1 \in U$, then there exists a NTP basis of U.

A space $U \subset C(\mathbf{R})$ is invariant under translations if $u_\tau(t) := u(t-\tau)$, $t \in \mathbf{R}$, belongs to U for any $u \in U$ and $\tau \in \mathbf{R}$. If $U \subset C^1(\mathbf{R})$ is invariant under translations, then $h^{-1}(u_{-h} - u) \in U$ for any $u \in U$ and $h \neq 0$ and, taking limits as $h \to 0$, we deduce that $DU \subseteq U$.

If $U \subset C^n(\mathbf{R})$, $\dim U = n+1$, is invariant under translations, the *critical length* is defined (see Carnicer et al., 2004) as the positive number $\ell(U)$ such that U is extended Chebyshev on an interval I if and only if I does not contain an interval of the form $[a, a+\ell(U)]$.

The following result relates critical lengths with the fact that the Greville abscissae are strictly increasing.

Theorem 22. *Let $U \subset C^n(\mathbf{R})$, $\dim U = n+1$, $n \geq 2$, be a space invariant under translations such that $1, t \in U$. If $b - a < \ell(DU)$, then U is extended Chebyshev with an NTP basis satisfying the endpoint interpolation property on $[a,b]$. Moreover, if $b - a < \ell(D^2U) \leq \ell(DU)$, then the Greville abscissae with respect to any NTP basis of U on $[a,b]$ with the endpoint interpolation property are strictly increasing.*

Proof. From the definition of critical length, if $b - a < \ell(DU)$, then DU is an extended Chebyshev space on $[a,b]$. By Theorem 4.1 of Carnicer et al. (2004) we have that U is an extended Chebyshev space with a normalized B-basis on $[a,b]$. From Theorem 5 and Proposition 4.5 of Carnicer and Peña (1994), it follows that the NTP B-basis of U satisfies the endpoint interpolation property.

By Corollary 4.1(i) of Carnicer et al. (2004), $\ell(D^2U) \leq \ell(DU)$. If $b - a < \ell(D^2U)$, then D^2U is an extended Chebyshev space. By Theorem 2.4 (iii) of Carnicer et al. (2004), D^2U has a TP basis on $[a,b]$. By Theorem 20, the Greville abscissae with respect to any NTP basis of U on $[a,b]$ with the endpoint interpolation property are strictly increasing. $\square$

We end by applying the above result to the particular case of cycloidal spaces. The $(n+1)$-dimensional *cycloidal space* is defined as

$$C_n := \langle \cos t, \sin t, 1, t, \dots, t^{n-2} \rangle, \quad n \geq 2.$$

For $n = 1$, it is convenient to define $C_1 := \langle \cos t, \sin t \rangle$. Clearly $\ell(C_1) = \pi$. Observe that $D^2 C_n = C_{n-2}$, $n \geq 3$.

In Theorem 25 of Aldaz et al. (2009) and Example 2.17 of Mazure (2009), it is shown that the Greville abscissae of the normalized B-basis of the space $C_3 = \langle \cos t, \sin t, 1, t \rangle$ on the interval $[a, b]$ are strictly increasing if $b - a < \pi = \ell(C_1)$. In the case that $b - a = \pi$, two Greville abscissae coincide. Finally in the cases that $\pi < b - a < 2\pi$, they show that the Greville abscissae are not monotonic. Using Theorem 22, we can deduce the following result valid for any cycloidal space.

Corollary 23. *If $b - a < \ell(C_{n-2})$, $n \geq 3$, then the Greville abscissae for any NTP basis of C_n on $[a, b]$ with the endpoint interpolation property are strictly increasing.*

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