Bending and buckling of nonlocal strain gradient elastic beams

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ABSTRACT

Featured by the two material length parameters in the nonlocal strain gradient theory, it is still unknown that what are the boundary conditions of nonlocal strain gradient beams, since the equations of motion and boundary conditions of these beam models appear in the same form as those of the classical ones. Based on the weighted residual approaches, this paper provides the boundary value problems of Euler–Bernoulli beams within the framework of the nonlocal strain gradient theory in conjunction with the von Kármán nonlinear geometric relation. The closed-form solutions for bending and buckling loads of nonlocal strain gradient beams are obtained. Numerical results show that the higher-order boundary conditions have no effect on the static bending deflection of beams for the cases studied. However, the higher-order boundary conditions and the material length parameters have a significant effect on the buckling loads. Finally, when the two material length parameters are the same, the buckling loads can not always reduce to the classical solutions, the findings of which violate our expectations. The results provided in this work are expected to be helpful for the applications of this theory to the analysis of engineering structures.

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1. Introduction

Engineering structures such as beams, plates and shells have been widely used in micro- and nano-sized sensors, actuators, atomic force microscopes. In these applications, size effects of material properties are observed at small sizes both in experimental works [1–4] and in numerical simulations [5–7]. The aforementioned works show that the materials exhibit either stiffening behaviors or softening behaviors in comparison to the bulk cases. Therefore, continuum theories that can capture the size effects of materials at small sizes have attracted considerable attention in the research communities with the view toward a better understanding and characterization of materials.

Based on the concept that the stress at a reference point is not only a function of the reference point, but also the strain at all points of the body, Eringen [8] developed an elasticity theory for the applications in surface waves. With the emerging of carbon nanotubes and graphene sheets, this theory have been extended to the study of the static and dynamic behaviors of structures in terms of rods [9–14], beams [14–25], plates [26–33] and shells [34–38]. For more details, the interested reader may refer to the recent reviews by Arash and Wang [39] and Eltaher, et al. [40].

In general, the use of this theory results in the softening effect when it is compared with the classical elasticity theory. However, two issues violate the softening phenomena. The first issue is that the bending solutions of nonlocal models in some cases are found to be the same as the classical solutions. In other words, the size effects vanish for cantilever beams subjected to concentrated forces [41]. To address this issue, Challamel and Wang [42] proposed a gradient elastic model as well as an integral nonlocal elastic model that is based on combining the local and the nonlocal curvatures in the constitutive relation. After this, several fresh ideas are raised to clarify this issue [16,43–46]. For example, Khodabakhshi and Reddy [43] developed a unified integro-differential nonlocal elasticity model and used this model to the bending of Euler–Bernoulli beams. Fernández-Sáez et al. [46] investigated the bending problems of Euler–Bernoulli beams using the Eringen integral constitutive equation. The closed-form bending solutions of Euler–Bernoulli beams and Timoshenko beams subjected to different loading and boundary conditions were carried out by Tuna and Kirca [16]. It appears that the first issue can be solved with the aid of the integro-differential nonlocal elasticity theory. The second issue is that one can only obtain a few natural frequencies of free vibrations of cantilever beams and that the counterintuitive stiffening effect is observed. This issue has been analytically solved by Xu et al. [47] using the weighted residual approaches (WRAs). In their work, they reformulated the variational-consistent boundary

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conditions, and presented the closed-form frequency solutions for Euler–Bernoulli beams and Timoshenko beams. The solutions of the above-mentioned issues demonstrate that, when one uses the nonlocal elasticity, the boundary conditions should be correctly employed rather than simply replacing the classical force resultants by the nonclassical force resultants in the equilibrium equations.

Within the framework of strain gradient elasticity theory, it emphasizes that materials constituting the body can be considered as atoms with higher-order deformation mechanism at small scales. Significant contributions in this field can be found in Mindlin and Tiersten [48], Toupin [49], recently in Yang et al. [50], Lam et al. [1] and Zhou et al. [51] and literature therein. This theory has been adopted to solve boundary value problems of static and dynamic behaviors of structures. For example, Papargyri-Beskou et al. [52] investigated the effects of the material length parameters on the bending and buckling of Euler–Bernoulli beams. Since then, the strain gradient theory has been widely used in modelling the micro- and nano-sized beams [53–67], plates [32,68–79] and shells [80–83]. These works show a stiffening effect for structures with characteristic sizes reducing to small sizes.

In order to bring both of the length scales into a combined elastic theory such that the stiffening effects and the softening effects of materials can be well described, Lim et al. [84] proposed a higher-order nonlocal strain gradient theory and applied the nonlocal strain gradient beam models to the study of the wave propagation. After that, several works dealing with buckling [85], free vibration [86,87] and wave propagation [88] of beams are reported. In these works, they emphasize that the classical results will be obtained for the same material length parameters. Since the partial differential order of the governing equation(s) of motion increases, the boundary value problems of structures modelled by the nonlocal strain gradient theory should be treated carefully. However, similar works have not been reported in the literature. For more details of nonlocal strain gradient models, one can refer to Papargyri-Beskou et al. [52], Li et al. [60], Akgöz and Civalek [57], Lazopoulos and Lazopoulos [59], Liang et al. [65] and Xu and Deng [66] for developing appropriate method to solve the boundary value problems.

The present paper is motivated by the fact that the higher-order boundary conditions induced by the nonlocal strain gradient theory should play a significant role on the buckling behaviors of Euler–Bernoulli beams. Therefore, the objective of the present work is to use the WRAs to derive the variational-consistent boundary conditions of nonlocal strain gradient beams, and to present the closed-form buckling solutions for beams subjected to various boundary conditions in which the effect of higher-order boundary conditions on the buckling loads is highlighted.

The structure of this paper is as follows. Section 2 briefly summarizes the nonlocal strain gradient theory. In Section 3, the governing equations of motion of nonlocal strain gradient Euler–Bernoulli beams in conjunction with the von Kármán nonlinear geometric relation are given, and the variational-consistent boundary conditions are derived by the WRAs. After the closed-form solutions of beam bending problems given in Section 4, the buckling solutions for beams subjected to three typical boundary conditions are addressed in Section 5. Finally, the conclusions are drawn in Section 6.

2. Nonlocal strain gradient theory

Motivated by the observations that materials at small scales exhibit either softening behaviors or stiffening behaviors, Lim et al. [84] developed an elastic theory which combines both the nonlocal elasticity theory and the strain gradient theory. Within

the framework of this theory, the concept of the higher-order nonlocal strain gradient elasticity is proposed

$$\mathbf{t} = \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\sigma}^*, \tag{1}$$

where $\mathbf{t}$ is the total stress tensor of nonlocal strain gradient theory, ∇ is the gradient symbol. The stress tensor $\boldsymbol{\sigma}$ and higher-order stress tensor $\boldsymbol{\sigma}^*$ are given by

$$\boldsymbol{\sigma} = \int_V K_0(\mathbf{y}, \mathbf{x}, l_{e0}) \mathbf{C} : \boldsymbol{\varepsilon}^{\mathbf{y}}(\mathbf{y}) \, dV \tag{2}$$

$$\boldsymbol{\sigma}^* = l_m^2 \int_V K_1(\mathbf{y}, \mathbf{x}, l_{e1}) \mathbf{C} : \nabla \boldsymbol{\varepsilon}^{\mathbf{y}}(\mathbf{y}) \, dV \tag{3}$$

where $\boldsymbol{\varepsilon}$ is the classical strain tensor, $\mathbf{C}$ is the usual fourth-order elasticity tensor, l_m is the internal length parameter, l_{e0}, l_{e1} are nonlocal parameters, $K_i(\mathbf{y}, \mathbf{x}, l_{ei})$, $i = 0, 1$ is the attenuation kernel function.

Mathematically, it is difficult to solve the above integral constitutive equations, Lim et al. [84] then, following the Eringen's method, introduced a simple constitutive equation

$$(1 - l_e^2 \nabla^2)\mathbf{t} = \mathbf{C} : \boldsymbol{\varepsilon} - l_m^2 \mathbf{C} : \nabla^2 \boldsymbol{\varepsilon} \tag{4}$$

For one-dimensional problems, the above constitutive equation reduces to

$$\left(1 - l_e^2 \frac{d^2}{dx^2}\right) t_{xx} = E\varepsilon_{xx} - l_m^2 E\varepsilon_{xx,xx}. \tag{5}$$

Note that Eq. (5) contains two material length parameters. The first one indicates the nonlocal effect, and the second one denotes the size effect due to the higher-order strain gradient. Additionally, the nonlocal elasticity [8] and the strain gradient theory [89–91] can be obtained by taking $l_m = 0$ and $l_e = 0$, respectively.

3. Basic equations of nonlocal strain gradient beams

For preliminaries, we first present in Section 3.1 the main procedures developed in the literature to the boundary value problems of the nonlocal strain gradient beams. How the boundary conditions are obtained can be easily identified. Then, we use the WRAs to formulate the variational-consistent boundary conditions in Section 3.2.

3.1. Governing equations: A summary

We consider an elastic beam of length L, width b and thickness h. The x-axis is taken along the length of the beam, and z-axis is along the thickness of the beam. According to the Euler–Bernoulli beam theory, the displacements (u_1, u_2, u_3) along the (x, z) coordinate directions are given by

$$u_1(x,z) = u(x) - zw', \; u_2(x,z) = 0, \; u_3(x,z) = w(x), \tag{6}$$

where u, w are the axial and the transverse displacements of the beam mid-plane; the prime denotes the spatial differentiation with respect to variable x.

The only non-vanishing strain for a beam under large displacements can be captured by the von Kármán nonlinear strain, i.e.,

$$\varepsilon_{xx} = u_1' + \frac{1}{2}u_3'^2 = u' + \frac{1}{2}w'^2 - zw'', \tag{7}$$

where ε_{xx} is the longitudinal strain.

Next, we will present the detailed derivation of the governing equation and boundary conditions. With this aim at hand, we first give the following virtual work of the strain energy as follows

$$\delta U = \int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xx}^*\delta\varepsilon_{xx,x}) \, dV$$

$$= \int_V (\sigma_{xx}\delta\varepsilon_{xx} - \sigma_{xx,x}^*\delta\varepsilon_{xx}) \, dV + \left[\int_A \sigma_{xx}^*\delta\varepsilon_{xx} \, dA\right]\bigg|_0^L$$

$$= \int_0^L [N_c(\delta u' + w'\delta w') - M_c\delta w''] \, dx$$

$$+ [N_n(\delta u' + w'\delta w') - M_n\delta w'']\big|_0^L \tag{8}$$

where V, L and A are the volume, the length and the cross-sectional area of the beam. The stress resultants defined in Eq. (8) are given by

$$(N_c, M_c) = \int_A t_{xx}(1, z) \, dA, \ (N_n, M_n) = \int_A \sigma_{xx}^*(1, z) \, dA. \tag{9}$$

The virtual work done by the external forces reads

$$\delta V = \int_0^L (f\delta u + q\delta w) \, dx, \tag{10}$$

where $f(x)$ and $q(x)$ are the distributed axial and transverse loads, respectively.

The variational principle states that the first variation of the total potential energy of the beam vanishes, i.e.,

$$\delta U - \delta V = 0. \tag{11}$$

Substitution of the expressions for δU and δV from Eqs. (8) and (10) into Eq. (11), and integration by parts with respect to x, we arrive at the equations of motion of the beam as

$$\begin{aligned} \delta u : & \quad N_c' + f = 0 \\ \delta w : & \quad M_c'' + q + (N_c w')' = 0, \end{aligned} \tag{12}$$

and boundary conditions at the beam ends as

$$\begin{aligned} \delta u : & \quad \text{either } N_c = 0 \text{ or } u = 0 \\ \delta u' : & \quad \text{either } N_n = 0 \text{ or } u' = 0 \\ \delta w : & \quad \text{either } M_c' + N_c w' = 0 \text{ or } w = 0 \\ \delta w' : & \quad \text{either } M_c - N_n w' = 0 \text{ or } w' = 0 \\ \delta w'' : & \quad \text{either } M_n = 0 \text{ or } w'' = 0. \end{aligned} \tag{13}$$

The nonlocal strain gradient constitutive equation as a function of displacements can be written, by taking into account of Eq. (7), as

$$t_{xx} - l_e^2 t_{xx}'' = E(z)\left(1 - l_m^2 \frac{d^2}{dx^2}\right)\left(u' + \frac{1}{2}w'^2 - zw''\right). \tag{14}$$

Then, the stress resultants can be obtained directly by combining Eqs. (9) and (14), i.e.,

$$N_c - l_e^2 N_c'' = A_{xx}\left(u' + \frac{1}{2}w'^2\right) - A_{xx}l_m^2(u''' + w'w''' + w''^2) \tag{15}$$

$$M_c - l_e^2 M_c'' = -D_{xx}w'' + D_{xx}l_m^2 w'''' \tag{16}$$

where

$$(A_{xx}, D_{xx}) = \int_A [E(z) \times (1, z)] \, dA. \tag{17}$$

We then combine the equilibrium Eq. (12) and Eqs. (15) and (16) to yield the following stress resultants

$$N_c = A_{xx}\left(u' + \frac{1}{2}w'^2\right) - A_{xx}l_m^2(u''' + w'w''' + w''^2) - l_e^2 f' \tag{18}$$

$$M_c = -D_{xx}w'' + D_{xx}l_m^2 w'''' - l_e^2[q + (N_c w')'] \tag{19}$$

Finally, we obtain the equations of motion in terms of displacements by introducing Eqs. (18) and (19) into the equilibrium Eq. (12), i.e.,

$$\left[A_{xx}\left(u' + \frac{1}{2}w'^2\right)\right]' - [A_{xx}l_m^2(u''' + w'w''' + w''^2)]' - l_e^2 f'' + f = 0, \tag{20}$$

$$D_{xx}l_m^2 w'''''' - D_{xx}w'''' + q - l_e^2 q'' + \left(1 - l_e^2 \frac{d^2}{dx^2}\right)(N_c w')' = 0, \tag{21}$$

where N_c is given by Eq. (18).

One can see from Eq. (21) that only the presence of the strain gradient parameter l_m raises the order of the differential equation of motion from four to six. Therefore, it implies from the mathematical points of view that in addition to the classical boundary conditions, higher-order boundary conditions must also be enforced for a well-posed boundary value problem of nonlocal strain gradient beams. The formulation of the boundary value problems of nonlocal strain gradient beams is presented in Section 3.2.

For an elastic homogeneous beam with a rectangular cross section, we have $E(z) = E$, this then allows us to yield

$$A_{xx} = EA, D_{xx} = EI, \tag{22}$$

where I is the second moment of inertia of the beam.

3.2. Variational-consistent boundary conditions

For convenience of the illustration, we define the following two classical stress resultants as

$$N_{cl} = A_{xx}\left(u' + \frac{1}{2}w'^2\right), \tag{23}$$

$$M_{cl} = -D_{xx}w'' \tag{24}$$

These two expressions enable us to re-write the equations of motion in terms of stress resultants as

$$N_{cl}' - l_m^2 N_{cl}''' - l_e^2 f'' + f = 0, \tag{25}$$

$$M_{cl}'' - l_m^2 M_{cl}'''' + q - l_e^2 q'' + \left(1 - l_e^2 \frac{d^2}{dx^2}\right)(N_c w')' = 0. \tag{26}$$

In order to derive the true boundary conditions of the equations of motion for a nonlocal strain gradient beam. We integrate over the beam length the summation between the product of Eq. (25) and δu, and the product of Eq. (26) and δw to obtain

$$0 = \int_0^L (N_{cl}' - l_m^2 N_{cl}''' - l_e^2 f'' + f)\delta u \, dx$$

$$+ \int_0^L \left[M_{cl}'' - l_m^2 M_{cl}'''' + q - l_e^2 q'' + \left(1 - l_e^2 \frac{d^2}{dx^2}\right)(N_c w')'\right]\delta w \, dx. \tag{27}$$

Then, by utilizing the integration by parts with respect to the right hand side of Eq. (27), we have

$$\begin{aligned} 0 &= \int_0^L (N_{cl}' - l_m^2 N_{cl}''' - l_e^2 f'' + f)\delta u \, dx \\ &+ \int_0^L \left[M_{cl}'' - l_m^2 M_{cl}'''' + q - l_e^2 q'' + \left(1 - l_e^2 \frac{d^2}{dx^2}\right)(N_c w')'\right]\delta w \, dx \\ &= -\delta U^* + \delta V^* + [N\delta u]_0^L + [N_h\delta u']_0^L + [Q\delta w]_0^L + [M\delta w']_0^L + [M_h\delta w'']_0^L, \end{aligned} \tag{28}$$

where the virtual strain energy expressed in the last line of Eq. (28) is given by

$$\delta U^* = \int_0^L [N_{cl}\delta u' + l_m^2 N'_{cl}\delta u'' - M_{cl}\delta w'' - l_m^2 M'_{cl}\delta w''' + N_c w' \delta w'$$
$$+ l_e^2 (N_c w')' \delta w''] \, dx. \tag{29}$$

The virtual external work is given by

$$\delta V^* = \int_0^L \left[\left(1 - l_e^2 \frac{\partial^2}{\partial x^2} \right) (f \delta u + q \delta w) \right] dx. \tag{30}$$

The nonclassical variational-consistent stress resultants (i.e., force resultant N, higher-order force resultant N_h, shear force resultant Q, bending moment resultant M and higher-order bending moment resultant M_h) introduced in Eq. (28) are defined by

$$N = N_{cl} - l_m^2 N''_{cl} = \left(1 - l_m^2 \frac{d^2}{dx^2} \right) [A_{xx}(u' + \tfrac{1}{2}w'^2)]$$

$$N_h = l_m^2 N'_{cl} = l_m^2 A_{xx}(u' + \tfrac{1}{2}w'^2)'$$

$$Q = M'_{cl} - l_m^2 M'''_{cl} + N_c w' - l_e^2 (N_c w')''$$
$$= -\left(1 - l_m^2 \frac{d^2}{dx^2} \right)(D_{xx} w'')' + \left(1 - l_e^2 \frac{d^2}{dx^2} \right)(N_c w')$$

$$M = -M_{cl} + l_m^2 M''_{cl} + l_e^2 (N_c w')' = \left(1 - l_m^2 \frac{d^2}{dx^2} \right)(D_{xx} w'') + l_e^2 (N_c w')'$$

$$M_h = -l_m^2 M'_{cl} = l_m^2 (D_{xx} w'')'. \tag{31}$$

It is shown that although the equations of motion are identical to those reported in the literature [85,87], the resulted boundary conditions are indeed different, especially for the higher-order boundary conditions (i.e., $(31)_2$ and $(31)_5$). It is emphasized that Eqs. (28) and (31) provide all possible boundary conditions of the nonlocal strain gradient Euler–Bernoulli beams. These boundary conditions require either prescribed boundary deformations (u, w, u', w', w''), or prescribed stress resultants (N, Q, N_h, M, M_h), or prescribed mixed boundary deformations and stress resultants. For example, either the deflections (u, w) or the stress resultants (N, Q), and the slope w' or moment M have to be prescribed for the lower-order boundary conditions. For the higher-order boundary conditions, we should specify either u' or higher-order stress resultants N_h, and w'' or higher-order stress resultants M_h to constitute a well-posed boundary value problems.

Table 1 presents the comparisons of the basic differences of the boundary stress resultants for the nonlocal strain gradient Euler–Bernoulli beams. Actually, different results will be obtained when one deals with boundary value problems using different higher-order boundary conditions.

On the other hand, Eq. (31) is the general form of the boundary stress resultants of the nonlocal strain gradient Euler–Bernoulli beams. As a result, these expressions can be reduced either to the nonlocal elasticity or to the strain gradient elasticity. For example, when the nonlocal parameter vanishes, the boundary stress resultants reduce to the same results given by Akgöz and Civalek [58] and Xu and Deng [92].

4. Bending solutions

In Section 3, the classical Euler–Bernoulli beam theory has been generalized to the development of the nonlocal strain gradient beam theory that features the two material length parameters. Here, we study the boundary value problems of the model to illustrate the size effect of beams. For simplicity, we omit the geometric nonlinearity (i.e., $N_c = 0$) and consider a simply supported beam subjected to an arbitrary distributed load $q(x)$ along the beam length. Based on these considerations, we can easily formulate the following boundary value problems according to Eqs. (21) and (31) as

$$EI l_m^2 w'''''' - EI w'''' + q - l_e^2 q'' = 0 \tag{32}$$

subjected to the following boundary stress resultants

$$Q = -\left(1 - l_m^2 \frac{d^2}{dx^2} \right) EI w''' = 0$$
$$M = \left(1 - l_m^2 \frac{d^2}{dx^2} \right) EI w'' = 0 \tag{33}$$
$$M_h = l_m^2 EI w''' = 0.$$

4.1. Nondimensional boundary value problems

The boundary value problems in Eqs. (32) and (33) can be easily converted into the nondimensional forms as

$$W'''''' - \bar{l}_2^2 W'''' + \left(1 - \tau_1^2 \frac{d^2}{dX^2} \right) \bar{q}(X) = 0 \tag{34}$$

and

$$\bar{Q} = \frac{QL^2}{EI} = -\left(1 - \bar{l}_2^2 \frac{d^2}{dX^2} \right) W''' = 0$$
$$\bar{M} = \frac{ML}{EI} = \left(1 - \bar{l}_2^2 \frac{d^2}{dX^2} \right) W'' = 0 \tag{35}$$
$$\bar{M}_h = \frac{M_h \bar{l}_2}{EI} = W''' = 0,$$

where we have introduced the following dimensionless parameters:

$$X = x/L, W(X) = w(x)/L, \bar{l}_2 = L/l_m, \tau_1 = l_e/L, \bar{q}(X)$$
$$= q(x)L^3 \bar{l}_2^2 / EI. \tag{36}$$

For an arbitrary load $q(x)$, it can be expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(m\pi x/L) \tag{37}$$

where Q_m is the amplitude given by

$$Q_m = \frac{2}{L} \int_0^L q(x) \sin(m\pi x/L) \, dx. \tag{38}$$

The detailed expressions of Q_m are calculated as follows

$$Q_m = \begin{cases} q_0 \ (m = 1) & \text{sinusoidal load of intensity } q_0 \\ \frac{4}{m\pi} q_0 \ (m = 1, 3, 5, \ldots) & \text{uniform load of intensity } q_0. \end{cases} \tag{39}$$

Table 1
Boundary stress resultants for the nonlocal strain gradient Euler–Bernoulli beams.

	N	N_h	Q	M	M_h
Li et al. [85,86]	$N_c = 0$	$N_n = 0$	$M'_c = 0$	$M_c = 0$	$M_n = 0$
Şimşek [87]	$N_c = 0$	$N_n = 0$	$M'_c + N_c w' = 0$	$M_c - N_n w' = 0$	$M_n = 0$
Present work (see Eq. (31))	$N_{cl} - l_m^2 N''_{cl} = 0$	$N_{cl} = 0$	$M'_{cl} - l_m^2 M'''_{cl} + N_c w' - l_e^2 (N_c w')'' = 0$	$-M_{cl} + l_m^2 M''_{cl} + l_e^2 (N_c w')' = 0$	$M'_{cl} = 0$
$l_m = 0$ [15,47,93]	$N_{cl} = 0$	–	$M'_{cl} + N_c w' - l_e^2 (N_c w')'' = 0$	$-M_{cl} + l_e^2 (N_c w')' = 0$	–
$l_e = 0$ [58,92]	–	–	$M'_{cl} - l_m^2 M'''_{cl} + N_c w' = 0$	$-M_{cl} + l_m^2 M''_{cl} = 0$	$M'_{cl} = 0$
$l_e = 0$ [94]	$N_{cl} - l_m^2 N''_{cl} = 0$	$N_{cl} = 0$	$M'_{cl} - l_m^2 M'''_{cl} + N_c w' - l_m^2 N''_c w' = 0$	$-M_{cl} + l_m^2 M''_{cl} = 0$	$M'_{cl} = 0$

Then, we obtain the dimensionless load from Eq. (36) as

$$\bar{q}(X) = \frac{L^3 \bar{l}_2^2}{EI} \sum_{m=1}^{\infty} Q_m \sin(m\pi X). \tag{40}$$

4.2. Closed-form solutions

The equation of motion of the nonlocal strain gradient beam is the linear nonhomogeneous six-order ordinary differential equation, the complete solution of Eq. (34) for W includes the general solution and the particular solution, i.e.,

$$W = C_0 + C_1 X + C_2 X^2 + C_3 X^3 + C_4 \sinh(\bar{l}_2 X) + C_5 \cosh(\bar{l}_2 X)$$
$$+ \sum_{m=1}^{\infty} Q^* \sin(m\pi X) \tag{41}$$

where C_i, $(i = 0,1,2,\ldots,5)$ are integration constants to be determined by the specified boundary conditions, and $Q^* = \frac{1+\tau_1^2(m\pi)^2}{(m\pi)^4[\bar{l}_2^2+(m\pi)^2]} \frac{L^3 \bar{l}_2^2}{EI} Q_m$.

4.2.1. Case 1

For simply supported boundary conditions, the lower-order boundary conditions are

$$W = 0, \bar{M} = 0 \text{ at } X = 0 \text{ and } 1. \tag{42}$$

The higher-order boundary conditions are

$$W'' = 0 \text{ at } X = 0 \text{ and } 1. \tag{43}$$

Substitution of Eq. (41) into Eqs. (42) and (43) yields six linear algebraic equations for C_i, $(i = 0,1,2,\ldots,5)$. Solving these equations, we can determine these coefficients as

$$C_0 = C_2 = C_5 = 0, C_1 = -\frac{\sin(m\pi)(\bar{l}_2^2 - 1)(5\bar{l}_2^2 + 6)}{6\bar{l}_2^4},$$

$$C_3 = \frac{\sin(m\pi)(\bar{l}_2^2 - 1)}{\bar{l}_2^2}, \quad C_4 = -\frac{\sin(m\pi)}{\bar{l}_2^4 \sinh(\bar{l}_2)}. \tag{44}$$

Then, substituting these expressions back into Eq. (41), we can determine the bending solution of the nonlocal strain gradient beams.

4.2.2. Case 2

For the second choice of the higher-order boundary conditions, we have

$$W''' = 0 \text{ at } X = 0 \text{ and } 1. \tag{45}$$

Solving the boundary value problems of the nonlocal strain gradient beams, we have

$$C_0 = -\frac{\sin(m\pi)(-\bar{l}_2^2 + \cosh\bar{l}_2 - 1)}{\bar{l}_2^5 \sinh\bar{l}_2},$$

$$C_1 = -\frac{\sin(m\pi)(-6\bar{l}_2^2\cosh\bar{l}_2 + 6\bar{l}_2^2 + 6\bar{l}_2^5\sinh\bar{l}_2 + \bar{l}_2^3\sinh\bar{l}_2 - 12\cosh\bar{l}_2 + 12)}{6\bar{l}_2^5 \sinh\bar{l}_2},$$

$$C_2 = 0,$$

$$C_3 = \frac{\sin(m\pi)}{6\bar{l}_2^2}, \tag{46}$$

$$C_4 = -\frac{\sin(m\pi)}{\bar{l}_2^5},$$

$$C_5 = -\frac{\sin(m\pi)(\bar{l}_2^2 - \cosh\bar{l}_2 + 1)}{\bar{l}_2^5 \sinh\bar{l}_2}.$$

It can be verified from the coefficients given by Eqs. (44) and (46) that the all the coefficients will be identically zero for integer values of m. In other words, the proposed simply supported boundary conditions do not influence the bending deflection of nonlocal strain gradient beams subjected to the distributed load. As a result, Case 1 and Case 2 will result in the same expression for bending deflections, i.e.,

$$W = \sum_{m=1}^{\infty} \frac{1 + \tau_1^2(m\pi)^2}{(m\pi)^4[\bar{l}_2^2 + (m\pi)^2]} \frac{L^3 \bar{l}_2^2}{EI} Q_m \sin(m\pi X) \tag{47}$$

where Q_m is given by Eq. (39). It is worth mentioning that the inclusion of the material length parameter l_m has the ability to stiffen the beam, and consequently, it reduces the beam bending deflection; whereas, the material length parameter l_e has the opposite behavior.

Note that once the strain gradient parameter vanishes (i.e., $\bar{l}_2 \to \infty$), the above equation for bending deflection reduces to a nonlocal form

$$W = \sum_{m=1}^{\infty} \frac{1 + \tau_1^2(m\pi)^2}{(m\pi)^4} \frac{L^3}{EI} Q_m \sin(m\pi X). \tag{48}$$

Additionally, when the two material length parameters are the same, the bending deflection reduces to the classical form.

4.3. Numerical results

With the purpose of studying how the material length parameters affect the bending behaviors of the nonlocal strain gradient beams, we define, for convenience, the following bending deflection ratio

$$\gamma_1 = \frac{w}{w_{cl}} = \frac{W}{\frac{5q_0 L^3}{384EI}} \tag{49}$$

where w_{cl} is the maximum deflection of the classical solution for beams subjected to a uniform load.

Fig. 1 depicts the bending profiles of the size-dependent beams subjected to the uniform load for various values of the material
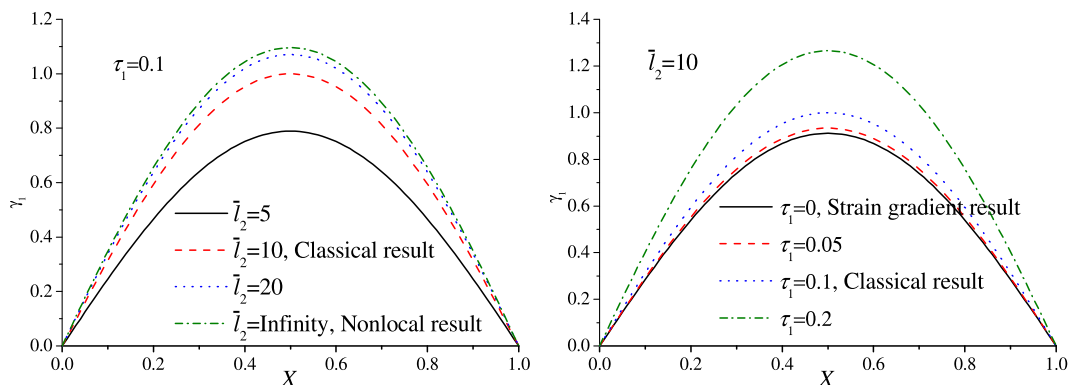

Fig. 1. The bending profiles of the size-dependent beams subjected to the uniform load. (a) $\tau_1 = 0.1$ and (b) $\bar{l}_2 = 10$.

Table 2
The bending deflection ratio γ_1 (or γ_2) for a simply supported beam under distributed loads. $\bar{l}_2 \to \infty$ denotes results of a nonlocal beam model.

$\bar{l}_2$	τ_1				
	0[a]	0	0.05	0.1	0.2
5	0.7188	0.7188	0.7364	0.7891	1.0000
10	0.9116	0.9116	0.9337	1.0000	1.2653
20	0.9765	0.9765	1.0000	1.0706	1.3528
∞	1.0000	1.0000	1.0240	1.0960	1.3840

[a] Numerical results of Eq. (29) in Akgöz and Civalek [57].

length parameters. We mention that the values $\bar{l}_2\tau_1 = 1$, $\bar{l}_2 \to \infty$ and $\tau_1 = 0$ correspond to the classical case, nonlocal case and strain gradient case, respectively. As observed in Fig. 1, the values of deflections of the beam increase with the increasing $\bar{l}_2$ and τ_1. Additionally, we can see that the deflections are more sensitive for small values of $\bar{l}_2$ and large values of τ_1. Furthermore, we can observe that the nonlocal model overestimates the bending deflections, whereas the strain gradient model underestimates the bending deflections. These observations demonstrate that the proposed nonlocal strain gradient beam model can capture the structures featuring either stiffening or softening behaviors.

To investigate the bending behaviors of nonlocal strain gradient beams subjected to the sinusoidal load of intensity q_0, we define the following bending deflection ratio

$$\gamma_2 = \frac{w}{w_{cl}} = \frac{W}{\frac{q_0 L^3}{\pi^4 EI}} \tag{50}$$

where w_{cl} is the maximum deflection of the classical solution for beams subjected to a sinusoidal load. It is interestingly found that they exhibit the identical behaviors compared to the uniform load case, and consequently, the bending profiles of the size-dependent beams subjected to the sinusoidal load can also be described in Fig. 1.

Table 2 lists the numerical results of the deflection ratios of a simply supported beam under distributed loads as functions of the normalized material length parameters. Again, the effects of material length parameters are to make the beam behave either stiffer or softer, depending on the competitions of the two material length parameters. Additionally, our numerical results reduce to the strain gradient beam model developed by Akgöz and Civalek [57].

5. Buckling solutions

The boundary value problems of the buckling cases of a nonlocal strain gradient beam will be studied in this section to highlight the effects of the material length parameters on the buckling loads. We consider beams with three common boundary conditions, i.e., clamped-free (CF) boundary conditions, simply supported (SS) boundary conditions and clamped-clamped (CC) boundary conditions. Since each boundary value problem has two alternative higher-order boundary conditions, we will study the differences of each higher-order boundary conditions adopted to gain insight into the differences of the buckling results.

For the buckling case, we take $q = 0$ and $N_c = -N_b$. Then, the governing equation (21) of a beam becomes

$$EIl_m^2 w'''''' - EIw'''' - \left(1 - l_e^2 \frac{d^2}{dx^2}\right)(N_b w'') = 0 \tag{51}$$

subjected to the following boundary conditions at the beam ends

$$Q = -\left(1 - l_m^2 \frac{d^2}{dx^2}\right)EIw''' - \left(1 - l_e^2 \frac{d^2}{dx^2}\right)(N_b w') = 0 \text{ or } w = 0$$

$$M = \left(1 - l_m^2 \frac{d^2}{dx^2}\right)EIw'' - l_e^2 N_b w'' = 0 \text{ or } w' = 0 \tag{52}$$

$$M_h = EIl_m^2 w''' = 0 \text{ or } w'' = 0.$$

Analogously, when the strain gradient parameter l_m vanishes, the above boundary value problem reduces to the nonlocal Euler–Bernoulli theory by Xu et al. [47]. On the other hand, when the nonlocal parameter l_e equals to zero, the above boundary value problem reduces to the similar form of Papargyri-Beskou et al. [52] and Akgöz and Civalek [58].

5.1. Nondimensional boundary value problems

Analogous to that presented in Section 4.1, we can introduce an extra dimensionless parameter $\bar{N} = N_b L^2 / EI$ to re-write the boundary value problem expressed in Eqs. (51) and (52) as

$$W'''''' + a_4 W'''' + a_2 W'' = 0 \tag{53}$$

and

$$\bar{Q} = \frac{QL^2}{EI} = -\left(1 - \bar{l}_2^{-2} \frac{d^2}{dX^2}\right)W''' - \left(1 - \tau_1^2 \frac{d^2}{dX^2}\right)(\bar{N}W') = 0 \text{ or } W = 0$$

$$\bar{M} = \frac{ML}{EI} = \left(1 - \bar{l}_2^{-2} \frac{d^2}{dX^2}\right)W'' - \tau_1^2 \bar{N}W'' = 0 \text{ or } W' = 0$$

$$\bar{M}_h = \frac{M_h \bar{l}_2}{EI} = W''' = 0 \text{ or } W'' = 0, \tag{54}$$

where $a_4 = \bar{l}_2^2(\bar{N}\tau_1^2 - 1), a_2 = -\bar{N}\bar{l}_2^2$.

We state that all the other dimensionless parameters are defined by Eq. (36).

5.2. Closed-form solutions

The general solution for the sixth-order ordinary differential equation from Eq. (53) is

$$W = B_1 + B_2 X + B_3 \sinh(\alpha X) + B_4 \cosh(\alpha X) + B_5 \sin(\beta X) + B_6 \cos(\beta X) \tag{55}$$

where B_i, $(i = 1,2,...,6)$ are the integration constants to be determined by the specified boundary conditions, and

$$\alpha = \left[\frac{\bar{l}_2^2(1-\bar{N}\tau_1^2) + \bar{l}_2\sqrt{\bar{l}_2^2(1-\bar{N}\tau_1^2)^2 + 4\bar{N}}}{2}\right]^{1/2},$$

$$\beta = \left[\frac{-\bar{l}_2^2(1-\bar{N}\tau_1^2) + \bar{l}_2\sqrt{\bar{l}_2^2(1-\bar{N}\tau_1^2)^2 + 4\bar{N}}}{2}\right]^{1/2}. \tag{56}$$

Next, we will solve various boundary value problems of a nonlocal strain gradient beam with a view toward evaluating how the material length parameters affect the buckling behaviors of a nonlocal strain gradient beam.

5.2.1. CF boundary conditions

For a cantilever beam clamped at the left end and free at the right end, the lower-order boundary conditions are

$$W(0) = W'(0) = 0,$$

$$\bar{Q}(1) = -\left(1 - \bar{l}_2^{-2} \frac{d^2}{dX^2}\right)W'''(1) - \left(1 - \tau_1^2 \frac{d^2}{dX^2}\right)[\bar{N}W'(1)] = 0,$$

$$\bar{M}(1) = \left(1 - \bar{l}_2^{-2} \frac{d^2}{dX^2}\right)W''(1) - \tau_1^2 \bar{N}W''(1) = 0. \tag{57}$$

The higher-order boundary conditions have two alternative expressions. The first ones (BC1) we adopt are

$$W''(0) = 0, \ \bar{M}_h(1) = W'''(1) = 0. \tag{58}$$

It is worth mentioning that Eq. (58)$_1$ indicates the vanishing of the curvature at the clamped end of the beam, and Eq. (58)$_2$ represents the vanishing of the higher-order moment at the free end of the beam.

We then introduce the general solution in Eq. (55) into the boundary conditions in Eqs. (57) and (58) to yield a set of linear algebraic equations for coefficients B_i, ($i = 1,2,\ldots,6$). The necessary condition for nonzero solution of coefficients B_i, ($i = 1,2,\ldots,6$) is the vanishing determinant of its coefficient matrix, i.e.,

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \alpha & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha^2 & 0 & -\beta^2 \\ 0 & 0 & \alpha^3\cosh\alpha & \alpha^3\sinh\alpha & -\beta^3\cos\beta & \beta^3\sin\beta \\ 0 & 0 & -a_2\sinh\alpha & -a_2\cosh\alpha & -a_2\sin\beta & -a_2\cos\beta \\ 0 & a_2 & 0 & 0 & 0 & 0 \end{vmatrix} = 0. \tag{59}$$

Or alternatively,

$$2\alpha^2\beta^2 + (\alpha^4 + \beta^4)\cosh\alpha\cos\beta + \alpha\beta(\alpha^2 - \beta^2)\sinh\alpha\sin\beta = 0. \tag{60}$$

For the pure strain gradient model (i.e., $\tau_1 = 0$), we emphasize that the above characteristic equation is not the same as the one given by Akgöz and Civalek [58].

The second ones (BC2) we adopt are

$$\bar{M}_h(0) = \bar{M}_h(1) = 0 \Rightarrow W'''(0) = W'''(1) = 0. \tag{61}$$

Analogously, the buckling load is determined by the following equation

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \alpha & 0 & \beta & 0 \\ 0 & 0 & \alpha^3 & 0 & -\beta^3 & 0 \\ 0 & 0 & \alpha^3\cosh\alpha & \alpha^3\sinh\alpha & -\beta^3\cos\beta & \beta^3\sin\beta \\ 0 & 0 & -a_2\sinh\alpha & -a_2\cosh\alpha & -a_2\sin\beta & -a_2\cos\beta \\ 0 & a_2 & 0 & 0 & 0 & 0 \end{vmatrix} = 0. \tag{62}$$

Or its equivalent form

$$\alpha^3\cos\beta\sinh\alpha - \beta^3\cosh\alpha\sin\beta = 0. \tag{63}$$

Lazopoulos and Lazopoulos [59] studied the buckling of the pure stain gradient beams, they used in their work the following higher-order boundary conditions (BC3)

$$W''(0) = W''(1) = 0. \tag{64}$$

The boundary value problem in this case can be obtained by solving the following characteristic equation

$$\alpha\sinh\alpha\cos\beta + \beta\cosh\alpha\sin\beta = 0. \tag{65}$$

Note that our expression in Eq. (65) is in the same form as that of Lazopoulos and Lazopoulos [59], but with different parameters. More specially, their result for pure strain gradient models is here generalized to the nonlocal strain gradient beam models (see comparisons shown in Table 3).

5.2.2. SS boundary conditions

We next consider a nonlocal strain gradient beam with SS boundary conditions at the beam ends. The lower-order boundary conditions read

$$W = 0, \ \bar{M} = -\bar{l}_2^2(W'''' + a_4W'') = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{66}$$

The higher-order boundary conditions (BC1) are given by

$$W'' = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{67}$$

Noteworthy, Eq. (67) can be used to simplify Eq. (66)$_2$, i.e.,

$$W'''' = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{68}$$

Then, we insert the general solution from Eq. (55) into Eqs. (66)$_1$, (67) and (68) to yield the following characteristic equation

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \alpha^2 & 0 & -\beta^2 \\ 0 & 0 & 0 & \alpha^4 & 0 & \beta^4 \\ 1 & 1 & \sinh\alpha & \cosh\alpha & \sin\beta & \cos\beta \\ 0 & 0 & \alpha^2\sinh\alpha & \alpha^2\cosh\alpha & -\beta^2\sin\beta & -\beta^2\cos\beta \\ 0 & 0 & \alpha^4\sinh\alpha & \alpha^4\cosh\alpha & \beta^4\sin\beta & \beta^4\cos\beta \end{vmatrix} = 0. \tag{69}$$

Table 3
Comparisons of the first five dimensionless buckling loads, $\bar{N}$, under three typical boundary conditions ($\tau_1 = 0$).

	CF boundary conditions with a BC1 case		CF boundary conditions with a BC3 case		SS boundary conditions with a BC2 case	
n	Akgöz and Civalek [58]	Present work	Lazopoulos and Lazopoulos [59]	Present work	Akgöz and Civalek [58]	Present work
$\bar{l}_2 = 5$						
1	2.71092	3.97027	4.20110	4.20110	13.76597	11.14825
2	41.93195	49.62049	61.14888	61.14888	101.82024	67.30866
3	213.88673	228.78025	288.78071	288.78071	404.43189	258.96628
4	705.60072	727.17906	897.50034	897.50034	1155.38276	770.34915
5	1797.61210	1825.66037	2191.24917	2191.24917	2681.96739	1881.66603
$\bar{l}_2 = 10$						
1	2.52828	3.10130	3.12159	3.12159	10.84370	10.46073
2	27.13795	31.92166	33.41908	33.41908	55.06387	49.19484
3	99.73545	111.30482	121.41988	121.41988	167.72780	140.00742
4	267.07717	286.11449	319.46022	319.46022	407.28094	327.41546
5	599.29764	625.59821	703.52068	703.52068	855.54693	679.93418
$\bar{l}_2 = 20$						
1	2.48262	2.74904	2.75100	2.75100	10.11313	10.06426
2	23.43944	25.82298	25.98219	25.98219	43.37478	42.58753
3	71.19763	77.69890	78.91961	78.91961	108.55178	104.55529
4	157.44628	169.72744	174.33277	174.33277	220.25549	207.72263
5	299.71903	318.89993	331.07539	331.07539	398.94181	368.98595
$\bar{l}_2 \to \infty$						
1	2.46740	2.46740	2.46740	2.46740	9.86960	9.86960
2	22.20661	22.20661	22.20661	22.20661	39.47842	39.47842
3	61.68503	61.68503	61.68503	61.68503	88.82644	88.82644
4	120.90265	120.90265	120.90265	120.90265	157.91367	157.91367
5	199.85949	199.85949	199.85949	199.85949	246.74011	246.74011

The above expression can be simplified as

$$-\alpha^4\beta^4(\alpha^2 + \beta^2)^2 \sinh \alpha \sin \beta = 0 \Rightarrow \sin \beta = 0. \tag{70}$$

By combining Eqs. (70) and (56)$_2$, we can obtain the analytical solution for the buckling load as

$$\bar{N} = \frac{n^2\pi^2(\bar{l}_2^2 + n^2\pi^2)}{\bar{l}_2^2(1 + n^2\pi^2\tau_1^2)}. \tag{71}$$

Note that when the strain parameter vanishes (i.e., $\bar{l}_2 \to \infty$), the above solution reduces to the nonlocal results [14,15,47]. On the other hand, when the nonlocal parameter is zero (i.e., $\tau_1 = 0$), the above result reduces to that of the strain gradient beam model by Akgöz and Civalek [58] and a similar form by Li and Hu [85]. Moreover, the above result can be further reduced to the classical buckling load in the absence of all the material length parameters. The buckling solution in Eq. (71) indicates that the buckling load increases with the decreasing of the nonlocal parameter τ_1 and the strain gradient parameter $\bar{l}_2$.

The second higher-order boundary conditions (BC2) we adopt are

$$W''' = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{72}$$

Substituting Eq. (55) into Eqs. (66) and (72) yields a set of linear algebraic equations. The buckling load of the beam can be determined by solving the following characteristic equation

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -a_2 & 0 & -a_2 \\ 0 & 0 & \alpha^3 & 0 & -\beta^3 & 0 \\ 1 & 1 & \sinh\alpha & \cosh\alpha & \sin\beta & \cos\beta \\ 0 & 0 & -a_2\sinh\alpha & -a_2\cosh\alpha & -a_2\sin\beta & -a_2\cos\beta \\ 0 & 0 & \alpha^3\cosh\alpha & \alpha^3\sinh\alpha & -\beta^3\cos\beta & -\beta^3\sin\beta \end{vmatrix} = 0. \tag{73}$$

Or its simplified form as

$$2\alpha^3\beta^3 + (\beta^6 - \alpha^6)\sinh\alpha\sin\beta - 2\alpha^3\beta^3\cosh\alpha\cos\beta = 0. \tag{74}$$

For the reduced case of the pure strain gradient model, it is again stated that the above characteristic equation is not the same as that of Akgöz and Civalek [58].

5.2.3. CC boundary conditions

Finally, we consider a beam with CC boundary conditions. The lower-order boundary conditions of the beam can be written as

$$W = W' = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{75}$$

The first choice of the high-order boundary conditions (BC1) are

$$W'' = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{76}$$

We introduce Eq. (55) into Eqs. (75) and (76), and carry out a similar procedure presented in Sections 5.2.1 and 5.2.2. After some mathematical manipulations, we can obtain the following characteristic equation

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \alpha & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha^2 & 0 & -\beta^2 \\ 1 & 1 & \sinh\alpha & \cosh\alpha & \sin\beta & \cos\beta \\ 0 & 1 & \alpha\cosh\alpha & \alpha\sinh\alpha & \beta\cos\beta & -\beta\sin\beta \\ 0 & 0 & \alpha^2\sinh\alpha & \alpha^2\cosh\alpha & -\beta^2\sin\beta & -\beta^2\cos\beta \end{vmatrix} = 0. \tag{77}$$

Its compact form is

$$2\beta(\alpha^2 + \beta^2)\sin\beta(\cosh\alpha - 1) + 2\alpha(\alpha^2 + \beta^2)\sinh\alpha(\cos\beta - 1)$$
$$+ \alpha\beta(\alpha^2 - \beta^2)\sinh\alpha\sin\beta - 2\alpha^2\beta^2\cosh\alpha\cos\beta + 2\alpha^2\beta^2 = 0 \tag{78}$$

from which the buckling load can be determined.

The second higher-order boundary conditions (BC2) are

$$W''' = 0 \text{ at } X = 0 \text{ and } X = 1. \tag{79}$$

The buckling solution of the above boundary value problems can be solved by

$$\begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & \alpha & 0 & \beta & 0 \\ 0 & 0 & \alpha^3 & 0 & -\beta^3 & 0 \\ 1 & 1 & \sinh\alpha & \cosh\alpha & \sin\beta & \cos\beta \\ 0 & 1 & \alpha\cosh\alpha & \alpha\sinh\alpha & \beta\cos\beta & -\beta\sin\beta \\ 0 & 0 & \alpha^3\cosh\alpha & \alpha^3\sinh\alpha & -\beta^3\cos\beta & \beta^3\sin\beta \end{vmatrix} = 0. \tag{80}$$

Its simplified form is

$$2\alpha^3\cos\beta\sinh\alpha + \alpha\beta(\alpha^2 + \beta^2)\sinh\alpha\sin\beta$$
$$- 2\beta^3\cosh\alpha\sin\beta - 2\alpha^3\sinh\alpha + 2\beta^3\sin\beta = 0. \tag{81}$$

For the nonlocal beam model (i.e., $\bar{l}_2 \to \infty$), we can write down, according to Eq. (53), the following governing equation:

$$a_4 W'''' + a_2 W'' = 0 \tag{82}$$

subjected to boundary conditions which are the same as Eq. (75).

This boundary value problems allow us to obtain the following characteristic equation

$$\sin Q\left(\tan\frac{Q}{2} - \frac{Q}{2}\right) = 0, \tag{83}$$

where $Q = \sqrt{a_2/a_4} = \sqrt{\bar{N}/(1 - \bar{N}\tau_1^2)}$.

Eq. (83) further indicates that the buckling load is determined by

$$\sin Q = 0 \Rightarrow \bar{N} = \frac{n^2\pi^2}{1 + n^2\pi^2\tau_1^2} \quad \text{or} \quad \tan\frac{Q}{2} = \frac{Q}{2}. \tag{84}$$

The numerical results for this case can be clearly seen in Table 3 for $\tau_1 = 0$.

5.3. Numerical results

In order to illustrate the analytical solutions for buckling problems of a nonlocal strain gradient beam, we present some numerical examples in this section[1]. We first attempt to demonstrate the efficiency and accuracy of the present closed-form buckling solutions. To this end, shown in Table 3 is a comparison between our numerical results of Eq. (65) without considering the nonlocal parameter and those of Eq. (58) in Lazopoulos and Lazopoulos [59]. As expected, the two numerical results have the same buckling values.

Akgöz and Civalek [58] recently studied the buckling behaviors of a strain gradient beam and obtained closed-form solutions for buckling loads under CF and SS boundary conditions, i.e.,

$$\bar{N} = \lambda^2(\bar{l}_2^2 + \lambda^2)/\bar{l}_2^2 \tag{85}$$

where $\lambda = (2n - 1)\pi/2$ is for CF boundary conditions and $\lambda = n\pi$ is for SS boundary conditions.

However, their numerical results are not the same as the present results. To demonstrate these differences in detail, we tabulate the numerical results of a strain gradient beam in Table 3 for $\tau_1 = 0$. As observed in this table, all the buckling loads decrease

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[1] For the bulking studies of engineering structures, analytical solutions for the boundary value problems are normally challenging with mathematical difficulties. As a result, the first several eigenvalues are presented in this paper to investigate, on the one hand, the effect of physical parameters on the buckling of engineering structures, and on the other hand, to serve as benchmark results for possible comparisons with other numerical methods.

with the increasing of the material length parameter $\bar{l}_2$. In other words, the strain gradient beams are stiffer than the classical beams. However, only for the classical buckling solutions when $\bar{l}_2 \to \infty$ can we have the same numerical results. More specially, the differences between the numerical results of Akgöz and Civalek [58] and present work are remarkable for higher values of the strain gradient parameter and the buckling mode numbers. These differences will be illustrated later with the purpose of clarifying the effects of two material length parameters and boundary conditions on the buckling of nonlocal strain gradient beams.

5.3.1. Effect of the strain gradient parameter

In the literature, numerous works have studied the boundary value problems of strain gradient beams and nonlocal beams. For the former beam models, there exist alternative higher-order boundary conditions, and as a result, effects of the higher-order boundary conditions on the static and dynamic behaviors of beams should be carefully investigated to demonstrate the main differences. However, recent works relating to this issue have not been well documented [58,59]. On the other hand, the nonlocal beam models will also introduce nonclassical boundary conditions, as suggested by the modified variational principles [47,93,95–97]. These nonclassical boundary conditions certainly affect the mechanical behaviors of beams, and thus worth a further study. The nonlocal strain gradient beam models, as a combination of these two elastic theories, feature the above two beam models. Therefore, the boundary value problems of the proposed beam models should be carefully studied.

Fig. 2 plots the buckling loads of nonlocal strain gradient beams as functions of the material length parameters for different mode numbers and higher-order boundary conditions. It is found that

- The buckling loads decrease with the increase of the dimensionless strain gradient parameter, which, remembering Eq. (36)$_3$, indicate that the inclusion of the strain gradient parameter makes the beams stiffer than that of the classical beams.
- For CF beams, the buckling loads calculated by BC3 are slightly larger than those calculated by BC1, and are remarkably larger than those calculated by BC2. From the mechanical points of view, the stiffer the higher-order boundary conditions, the larger the buckling loads. As a result, the explanations for these observations are that higher-order boundary condition $W''(1) = 0$ from Eq. (64)$_2$ in BC3 is stiffer than that of $W'''(1) = 0$ from Eq. (58)$_2$ in BC1, and $W''(0) = 0$ from Eq. (58)$_1$ in BC1 is stiffer than that of $W'''(0) = 0$ from Eq. (61)$_1$ in BC2. For SS and CC boundary conditions, the buckling loads predicted by BC1 are larger than those predicted by BC2, as also evidenced by the expressions of the higher-order boundary conditions (see, e.g., Eqs. (67) and (72), and Eqs. (76) and (79)).
- The influence of the adopted higher-order boundary conditions on the critical buckling loads is more remarkable for beams with CC boundary conditions. This means that the critical buckling loads for beams with clamped ends are more sensitive than those with other constrains.

5.3.2. Effect of the nonlocal parameter

Fig. 3 displays the buckling loads of nonlocal strain gradient beams as functions of the material length parameters for different mode numbers and higher-order boundary conditions. Similar conclusions in accord with Section 5.3.1 can be drawn. Noteworthy, there exists one opposite conclusion that the inclusion of the nonlocal parameter makes the nonlocal strain gradient beams softer; this conclusion is the same with those reported for nonlocal beams in the literature [14,15,19,45–47,93,98].
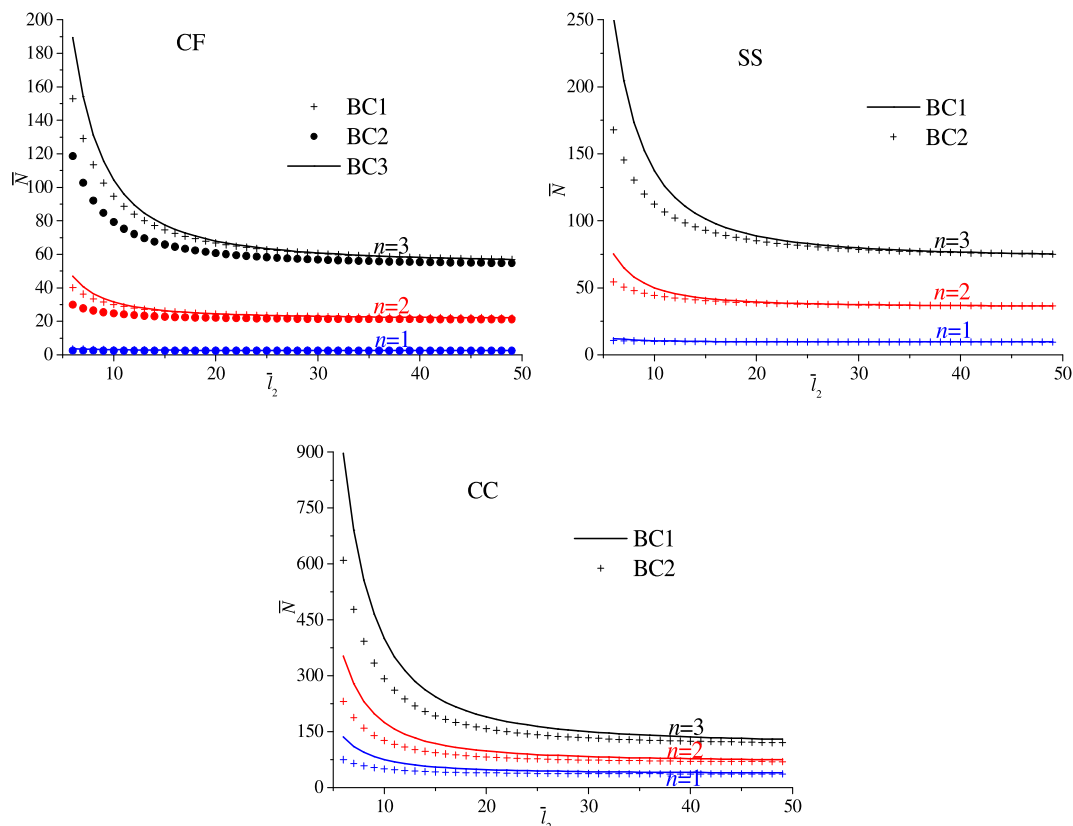

Fig. 2. Effects of the higher-order boundary conditions on the first three buckling loads of nonlocal strain gradient beams for $\tau_1 = 0.05$. (a) CF boundary conditions, (b) SS boundary conditions and (c) CC boundary conditions.
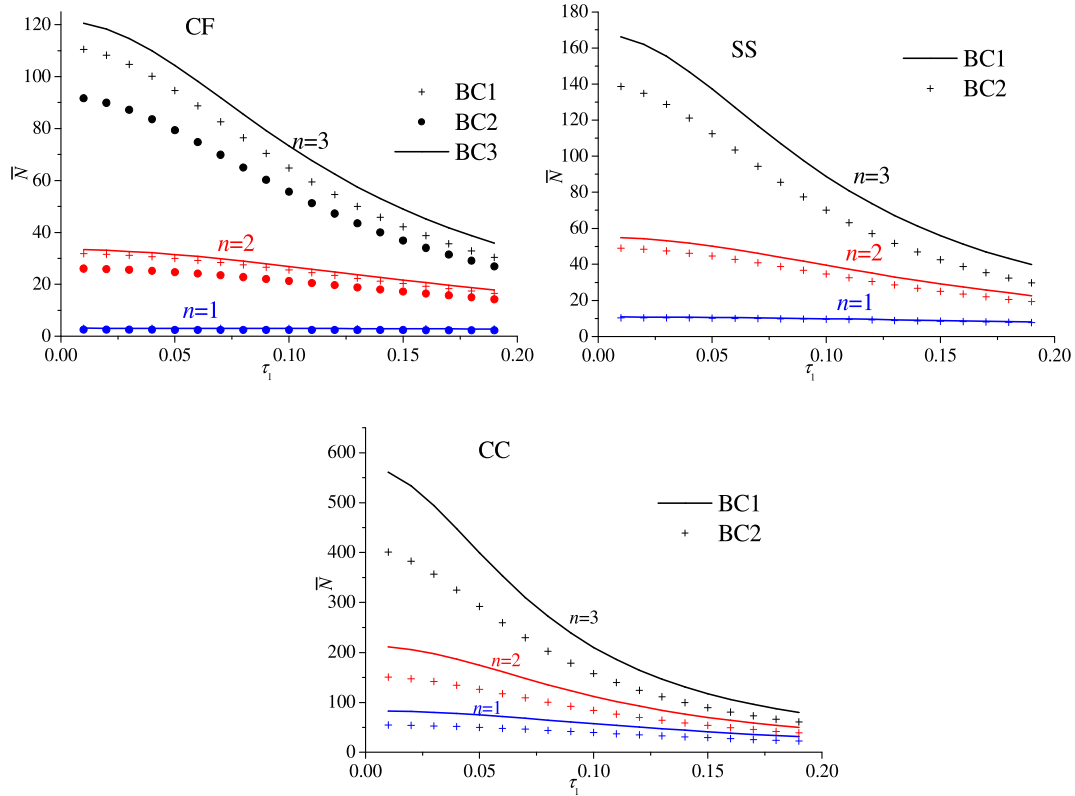
Fig. 3. Effects of the higher-order boundary conditions on the first three buckling loads of nonlocal strain gradient beams for $\bar{l}_2 = 10$. (a) CF boundary conditions, (b) SS boundary conditions and (c) CC boundary conditions.

Table 4
Comparisons of buckling parameters between the nonlocal strain gradient beam models with the classical solutions for the same values of the two material length parameters ($\bar{l}_2^{-1} = \tau_1$).

	τ_1						
	0	0.05		0.1		0.2	
n	Classical results[a]	BC1	BC2	BC1	BC2	BC1	BC2
CF boundary conditions							
1	2.46740	2.73135	2.46588	3.01925	2.45533	3.53802	2.37622
2	22.20661	24.39820	22.08452	25.52183	21.29312	22.56250	17.49515
3	61.68503	66.81101	60.75967	64.81754	55.65779	57.16417	44.62507
4	120.90265	128.39493	117.45737	119.02403	103.69327	113.51766	91.51941
SS boundary conditions							
1	9.86960	9.86960	9.82132	9.86960	9.50487	9.86960	7.79622
2	39.47842	39.47841	38.72520	39.47841	34.59492	39.47841	23.32515
3	88.82644	88.82644	85.1725	88.82644	69.98060	88.82644	48.70934
4	157.91367	157.91351	147.07911	157.91351	116.07657	157.91351	94.10167
CC boundary conditions							
1	39.47842	48.33421	39.47842	57.46162	39.47842	69.41978	39.47842
2	80.76291	98.06520	82.01347	112.13209	84.00135	124.76962	86.22035
3	157.91367	189.68094	157.91367	210.26177	157.91367	226.76438	157.91367
4	238.71806	283.77547	241.54126	307.28378	244.02449	322.48482	245.58726

[a] Numerical results calculated by Eqs. (84) and (85).

5.3.3. Comments on previous works

In the literature, numerous works demonstrate that, when the two material length parameters are the same, the classical solutions characterizing the buckling [85], vibration [86,87] and wave propagation [84,88] will be obtained. However, our closed-form solutions for buckling problems of nonlocal strain gradient beams are proved this to be not always the case. As a result, it is of great importance to address the issue: does the nonlocal strain gradient solutions reduce to the classical solutions when the two material length parameters are the same, at least for beam buckling problems.

Table 4 lists the calculated values of the buckling loads of nonlocal strain gradient beams as functions of the same material length parameters. It is observed from this table that

When the material length parameters are the same but with vanishing values (i.e., $l_2 = \tau_1 = 0$), our numerical results for three typical boundary conditions can recover to the classical solutions. The classical solutions can also be seen for SS beams with the first choice of the higher-order boundary conditions (BC1).

When the nonzero material length parameters equal to each other (i.e., $\bar{l}_2^{-1} = \tau \neq 0$), the buckling solutions do not recover to the classical solutions; the values of the critical buckling load

increase with the increasing τ_1. However, for the higher modes, the values of the buckling load appear to have the opposite tendency for the CF and CC boundary conditions.

For the same material length parameters, the values of the buckling loads of the nonlocal strain gradient beams are not the same. The possible reason is that the classical buckling modes do not satisfy the buckling modes of strain gradient beams. In general, the values of BC1 are larger than those of BC2.

In general, it is found from this table that the classical and the nonclassical solutions are not always the same when the two material length parameters are equal to each other. This conclusion is clearly not in accord with those published works [84–88]. Furthermore, the present closed-form solutions are useful for the choices of the shape functions applied in the finite element methods and Garlerkin methods to highlight the differences of higher-order boundary conditions considered.

6. Conclusions

Based on the nonlocal strain gradient theory, the governing equations of motion of nonlinear Euler–Bernoulli beams are derived. Then, these equations are used to determine all possible (lower-order and higher-order) boundary conditions by the WRAs. These derivations, on the one hand, allow us to derive the boundary conditions for nonlocal beams, and on the other hand, they enable us to obtain the boundary conditions for the strain gradient beams. The bending deflections of a nonlocal strain gradient beam subjected to the distributed load are analytically obtained. Meanwhile, the boundary value problems of the buckling behaviors of nonlocal strain gradient beams are studied. In conclusion, the main findings of this work are summarized as follows:

- Reformulation of the boundary value problems of nonlinear Euler–Bernoulli beams (see Eqs. (20), (21) and (31)) within the framework of the nonlocal strain gradient theory.
- The bending deflections of nonlocal strain gradient beams subjected to the distributed load are found to be independent of the choices of higher-order boundary conditions. The stiffening and softening behaviors of beams are observed by comparing the effect of the two material length parameters.
- The buckling loads are affected by the choices of the higher-order boundary conditions. When the two material length parameters are the same, the buckling loads of nonlocal strain gradient beams are not always the same; this conclusion is not the same as those reported in the literature.

The authors believe that by utilizing the WRAs, the present work can be extended to the studies of dynamic boundary value problems for rods, beams, plates and shells. As a result, further work is needed to provide WRAs as a useful tool for the engineering structures such as functionally graded materials and piezoelectric materials.

Acknowledgements

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