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Analytical solutions for estimating tsunami propagation speeds

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ABSTRACT

Recent studies suggest that the tsunami speed can be slowed down by around 1% due to Earth elasticity, water compressibility and density stratification. Analytical solutions of wave dispersion relationship, accounting for such effects, were found in previous studies. In this paper, we investigate the additional effects of water viscosity, ocean stratification due to temperature/salinity and numerical dispersion. Theoretical solutions are derived and checked with known solutions. All the formulas are then simplified for long tsunamis waves so that the propagation speed can be calculated explicitly. The simplified solutions are evaluated using realistic geophysical parameters. For a typical tsunami wavelength of ~200km, the viscous effect is found to be negligible; ocean stratification due to temperature/salinity causes significant speed reduction because of the high density change rate, which has been ignored before. We also evaluate the numerical dispersion of tsunami simulations, which is shown to be potentially comparable to physical dispersion.

1. Introduction

From recent large tsunami events it has been observed that there exists a systematic delay of arrival time compared to the prediction of shallow water equations (e.g., Rabinovich et al., 2011; Wei et al., 2008; Hébert et al., 2009; Saito et al., 2010; Kato et al., 2011; Fujii and Satake, 2013; Kimura et al., 2013; Watada et al., 2014). Watada et al. (2014) summarized the observations and showed that the discrepancy between observation and prediction, which has the order of around 1%, can be explained by the effects of Earth elasticity, water compressibility and geopotential variations. Previous studies have led to the theoretical solutions accounting for one or more of those effects. Mallard et al. (1977) studied the problem of water waves propagating over an elastic bed, using a model that consists of a single layer of incompressible potential fluid over half-space homogeneous elastic earth. The dispersion relation of the small amplitude water waves in such a model was given. Dawson (1978) reviewed the problem but took into account the solid inertia and suggested its importance for cases that include thick soft sediment. Dalrymple and Liu (1978) studied the problem of viscous water waves propagating over a mud bottom using a perturbation method, and derived the dispersion relation and the decay rate of wave amplitude. Ward (1980) presented the theory of tsunami generation and propagation on a spherically symmetric, self-gravitating, elastic Earth in terms of normal modes. Okal (1982) studied the asymptotic behavior of the gravity modes of an incompressible spherical oceanic layer surrounding a rigid Earth, as its radius goes to infinity. Okal showed that the flat-layered ocean tsunami solution and its dispersion is an asymptotic limit of the normal modes of a spherical oceanic shell and that only one branch of tsunami modes exists. Comer (1984) considered water waves in an incompressible fluid within a uniform gravitational field overlying an elastic earth in which the gravitational forces are ignored. Comer concluded that elastic forces are far more important than inertia and they also derived the dispersion relation for the water waves when the Poisson's ration of the Earth is 0.25. Panza et al. (2000) derived solutions for a model with multi-layered compressible inviscid fluid on top of a multi-layered solid half-space Earth, with the compressibility of the fluid treated as elastic solid. Tsai et al. (2013) derived theoretical tsunami propagation speeds accounting for Earth elasticity and water compressibility and stratification based on a method of conservation of potential and kinetic energy. Watada (2013) derived the theoretical dispersion relationship accounting for water compressibility and ocean stratification due to compression of gravity. Allgeyer and Cummins (2014) conducted numerical simulations to

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investigate the effects of elastic Earth. Watada et al. (2014) investigated the data from the 2010 Chilean and 2011 Tohoku tsunamis and showed that the systematic arrival time delay could be explained by the effects of Earth elasticity, water compressibility and geopotential variations.

In this study, we use a method which is slightly different from Watada (2013) to investigate additional secondary effects on tsunami propagation speeds, i.e., the water viscosity and ocean stratification due to temperature/salinity. For completeness the known second order effects, such as the Earth elasticity and water compressibility will be included in the solutions. In addition, we simplify all the theoretical dispersion relationships under the assumption of long waves so that the tsunami propagation speeds can be calculated explicitly. The simplifications are verified for typical tsunami wavelength of 50–1000 km (or wave period 260–5000 s) using realistic geophysical parameters. The relative importance of the second order processes in affecting the tsunami propagation speeds are assessed. Finally, the numerical dispersions of typical tsunami simulating algorithms are evaluated and compared with the above physical effects.

2. Governing equations and boundary conditions

We consider 2D problems in x and z, with x in the horizontal direction to the right and z pointing upwards (Fig. 1).

2.1. Fluid

The linearized governing equations for compressible fluids are derived from the complete Navier-Stokes equations by assuming slight compressibility. They are written as:

$$\begin{cases} \frac{\partial p_1}{\partial t} + B\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - \rho_0 g w = 0 \\ \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p_1}{\partial x} + 2\mu\left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] + \frac{\partial \sigma_{xz}}{\partial z} \\ \rho_0 \frac{\partial^2 w}{\partial t^2} = \left(\frac{\rho_0^2 g^2}{B} + g\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\right) w - \frac{\rho_0 g}{B}\frac{\partial p_1}{\partial t} - \frac{\partial^2 p_1}{\partial t \partial z} + \frac{\partial^2 \sigma_{xz}}{\partial t \partial x} + \frac{\partial^2 \sigma_{zz}}{\partial t \partial z} \\ \sigma_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \sigma_{zz} = 2\mu\left[\frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right] \end{cases} \quad (1)$$

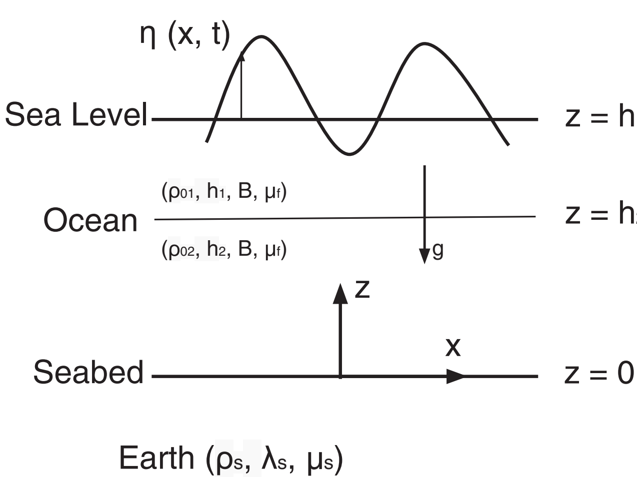

Fig. 1. The 2D model: a tsunami wave $\eta(x,t)$ is propagating over an elastic bed. The z axis origins at the seabed and points upwards; the x axis is in the horizontal direction. The ocean is assumed to be stratified with two layers, with the background density and water depth denoted as (ρ_{01}, h_1) and (ρ_{02}, h_2), respectively. The total water depth $h = h_1 + h_2$. The fluid is assumed to be viscous and slightly compressible, with constant bulk modulus B and viscosity μ_f; Earth is elastic with Lamè constants λ_s and μ_s. The constant gravitational acceleration is g.

where $u(x,z,t)$ and $w(x,z,t)$ are horizontal and vertical velocities, σ_{xz} and σ_{zz} are components of the stress tensor, μ_f is the fluid viscosity and B is the bulk modulus of the fluid. The subscript f of μ_f is omitted for simplicity. The subscript is used to distinguish between the viscosity in the fluid μ_f and the shear modulus in the solid μ_s only when necessary. The total pressure p is defined as the sum of static pressure p_0 and dynamic pressure p_1: $p(x,z,t) = p_0(z) + p_1(x,z,t)$; and the total density ρ is defined as the sum of the background density ρ_0 and density perturbation ρ_1 due to pressure change: $\rho(x,z,t) = \rho_0(z) + \rho_1(x,z,t)$. It is required that $\mathrm{d}p_0(z)/\mathrm{d}z = -\rho_0(z)g$.

Note that it has been assumed that the bulk modulus is constant and that the fluid is only slightly compressible, i.e., $\rho_1(x,z,t) \ll \rho_0(z)$ and $B = \rho \mathrm{d}p/\mathrm{d}\rho$, so we have $\mathrm{d}\rho/\mathrm{d}t = (\rho/B)\mathrm{d}p/\mathrm{d}t$, which leads to, by keeping only the leading order terms,

$$\frac{\partial \rho_1}{\partial t} = \frac{\rho_0}{B}\frac{\partial p_1}{\partial t} - \frac{\rho_0^2 g}{B} w - \frac{\mathrm{d}\rho_0}{\mathrm{d}z} w. \quad (2)$$

The total stress tensor is expressed as

$$\begin{aligned} \mathbf{s} &= -(p_0 + p_1)\mathbf{I} + \begin{bmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{bmatrix} \\ &= -(p_0 + p_1)\mathbf{I} + 2\mu \begin{bmatrix} \frac{\partial u}{\partial x} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{\partial w}{\partial z} - \frac{1}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \end{bmatrix}, \end{aligned} \quad (3)$$

where $\mathbf{I}$ is the identity tensor. The equations are similar to those proposed by Watada (2009), but the shear stress components are included in order to evaluate the viscous effects.

Adopting a plane wave solution in x direction, we have

$$\begin{cases} \eta(x,t) = A\, \mathrm{e}^{i(kx-\omega t)} \\ u(x,z,t) = \hat{u}(z)\, \mathrm{e}^{i(kx-\omega t)} \\ w(x,z,t) = \hat{w}(z)\, \mathrm{e}^{i(kx-\omega t)} \\ p_1(x,z,t) = \hat{p}_1(z)\, \mathrm{e}^{i(kx-\omega t)} \\ \sigma_{xz}(x,z,t) = \hat{\sigma}_{xz}(z)\, \mathrm{e}^{i(kx-\omega t)} \\ \sigma_{zz}(x,z,t) = \hat{\sigma}_{zz}(z)\, \mathrm{e}^{i(kx-\omega t)}, \end{cases} \quad (4)$$

where A, k and ω are the constant wave amplitude, wave number and angular frequency respectively. Substituting the assumed solution (4) into the governing Eq. (1), we obtain a set of first order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \hat{u}(z) \\ \hat{w}(z) \\ \hat{p}_1(z) \\ \hat{\sigma}_{xz}(z) \end{bmatrix} = \begin{bmatrix} 0 & -ik & 0 & \frac{1}{\mu} \\ -ik & \frac{\rho_0 g}{B} & i\frac{\omega}{B} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 2k^2\mu - i\rho_0\omega & i\frac{2k\mu\rho_0 g}{3B} & ik - \frac{2k\mu\omega}{3B} & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{w} \\ \hat{p}_1 \\ \hat{\sigma}_{xz} \end{bmatrix} \quad (5)$$

and

$$\hat{\sigma}_{zz}(z) = -i\, 2k\mu\, \hat{u}(z) + \frac{4\mu\rho_0 g}{3B}\hat{w}(z) + i\frac{4\mu\omega}{3B}\, \hat{p}_1(z),$$

where the third row of the coefficient matrix is

$$\begin{cases} m_{31} = -i\frac{4k\mu\rho_0 g}{3B-i\,4\mu\omega} \\ m_{32} = \frac{1}{1-i\,\frac{4\mu\omega}{3B}}\left[i\omega\rho_0 - 2k^2\mu + \frac{4\mu g}{3B}\frac{d\rho_0}{dz} + \frac{4\mu\rho_0^2 g^2}{3B^2} + i\frac{\rho_0^2 g^2}{\omega B} + i\frac{g}{\omega}\frac{d\rho_0}{dz}\right] \\ m_{33} = \frac{1}{1-i\,\frac{4\mu\omega}{3B}}\left[i\frac{4\omega\mu\rho_0 g}{3B^2} - \frac{\rho_0 g}{B}\right] \\ m_{34} = -\frac{i\,k}{1-i\frac{4\mu\omega}{3B}}. \end{cases} \tag{6}$$

The analytical solutions of Eq. (5) can be found generally by solving the eigen-system of the coefficient matrix. However, a full 4 × 4 matrix does not have an explicit solution of the eigen-system, so in later sections we decouple the effects of viscosity, compressibility and background density change and obtain analytical solutions accounting for those effects separately.

2.2. Solid

In the solid, the governing equations are

$$\rho\ddot{\mathbf{U}} = \nabla\cdot\boldsymbol{\sigma}, \tag{7}$$

where ρ is the solid density, $\mathbf{U}(x,z,t)$ is the displacement vector and $\boldsymbol{\sigma}(x,z,t)$ is the stress tensor. The gravitational field is ignored. Note that the density will be denoted as ρ_f and ρ_s for fluid and solid density, respectively, when necessary. Assuming homogeneous elastic properties in the solid region and adopting plane wave solutions similar to Eq. (4), the governing equations in the solid can be converted to a set of ODEs, which are then solved by solving the eigen-system of coefficient matrix. The solutions in the solid are found to be

$$\begin{cases} U(x,z,t) = \left(C_a\, v_{11}\, e^{\lambda_1 z} + C_c\, v_{31}\, e^{\lambda_3 z}\right)\, e^{i(kx-\omega t)} \\ W(x,z,t) = \left(C_a\, v_{12}\, e^{\lambda_1 z} + C_c\, v_{32}\, e^{\lambda_3 z}\right)\, e^{i(kx-\omega t)} \\ \sigma_{xz}(x,z,t) = \left(C_a\, v_{13}\, e^{\lambda_1 z} + C_c\, v_{33}\, e^{\lambda_3 z}\right)\, e^{i(kx-\omega t)} \\ \sigma_{zz}(x,z,t) = \left(C_a\, v_{14}\, e^{\lambda_1 z} + C_c\, v_{34}\, e^{\lambda_3 z}\right)\, e^{i(kx-\omega t)}, \end{cases} \tag{8}$$

where $U(x,z,t)$ and $W(x,z,t)$ are displacements in x and z direction, C_a and C_c are constants to be determined from boundary conditions, and $\lambda_{1,2,3,4}$ and $\mathbf{v}_{1,2,3,4}$ are the eigen solution of the coefficient matrix, given by

$$\lambda_{1,2} = \pm k\sqrt{1-\frac{c^2}{\beta^2}}, \quad \lambda_{3,4} = \pm k\sqrt{1-\frac{c^2}{\alpha^2}}, \tag{9}$$

and

$$\begin{cases} \mathbf{v}_{1,2} = \left(i\sqrt{1-\frac{c^2}{\beta^2}},\ \pm 1,\ \pm ik\mu\left(2-\frac{c^2}{\beta^2}\right),\ 2k\mu\sqrt{1-\frac{c^2}{\beta^2}}\right) \\ \mathbf{v}_{3,4} = \left(i,\ \pm\sqrt{1-\frac{c^2}{\alpha^2}},\ \pm 2i\,k\mu\sqrt{1-\frac{c^2}{\alpha^2}},\quad k\mu\left(2-\frac{c^2}{\beta^2}\right)\right). \end{cases} \tag{10}$$

Here c is the tsunami wave phase speed defined as $c = \omega/k$, and α and β are P and S wave speeds in the solid:

$$c = \frac{\omega}{k}, \qquad \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}, \qquad \beta = \sqrt{\frac{\mu}{\rho}}. \tag{11}$$

λ_s and μ_s, with the subscripts omitted for simplicity, are the Lamè constants. Note that here we have applied the boundary condition at $z = -\infty$, which states $U \to 0$ and $W \to 0$, and therefore the terms associated with the negative eigenvalues λ_2 and λ_4 vanish in the solutions (8).

2.3. Boundary conditions

The boundary conditions on the free surface are:

$$z = h+\eta, \quad \text{fluid 1}: \quad \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = w, \quad \mathbf{n}\cdot\boldsymbol{\sigma} = 0, \tag{12}$$

where $\mathbf{n}$ is the normal direction of the free surface: $\mathbf{n} = (\partial\eta/\partial x, -1)$. They are linearized to be (e.g., Mei, 1989, p. 10)

$$z = h, \quad \text{fluid 1}: \quad \begin{cases} \frac{\partial\eta}{\partial t} = w \\ \sigma_{xz} = 0 \\ \rho_0 g\eta - p_1 + \sigma_{zz} = 0. \end{cases} \tag{13}$$

Similarly, the linearized boundary conditions at the fluid-fluid interface are

$$z = h_2: \quad \begin{cases} u|_{\text{fluid 1}} = u|_{\text{fluid 2}}, \qquad w|_{\text{fluid 1}} = w|_{\text{fluid 2}} \\ \sigma_{xz}|_{\text{fluid 1}} = \sigma_{xz}|_{\text{fluid 2}} \\ -p_1 + \sigma_{zz}|_{\text{fluid 1}} = -p_1 + \sigma_{zz}|_{\text{fluid 2}}. \end{cases} \tag{14}$$

The linearized boundary conditions at the fluid-solid interface are

$$z = 0: \quad \begin{cases} u|_{\text{fluid 2}} = \frac{\partial U}{\partial t}|_{\text{solid}}, \qquad w|_{\text{fluid 2}} = \frac{\partial W}{\partial t}|_{\text{solid}} \\ \sigma_{xz}|_{\text{fluid 2}} = \sigma_{xz}|_{\text{solid}} \\ -p_1 + \sigma_{zz}|_{\text{fluid 2}} = \sigma_{zz}|_{\text{solid}}. \end{cases} \tag{15}$$

Note that the static pressure p_0 in the fluid is ignored in the balance of stress due to the cancellation of the equivalent pressure in the solid. The boundary conditions at $z = -\infty$ are

$$z = -\infty, \quad \text{solid}: \quad U = 0, \quad W = 0, \tag{16}$$

which have already been applied to obtain the solution (8) in the solid.

3. Dispersion relationships

In this section, we first derive the analytical solutions in the fluid that separately account for the effects of Earth elasticity, water viscosity, compressibility and ocean stratification due to compression and temperature/salinity. The solutions are generally found by solving the eigen-system of coefficient matrix of Eq. (5) in the fluid. They are then applied to the boundary conditions along with the analytical solutions in the solid. For each case, the number of boundary conditions is one more than the number of constants in the solutions, and the extra boundary condition leads to the dispersion relationship. Thereafter, the theoretical dispersion relationships are approximated to be an explicit form under the assumption of long waves. Note that the solutions for Earth elasticity, water compressibility and ocean stratification due to compression have been given by previous studies (Comer, 1984; Tsai et al., 2013; Watada, 2013). Here we re-derive them for completeness and show the simplifications of the full dispersion relationships and associated assumptions.

3.1. Earth elasticity

By neglecting all other effects except for Earth elasticity, the fluid is inviscous ($\mu = 0$), incompressible ($B = \infty$) and the density is constant over the entire region ($\rho_{10} = \rho_{20} = \rho_0 = \text{const}$). The ODEs in the fluid (5) are simplified to

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\hat{w}(z)\\ \hat{p}_1(z)\end{bmatrix} = \begin{bmatrix}0 & -i\frac{k^2}{\rho_0\omega}\\ i\rho_0\omega & 0\end{bmatrix}\begin{bmatrix}\hat{w}\\ \hat{p}_1\end{bmatrix}, \tag{17}$$

and

$$\begin{cases}\hat{u}(z) = \frac{k}{\rho_0\omega}\,\hat{p}_1(z)\\ \hat{\sigma}_{xz} = 0\\ \hat{\sigma}_{zz} = 0.\end{cases} \tag{18}$$

By solving the eigen-system of the coefficient matrix, the solutions can be readily obtained. They are written in hyperbolic functions as conventionally used in hydrodynamics:

$$\begin{cases}u(x,z,t) = i\ \left(\mathrm{C}_1\cosh kz + \mathrm{C}_2\sinh kz\right)\ \mathrm{e}^{i(kx-\omega t)}\\ w(x,z,t) = \left(\mathrm{C}_1\sinh kz + \mathrm{C}_2\cosh kz\right)\ \mathrm{e}^{i(kx-\omega t)}\\ p_1(x,z,t) = i\frac{\rho_0\omega}{k}\ \left(\mathrm{C}_1\cosh kz + \mathrm{C}_2\sinh kz\right)\ \mathrm{e}^{i(kx-\omega t)}\\ \sigma_{xz}(x,z,t) = 0\\ \sigma_{zz}(x,z,t) = 0,\end{cases} \tag{19}$$

where C_1 and C_2 are constants to be determined from boundary conditions. The static pressure in the fluid is simply $p_0(z) = -\rho_0 g(z-h)$, derived from $\mathrm{d}p_0/\mathrm{d}z = -\rho_0 g$.

By neglecting water stratification, the boundary conditions at the fluid-fluid interface (14) are ignored; by neglecting water viscosity, the horizontal velocity at the fluid-solid interface is not necessarily continuous. Substituting the solution in the fluid (19) and the solution in the solid (8) into the boundary conditions at the free surface (13) and fluid-solid interface (15), given that $\eta(x,t) = A\mathrm{e}^{i(kx-\omega t)}$, we obtain

$$\begin{cases}\sinh(kh)\,\mathrm{C}_1 + \cosh(kh)\,\mathrm{C}_2 = -i\omega A\\ \omega\cosh(kh)\,\mathrm{C}_1 + \omega\sinh(kh)\,\mathrm{C}_2 = -i\,kgA\\ \mathrm{C}_2 + i\omega v_{12}\mathrm{C}_a + i\omega v_{32}\mathrm{C}_c = 0\\ v_{13}\mathrm{C}_a + v_{33}\mathrm{C}_c = 0\\ i\,\rho_0\omega\mathrm{C}_1 + kv_{14}\mathrm{C}_a + kv_{34}\mathrm{C}_c = 0.\end{cases} \tag{20}$$

The first two equations are inhomogeneous and can be converted into a homogeneous one, resulting in a set of homogeneous equations for C_1, C_2, C_a and C_b. Thereafter, to have non-trivial solutions, the determinant of the coefficient matrix has to be zero, which leads to the dispersion relationship. After some manipulation, we obtain

$$c^2 = c_0^2 - \mathrm{F}\left(c_0^2\coth(kh) - c^2\tanh(kh)\right), \tag{21}$$

where c_0 is the standard dispersive wave speed and

$$c_0^2 = \frac{g}{k}\tanh(kh), \qquad \mathrm{F} = \frac{\rho_f}{\rho_s}\frac{\sqrt{1-\xi_\alpha}\,\xi_\beta^2}{4\sqrt{1-\xi_\alpha}\sqrt{1-\xi_\beta} - (2-\xi_\beta)^2},$$
$$\xi_\alpha = c^2/\alpha^2, \qquad \xi_\beta = c^2/\beta^2. \tag{22}$$

Again, α and β are the *P* and *S* wave speeds in the solid respectively, given by Eq. (11).

Eq. (21) can be solved numerically. Alternatively, it can be simplified under the assumption that the tsunami wave speed c is much smaller than the seismic wave speeds in the solid, i.e., $c^2 \ll \alpha^2$ and $c^2 \ll \beta^2$, or, $\xi_\alpha \ll 1$ and $\xi_\beta \ll 1$. Expanding F near $\xi_\alpha = 0$ and $\xi_\beta = 0$ and only keeping the leading order terms, we find

$$\mathrm{F} \approx \frac{\rho_f}{2\rho_s\left(1-\frac{\beta^2}{\alpha^2}\right)}\,\frac{c^2}{\beta^2}, \tag{23}$$

where $0 \le \xi_\alpha/\xi_\beta \le 0.5$. Thus, F is a small variable with order of the squared ratio of tsunami wave speed and P or S wave speed, and the correction of the tsunami wave speed due to Earth elasticity has the same order. Substituting the simplified F into the full dispersion relationship (21), we obtain a quadratic equation for c^2. It is then solved analytically and simplified given that $c_0^2 \ll \beta^2$, leading to

$$c^2 \approx \left[\frac{1}{1+\xi_e \coth(kh)} + \frac{\xi_e \tanh(kh)}{(1+\xi_e \coth(kh))^3}\right] c_0^2, \tag{24}$$

where

$$\xi_e = \frac{\rho_f}{2\rho_s\left(1-\frac{\beta^2}{\alpha^2}\right)} \frac{c_0^2}{\beta^2} \ll 1. \tag{25}$$

If we further assume that $\xi_e \coth(kh) \ll 1$, i.e.,

$$\xi_e = \frac{\rho_f}{2\rho_s\left(1-\frac{\beta^2}{\alpha^2}\right)} \frac{c_0^2}{\beta^2} \ll \tanh(kh), \tag{26}$$

the dispersion relationship can be simplified to

$$c \approx \left[1 - \frac{\coth(kh)-\tanh(kh)}{4\left(1-\frac{\beta^2}{\alpha^2}\right)} \frac{\rho_f}{\rho_s} \frac{c_0^2}{\beta^2}\right] c_0. \tag{27}$$

Again, α and β are the P and S wave speeds in the solid, given by Eq. (11). The solution is the same as Comer's (1984) analytical result obtained by assuming $\lambda = \mu$. Note that solution (27) requires that the squared ratio of tsunami and seismic speeds c_0^2/β^2 should be much smaller than the ratio of water depth and wavelength (kh) (26), which is stronger than the assumption (25) required by solution (24). Later we will verify that, for typical tsunami wavelength around 50–1000 km (or wave period around 260–5000 s), solution (27) is a good approximation of the full solution (21).

By taking the assumption of long waves, i.e., $kh \ll 1$, $\tanh(kh) \approx kh$ and $\coth(kh) \approx 1/kh$, we can separate the effects of Earth elasticity and wave dispersion, and solution (27) is revised to

$$c \approx \left[1 - \frac{1}{4\left(1-\frac{\beta^2}{\alpha^2}\right) kh} \frac{\rho_f}{\rho_s} \frac{gh}{\beta^2}\right] \sqrt{gh}. \tag{28}$$

This result shows that the reduction of tsunami propagation speeds increases approximately linearly with the wavelength, and the order of the correction is around $\frac{c_0^2}{\beta^2} \frac{1}{kh}$.

3.2. Water viscosity

To evaluate the effect of water viscosity, we neglect water compressibility and density stratification. By taking $B = \infty$ and $\rho_0 = \text{const}$, the ODEs in the water (5) are simplified to

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} \hat{u}(z) \\ \hat{w}(z) \\ \hat{p}_1(z) \\ \hat{\sigma}_{xz}(z) \end{bmatrix} = \begin{bmatrix} 0 & -ik & 0 & \frac{1}{\mu} \\ -ik & 0 & 0 & 0 \\ 0 & -(2k^2\mu - i\rho_0\omega) & 0 & -ik \\ 2k^2\mu - i\rho_0\omega & 0 & ik & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{w} \\ \hat{p}_1 \\ \hat{\sigma}_{xz} \end{bmatrix},$$

and

$$\hat{\sigma}_{zz}(z) = -i\, 2k\mu\, \hat{u}(z).$$

Again, the solutions can be obtained by solving the eigen-system of the coefficient matrix, and they are

$$\begin{cases} u(x,z,t) = i\left(C_1\cosh(kz) + C_2\sinh(kz) + C_3\frac{\lambda}{k}\,e^{\lambda(z-h)} - C_4\frac{\lambda}{k}\,e^{-\lambda z}\right) e^{i(kx-\omega t)} \\ w(x,z,t) = \left(C_1\sinh(kz) + C_2\cosh(kz) + C_3\,e^{\lambda(z-h)} + C_4\,e^{-\lambda z}\right) e^{i(kx-\omega t)} \\ p_1(x,z,t) = i\frac{\rho\,\omega}{k}\left(C_1\cosh(kz) + C_2\sinh(kz)\right) e^{i(kx-\omega t)} \\ \sigma_{xz}(x,z,t) = i\,2k\mu\left(C_1\sinh(kz) + C_2\cosh(kz) + C_3\frac{k^2+\lambda^2}{2k^2}\,e^{\lambda(z-h)} + C_4\frac{k^2+\lambda^2}{2k^2}\,e^{-\lambda z}\right) \\ \sigma_{zz}(x,z,t) = 2k\mu\left(C_1\cosh(kz) + C_2\sinh(kz) + C_3\frac{\lambda}{k}\,e^{\lambda(z-h)} - C_4\frac{\lambda}{k}\,e^{-\lambda z}\right) e^{i(kx-\omega t)}, \end{cases} \tag{29}$$

where

$$\lambda^2 = k^2 - i\,\frac{\rho_0\omega}{\mu}, \tag{30}$$

and C_1, C_2, C_3 and C_4 are constants to be determined from boundary conditions. The static pressure is simply $p_0(z) = -\rho_0 g(z-h)$. Note that λ here is a complex wavenumber. The solution is essentially the same as that obtained by Dalrymple and Liu (1978), who used a perturbation method to investigate a two-layer model of viscous fluids.

By neglecting Earth elasticity, water compressibility and stratification, the boundary conditions at the fluid-fluid interface (14) are ignored; the boundary conditions at the fluid-solid interface (15) are simplified to that the horizontal and vertical fluid velocities vanish. Substituting the solution (29) into the boundary conditions at the free surface (13) and fluid-solid interface (15), we obtain

$$\begin{cases} \sinh(kh)C_1 + \cosh(kh)C_2 + C_3 + e^{-\lambda h}C_4 = -i\omega A \\ 2k^2\sinh(kh)C_1 + 2k^2\cosh(kh)C_2 + (\lambda^2+k^2)C_3 + (\lambda^2+k^2)e^{-\lambda h}C_4 = 0 \\ (2k^2\mu - i\,\rho_0\omega)\cosh(kh)C_1 + (2k^2\mu - i\,\rho_0\omega)\sinh(kh)C_2 + 2k\lambda\mu C_3 - 2k\lambda\mu e^{-\lambda h}C_4 = -k\rho_0 gA \\ kC_1 + \lambda e^{-\lambda h}C_3 - \lambda C_4 = 0 \\ C_2 + e^{-\lambda h}C_3 + C_4 = 0. \end{cases}$$

Eliminating the constants on the R.H.S., the equations can be converted into homogeneous ones. Seeking non-trivial solutions, the determinant of the coefficient matrix has to be zero, which leads to

$$\begin{aligned} &\left[\rho_0 gk^2(k^2-\lambda^2)\right]\left(1-e^{-2\lambda h}\right)\cosh(kh) - \left[\rho_0\lambda\omega^2(k^2+\lambda^2) + 2ik^2\lambda\mu\omega(3k^2+\lambda^2)\right]\left(1+e^{-2\lambda h}\right)\cosh(kh) \\ &\quad + \left[\rho_0 k\omega^2(k^2+\lambda^2) + 2ik^3\mu\omega(k^2+3\lambda^2)\right]\left(1-e^{-2\lambda h}\right)\sinh(kh) - \left[\rho_0 gk\lambda(k^2-\lambda^2)\right]\left(1+e^{-2\lambda h}\right)\sinh(kh) \\ &\quad + 4k^2\lambda\omega\left(3ik^2\mu + i\lambda^2\mu + \rho_0\omega\right)e^{-\lambda h} = 0. \end{aligned} \tag{31}$$

We define

$$\xi_v = k\sqrt{\frac{\mu}{\rho_0}}(gk)^{-\frac{1}{4}} \ll 1, \tag{32}$$

and further simplify the results by expanding the dispersion relationship near $\xi_v = 0$. First,

$$\begin{aligned} \lambda^2 &= k^2 - i\frac{\rho_0\omega}{\mu} = k^2 - i\,\frac{\omega}{\sqrt{gk}\,\xi_v^2}k^2, \\ \lambda &= \sqrt{k^2 - i\frac{\rho_0\omega}{\mu}} = (1-i)\frac{k\sqrt{\omega}}{\sqrt{2}(gk)^{\frac{1}{4}}}\,\frac{1}{\xi_v} + (1+i)\frac{k(gk)^{\frac{1}{4}}}{\sqrt{8\omega}}\,\xi_v + O\left(\xi_v^3\right). \end{aligned} \tag{33}$$

So we have $e^{-\lambda h} = 0$ and $e^{-2\lambda h} = 0$ if keeping only the leading order small terms. Substituting λ into the dispersion relationship and only keeping the leading order terms, we obtain

$$\omega^2 = gk\tanh(kh) - (1+i)\frac{\sqrt{2}}{\sinh(2kh)\sqrt[4]{\tanh(kh)}}gk\tanh(kh)\,\xi_v + O(\xi_v^2), \tag{34}$$

or written in phase speed:

$$c^2 \approx \left[1 - (1+i)\frac{\sqrt{2}}{\sinh(2kh)\sqrt[4]{\tanh(kh)}}k\sqrt{\frac{\mu}{\rho_0}}(gk)^{-\frac{1}{4}}\right]c_0^2. \tag{35}$$

The dispersion relationship (35) is complex, i.e., the imaginary part is not trivial compared to the real part, indicating that water viscosity not only affects the wave speed but also damps the wave amplitude. The result is consistent with the simplified result obtained by Mei and Liu (1973) who used a perturbation method to investigate the damping of waves in a channel. Under the assumption of long waves, i.e., $kh \ll 1$, $\tanh(kh) \approx kh$ and $\coth(kh) \approx 1/kh$, the effects of viscosity are separated from wave dispersion and solution (35) is revised to

$$c^2 \approx \left[1 - (1+i)\frac{1}{\sqrt{2}(kh)^{\frac{5}{4}}}k\sqrt{\frac{\mu}{\rho_0}}(gk)^{-\frac{1}{4}}\right]gh. \tag{36}$$

3.3. Water compressibility

If we only consider the effects of water compressibility, by taking $\mu = 0$ and $\rho_0 = \text{const}$, the ODEs in the water (5) are simplified to

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\hat{w}(z)\\ \hat{p}_1(z)\end{bmatrix} = \begin{bmatrix}\frac{\rho_0 g}{B} & -i\frac{k^2}{\rho_0\omega} + i\frac{\omega}{B}\\ i\rho_0\omega + i\frac{\rho_0^2 g^2}{\omega B} & -\frac{\rho_0 g}{B}\end{bmatrix}\begin{bmatrix}\hat{w}\\ \hat{p}_1\end{bmatrix}, \tag{37}$$

and

$$\begin{cases}\hat{u}(z) = \frac{k}{\rho_0\omega}\hat{p}_1(z)\\ \hat{\sigma}_{xz}(z) = 0\\ \hat{\sigma}_{zz}(z) = 0.\end{cases} \tag{38}$$

Similarly, we solve the eigen-system of the coefficient matrix and derive the solutions. They are

$$\begin{cases}u(x,z,t) = kd_1\left(C_1 e^{\lambda z} + C_2 e^{-\lambda z}\right)\ e^{i(kx-\omega t)}\\ w(x,z,t) = -i\left[C_1\left(\lambda + \frac{\rho_0 g}{B}\right)e^{\lambda z} + C_2\left(-\lambda + \frac{\rho_0 g}{B}\right)e^{-\lambda z}\right]\ e^{i(kx-\omega t)}\\ p_1(x,z,t) = \rho_0\omega d_1\left(C_1 e^{\lambda z} + C_2 e^{-\lambda z}\right)\ e^{i(kx-\omega t)}\\ \sigma_{xz}(x,z,t) = 0\\ \sigma_{zz}(x,z,t) = 0,\end{cases} \tag{39}$$

where

$$\lambda = k\sqrt{d_1 - \frac{\rho_0}{B}\frac{\omega^2}{k^2}}, \qquad d_1 = 1 + \frac{\rho_0 g^2}{B\omega^2}, \tag{40}$$

and C_1 and C_2 are constants to be determined from boundary conditions.

By neglecting Earth elasticity and water stratification, the boundary conditions at the fluid-fluid interface (14) are ignored; the boundary conditions at the fluid-solid interface (15) are simplified to that the vertical fluid velocity vanishes. Substituting the solution (39) into the boundary conditions at the free surface (13) and fluid-solid interface (15), we obtain

$$\begin{cases}C_1\left(\lambda + \frac{\rho_0 g}{B}\right)e^{\lambda h} + C_2\left(-\lambda + \frac{\rho_0 g}{B}\right)e^{-\lambda h} = \omega A\\ C_1 e^{\lambda h} + C_2 e^{-\lambda h} = \frac{gA}{\omega d_1}\\ C_1\left(\lambda + \frac{\rho_0 g}{B}\right) + C_2\left(-\lambda + \frac{\rho_0 g}{B}\right) = 0.\end{cases} \tag{41}$$

Again, by eliminating the constants on the R.H.S. and converting the equations into homogeneous ones, the determinant of the coefficient matrix must be zero, which leads to

$$c^2 = \frac{\omega^2}{k^2} = g\frac{\tanh(\lambda h)}{\lambda}, \tag{42}$$

where λ is given by Eq. (40). The result (42) is the same as Eqs. (26) and (10) derived from a propagator matrix method in Watada (2013).

The formula (42) is implicit in terms of phase speed, so the solution is only available from numerical approaches. Here we simplify it under specific assumptions. First, we define

$$\xi_c = \frac{\omega^2\rho_0}{k^2 B} = \frac{c^2}{c_s^2} \ll 1, \tag{43}$$

and expand λ near $\xi_c = 0$:

$$\lambda = k\sqrt{d_1 - \xi_c} = k\sqrt{d_1}\left(1 - \frac{\xi_c}{2d_1}\right) + O(\xi_c^2), \tag{44}$$

where c_s is the sound speed and $c_s^2 = B/\rho_0$. Substituting the expanded λ into the full dispersion relationship (42) and keeping only the leading order terms, we obtain

$$c^2 \approx \frac{g}{k}\frac{\tanh\left(\sqrt{d_1}kh\right)}{\sqrt{d_1}} + \frac{g}{k\ 2d_1}\left[\frac{\tanh\left(\sqrt{d_1}kh\right)}{\sqrt{d_1}} - kh + kh\left(\tanh(\sqrt{d_1}kh)\right)^2\right]\xi_c. \tag{45}$$

For long waves, if we further assume that

$$\xi_d = \sqrt{d_1}kh = \sqrt{1 + \frac{\rho_0 g^2}{B\omega^2}}\, kh \ll 1, \tag{46}$$

by expanding Eq. (45) near $\xi_d = 0$ and $kh = 0$ and keeping only the leading order terms, the dispersion relationship (45) is simplified to

$$c \approx \left[1 - \frac{\rho_0 gh}{6B}\right] c_0, \tag{47}$$

where $c_0^2 = g/k\tanh(kh)$. The simplification requires the condition of Eq. (46), which is essentially equivalent to Eq. (43) for long waves. It is shown as follows:

$$\xi_d = \sqrt{(kh)^2 + \frac{(gh)^2}{c_s^2 c^2}} \approx \sqrt{(kh)^2 + \frac{c^2}{c_s^2}}, \tag{48}$$

where we have used $c^2 \approx gh$ for long waves. Later we will verify that the simplified solution (47) agrees well with the full solution (42) for typical tsunami wavelength of 50–1000 km (or wave period 260–5000 s).

By taking $c_0 = \sqrt{gh}$ we can separate the effects of compressibility and wave dispersion, leading to the dispersion relationship accounting for water compressibility only:

$$c \approx \left[1 - \frac{\rho_0 gh}{6B}\right] \sqrt{gh}. \tag{49}$$

3.4. Water stratification

In this section two kinds of ocean density stratification are evaluated: the increase of density due to compression of gravity and the ocean stratification due to temperature/salinity. With water being compressible, the ocean density increases slightly and gradually with depth under gravitational compression, and it is larger at the seabed than surface. In addition, the vertical ocean column is formed to two layers due to temperature/salinity, with the less dense layer on top. The first kind has a larger absolute variation of density, while the second has a smaller density change but a larger changing rate over a small depth. Previous studies have ignored the second kind. We will show that the effects of the second kind on tsunami propagation speeds can be the same order as the first kind.

3.4.1. Density stratification due to compression of gravity

Considering a background water density $\rho_0(z)$ but neglecting water viscosity ($\mu = 0$) and compressibility ($B = \infty$), the equations in the water (5) are simplified to

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix}\hat{w}(z)\\ \hat{p}_1(z)\end{bmatrix} = \begin{bmatrix} 0 & -i\frac{k^2}{\rho_0\omega} \\ i\rho_0\omega + i\frac{g}{\omega}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} & 0\end{bmatrix}\begin{bmatrix}\hat{w}\\ \hat{p}_1\end{bmatrix}, \tag{50}$$

and

$$\begin{cases}\hat{u}(z) = \frac{k}{\rho_0\omega}\hat{p}_1(z)\\ \hat{\sigma}_{xz}(z) = 0\\ \hat{\sigma}_{zz}(z) = 0.\end{cases} \tag{51}$$

In this situation, the coefficient matrix depends on z because ρ_0 is a function of z. Hence we are not able to find the solution by simply solving the eigen-system of the matrix as in the previous sections. By eliminating $\hat{p}_1$, we obtain

$$\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\frac{\mathrm{d}\hat{w}}{\mathrm{d}z} - k^2\left(1 + \frac{g}{\omega^2}\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\right)\hat{w} = 0. \tag{52}$$

This equation has simple analytical solutions when $1/\rho_0 \mathrm{d}\rho_0/\mathrm{d}z = \text{const}$. Since the density variation in the ocean is very small in depth, the following three density profiles are similar: (1) $1/\rho_0 \mathrm{d}\rho_0/\mathrm{d}z = -1/H = \text{const}$; (2) density profile under compression of gravity; (3) density profile that increases linearly in depth. For those three profiles, $\rho_0(z)$ is respectively expressed as

$$\begin{cases}\rho_0(z) = \rho_b\left(\frac{\rho_t}{\rho_b}\right)^{\frac{z}{h}} = \bar{\rho}\left[1 + \left(\frac{1}{2} - \frac{z}{h}\right)\frac{\Delta\rho}{\bar{\rho}} + O\left(\frac{\Delta\rho}{\bar{\rho}}\right)^2\right]\\ \rho_0(z) = \frac{\rho_b}{\left(\frac{\rho_b}{\rho_t} - 1\right)\frac{z}{h} + 1} = \bar{\rho}\left[1 + \left(\frac{1}{2} - \frac{z}{h}\right)\frac{\Delta\rho}{\bar{\rho}} + O\left(\frac{\Delta\rho}{\bar{\rho}}\right)^2\right]\\ \rho_0(z) = -\frac{\rho_b - \rho_t}{h}z + \rho_b = \bar{\rho}\left[1 + \left(\frac{1}{2} - \frac{z}{h}\right)\frac{\Delta\rho}{\bar{\rho}} + O\left(\frac{\Delta\rho}{\bar{\rho}}\right)^2\right],\end{cases} \tag{53}$$

where ρ_b and ρ_t are the fluid density at the bottom and top of the ocean, and

$$H = \frac{h}{\log(\rho_b/\rho_t)} = h\frac{\bar{\rho}}{\Delta\rho}\left[1 + O\left(\frac{\Delta\rho}{\rho}\right)^2\right],$$
$$B = gh\frac{\rho_b\rho_t}{\rho_b - \rho_t} = \bar{\rho}gh\frac{\bar{\rho}}{\Delta\rho}\left[1 + O\left(\frac{\Delta\rho}{\rho}\right)^2\right],$$
$$\bar{\rho} = \frac{\rho_b + \rho_t}{2}, \qquad \Delta\rho = \rho_b - \rho_t \ll \bar{\rho}. \tag{54}$$

Eq. (53) suggests that the three profiles are the same to the order of $\Delta\rho/\bar{\rho}$, and the solutions for these three profiles are therefore expected to be the same to the order of $\Delta\rho/\bar{\rho}$. Taking the first density profile, i.e., $1/\rho_0 d\rho_0/dz = -1/H$, the solutions in the fluid are found to be

$$\begin{cases} u(x,z,t) = i\,\frac{1}{k}\left(C_1\lambda_1 e^{\lambda_1 z} + C_2\lambda_2 e^{\lambda_2 z}\right)\ e^{i(kx-\omega t)} \\ w(x,z,t) = \left(C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}\right)\ e^{i(kx-\omega t)} \\ p_1(x,z,t) = i\,\frac{\rho_0\omega}{k^2}\left(C_1\lambda_1 e^{\lambda_1 z} + C_2\lambda_2 e^{\lambda_2 z}\right)\ e^{i(kx-\omega t)} \\ \sigma_{xz}(x,z,t) = 0 \\ \sigma_{zz}(x,z,t) = 0, \end{cases} \tag{55}$$

where

$$\lambda_1 = \frac{1}{2H} + \sqrt{k^2 + \frac{1}{4H^2} - \frac{gk^2}{\omega^2 H}}, \quad \lambda_2 = \frac{1}{2H} - \sqrt{k^2 + \frac{1}{4H^2} - \frac{gk^2}{\omega^2 H}}, \tag{56}$$

and C_1 and C_2 are constants to be determined from boundary conditions.

By neglecting Earth elasticity, water viscosity and multiple layers in the fluid, the boundary conditions at the fluid-fluid interface (14) are ignored; the boundary conditions at the fluid-solid interface (15) are simplified to that the vertical fluid velocity vanishes. Substituting the solution (55) into the boundary conditions at the free surface (13) and fluid-solid interface (15), we obtain

$$\begin{cases} C_1 e^{\lambda_1 h} + C_2 e^{\lambda_2 h} = -i\omega A \\ C_1\lambda_1 e^{\lambda_1 h} + C_2\lambda_2 e^{\lambda_2 h} = -i\frac{gAk^2}{\omega} \\ C_1 + C_2 = 0, \end{cases} \tag{57}$$

Eliminating the constants in the first two equations and converting the system into homogeneous equations, the determinant of the coefficient matrix has to be zero, leading to:

$$c^2 = \frac{\omega^2}{k^2} = g\frac{\tanh(\lambda h)}{\lambda + \frac{1}{2H}\tanh(\lambda h)}, \tag{58}$$

where

$$\lambda = \sqrt{k^2 + \frac{1}{4H^2} - \frac{gk^2}{\omega^2 H}}. \tag{59}$$

The result (58) is the same as Eqs. (25) and (10) by Watada (2013). Note that there are c^2 terms on the R.H.S. of Eq. (25) of Watada (2013).

To simplify the full dispersion relationship (58), we assume

$$\xi_s = \frac{h}{H} \ll 1, \tag{60}$$

which can be justified by $H \approx B/\bar{\rho}g \approx 200$ km $\gg h$ (Eq. (54), taking $B \approx 2\times10^9$Pa, $\bar{\rho} \approx 1\times10^3$ kg/m^3 and $g \approx 10$ ms/^{2}). According to Eq. (54), ξ_s is also determined to be the same as $\Delta\rho/\bar{\rho}$ to the leading order. By substituting $\omega^2 \approx ghk^2$ for long waves, it is found that

$$(\lambda h)^2 = k^2h^2 + \frac{h^2}{4H^2} - \frac{gk^2h^2}{\omega^2 H} \;\approx\; k^2h^2 + \frac{\xi_s^2}{4} - \xi_s \ll 1. \tag{61}$$

Therefore the full dispersion relationship (58) is expanded near $\lambda h = 0$ and then near $\xi_s = 0$ to have

$$c^2 \;\approx\; \left[gh - \frac{1}{3}(gh)(kh)^2\right] - \left(\frac{1}{2} - \frac{ghk^2}{3\omega^2}\right)(gh)\xi_s, \tag{62}$$

where the crossing terms of kh and ξ_s are neglected. Substituting $\omega^2 = ghk^2$ to the right hand side by neglecting small order crossing terms, we obtain

$$c \approx \left[1 - \frac{1}{12}\frac{h}{H}\right] c_0, \tag{63}$$

where $c_0^2 = g/k \tanh(kh)$. Replacing H in Eq. (63) with $\bar{\rho}$ and $\Delta\rho$ from Eq. (54), Eq. (63) can be written in terms of bulk modulus or fluid density as follows:

$$c \approx \left[1 - \frac{1}{12}\frac{h}{H}\right] c_0 \approx \left[1 - \frac{1}{12}\frac{\bar{\rho}gh}{B}\right] c_0 \approx \left[1 - \frac{1}{12}\frac{\Delta\rho}{\bar{\rho}}\right] c_0. \tag{64}$$

By taking $c_0 = \sqrt{gh}$ we can separate the effects of stratification and wave dispersion, leading to the dispersion relationship accounting for water stratification only:

$$c \approx \left[1 - \frac{1}{12}\frac{h}{H}\right] \sqrt{gh} \approx \left[1 - \frac{1}{12}\frac{\bar{\rho}gh}{B}\right] \sqrt{gh} \approx \left[1 - \frac{1}{12}\frac{\Delta\rho}{\bar{\rho}}\right] \sqrt{gh}. \tag{65}$$

3.4.2. Density stratification due to temperature/salinity

For this kind of density stratification, the model consists of two layers of fluid, but water viscosity, compressibility and density stratification are ignored in each layer. The solutions (19) are applied to the boundary conditions (13–15)to derive the dispersion relationship. By neglecting Earth elasticity, water viscosity, compressibility and background density change, the boundary conditions at the fluid-fluid interface (14) do not include the continuity of horizontal velocity; the boundary conditions at the fluid-solid interface (15) are simplified to that the vertical fluid velocity vanishes. Substituting the solution (19) into the boundary conditions (13–15), we obtain

$$\begin{cases} \sinh(kh)C_1 + \cosh(kh)C_2 = -i\omega A \\ \omega\cosh(kh)C_1 + \omega\sinh(kh)C_2 = -ikgA \\ \rho_1\cosh(kh_2)C_1 + \rho_1\sinh(kh_2)C_2 - \rho_2\cosh(kh_2)C_1' - \rho_2\sinh(kh_2)C_2' = 0 \\ \sinh(kh_2)C_1 + \cosh(kh_2)C_2 - \sinh(kh_2)C_1' - \cosh(kh_2)C_2' = 0 \\ C_2' = 0, \end{cases} \tag{66}$$

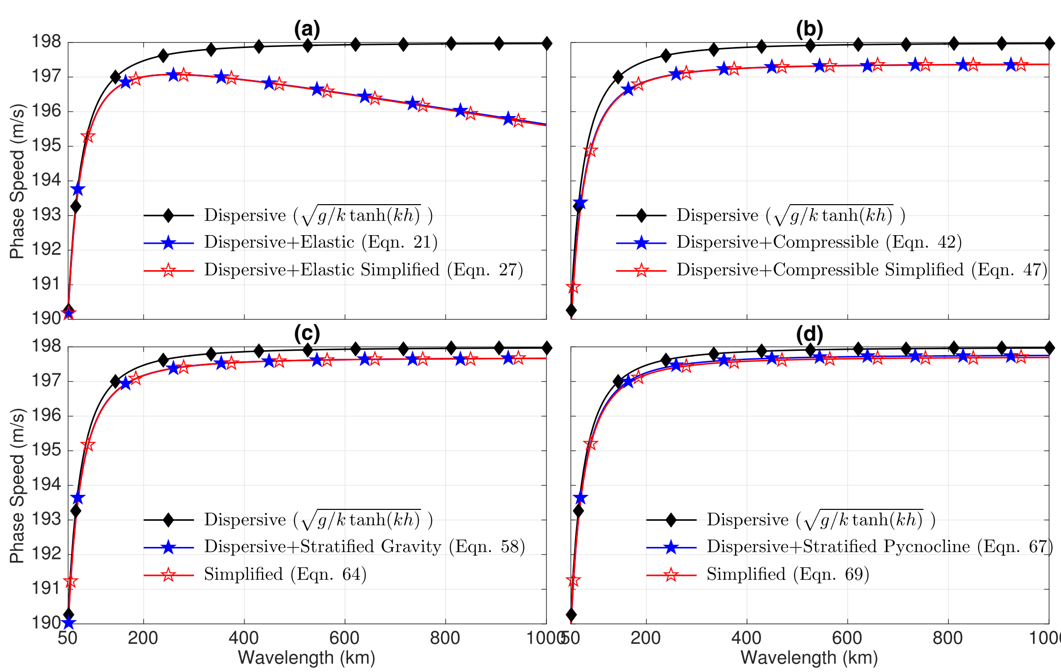

Fig. 2. Verification of the simplified dispersion relationships: (a) Earth elasticity; (b) water compressibility; (c) density stratification due to compression; (d) density stratification due to temperature/salinity. Note that all the speeds include the effects of dispersion, so the difference between the blue or red curves and the black curves is the correction and the difference between the blue and red curves is the error caused by simplification. For instance, the black curve in (a) gives the propagation speeds of dispersive tsunami waves, the blue curve shows the speeds of such waves under the effects of Earth elasticity, and the red curve denotes the simplified formula to calculate the blue. The error of simplification is so small that the red curves overlap with the blue ones. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $h = h_1 + h_2$, C_1 and C_2 are coefficients for fluid 1, C_1' and C_2' are coefficients for fluid 2. Eliminating the constants in the first two equations and converting the system into homogeneous equations, the determinant of the coefficient matrix has to be zero, leading to:

$$c^2 = \frac{\omega^2}{k^2} = \quad = \frac{g}{k}\,\frac{\tanh(kh_1) + \frac{\rho_1}{\rho_2}\tanh(kh_2)}{1 + \frac{\rho_1}{\rho_2}\tanh(kh_1)\tanh(kh_2)}. \tag{67}$$

The result can be verified by letting $\rho_1 = \rho_2$, which leads to $\omega^2 = gk\tanh(k(h_1 + h_2))$.

Assuming that the density difference is very small, i.e.,

$$\frac{\Delta\rho}{\bar{\rho}} = 2\,\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \ll 1, \tag{68}$$

by expanding the dispersion relationship near $\Delta\rho/\bar{\rho} = 0$ and only keeping the leading order terms, the dispersion relationship is simplified to

$$c \approx \left[1 - \frac{\Delta\rho}{2\bar{\rho}}\,\frac{\sinh(2kh_2)}{\sinh(2kh)}\right]c_0, \tag{69}$$

where $c_0^2 = g/k\tanh(k(h_1 + h_2)) = g/k\tanh(kh)$.
Taking $c_0 = \sqrt{gh}$, we obtain the dispersion relationship accounting for density stratification due to temperature/salinity:

$$c \approx \left[1 - \frac{\Delta\rho\ h_2}{2\bar{\rho}\ h}\right]\sqrt{gh}. \tag{70}$$

The density change $\Delta\rho$ of such stratifications is normally smaller than that of stratifications due to compression of gravity, but the coefficient of the correction term is larger (1/2 in Eq. (70) and 1/12 in Eq. (65)). Therefore it is expected that such stratifications can slow down the tsunami waves by the same order.

4. Verification of the simplifications

In this section we verify the simplifications of the dispersion relationships, that is, (a) the full dispersion relationship accounting for Earth elasticity (21) and the simplified formula (27); (b) the full dispersion relationship accounting for water compressibility (42) and the simplified formula (47); (c) the full dispersion relationship accounting for density stratification due to compression (58) and the simplified formula (64); (d) the full dispersion relationship accounting for density stratification due to temperature/salinity (67) and the simplified formula (69). The viscous effects are very small for tsunami waves and the verification for its simplified dispersion relationship (35) is omitted.

The geophysical parameters adopted are as follows: water depth $h = 4000$ m, water density $\rho_0 = \rho_t = 1025$ kg/m^3, water viscosity $\mu_f = 1.0 \times 10^{-3}$ Pa s, water bulk modulus $B = 2.2$ GPa, solid density $\rho_s = 3500$ kg/m^3, Earth Poisson's ratio $\nu_s = 0.25$, Earth shear modulus $\mu_s = 50$ GPa, water stratification $h_1 = 1000$ m, $h_2 = 3000$ m, $\Delta\rho = 3.0$ kg/m^3, $\rho_1 = \rho_0$, $\rho_2 = \rho_2 + \Delta\rho$ and constant gravity acceleration of $g = 9.8$ m/s^2.

The full dispersion relationships are solved using the Newton's iteration method, and we vary the initial guess in the range of $c_0 \pm 20$ m/s to ensure that the solution is independent of the initial guess. The results are plotted along with the simplified results in Fig. 2. It shows that, for typical tsunami wavelength of 50–1000 km (or wave period 260–5000 s, the error between the full and simplified solutions is negligible compared to the correction itself. Thus, the simplified solution is a very good approximate of the full.

5. Numerical dispersion and relative importance of secondary effects

When simulating tsunami waves, numerical dispersion is inevitably introduced. The shallow water wave equations are written as (Cho, 1995)

$$\begin{cases} \frac{\partial\eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \\ \frac{\partial P}{\partial t} + gh\frac{\partial\eta}{\partial x} = 0 \\ \frac{\partial Q}{\partial t} + gh\frac{\partial\eta}{\partial y} = 0, \end{cases} \tag{71}$$

where $\eta(x, y, t)$ is the water elevation, $P(x, y, t)$ and $Q(x, y, t)$ are the horizontal fluxes. As an example, the leap-frog numerical method can be used to solve the equations using the following scheme:

$$\begin{cases} \frac{\eta_{i,j}^{n+1/2} - \eta_{i,j}^{n-1/2}}{\Delta t} + \frac{P_{i+1/2,j}^{n} - P_{i-1/2,j}^{n}}{\Delta x} + \frac{Q_{i,j+1/2}^{n} - Q_{i,j-1/2}^{n}}{\Delta y} = 0 \\ \frac{P_{i+1/2,j}^{n+1} - P_{i+1/2,j}^{n}}{\Delta t} + gh\frac{\eta_{i+1,j}^{n+1/2} - \eta_{i,j}^{n+1/2}}{\Delta x} + \frac{\gamma gh}{12\Delta x} \\ +\left[\left(\eta_{i+1,j+1}^{n+1/2} - 2\eta_{i+1,j}^{n+1/2} + \eta_{i+1,j-1}^{n+1/2}\right) - \left(\eta_{i,j+1}^{n+1/2} - 2\eta_{i,j}^{n+1/2} + \eta_{i,j-1}^{n+1/2}\right)\right] = 0 \\ \frac{Q_{i,j+1/2}^{n+1} - Q_{i,j+1/2}^{n}}{\Delta t} + gh\frac{\eta_{i,j+1}^{n+1/2} - \eta_{i,j}^{n+1/2}}{\Delta y} + \frac{\gamma gh}{12\Delta y} \\ +\left[\left(\eta_{i+1,j+1}^{n+1/2} - 2\eta_{i,j+1}^{n+1/2} + \eta_{i-1,j+1}^{n+1/2}\right) - \left(\eta_{i+1,j}^{n+1/2} - 2\eta_{i,j}^{n+1/2} + \eta_{i-1,j}^{n+1/2}\right)\right] = 0, \end{cases} \tag{72}$$

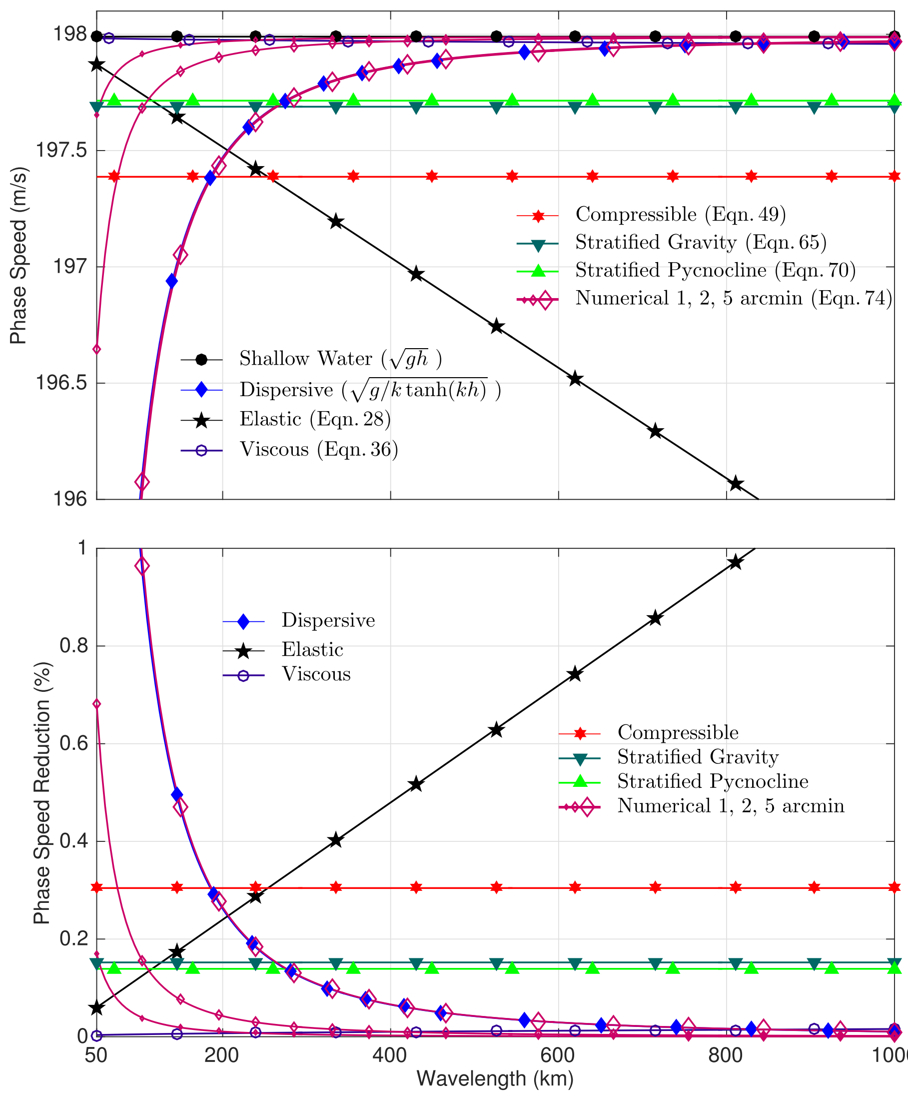

Fig. 3. Speed reduction and percentage due to secondary effects and numerical dispersion. In the top panel, the tsunami speeds in shallow water ($\sqrt{gh}$) and under effects of wave dispersion ($\sqrt{g/k\tanh(kh)}$,), Earth elasticity (Eq. (28)), water viscosity (Eq. (36)), water compressibility (Eq. (49)), water density change due to gravitational compression (Eq. (65)), ocean stratification due to temperature/salinity (Eq. (70)) and numerical dispersion (Eq. (74)) are plotted. In the bottom panel, the percentage of speed reduction under such effects with respect to the shallow water wave speed $\sqrt{gh}$ is plotted. The effect of water viscosity on speed reduction is minor and it is hardly seen from the plot. The numerical dispersion is close to the physical dispersion when the spatial grid size is 5 arc min.

where the superscripts denote the time step, the subscripts show the spatial grids, Δt is the time step, Δx and Δy are the spatial grid size and γ is a constant parameter. The scheme is the same as used by Cho (1995) if $\gamma = 1$ and the same as Imamura and Goto (1988) if $\gamma = 0$. Taking $\gamma = 1$ and $\Delta x = \Delta y$, the modified equation associated with the numerical scheme can be obtained after lengthy but straightforward algebra (Cho, 1995):

$$\frac{\partial^2\eta}{\partial t^2} - gh\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right) - gh\frac{\Delta x^2}{12}\left(1-\frac{gh\Delta t^2}{\Delta x^2}\right) \times\left(\frac{\partial^4\eta}{\partial x^4}+2\frac{\partial^4\eta}{\partial x^2\partial y^2}+\frac{\partial^4\eta}{\partial y^4}\right) = O(\Delta x^3\ldots). \qquad (73)$$

The first two terms in the equation are the exact shallow water wave equations, and the third term is the dispersive term induced by the numerical scheme. Adopting a plane wave solution, i.e., assuming $\eta(x,y,t) = Ae^{i(k_x x+k_y y-\omega t)}$, where A is constant, $k = \sqrt{k_x^2+k_y^2}$ is the wave number and ω is the angular frequency, and substituting $\eta(x,y,t)$ into Eq. (73), we find that the wave speed for such a governing equation is

$$c \approx \left[1-\frac{\Delta x^2}{24}\left(1-\frac{gh\Delta t^2}{\Delta x^2}\right)k^2\right]\sqrt{gh}. \qquad (74)$$

Thus, the wave speed is no longer $\sqrt{gh}$ and the modification depends the grid size and time step. Using C.F.L. number of 0.5, i.e., $\Delta t\sqrt{gh}/\Delta x = 0.5$, the numerical dispersion can be estimated purely according to the grid size:

$$c \approx \left[1-\frac{k^2\Delta x^2}{32}\right]\sqrt{gh}. \qquad (75)$$

In Fig. 3, we plot the speed reduction and percentage due to physical dispersion, Earth elasticity, water viscosity, compressibility, density stratification due to compression of gravity and temperature/salinity and numerical dispersion. The range of wavelength is from 50 to 1000 km, corresponding to wave period of 260–5000 s. The dispersion relationships used are the simplified formulas excluding dispersion, i.e., Eqs. (28), (36), (49), (65), (70) and (74). Fig. 3 shows that, the correction due to Earth elasticity increases roughly linearly with wavelength; the correction due to water viscosity is negligible; the correction due to water compressibility and density stratification is also constant regardless the wavelength. For a typical tsunami wavelength of ~200 km, the correction of dispersion for shallow water approximations is around 0.26%, the correction of Earth elasticity is around 0.24%, the correction of water viscosity is around 0.01%, the correction of water compressibility is around 0.30%, the correction of density stratification due to compression is around 0.15%, and the correction of density stratification due to temperature/salinity is around 0.14%. The numerical correction is 0.01% for $\Delta x = 1$ arc min, 0.04% for $\Delta x = 2$ arc min and 0.27% for $\Delta x = 5$ arc min.

6. Conclusions

In this study we derive the theoretical solutions of dispersion relationships accounting for the effects of water viscosity, ocean stratification due to temperature/salinity and numerical dispersion. We find that the water viscosity hardly affects the speed of tsunami waves, but the density stratification due to temperature/salinity is capable of slowing down the tsunami waves by the same order as the stratification due to gravitational compression, because such a stratification has a sudden change of density over a small depth. The numerical dispersion is comparable to the physical dispersion if the spatial grid size is around 5 arc min in the numerical simulation. It is noted that 5 arc min has been commonly used for global tsunami warning purposes. Therefore, we suggest that if the correction of Earth elasticity, water compressibility and density stratification due to compression (e.g., Allgeyer and Cummins, 2014; Watada, 2013; Watada et al., 2014) were deemed to be significant, the effects of stratification due to temperature/salinity and numerical dispersion should also be taken into account. In this study we also simplify the theoretical dispersion relationships so that the simplified formulas are explicit and easy to adopt. They are Eqs. (28), (36), (49), (65), (70) and (74). The simplifications are verified to be good approximations of the full dispersion relationships for typical tsunami wavelength of 50–1000 km (or wave period 260–5000 s). Since all the effects slow down the tsunami speeds by a small order (<1%), it is expected that the combination of them in a linear system is simply the linear summation, and the crossing terms of the coupling effects are one order even smaller. Using the typical physical parameters given in Section 4, for tsunami waves with wavelength of ~200 km, the effects that cause significant speed reduction are wave dispersion (0.26%), Earth elasticity (0.24%), water compressibility (0.30%), density stratification due to compression (0.15%) and density stratification due to temperature/salinity (0.14%). The total reduction is around 1.1%.

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