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Strain limits vs. reinforcement ratio limits – A collection of new and old formulas for the design of reinforced concrete sections

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ABSTRACT

This paper presents a formulation for the design of reinforced concrete flexural members. The formulation yields exactly the same results as the current American Concrete Institute (ACI) design approach but it is based entirely on the concept of *reinforcement ratios.* This is in contrast to the current ACI approach which relies on strain limits [1]. A formulation based on reinforcement ratios is simpler and more intuitive and therefore has important pedagogical advantages. The formulation presented here can be thought of as an attempt to reconcile the new approach to design introduced by the ACI code in 2002, with the traditional approach to design that was in use from 1963 to 2002. The traditional approach to design of reinforced concrete sections uses the concept of reinforcement ratios. The new ACI approach, referred to here as the *unified design method* (UDM), requires consideration of rather cumbersome strain limits and/or geometric strain relationships. In this paper, it is shown that the UDM approach can be formulated much in the same way as the traditional approach, as long as a series of formulas involving reinforcement ratios are introduced. These formulas are presented in this paper. Many of them are well known, but some are new. In particular, a new formula for the *compression-controlled* reinforcement ratio limit, and a new direct procedure for the design of *transition-zone* sections are presented. The formulation presented in this paper should prove useful both for the instructor in the classroom, and for the practicing structural engineer. Derivation details for many of the formulas in the paper are given and several numerical examples to illustrate their use are provided at the end.

Introduction

In 2002, the ACI introduced a new approach to the design of structural members subject to bending. This change was introduced through the ACI-318-02 publication [2]. The motivation for the change was to achieve uniformity between the design procedures used for the design of pre-stressed, reinforced, and compression members [3]. As a consequence of this, the new approach to design has been termed unified design method (UDM) by some authors [4,5]. This name will be adopted in this paper. The UDM introduced the ideas of *tension-controlled, compression-controlled*, and, *transition-zone* beam sections. This characterization of reinforced concrete sections requires consideration of strain limits and/or geometric strain

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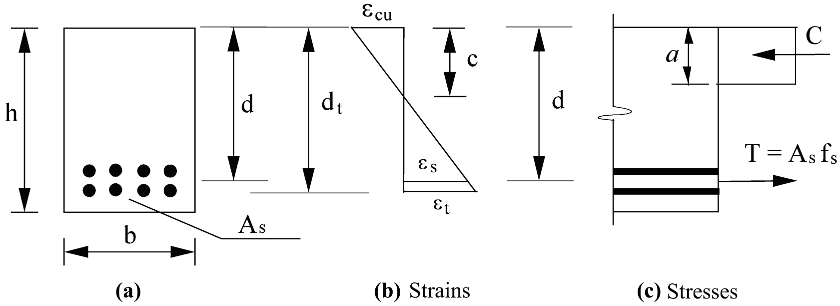

Fig. 1. Schematic representation of stresses and strains in a typical reinforced concrete section.

relationships. In the author's opinion, this characterization lacks intuitive and pedagogical appeal. The traditional approach to design of reinforced concrete sections relies instead on the concept of reinforcement ratios. The traditional approach can still be used (as of the ACI-318 2011 edition [1]) but it has been relegated to an appendix in the ACI code; and it is likely to disappear from it in subsequent editions [6]. The objective of this paper is to introduce a series of formulas for the design of flexural members that relies entirely on reinforcement ratios.

Tension-controlled, transition-zone, and compression-controlled sections

Fig. 1 shows the usual representation of the flexural stresses and strains in a typical reinforced concrete section. The notation and nomenclature are the usual, with c representing the depth of the neutral axis of the section; d representing the distance from the top of the section to the centroid of the main reinforcement; d_t the distance from the top of the beam to the bottom layer of the tensile reinforcement; a, the depth of the compression block; A_s, the area of the main reinforcement; and f_s, the stress in the tension steel.

Referring to Fig. 1b, we observe that:

$$\frac{c}{d_t} = \frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_t} \tag{1}$$

On the other hand, from the stress diagram (Fig. 1c), we get the familiar expression for the depth of the compression block a, as:

$$a = \frac{A_s f_s}{0.85 f'_c b} \tag{2}$$

where f'_c represents the compressive strength of concrete, as usual.

Comparing now (1) and (2), recalling that $a = \beta_1 c$, and that the reinforcement ratio is defined as $\rho \equiv \frac{A_s}{bd}$, (see e.g., [7]) we conclude that:

$$\rho = \frac{0.85\beta_1 f'_c}{f_s}\left(\frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_t}\right)\frac{d_t}{d} \tag{3}$$

Eq. (3) is a completely general relationship for singly reinforced concrete beam sections. This equation can be specialized to fit a number of important situations as it will be shown below.

Tension-controlled sections

The ACI-318-02 defines a tension-controlled section as a section such that the strain ε_t in the lowermost layer of steel is greater than or equal to 0.005. This is to ensure that the main steel yields well before the concrete crushes, providing enough ductility for the section even for seismic zones. This definition is inspired by the yield strain of grade 60 steel (G60 for short[1]) which is approximately equal to 0.002[2]. (The definition of tension-controlled section is the same for other grades of steel despite the fact that ε_y is different: for G75 steel for instance, $\varepsilon_y = 0.0026$). The European code (known as EC2) has similar provisions to guarantee adequate ductility of the sections (see e.g., [8,9]).

Now, by substituting $\varepsilon_{cu} = 0.003$ and $\varepsilon_t = 0.005$ in (1), one obtains:

$$\frac{c}{d_t} = 0.375 \tag{4}$$

[1] The grade is equal to the yield strength of the steel in the corresponding units. For instance, in metric units G420 steel has a yield strength of 420 MPa (i.e., $f_y = 420$ MPa). This steel is practically equivalent to G60 steel which has a nominal yield strength of 60 ksi (i.e., $f_y = 60$ ksi).

[2] Actually, $\varepsilon_y = 0.00207$ for G60 steel (assuming a modulus of elasticity $E_s = 29{,}000$ ksi).

Eq. (4) is the most commonly used equation to test for tension-controlled sections in the literature (see e.g., [7]). It is the purpose of this paper to replace expressions of the type (4) with equivalent expressions in terms of the reinforcement ratio ρ. To obtain the tension-controlled limit, it suffices to substitute ε_t = 0.005 and $f_s = f_y$ in (3). The expression for the tension-controlled reinforcement ratio is then obtained as[3]:

$$\rho_{tcl} = \frac{2.55\beta_1 f_c'}{8 f_y}\frac{d_t}{d} \tag{5}$$

Eq. (5) assumes that the steel at a depth d from the top of the beam (i.e., at the level of the centroid of the main reinforcement) has also yielded when $\varepsilon_t = 0.005$.[4]

Compression-controlled sections and transition-zone sections

Beam sections for which $\varepsilon_t \le \varepsilon_y$ are classified as compression-controlled sections and are not permitted by the ACI code as of the 2002 edition. Occasionally however, it might still be required to find the moment carrying capacity of one of these sections since it may have been designed in a non-compliant way by mistake, or using a different standard. Or the section may have been designed during the period from 1999 to 2002 in which the ACI code was not clear about whether these sections were allowed or not (see [6], page 127). The strength reduction factor for compression-controlled sections (with ties) is $\phi = 0.65$.

Beam sections for which:

$$\varepsilon_y \le \varepsilon_t \le 0.005 \tag{6}$$

are considered transition-zone sections. For transition-zone sections with ties, the strength reduction factor ϕ must be linearly interpolated between the compression-controlled limit value of 0.65 and the tension-controlled limit value of 0.9, as given by the next equation[5]:

$$\phi = 0.65 + \frac{250}{3}(\varepsilon_t - 0.002) \tag{7}$$

Now, the compression-controlled limit corresponds to $\varepsilon_t = \varepsilon_y$. By substituting $\varepsilon_t = \varepsilon_y \equiv \frac{f_y}{E_s}$ in Eq. (1), we obtain:

$$\frac{c}{d_t} = \frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} + f_y} \tag{8}$$

which upon substitution of the values $\varepsilon_{cu} = 0.003$ [6], and E_s = 29,000 ksi (200,000 MPa), becomes[6]:

$$\frac{c}{d_t} = \frac{87}{87 + f_y} \tag{9}$$

in US customary units, and

$$\frac{c}{d_t} = \frac{600}{600 + f_y} \tag{10}$$

in metric units.

For G60 (G420) steel the compression-controlled limit corresponds to $\varepsilon_y = 0.00207$ (0.0210). By substituting $\varepsilon_y = 0.00207$ (0.0210) and $\varepsilon_{cu} = 0.003$ in Eq. (1), we obtain,

$$\frac{c}{d_t} = 0.5918 \tag{11}$$

in US customary units, and

$$\frac{c}{d_t} = 0.5882 \tag{12}$$

in metric units.

If $\varepsilon_t = \varepsilon_y$ is approximated as 0.002, either of the previous two formulas can be approximated as:

$$\frac{c}{d_t} = \frac{3}{5} = 0.6 \tag{13}$$

[3] Ref. [10] suggests a tension-controlled limit of 0.0066 (instead of 0.005) for high-strength steels. All of the formulas presented in this paper can be easily modified to reflect any desired change in this or any other strain limit.

[4] There is a slight possibility that this will not happen. The steel at the level of the main reinforcement will not yield (despite the fact that $\varepsilon_t = 0.005$) in the very unusual circumstance that the ratio d/d_t is less than $\frac{f_y + E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} + 0.005E_s}$. This value is approximately equal to 0.6336 for G60 steel.

[5] This formula is for G60 (G420) steel only. For other steel grades see e.g., [6].

[6] These formulas are unit dependent. For the metric formula, stresses must be in MPa. For the US customary units formula, stresses must be in ksi.

Eq. (13) is the expression most commonly used to test for compression-controlled sections in the literature (see e.g., [6]). The linear interpolation formula for ϕ (Eq. (7)) can also be written in terms of the c/d_t ratio as follows[7]:

$$\phi = 0.65 + 0.25\left(\frac{d_t}{c} - \frac{5}{3}\right) \tag{14}$$

It is also useful to write this formula in terms of the depth of the compression block a, as:

$$\phi = \frac{0.7}{3} + \frac{0.25\beta_1 d_t}{a} \tag{15}$$

It can be concluded now that transition-zone sections are characterized by:

$$0.375 \le \frac{c}{d_t} \le 0.6 \tag{16}$$

Eq. (16) is valid for G60 (G420) steel only. A more general, and slightly more accurate expression to characterize transition-zone sections can be obtained by using Eq. (8) instead of Eq. (13) as the upper limit in (16) as follows:

$$\frac{3}{8} \le \frac{c}{d_t} \le \frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} + f_y} \tag{17}$$

This formula is of course general and *unit independent*. In what follows, only general formulas will be given. To specialize any of the these formulas either to the US customary system or the metric system, it suffices to replace the term $E_s\varepsilon_{cu}$ with the quantity "87" for the US customary system and with the quantity "600" for the metric system.

In terms of strains, the expression that characterizes transition-zone sections for G60 steel becomes:

$$0.002 \le \varepsilon_t \le 0.005 \tag{18}$$

To ensure enough ductility however, the ACI code further restricts the value of ε_t to a *minimum* value of 0.004. When this value is substituted in Eq. (1), it can be concluded that the value of the *maximum* allowable c/d_t ratio is:

$$\frac{c}{d_t} = \frac{3}{7} \approx 0.4286 \tag{19}$$

which means that the range of *permitted* (or allowed) transition-zone sections is actually:

$$0.375 \le \frac{c}{d_t} \le 0.4286 \tag{20}$$

which is of course a subinterval of (16).

When $c/d_t = 3/7$ is used in conjunction with Eq. (14), the corresponding range of permitted ϕ values is obtained as:

$$0.8167 \le \phi \le 0.9 \tag{21}$$

When $\varepsilon_t = 0.004$, together with the value of $\varepsilon_{cu} = 0.003$, and $f_s = f_y$, are substituted into (3), an expression for the maximum permitted value of the reinforcement ratio is obtained as:

$$\rho^*_{\max} = \frac{2.55\beta_1 f'_c}{7f_y}\frac{d_t}{d} \tag{22}$$

where an asterisk is used to distinguish this ratio from the traditional maximum reinforcement ratio ($\rho_{\max} = 0.75\rho_b$ [7,11]). The permitted range of transition-zone sections, in terms of reinforcement ratios, can now be written as:

$$\rho_{tcl} \le \rho \le \rho^*_{\max} \tag{23}$$

which is completely equivalent to Eq. (21).

Now, by comparing Eq. (5) with Eq. (22), it can be concluded that:

$$\rho^*_{\max} = \frac{8}{7}\rho_{tcl} \tag{24}$$

It is interesting to compare Eq. (24) with the traditional expression for the balanced reinforcement ratio. When this is done, the following relationship is obtained:

$$\rho^*_{\max} = \frac{3}{7}\left(\frac{E_s\varepsilon_{cu} + f_y}{E_s\varepsilon_{cu}}\right)\frac{d_t}{d}\rho_b \tag{25}$$

where ρ_b is the traditional balanced reinforcement ratio given by:

$$\rho_b = \frac{0.85\beta_1 f'_c}{f_y}\left(\frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} + f_y}\right) \tag{26}$$

7 This is for G60 (G420) steel only. For other steel grades see e.g., [6].

For G60 steel and for the particular case when $d_t = d$ (one layer of main reinforcement steel), Eq. (25) becomes:

$$\rho^*_{\max} \approx 0.73\rho_b \tag{27}$$

which compares very well with the traditional value of the maximum reinforcement ratio [7,11]:

$$\rho_{\max} = 0.75\rho_b \tag{28}$$

Compression controlled limit

In order to find an expression for the reinforcement ratio that corresponds to the compression-controlled limit, it is necessary to first establish a relationship between the strain in the lowermost layer of steel ε_t and the strain at the centroid of the main steel reinforcement ε_s. The reason for this is that when the lowermost steel is exactly at the yield stress (compression-controlled limit), the stress at the centroid of the main steel reinforcement is less than the yield stress.

Referring to Fig. 1b, and using similar triangles, we obtain:

$$\varepsilon_s = \frac{d-c}{d_t-c}\varepsilon_t \tag{29}$$

or, in terms of stresses:

$$f_s = \frac{d-c}{d_t-c}f_t \tag{30}$$

When the lowermost reinforcement is exactly at the yield point, the stress at the centroid of the main reinforcement is then:

$$f_{sccl} = \frac{d-c}{d_t-c}f_y \tag{31}$$

Now, making use of Eq. (1), and multiplying numerator and denominator of (37) by E_s, we get:

$$f_{sccl} = \frac{f_y d - E_s\varepsilon_{cu}(d_t-d)}{d_t} \tag{32}$$

Substituting Eq. (32) into Eq. (3), and multiplying and dividing again by E_s, we finally obtain:

$$\rho_{ccl} = \frac{0.85\beta_1 f'_c d_t}{f_y d - E_s\varepsilon_{cu}(d_t-d)}\left(\frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu}+f_y}\right)\frac{d_t}{d} \tag{33}$$

In Eqs. (31)–(33) the subindex *ccl* stands for compression-controlled.

The range of transition-zone sections can now be written exclusively in terms of reinforcement ratio limits as:

$$\rho_{tcl} \le \rho \le \rho_{ccl} \tag{34}$$

Note that when $d_t = d$, the expression for the compression-controlled limit (Eq. (33)) reduces to the expression for the traditional balanced reinforcement ratio (Eq. (26)).

Fig. 2 illustrates schematically the relative values of all reinforcement limits together with their meaning in terms of the behavior of the corresponding sections. It is important to note (as it can be seen in Fig. 2) that for the normally occurring case when $d_t > d$, the value of ρ_{ccl} is larger than ρ_b. This means that for many transition-zone sections, the main tensile steel reinforcement does not yield. It is then necessary to solve a quadratic equation for a or for f_s to find the moment capacity of the section. This is a well-known procedure but it is outlined in the next section for completeness.

Determination of the moment carrying capacity of a section when the tensile reinforcement does not yield[8]

When $\rho > \rho_b$, then $f_s < f_y$ and the value of f_s is unknown. To determine the moment carrying capacity of this type of section it is necessary to first determine the value of f_s or the (also unknown) value of the depth of the compression block a. Referring to Fig. 1b, and using similar triangles, it can be easily established that:

$$f_s = E_s\varepsilon_{cu}\left(\frac{\beta_1 d}{a} - 1\right) \tag{35}$$

[8] These reinforced concrete sections are not allowed by the current ACI-318-11 code since $\rho > \rho_b$ means also that $\rho > \rho^*_{\max}$. However, as mentioned before, it might still be necessary to find the moment capacity of these sections in practice.

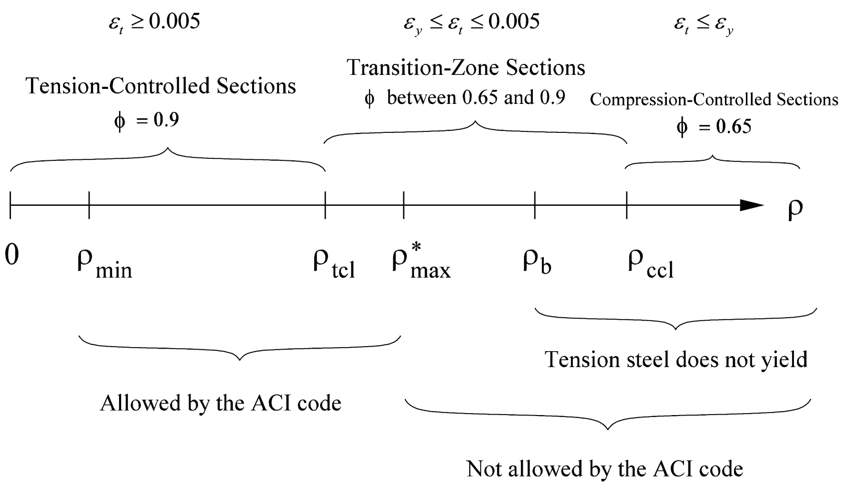

Fig. 2. Schematic Representation of the relative values of the reinforcement limits presented in this paper. In this figure ρ_{min} represents the minimum reinforcement ratio for beams given by: $\rho_{min} = \frac{3\sqrt{f_c'}}{f_y} (\geq \frac{200}{f_y})$ (see e.g., [7]).

as long as $f_s \leq f_y$. Substituting now the value of f_s from Eq. (35) into Eq. (2), the following quadratic equation for a can be readily obtained (see e.g. [6]):

$$\left(0.85f_c'\right) a^2 + (E_s\varepsilon_{cu} d\rho)a - E_s\varepsilon_{cu}\rho\beta_1 d^2 = 0 \tag{36}$$

Once this quadratic equation is solved for a, the stress in the steel can be found from (35). The nominal moment carrying capacity of the section can then be determined by using the well-known equation:

$$M_n = A_s f_s \left(d - \frac{a}{2}\right) \tag{37}$$

Alternatively, instead of (36), a quadratic equation for f_s can be derived as follows: Eq. (1) can be written in terms of ε_s and d instead of ε_t and d_t as, (see Fig. 1b):

$$\frac{c}{d} = \frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_s} \tag{38}$$

Now, by following a procedure similar to the one used to obtain Eq. (3) above, the following formula is obtained (see Eq. (3)):

$$\rho = \frac{0.85\beta_1 f_c'}{f_s}\left(\frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_s}\right) \tag{39}$$

By multiplying and dividing the term between parentheses in (39) by E_s, we obtain:

$$\rho = \frac{0.85\beta_1 f_c'}{f_s}\left(\frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} + f_s}\right) \tag{40}$$

This is also a completely general expression for the reinforcement ratio of singly reinforced sections. As a matter of fact, if $f_s = f_y$ is substituted into this equation, the expression for the balanced reinforcement ratio (Eq. (26)) is obtained.

Eq. (40) can be rearranged to yield a quadratic equation in f_s, as follows:

$$\rho f_s^2 + E_s\varepsilon_{cu}\rho f_s - 0.85E_s\varepsilon_{cu}\beta_1 f_c' = 0 \tag{41}$$

Solving (41) for f_s, and recalling that f_s must be positive, we obtain:

$$f_s = \sqrt{\left(\frac{E_s\varepsilon_{cu}}{2}\right)^2 + \frac{0.85E_s\varepsilon_{cu}\beta_1 f_c'}{\rho}} - \frac{E_s\varepsilon_{cu}}{2} \tag{42}$$

Once the stress in the steel is known, the value of a can be found from Eq. (2), and the nominal moment carrying capacity can be found using Eq. (37).

Design aspects

Most authors use a trial and error procedure for the design of reinforced concrete sections (see e.g., [6,7]). The author prefers to use a quadratic formula for the reinforcement ratio ρ. This formula is derived next. Similar formulas can be found in the literature (see e.g., [4], page 100). We start with the relationship for the nominal moment carrying capacity of a singly reinforced section with the tension steel yielding (Eq. (37) with $f_s = f_y$):

$$M_n = A_s f_y \left(d - \frac{a}{2} \right) \tag{43}$$

or, by using Eq. (2) with $f_s = f_y$, and multiplying both sides of (43) by ϕ:

$$\phi M_n = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b} \right) \tag{44}$$

Recognizing now that the most economical design results when $\phi M_n = M_u$, substituting ρbd for A_s in Eq. (44), and reorganizing terms, we get the following quadratic equation for ρ:

$$f_y \rho^2 - 1.7 f'_c \rho + \frac{1.7 M_u f'_c}{\phi b d^2 f_y} = 0 \tag{45}$$

whose solution is:

$$\rho = \frac{0.85 f'_c - \sqrt{(0.85 f'_c)^2 - \frac{1.7 f'_c M_u}{\phi b d^2}}}{f_y} \tag{46}$$

The minus sign before the square root has been chosen in (46) to obtain a reinforcement ratio that is less than one. Eq. (46) is very useful as long as ϕ is equal to 0.9 (i.e., for tension-controlled sections). If $0.8167 \leq \phi < 0.9$ (i.e., transition-zone sections), a different procedure is necessary. This is a new procedure that is described next.

Direct procedure for the design of transition-zone sections

For transition-zone sections, the value of ϕ must be interpolated between 0.65 and 0.9. (see Eq. (14) or Eq. (15). This means that the value of ϕ in Eq. (46) depends implicitly on ρ. For these sections, it is still possible to use Eq. (46) as a first approximation, and then iterate until a sufficiently accurate result is obtained for ρ and ϕ. Special design aids that rely on a series of charts have also been devised for this case [5,12]. Ref. [13] also presents an iterative procedure to design transition-zone sections. A new, direct, and exact approach that does not require iterations is presented next.

It turns out that it is possible to derive an equation that can be solved directly for ρ to obtain the exact combination of ρ and ϕ for transition-zone sections. Remarkably enough, in spite of the apparent high nonlinearity of the relationship between ρ and ϕ, the resulting equation is also a *quadratic* equation. The first step is to use Eq. (2) in conjunction with Eq. (15) and the definition of ρ, to write ϕ in terms of ρ as:

$$\phi = \frac{0.7}{3} + \frac{0.85 \beta_1 f'_c d_t}{4 \rho f_y d} \tag{47}$$

Substituting this value into Eq. (45) we get:

$$f_y \rho^2 - 1.7 f'_c \rho + \frac{20.4 \rho M_u f'_c}{2.8 \rho f_y b d^2 + 2.55 \beta_1 f'_c b d d_t} = 0 \tag{48}$$

which, after some algebraic manipulation becomes:

$$2.8 f_y^2 b d^2 \rho^3 + (2.55 \beta_1 f'_c f_y b d d_t - 4.76 f'_c f_y b d^2) \rho^2 + (20.4 M_u - 4.335 \beta_1 f'_c b d d_t) f'_c \rho = 0 \tag{49}$$

This is a cubic equation in ρ with the trivial solution $\rho = 0$. We are then left with the following *quadratic* equation to solve for the nontrivial reinforcement ratio ρ:

$$\rho^2 - \left(\frac{1.7 f'_c}{f_y} - \frac{2.55 \beta_1 f'_c d_t}{2.8 f_y d} \right) \rho + \left(\frac{5.1 f'_c M_u}{0.7 b d^2 f_y^2} - \frac{1.08375 \beta_1 (f'_c)^2 d_t}{0.7 f_y^2 d} \right) = 0 \tag{50}$$

The value of ρ found from this equation provides the design moment $\phi M_n = M_u$ that corresponds to the exact interpolated value of ϕ given by Eq. (15) (see Example 2 below).

Analysis and design of T- and doubly-reinforced sections

With only minor modifications, the formulas that are used to analyze and/or design singly reinforced sections can be used to analyze and/or design T- and doubly-reinforced sections.

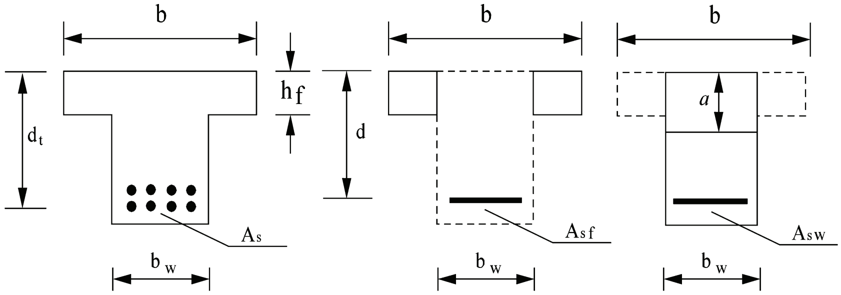

Fig. 3. Schematic representation of stresses and strains in a typical T-beam reinforced concrete section.

T-sections

When dealing with true T-sections (i.e., sections for which the depth of the compression block a is larger than the thickness of the flange h_f), it is customary to idealize the actual T-section as two separate sections that, when superimposed, reproduce the original section. These two sections are traditionally called the F-beam (corresponding to the flanges) and the W-beam (corresponding to the web) (Fig. 3). Within this framework, the area of main reinforcement steel that is balanced exactly by the compressive forces on the flanges is denoted by A_{sf}. The area of reinforcement steel that corresponds to the W-beam is then given by:

$$A_{sw} = A_s - A_{sf} \tag{51}$$

where A_s is the total area of main reinforcement of the section.

It is convenient to define reinforcement ratios corresponding to the F-beam and to the W-beam as $\rho_f \equiv \frac{A_{sf}}{b_w d}$, and $\rho_w \equiv \frac{A_{sw}}{b_w d}$. Note that in both of these equations, the denominator involves the web width b_w. This ensures that one can write:

$$\rho = \rho_w + \rho_f \tag{52}$$

where $\rho = \frac{A_s}{b_w d}$ is the total reinforcement ratio of the T-section.

To apply the formulas of reinforcement ratios presented above to the analysis and design of T-sections, it suffices to keep in mind that the quantity that must be compared with the different reinforcement limits is ρ_w (since this is the ratio that corresponds to the rectangular portion of the section). For instance, if $\rho_w \le \rho_{tcl}$, the section is tension-controlled; if $\rho_{tcl} \le \rho_w \le \rho_{ccl}$ the section is in the transition-zone, etc.

The remaining equations for the analysis and design of T-sections are standard and can be found in [6].

Doubly-reinforced sections

The typical notation and nomenclature for doubly-reinforced sections is shown in Fig. 4. In analogy with T-sections, doubly-reinforced sections are also considered to be composed of two parts that superimposed reproduce the original section. These two parts can be called Beam-1 and Beam-2. Beam-1 is composed of the compression reinforcement plus the portion of the tensile reinforcement that exactly balances the forces in the compression reinforcement. Beam-2 includes the compression block on the concrete and the portion of the tension reinforcement that balances the compressive forces on the concrete (Fig. 4).

The only difficulty with the analysis and design of doubly-reinforced sections is the fact that sometimes the compression reinforcement does not yield at failure. When this is the case, it is necessary to determine the value of the stress f_s' in the compression reinforcement at failure. Whether the compression reinforcement yields or not depends on the amount of tension reinforcement in the section (see e.g., [6]). This leads to the procedure outlined below.

The areas of steel reinforcement corresponding to the original beam section and to Beams 1 and 2 are related by $A_s = A_{s1} + A_{s2}$. Now, by the definition of Beam-1 (see Fig. 4), $A_{s1} = \frac{A_s' f_s'}{f_y}$ and therefore $A_{s2} = A_s - \frac{A_s' f_s'}{f_y}$. It is useful to define also, $\rho' = \frac{A_s'}{bd}$, and write:

$$\rho_2 = \rho - \rho' \frac{f_s'}{f_y} \tag{53}$$

where $\rho = \frac{A_s}{bd}$, as usual.

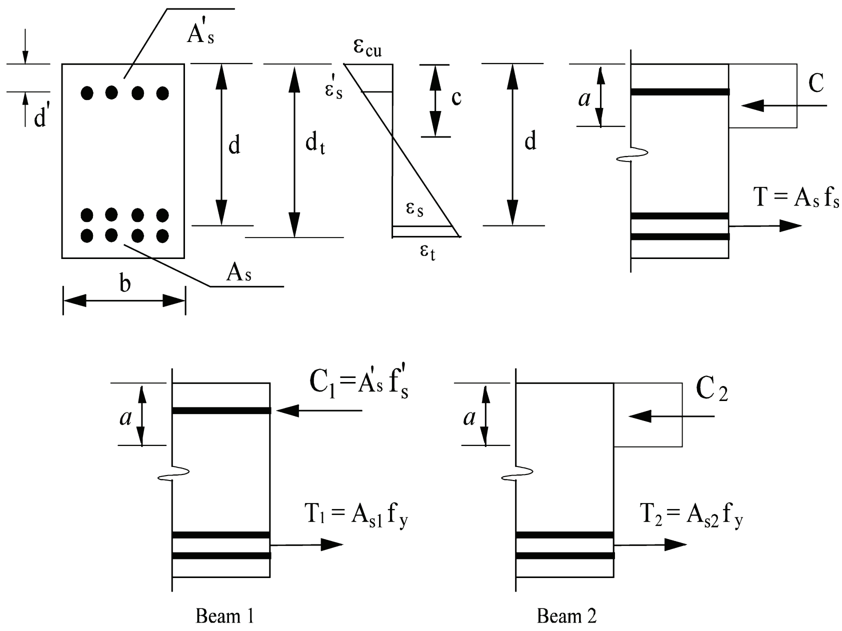

Fig. 4. Schematic representation of stresses and strains in a typical doubly-reinforced concrete section.

It can be shown (by a procedure similar to the one outlined in section "Tension-controlled, transition-zone, and compression-controlled sections"), that the limiting reinforcement ratio to determine whether the compression reinforcement has yielded at failure is (see e.g., [6]).

$$\rho_{cry} = \frac{0.85\beta_1 f_c'}{f_y}\left(\frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} - f_y}\right)\frac{d'}{d} \tag{54}$$

where the subindex *cry* stands for "compression reinforcement yielding". Now, it is important to emphasize that the quantity that must be compared with ρ_{cry} (and with any other reinforcement ratio limit for doubly reinforced sections) is ρ_2. The difficulty with this approach is of course that to be able to calculate ρ_2, f_s' must be known (see Eq. (53)). A way around this problem is to approximate ρ_2 as $\tilde{\rho}_2 \approx \rho - \rho'$, and perform the test using $\tilde{\rho}_2$ instead. This approximation does not pose any problems since $\rho_2 \geq \tilde{\rho}_2$. The test can then be performed as follows: if $\tilde{\rho}_2 > \rho_{cry}$, the compression reinforcement *yields*. This of course means that $f_s' = f_y$; if $\tilde{\rho}_2 < \rho_{cry}$, the compression reinforcement *does not yield*. This means that $f_s' < f_y$, and the stress in the compression reinforcement f_s' is unknown. When the compression reinforcement yields, the nominal moment carrying capacity of a doubly-reinforced section can be determined by simply adding the contributions from Beam-1 and Beam-2 in the standard way (see e.g., [6]).

When the compression reinforcement does not yield, it is necessary to determine *a* (the depth of the compression block) by means of the following quadratic equation:

$$(0.85f_c'b)a^2 - (A_sf_y - A_s'E_s\varepsilon_{cu})a - A_s'E_s\varepsilon_{cu}\beta_1 d' = 0 \tag{55}$$

The procedure to derive this equation is analogous to the one used in section "Determination of the moment carrying capacity of a section when the tensile reinforcement does not yield" to obtain Eq. (36).

Once Eq. (55) is solved for *a*, the stress in the compression steel can be found as:

$$f_s' = E_s\varepsilon_{cu}\left(1 - \frac{\beta_1 d'}{a}\right) \tag{56}$$

which is analogous to Eq. (35). The nominal moment carrying capacity of a doubly-reinforced section can then be determined in the standard way.

Alternatively, when the compression reinforcement does not yield, it can be shown that the stress in the compression steel can be found as:

$$f_s' = \left(\frac{E_s\varepsilon_{cu}}{2} + \frac{\rho f_y}{2\rho'}\right) - \sqrt{\left(\frac{E_s\varepsilon_{cu}}{2} + \frac{\rho f_y}{2\rho'}\right)^2 - E_s\varepsilon_{cu}\left(\frac{\rho f_y}{\rho'} - \frac{0.85\beta_1 f_c' d'}{\rho' d}\right)} \tag{57}$$

The derivation of this equation is similar to that of Eq. (42). It should be emphasized that when classifying, analyzing, or designing doubly-reinforced sections, the quantity that must be compared with the reinforcement limits is ρ_2 (or $\tilde{\rho}_2$)

since this is the ratio that corresponds to the rectangular portion of the section. For instance, if $\rho_2 \leq \rho_{tcl}$ the section is tension-controlled. if $\rho_{tcl} \leq \rho_2 \leq \rho_{ccl}$ the section is in the transition zone, etc.

Numerical examples

The following three numerical examples illustrate the use of some of the formulas and concepts presented above. Example 1 illustrates the use of Eq. (33); Example 2, that of Eq. (50); and Example 3, the design procedure for T- and doubly-reinforced sections.

Example 1

The purpose of this example is to illustrate the use of Eq. (33) and to demonstrate its accuracy. To do this, we will show that for a beam with $\rho = \rho_{ccl}$, the ratio c/d_t is exactly equal to 0.5918 as indicated by Eq. (12).

The compression controlled reinforcement ratio is given by Eq. (33). For the dimensions of the beam shown in Fig. 5, and assuming G60 steel and $f_c' = 3.6$ ksi, Eq. (33) yields:

$$\rho_{ccl} = 0.03776$$

Now, the balanced reinforcement ratio for this beam (see Eq. (26)) is:

$$\rho_b = 0.02566$$

On the other hand, the reinforcement ratio for this beam is (Area of a #8, 0.79 in^2; area of a #10, 1.27 in^2):

$$\rho = 0.037738$$

In other words, the reinforcement of the beam shown in Fig. 5 is, in practical terms, equal to the compression controlled limit. In addition, since this reinforcement ratio is larger than the balanced reinforcement ratio ($\rho_b = 0.02566$), the tension reinforcement does not yield at failure. Therefore, to find the value of the depth of the compression block it is necessary to solve Eq. (36) for a, as illustrated next.

For the material properties given above, and the dimensions and parameters of the beam shown in Fig. 5, Eq. (36) yields:

$$3.06a^2 + 59.10084a - 904.2429 = 0$$

The solution of this equation is $a = 10.06$ in. Therefore, $c = \frac{a}{\beta_1} = 11.835$ in. Therefore:

$$\frac{c}{d_t} = 0.59177$$

which corresponds almost exactly to the compression controlled limit of 0.5918 given by Eq. (12).

Alternatively, Eq. (42) can be used to solve for the stress in the steel f_s, as follows.

For $\rho = 0.03774$, Eq. (42) yields $f_s = 45.316$ ksi. Using now Eq. (3), a is found as: $a = 10.06$ in; and c is then $c = 11.8347$ in. Therefore:

$$\frac{c}{d_t} = 0.5917$$

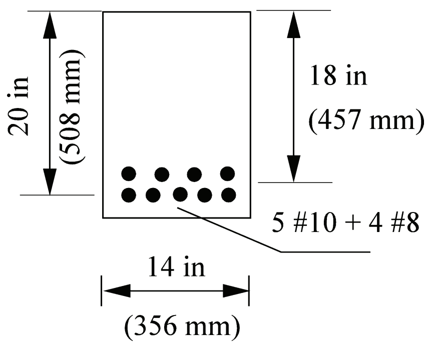

Fig. 5. Beam section for numerical Example 1.

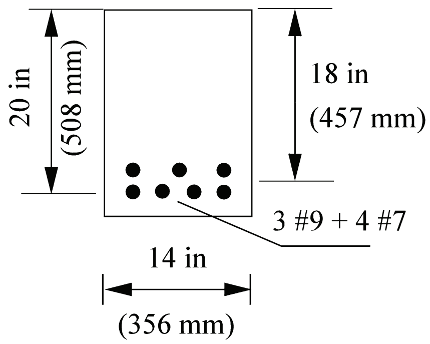

Fig. 6. Beam section for numerical Example 2.

Example 2

The purpose of this example is to illustrate the use of Eq. (50) for the design of transition-zone sections and to verify its validity.

Suppose that it is necessary to find the reinforcement of a section with the dimensions shown in Fig. 6, for the following data: $M_u = 338$ kips-ft; $f_c' = 4$ ksi, and $f_y = 60$ ksi.

A quick check, using Eq. (46) (with $\phi = 0.9$) yields $\rho \approx 0.02014$. On the other hand, the tension-controlled and maximum reinforcement ratios are, respectively:

$$\rho_{tcl} = 0.02007 \text{ and } \rho^*_{\max} = 0.02294$$

Therefore, the section is very likely in the transition-zone and ϕ is less than 0.9. Using now Eq. (50) to find ρ, the following quadratic equation is obtained:

$$\rho^2 - 0.056\rho + 0.00073992 = 0$$

which yields: $\rho = 0.02136$. This corresponds to an area of steel $A_s = 5.383$ in^2. For this area of steel, the values of a, ϕ, and M_n, are as follows (using Eqs. (3), (15) and (37)):

$$a = 6.7852 \text{ in}; \quad \phi = 0.86; \quad M_n = 393.16 \text{ kips-ft}$$

which corresponds to: $\phi M_n = 338.12$ kips-ft $\approx M_u$

The design can be completed by choosing 4 # 7 bars plus 3 # 9 bars to yield a total area of steel $A_s = 5.4$ in^2 (area of a #7, 0.6 in^2; area of a #9, 1.00 in^2), and a reinforcement ratio $\rho = 0.02143$ ($< \rho^*_{\max}$). This area of steel and reinforcement ratio correspond to $\phi = 0.858$, $M_n = 394.1$ kips-ft, and, $\phi M_n = 338.0$ kips-ft. The final design is shown in Fig. 6.

Example 3

The purpose of this example is to illustrate the procedures for the design of T- and doubly-reinforced sections outlined in section "Analysis and design of T- and doubly-reinforced sections" of this paper. The beam section for this example is shown in Fig. 7. This T-section is to be designed for the following data: $M_u = 360$ kips-ft; $f_c' = 4.0$ ksi, and $f_y = 60$ ksi. The first step is to

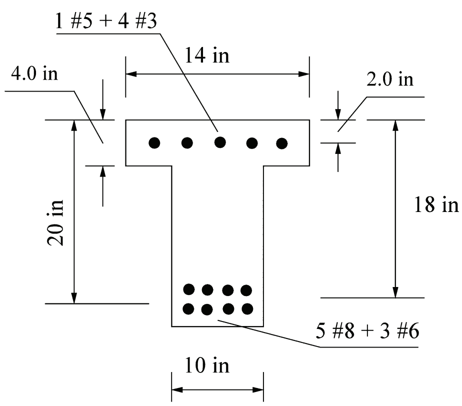

Fig. 7. Beam section for numerical Example 3.

check whether the section behaves as a rectangular section. To this end, Eq. (46) with $b = 14$ in is used. This Equation yields $\rho = 0.02185$. This corresponds to $A_s = 5.506\,\text{in}^2$. Therefore:

$$a = \frac{A_s f_y}{0.85 f_c' b} = 6.94 \quad \text{in} > h_f$$

This means that the section must be designed as a T-section. The following values from Eqs. (5) and (22) will be used in what follows:

$$\rho_{tcl} = 0.02007; \rho^*_{\max} = 0.02294$$

Following now the procedure outlined in section "T-sections" for the design of T-sections, the following values are obtained:

$$A_{sf} = 0.9067\, in^2; \quad M_{nf} = 72.53\,\text{kips-ft}; \quad M_{uf} = 65.28\,\text{kips-ft}\,(\text{assuming}\,\phi = 0.9); \quad M_{uw} = 294.72\,\text{kips-ft}$$

For $M_{uw} = 294.72$ kN.m and $b = b_w$, Eq. (46) yields $\rho_w = 0.02633$. This value is larger than $\rho^*_{\max}(= .02294)$ and therefore not permitted by the ACI code. This means that the web portion of the beam (the W-beam) must be doubly-reinforced (assuming that the dimensions of the section cannot be changed). To this end, let $\rho_2 = \rho_{tcl} = 0.02007$. This corresponds to $A_{s2} = 3.6126\,\text{in}^2$. Following now the procedure outlined in section "Doubly-reinforced sections" for the design of doubly-reinforced sections, the following quantities are obtained (see e.g., [6]):

$$a = 6.3752\,\text{in}$$
$$\Rightarrow M_{n2} = 267.6\,\text{kips-ft}$$
$$\Rightarrow M_{u2} = 240.84\,\text{kips-ft}$$
$$\Rightarrow M_{u1} = 53.88\,\text{kips-ft}$$
$$\Rightarrow M_{n1} = 59.87\,\text{kips-ft}$$

This results in a compression steel area $A_s' = 0.75\,\text{in}^2$. $\tilde{\rho}_2$ must now be checked against ρ_{cry} to find out whether the compression reinforcement yields or not. Using Eq. (54) we obtain $\rho_{cry} = 0.01724$. Therefore $\tilde{\rho}_2 > \rho_{cry}$ and the compression reinforcement yields. The total W-beam reinforcement is then $A_{sw} = A_{s2} + A_s' = 4.36\,\text{in}^2$. And the total tensile reinforcement for the entire T-section is $A_s = A_{sw} + A_{sf} = 5.27\,\text{in}^2$. Choose 5 # 8 plus 3 # 6 bars to yield a total reinforcement area of $A_s = 5.27\,\text{in}^2$ (area of a #6, 0.44 in^2; area of a #8, 0.79 in^2). Choose 4 # 3 plus 1 # 5 bars for the compression reinforcement (area of a #3, 0.11 in^2; area of a #5, 0.31 in^2). This corresponds to an area of compression reinforcement $A_s' = 0.75\,\text{in}^2$. The final design is shown in Fig. 7.

Concluding remarks

A series of ideas and formulas for the flexural design of reinforced concrete sections have been presented. They are an attempt to reconcile the so-called unified design method (UDM) with the ACI traditional approach to design. The UDM relies on strain limits for the characterization of the behavior of reinforced concrete sections. The traditional ACI approach to design relies on the simple concept of reinforcement ratios. The formulas and ideas presented in this paper bridge the gap between these two approaches by allowing the treatment of reinforced concrete sections in a much simpler way. These formulas and ideas should also prove useful in the classroom since the concept of reinforcement ratios is more intuitive and therefore more pedagogically appealing than that of strain limits. Many of the formulas presented in this paper are well known, but some are new. In particular, the formula for the compression-controlled reinforcement ratio limit and the procedure for the design of transition-zone sections are new.

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