Transportation Research Part E 61 (2014) 68–83

Contents lists available at ScienceDirect

Transportation Research Part E

journal homepage: www.elsevier.com/locate/tre

A latent class choice based model system for railway optimal pricing and seat allocation

Pratt Hetrakul [a,*], Cinzia Cirillo [b,1]

[a] Resource Systems Group, Inc., 55 Railroad Row, White River Junction, VT 05001, USA
[b] Department of Civil and Environmental Engineering, University of Maryland, 1173 Glenn L. Martin Hall, Bldg #088, College Park, MD 20742, USA

ARTICLE INFO

Article history:
Received 10 November 2012
Received in revised form 13 September 2013
Accepted 17 October 2013

Keywords:
Heterogeneity
Latent class model
Multinomial logit model
Pricing
Revenue management
Seat allocation

ABSTRACT

In this paper, discrete choice methods in the form of multinomial logit and latent class models are proposed to explain ticket purchase timing of passenger railway. The choice model and demand functions are incorporated into a revenue optimization problem which jointly considers pricing and seat allocation. The framework provides insightful policy implications in term of fare and capacity distribution derived from actual passenger behavior. It shows that accepting short-haul demand provides greater revenue than long-haul demand using the same capacity. Revenue improvement ranges from 16.24% to 24.96% in multinomial logit models and from 13.82% to 21.39% in latent class models respectively.

1. Introduction

1.1. Background and motivation

Demand forecasting is an essential component of revenue management (RM) model systems; it also plays a relevant role in seat allocation and optimization problems. With respect to pricing, the operators in charge of the reservation system need to know the distribution of the expected late-booking-high-fare demand in order to protect the right number of seats. For capacity allocation, it is necessary to predict the expected market size for each trip in order to provide efficient seat allocation strategies. Discrete choice analysis (DCA) is a standard approach for determining factors influencing decision making process. Recently, researchers in RM have argued that DCA enables for realistic representation of passenger response to RM policy. Moreover, passengers are usually characterized by a high level of taste heterogeneity (Hetrakul and Cirillo, 2013), that depends both on socio-demographic characteristics and different preferences over scheduling and pricing. Over the past decades considerable progress has been made in the characterization of unobserved taste heterogeneity for travel choice behavior (Wen and Lai, 2010). Latent class (LC) model is considered a valid approach to account for taste heterogeneity (Walker, 2001); LC model is a special case of mixture logit models (Train, 2003) in which the mixing distribution is discrete. This approach segments passengers with similar characteristics into classes of unknown size using a function of observable variables. Accounting for differences on individual taste offers a more realistic representation of passenger choice behavior and might lead to significant revenue improvement. To date a very limited number of studies applies this technique to RM and in particular to the pricing and seat allocation optimization problem for railway systems.

* Corresponding author. Tel.: +1 802 295 4999.
E-mail addresses: Pratt.Hetrakul@rsginc.com (P. Hetrakul), ccirillo@umd.edu (C. Cirillo).
[1] Tel.: +1 301 405 7768; fax: +1 301 405 2585.

1366-5545/$ - see front matter
http://dx.doi.org/10.1016/j.tre.2013.10.005

This paper aims at filling this gap by demonstrating that LC choice models can be effectively used to model railway passenger booking behavior and to represent the underlying taste heterogeneity. The case study based on intercity passenger railway internet booking data demonstrates that gain in revenues can be achieved when those techniques are made operational. The remaining of this paper is organized as follows. Previous studies on the use of demand modeling for revenue management are presented in Section 2. Section 3 describes the conceptual framework which incorporates passenger choice models and demand functions in RM revenue optimization. Section 4 presents a passenger choice model for ticket purchase timing decision. In Section 5, the passenger demand function used to estimate demand volume for each market is estimated. Section 6 describes the mathematical formulation and the optimization procedure adopted to incorporate both multinomial logit and latent class models into RM systems. Section 7 presents numerical results in term of seat allocation and fare strategy, revenue improvement, and capacity redistribution. Finally conclusions and future research directions are given in Section 8.

2. Literature review

2.1. Incorporating heterogeneous choice models in RM problem

Accounting for taste heterogeneity is essential for demand forecasting especially for railway RM because passenger preferences generally vary by departure time of day, day of week, and trip distance. A number of papers have investigated special classes of discrete choice model that accommodates taste heterogeneity. Bhat (1998) estimated an intercity travel mode choice model which accommodates variations in response to level of service measures due to observed and unobserved individual characteristics. The study emphasized the necessity to incorporate systematic and random variation in response to level-of-service variables. Greene and Hensher (2003) compared latent class (LC) with mixed logit (ML) model using stated preference data on long distance travel survey in 2000. Shen (2009) compared the difference between latent class and mixed logit models using two stated choice datasets on mode choice from Osaka, Japan using non-nested test to compare the model fits.

In the RM context, Carrier (2008) analyzed the choice of airline itinerary and fare product based on latent class (LC) model framework. In this model, passenger choice set was constituted from booking data, fare rules, and seat availability data. Instead of segmenting passenger by trip purpose, which is not available in booking data, the author utilizes variables such as frequent flyer membership, ticket distribution channel, and travel day of week for the class membership model. The approach is shown to provide a more distinct and intuitive segmentation across passengers. Teichert et al. (2008) applied the latent class model to explore preferences within airlines segments and analyzed respondents' profiles in terms of individual socioeconomic and trip characteristics. They concluded that the segmentation criterion currently applied by airlines does not adequately mirror the heterogeneity in customer's preference patterns. They suggested product marketing be aligned to passenger attitudes and socio-demographic profiles which are different across passenger segments. Wen and Lai (2010) used latent class model to identify airline passengers' potential segments and preferences for international air carriers using individual socioeconomic and trip characteristics as class membership variables. The latent class model is capable of representing heterogeneity across passenger segments which results in improved prediction accuracy over the multinomial logit model. Specifically, the willingness to pay for service attribute improvements is found to be substantially different across air routes and to vary by traveler segments.

Regardless of the number of research efforts on heterogeneity in choice behavior, the application of latent class choice model in RM problem is still relatively limited. Most of the studies which incorporated choice models in RM problem have assumed that customers are homogeneous in taste preferences. Studies which rely on this assumption include the work of Zhang and Adelman (2009), Topaloglu (2009); and Erdelyi and Topaloglu (2010) who incorporated customer choice models in the network RM pricing. In their setting, the price for each product is chosen from a discrete set, and the demand for each product depends on the price of the product only. However, given that RM relies on the premise that different customers are willing to pay different amounts for a product, accounting for passenger heterogeneity is expected to provide high yield toward RM strategy. More specifically, Garrow (2010) suggested that calibrating models by segments to distinguish between time-sensitive and price-sensitive customers can highly impact demand prediction accuracy and contribute to significant RM system performance when being incorporated in RM optimization problem.

Recently, a limited number of studies which incorporate heterogeneous passenger choice model in RM problem have primarily focused on choice-based deterministic linear programming (CDLP) problem. CDLP is a class of revenue optimization which solves for sets of product to be made available to the customers at different points in time during the sales horizon. In this context, Rusmevichientong et al. (2012) analyzed a model that captures the substitution between the products and preference heterogeneity. Each customer is assumed to belong to a particular class and the demand from each customer class is governed by a multinomial logit choice model with class-dependent parameters. This problem considers a set of different products and maximizes the expected profit across all customer classes. Méndez-Díaz et al. (2012) specified LC model which divides customers into segments based on choice of product alternatives considered by each customer. Their demand model allows product to overlap across segments and the preference parameters for each product alternative in the logit model are assumed to be known in advance. The authors prove that the latent class logit assortment problem is NP-Hard, and solve the choice-based deterministic linear program (CDLP) using branch and cut approximation method. The procedure is tested in the context of both capacitated and un-capacitated retail assortment problems.

2.2. Joint pricing and seat allocation problem

In the past few decades, the pricing and seat allocation problem has generated a number of studies, although often the two aspects have been treated separately. Traditional approaches have assumed that prices are fixed and only the optimal allocation of resources is computed. On the other hand pricing is usually determined on demand segmentation and optimal fare is calculated regardless capacity constraints. Nevertheless, the two problems are interrelated and complimentary to one another. As noted by McGill and van Ryzin (1999), the integration of pricing and inventory allocation decisions should receive more attention by analysts in RM. In this context, Weatherford (1997) emphasized the importance of considering prices as part of the optimization problem and suggested including them as decision variables in the seat allocation problem. The author considered a single flight leg with at least two fare products. The demand for each fare product was assumed to be normally distributed and represented by a linear demand function of the competing products' fare.

Kuyumcu and Garcia-Diaz (2000) studied pricing and seat allocation problems jointly for an airline network using historical data. The optimization problem aimed at maximizing the total revenues within the network. The study assumed that demand is normally distributed and that there is no interaction of demand across fare classes, and markets (origin destination). Fare was assumed to be an exogenous variable for the passenger decision process, as no explicit hypothesis regarding the relationship between demand and fare or any other product characteristics was made.

Bertsimas and de Boer (2002) analyzed the joint problem of pricing and seat allocation in a network setting. The authors assumed that demand for each fare product was uncertain and that expectation of the product demand only depends on the product's price. The numerical experiment suggested that coordination of pricing and seat allocation policies in the network and accounting for demand uncertainty can lead to significant revenue gains. It was also demonstrated that the underlying optimization problem is convex for certain types of demand distributions, thus tractable for large instances.

Cote et al. (2003) proposed a model with the capability of jointly solving the pricing and seat allocation problem in a network with competitor. The approach was based on a bi-level programming framework: the airline was assumed to know how its competitor would react and this behavior was explicitly integrated in the decision process. The decision variables include fares, but not seat allocation. The main assumption was that the demand for each fare product and itinerary combination was assumed to be fully known.

Ongprasert (2006) studied the seat allocation problem for intercity high speed rail services in Japan. The analysis includes: revenue maximization, average passenger load factor (APLF), and the number of passenger rejection. The choice model was estimated using nested logit model where the upper level consists of two transportation alternatives: high speed rail, and airlines and the lower level consists of fare product alternatives. The passenger choice model is incorporated in the seat allocation optimization problem which accounts for shared capacity of the railway network. Results show that seat allocation accounting for passenger choice behavior contributes to revenue improvement by offering discounted fare in the off peak trip.

Chew et al. (2008) developed a joint optimization model of pricing and seat allocation for a single product with a two period lifetime. Product price was assumed to increase as the time it perishes approaches, while demand is expressed as a linear function of price. To maximize the expected revenue, a discrete time dynamic programming model was developed to obtain the optimal prices and the optimal inventory allocations. Based on the concave property of the objective function, the authors used an iterative procedure to find the optimal solution. The problem was also extended to multiple time periods, where the concavity property no longer holds; for this case, several heuristics were suggested to solve the problem.

Cizaire (2011) developed several approaches to solve the joint problems of airline optimal fare and seat allocation. The underlying demand volume is modeled as a function of fares. The analysis proposes both deterministic and stochastic

Table 1
Review summary.

Authors	No. of product > 1	Demand model	No. of time period > 1	Multiple legs	Competitor	Joint pricing/ seat allocation	Simultaneous optimization
Weatherford (1997)	√	Linear function of price with cross elasticities	×	×	×	√	√
Kuyumcu and Garcia-Diaz (2000)	√	Normally distributed demand	×	√	×	√	×
Bertsimas and de Boer (2002)	√	Function of price	√	√	×	√	×
Cote et al. (2003)	√	Constant	×	√	√	√	×
Ongprasert (2006)	√	Nested logit	√	√	√	√	×
Chew et al. (2008)	×	Linear function of price	√	×	×	√	×
Cizaire (2011)	√	Function of price	√	×	×	√	√
This research	×	MNL and LC choice models, log-linear demand functions	√	√	×	√	√

approaches to model demand; in particular, heuristics were developed to solve the stochastic problem. The problem considers a multiple products, multiple time periods without network considerations.

In Table 1, we present a summary of the studies reviewed in this Section and we compare them to the study we propose. Comparisons is based on the number of products and time periods accounted, the presence of a competitor, the number of legs in the network, the approaches used to model demand and to optimize the problem of pricing and seat allocation.

3. Research framework

The proposed optimization framework solves a joint pricing and seat allocation problem for revenue management. The model system accounts for both passengers' response to RM policy and for demand volumes. The passenger choice models, estimated with discrete choice methods, predict the timing in which passengers purchase the ticket as a function of fare and other trip attributes (Section 4). The demand functions account for passengers deciding not to travel with this service or for induced demand due to advantageous fare policy. Passenger volumes are estimated using log-linear regression, where independent attributes are fare and trip attributes (Section 5). The passenger choice models and the demand functions are incorporated into a RM revenue optimization system that maximizes expected ticket revenue per each train trip in a network setting (Section 6).

4. Passenger choice model

In this section, we propose a disaggregate choice model for ticket purchase timing decision, based on the assumption that each individual purchases the ticket at the time that maximizes his/her utility. Given that this study is based on confirmed booking data, we assume that the mode choice decision has already been made by the passengers. Thus, the proposed model aims to capture the passengers' purchase timing decisions as a function of booking time and trip specific attributes.

4.1. Sample selection

The analysis is based on ticket reservation data for intercity passenger trips registered by a railway operator in March 2009. To reduce the complexity of the problem we consider here just northbound trips and coach class passengers which constitute the predominant part of the market (92%). Only confirmed reservations which contribute to the actual revenue are retained for the modeling exercise; the final dataset is constituted by 110,828 records.

Fig. 1 represents the number of reservations by number of days before departure. Data indicate that about 98 percent of the passengers make the reservation no earlier than 30 days before the departure date. The majority of the passengers book the ticket about one week before departure and a very high portion of passengers book the ticket within 2 days before departure.

Fig. 2 represents fare distribution by number of day before departure in major markets (Station 16 to Station 8, and Station 8 to Station 1). These two markets account for more than one third of the passenger demand in the northbound direction. It shows that fares primarily increase as time approaches departure. The same fare pattern is also observed in other markets.

The railway service includes 16 stations; which have been aggregated into 4 groups based on their geographical location as shown in Fig. 3. Stations in each group belong to the same metropolitan region and it is reasonable to assume that passengers' behavior is similar within these areas. Ten models are estimated for trips between and within the station groups; markets are segmented based on trip distances: long, medium, and short. According to a number of studies in travel demand modeling, travelers' behavior in long-distance journeys differs substantially from routine journey patterns or short trips

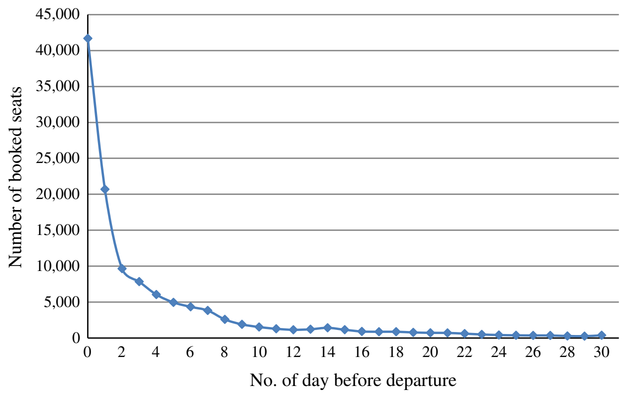

Fig. 1. Total number of booked seats prior to departure.

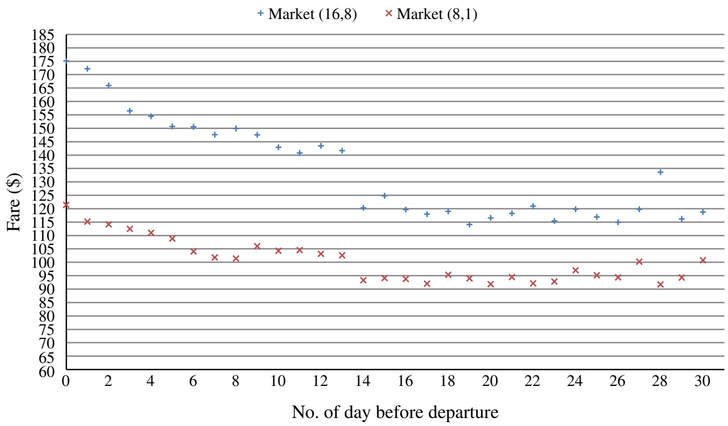

Fig. 2. Average fares by booking time of major markets.

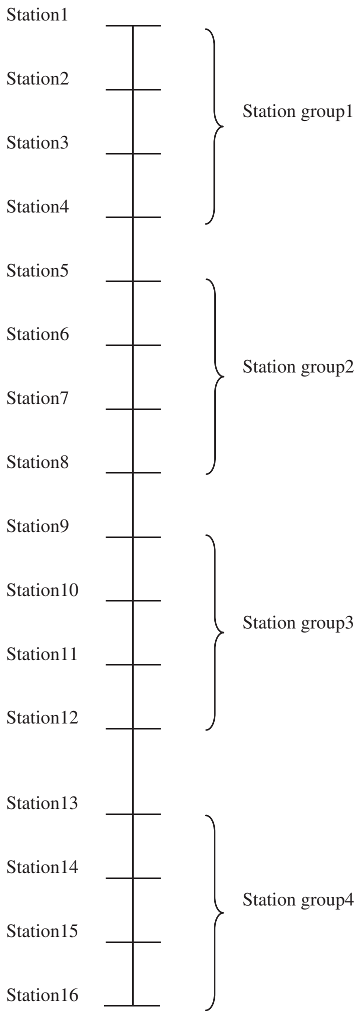

Fig. 3. Station group.

(Rohr et al., 2009). Due to space limitation, we only show estimation results for a long distance market from station group 4 to station group 2; more details on market segmentation and choice model estimation can be found in Hetrakul and Cirillo (2013).

4.2. Choice set generation

The fare of the railway service considered varies depending on departure time of day, day of week, how early the reservation is made in advance, and customer demand for each departure. Passengers decide when to purchase the ticket based on fare variation over the sale horizon and personal consideration about their trip. Since the data indicate that 98 percent of the tickets were purchased within 30 days before departure (Fig. 1), we assume that the choice set is constituted of 31 days, from 30 days before departure (booking day 1) until departure date (booking day 31). Based on the data set, for each reservation, we can only observe fare on the day ticket is purchased but not on other days in the sale horizon. To accommodate choice modeling, fares on other days in the sale horizon are approximated from the actual data by using averaged fare for each booking day in the sale horizon from the monthly data. The choice model is estimated using multinomial logit (MNL) and latent class (LC) formulations.

4.3. Multinomial logit (MNL) model specification

The independent variables that enter the final models are fare ($) and advance booking. Fare and advance booking variables are aimed to capture passenger tradeoff behavior between early booking with cheaper fare and late booking with higher fare. The model specification includes different price coefficients across booking periods in order to accommodate the assumption that passengers have different price sensitivities over the sale horizon. The 31 booking days are grouped into six booking periods such that booking days within the same booking period have approximately the same number of reservations. These six booking periods (k) are: (1) Booking day 1 to booking day 11, (2) Booking day 12 to booking day 20, (3) Booking day 21 to booking day 25, (4) Booking day 26 to booking day 29, (5) Booking day 30, and (6) Booking day 31. The resulting utility of passenger i booking the ticket on day j can be expressed as:

$$U_i(j) = (\beta_{advbk} \times advbk_j) + (\beta^k_{fare} \times fare_j) + \varepsilon_{ij} \quad (1)$$

where the independent variables and their associated index are: j = Booking day, $j \in \{1, \ldots, 31\}$; k = Booking period, $k \in \{1, \ldots, 6\}$; $advbk_j$ = number of day from booking day j to departure; $fare_j$ = fare of booking day j ($); ε_{ij} = a mutually independent noise term of individual i on choice j following a Gumbel distribution.

The probability of passenger i booking on day j can be calculated by using the logit probability formulation as:

$$\Pr(bkday_j) = \frac{\exp[V_i(j)]}{\sum_{l=1}^{31} \exp[V_i(l)]} \quad (2)$$

where $V_i(j)$ and $V_i(l)$ are deterministic utility (without the term ε_{ij}) of alternative j and l respectively.

4.4. Latent class (LC) model specification and estimation

In latent class model specification, we aim at overcoming the lack of individual specific variables by segmenting passenger behavior by trip characteristics. Passengers traveling at particular periods (i.e. time of day or day of week) are believed to be relatively homogeneous in their characteristics. In latent class model, let i represents individual and j represents alternative from $1, \ldots, J$ in the choice set C. The model form can be written as:

$$P_i(j|X_M, X_C) = \sum_{s=1}^{S} P(S|X_M)P(j|X_C, S) \quad \forall j \in C \quad (3)$$

where s is class index; $\{1, \ldots, S\}$; X_M is class membership explanatory variable; X_C is class specific choice model explanatory variable.

The utility function of alternative j given the customer i is in the class s can be written as:

$$U_{ij} = X_{Cj}\beta_C + \varepsilon_{ij} \quad (4)$$

The class specific choice probability of alternative j can be expressed as:

$$P(j|X_{Cj}, S) = \frac{e^{X_{Cj,s}\beta_{C,s}}}{\sum_{j=1}^{J} e^{X_{Cj,s}\beta_{C,s}}} \quad \forall s \in S, \forall j \in C \quad (5)$$

where $\beta_{C,s}$ are the unknown parameters of the class-specific choice model. The utility function of customer i belonging to class s can be written as:

$$U_{is} = X_{M,s}\beta_M + \varepsilon_{is} \quad (6)$$

The probability of belonging to the latent class s can be written as:

$$P(s|X_{Mb}) = \frac{e^{X_{M,s}\beta_M}}{\sum_{s=1}^{S} e^{X_{M,s}\beta_M}} \quad (7)$$

where β_M are the unknown parameters for class membership model. Class specific choice model specification of LC includes fare ($) and advance booking. The utility of choice j can be written as:

$$U_i(j,k) = (\beta_{advbk} \times advbk_j) + (\beta^k_{fare} \times fare_j) + \varepsilon_{ij} \tag{8}$$

For the class membership model, other elements of the booking data are extracted to segment demand and capture heterogeneity of behavior across passenger in the class membership model which are:

Departure time of day: Dummy variables are used to indicate whether the trip is taken on a particular time of day. We use six departure times (Jin, 2007) for the intercity trip which are (1) early morning (0:00 am–6:29 am), (2) a.m. peak (6:30 am–8:59 am), (3) a.m. off-peak (9:00 am–11:59 am), (4) p.m. off-peak (12:00 pm–15:59 pm), (5) p.m. peak (16:00 pm–18:29 pm), and (6) evening (18:30 pm–23:59 am). Five departure times of day (except evening) are used for the class membership model.

Departure day of week: Dummy variables are used to indicate whether the trip is taken on a particular day of week; this results into six dummy variables for the class membership model, one for each day of the week (except Sunday).

Thus, the utility function of customer i belonging to class s has the form:

$$U_i(s) = C_s + \sum_{t=1}^{5}(\beta^t_{TOD} \times TOD_i) + \sum_{d=1}^{6}(\beta^d_{DOW} \times DOW_i) + \varepsilon_{is} \tag{9}$$

For the choice model estimation, the MNL model is estimated with AMLET (Another Mixed Logit Estimation Tool) (Bastin, 2011). The LC model is estimated with Latent Gold Choice 4.5, a software package by Statistical Innovations specifically designed for latent class choice modeling (Vermunt and Magdison, 2005).

Estimation results relative to MNL and LC models for the long distance market are shown in Table 2. Fare coefficients are all negative (as expected) for the MNL model and for almost all the time periods considered in the LC specifications; positive values for the fare are not particularly significant and are left in our final specification for consistency. Regarding class characteristics, we found higher WTP for purchase delay in class 1 (average $46.17 per day) than class 2 (average $13.86 per day) for the majority of the booking periods, indicating that passengers from class 1 are willing to pay more for the possibility to

Table 2
Choice model estimation of selected market (long distance market).

	MNL		LC				
			Choice Model	Class1		Class2	
Variable	Est	T-Stat	Variable	Est	T-Stat	Est	T-Stat
advbk	−0.181	−52.400*	Advbk	−0.139	−21.091*	−0.899	−18.162*
fare.period1	−0.004	−5.100*	fare.period1	0.000	0.111	0.014	1.560
fare.period2	−0.010	−18.600*	fare.period2	−0.003	−1.684	−0.082	−2.069*
fare.period3	−0.011	−24.200*	fare.period3	−0.005	−2.778*	−0.049	−5.557*
fare.period4	−0.009	−22.400*	fare.period4	−0.001	−0.737	−0.069	−7.931*
fare.period5	−0.005	−11.700*	fare.period5	0.003	1.789	−0.077	−8.052*
fare.period6	−0.002	−6.200*	fare.period6	−0.010	−4.348*	−0.059	−6.941*
			Class Model	Class1		Class2	
			Class Size	0.619		0.381	
			Variable	Est	T-Stat	Est	T-Stat
			Intercept	0.181	4.845*	−0.181	−4.845*
			Monday	−0.402	−14.288*	0.402	14.288*
			Tuesday	−0.338	−11.586*	0.338	11.586*
			Wednesday	−0.375	−11.908*	0.375	11.908*
			Thursday	−0.286	−9.502*	0.286	9.502*
			Friday	−0.213	−7.690*	0.213	7.690*
			Saturday	−0.019	−0.433	0.019	0.433
			Early morning	1.085	19.204*	−1.085	−19.204*
			AM peak	0.985	22.737*	−0.985	−22.737*
			AM off peak	0.474	15.353*	−0.474	−15.353*
			PM off peak	0.096	3.491*	−0.096	−3.491*
			PM peak	0.113	4.036*	−0.113	−4.036*
No. of observation		37,373	No. of observation				37,373
Rho-squared:		0.2932	Rho-squared:				0.3034
Adjusted rho-squared:		0.2931	Adjusted rho-squared:				0.3032
Log-likelihood at optimal		−90,711	Log-likelihood at optimal				−89,402
Log-likelihood at zero		−128,338					
Log-likelihood at constant		−90,487					

* Statistically significant at 5% significance level.

change their travel plans. Based on this assumption, passengers in class 1 are believed to be business oriented travelers which account for 61.9% of this market; while the remaining passengers are mostly travelling for leisure. More specifically, the class membership model indicates that passengers departing from early morning to AM peak are predominantly class 1.

In this study we present results for a two class membership model only; models with up to six classes have been estimated and their goodness of fit compared. The model with five classes was found to be superior to the others; however, given the difficulty of applying the optimization framework proposed to five classes we decided to continue our analysis with two latent classes.

4.5. Model validation

To compare the prediction capabilities of the models, we perform the out of sample validation for the long distance market. The dataset (37,373 observations) is divided into two groups. The first group consists of approximately 80 percents of the data (30,125 observations) containing passengers traveling from the 1st day to 25th day of the month. Departure day was chosen as a cut point because we assumed that data from the first group is obtained prior to the second group. The model is re-estimated using data of the first group and the results are applied to predict the decision of the second group which contains passengers who traveled on day 26th to 31st of the month (7248 observations). The prediction capabilities of the four models are compared in Table 3. The root mean square error (RMSE) is used as the measure of error.

Based on Table 3, LC model results in the least error with the root mean square error (RMSE) of 54 compared to RMSE of MNL which is 65.

5. Demand function

In this section, the demand function to predict the passenger volume for each origin destination pair is estimated. The dataset for aggregate demand estimation is the same used for the disaggregate choice model of ticket purchase timing

Table 3
Out of sample validation for long distance market.

Choice	Actual	MNL	LC
Day1	42	8	11
Day2	21	10	13
Day3	25	12	15
Day4	23	15	18
Day5	32	18	20
Day6	42	22	24
Day7	45	26	28
Day8	41	32	32
Day9	60	39	38
Day10	50	48	45
Day11	50	59	57
Day12	58	29	37
Day13	66	35	43
Day14	95	43	50
Day15	61	52	58
Day16	86	62	67
Day17	103	73	77
Day18	86	86	89
Day19	73	102	102
Day20	88	120	118
Day21	133	119	103
Day22	143	139	123
Day23	148	166	154
Day24	254	198	204
Day25	257	237	298
Day26	261	329	337
Day27	459	394	402
Day28	608	471	490
Day29	503	557	617
Day30	1196	1373	1390
Day31	2139	2374	2188
Total	7248	7248	7248
	RMSE	65	54

decisions. We adopt a log-linear regression type for its desirable theoretical and practical properties; log-linear form restricts the estimated passenger demand (dependent variable) to be strictly positive and bounded at zero.

5.1. Specification

The independent variables which enter the final model include the intercept, the square of the advance booking ($advbk^2$), fare ($fare$), weekend dummy ($wknd$) indicating whether the departure day is on weekend, and booking period specific intercepts (denoted in Section 2.3) indicating whether the departure day is in a particular booking period. The square of the advance booking is motivated from the non-linear relationship between advance booking and number of passenger booking observed in the data. The weekend dummy accounts demand variation by departure day of week. The specification of the log-linear demand function can be expressed as follows:

$$\log[D_{od}(j)] = \alpha_0 + (\alpha_{advbksq} \times advbk_j^2) + (\alpha_{fare} \times fare_j) + (\alpha_{wknd} \times wknd) + \sum_{k=1}^{6}(\alpha_k \times bkperiod_j) + \varepsilon \tag{10}$$

$$D_{od}(j) = \exp\left\{\alpha_0 + (\alpha_{advbksq} \times advbk_j^2) + (\alpha_{fare} \times fare_j) + (\alpha_{wknd} \times wknd) + \sum_{k=1}^{6}(\alpha_k \times bkperiod_j) + \varepsilon\right\} \tag{11}$$

where $D_{od}(j)$ is the number of reservation on booking day j for origin o destination d; $advbk_j^2$ the square of the advance booking for booking day j; $fare_j$ the average fare ($) of booking day j for the observed departure day; $wknd$ the weekend dummy (1 if departure day is weekend, 0 otherwise); $bkperiod_j$ the booking period dummy (1 if booking day j is in period k, 0 otherwise); α_0 the intercept; ε is the error term.

The daily passenger demand for each origin destination pair is the summation of passenger reservation on each booking day j over the sale horizon:

$$D_{od} = \sum_{j=1}^{31} D_{od}(j) \tag{12}$$

Example of the demand function estimation is shown in Table 4.

6. Optimization procedure

Our optimization procedure incorporates both the passenger choice model in Section 4 and the demand function presented in Section 5. The framework assumes that the railway operator maximizes the expected revenue for each train trip and proposes fare strategy which varies on a daily basis. The formulation also allows for pricing and seat allocation to be solved simultaneously. The railway service in consideration departs hourly from 5:00AM to 7:00PM. In this analysis, we focus on trains that depart from the south end station at four different departure times which are: 5:00AM., 9:00AM., 1:00PM., and 4:00PM. on Friday, March13, 2009. These four departure times (named Train#1 to Train#4) are selected to represent railway traffic at different time periods across the day.

Table 4
Estimated demand function of OD (9,4).[a]

Variable	Coeff	Std Err.	T-Stat	$P > \|t\|$	[95% Confidence interval]	
$advbk^2$	−0.0018	0.0003	−6.46	<0.001	−0.0024	−0.0013
Fare	−0.005	0.0015	−3.36	0.001	−0.0079	−0.0021
Wkndmy	−0.2318	0.0664	−3.49	0.001	−0.3622	−0.1014
Period 1	−3.3826	0.2444	−13.84	<0.001	−3.8623	−2.9029
Period 2	−3.2735	0.1811	−18.08	<0.001	−3.6289	−2.9181
Period 3	−2.8051	0.1696	−16.54	<0.001	−3.138	−2.4723
Period 4	−1.8583	0.169	−10.99	<0.001	−2.1901	−1.5265
Period 5	−0.7223	0.2119	−3.41	0.001	−1.1381	−0.3065
Constant	6.4649	0.3018	21.42	<0.001	5.8726	7.0572
No. obs	886				R-squared	0.657
$F(8, 877)$	209.65				Adj R-squared	0.654
Prob > F	0				Root MSE	0.834

[a] Station number is based on station renumbering in Fig. 4.

6.1. Problem setting

The problem is formulated to optimize ticket revenue of coach class passenger for each train trip from the south end to the north end stations. This railway network has a total of 16 stations. However, in our optimization problem we exclude seven stations, because passenger demand in these markets is low and insufficient for estimation. The remaining nine stations are shown in Fig. 4, and are renumbered into station 1 to station 9. The optimization focuses on these remaining stations; seats currently occupied by the excluded stations and seats currently empty (Y) are not allowed to contribute to the revenue. These seats are extracted from the total seat capacity (Z) in the constraint. The decision variables are: fare for each origin destination pair on each booking day over the sale horizon ($fare^j_{od}$) and the fraction of demand to be accepted for each origin destination pair (α_{od}).

6.2. Demand conversion

The proposed demand function provides the total number of passengers that intends to travel. Given that the optimization framework is solved independently for each train, the passenger demand estimated has to be segmented by departure time. To this scope, conversion factors are obtained from historical data and by observing the distribution of daily passenger demand across different departure times of the day. Our analysis focuses on the train which departs from the south end station (station 9) at four different departure times. The corresponding departure time (t) of each intermediate station which loads passenger into each train is shown in Table 5.

The conversion factor is denoted as f^t_{od} where t represents departure time, o represents origin station, and d represents destination station. The passenger demand by departure time can be computed from estimated demand function as follows:

$$D^t_{od} = f^t_{od} \times D_{od} \tag{13}$$

where D^t_{od} is the number of passenger demand from origin o to destination d at departure time t; f^t_{od} the conversion factor from daily demand to demand by departure time; D_{od} is the estimated passenger daily demand from origin o to destination d obtained from the demand function.

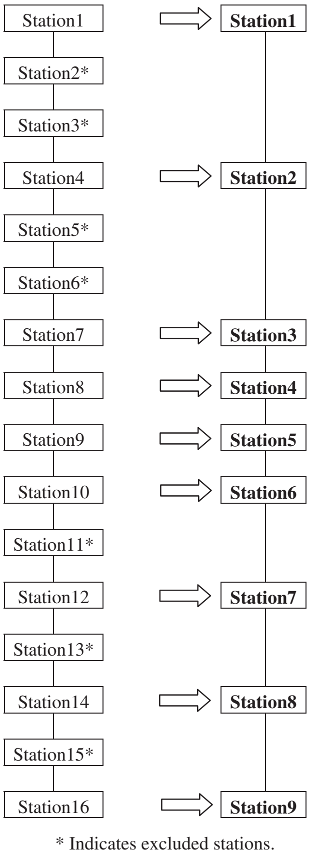

Fig. 4. Station renumbering.

Table 5
Departure time for each origin.

Departure time (t)				
Origin	Trian#1	Trian#2	Trian#3	Trian#4
9	5:00AM	9:00AM	1:00PM	4:00PM
8	5:30AM	9:30AM	1:30PM	4:30PM
7	6:35AM	10:35AM	2:35PM	5:35PM
6	7:19AM	11:19AM	3:19PM	6:19PM
5	7:30AM	11:30AM	3:30PM	6:30PM
4	8:00AM	12:00PM	4:00PM	7:00PM
3	8:44AM	12:44PM	4:44PM	7:44PM
2	10:55AM	2:55PM	6:55PM	9:55PM

6.3. Problem formulation

6.3.1. Notation

n	Number of stations (9 stations over the network)
o	Boarding station index
d	Alighting station index
D_{od}^t	Passenger demand from origin o to destination d at departure time t
α_{od}	Acceptance ratio; a fraction of demand (D_{od}^t) to be accepted
$fare_{od}^j$	Fare for origin o destination d on booking day j
Z	Total coach class seat capacity; equal to 260 (Railway Technology, 2011)
Y	Number of seats currently occupied by the excluded stations and seats currently empty
$revenue$	Revenue per train trip ($) from south end to north end station
$\Pr(j\|od)$	Probability that passenger purchases the ticket on booking day j for origin o destination d

6.3.2. MNL fare and seat allocation optimization

The problem is formulated as follows:

$$\max_{fare_{od}^j,\alpha_{od}} revenue = \sum_{o=1}^{n-1}\sum_{d=o+1}^{n}\left[\alpha_{od}D_{od}^t\sum_{j=1}^{31}\{\Pr(j|od)fare_{od}^j\}\right] \tag{14}$$

Subject to:

- Capacity constraint

$$\sum_{o=1}^{l}\sum_{d=l+1}^{n}\alpha_{od}D_{od}^t \leqslant Z - Y \tag{15}$$

$$0 \leqslant \alpha_{od} \leqslant 1 \quad \text{for all } l = \{1,\ldots,n-1\} \tag{16}$$

- Fare policy constraint

$$fare_{od}^- \leqslant fare_{od}^j \leqslant fare_{od}^+ \tag{17}$$

The purchase time probability $\Pr(j|od)$ is equivalent to the passenger share that purchases the ticket on the considered booking day. A MNL choice model is used in this optimization framework, consequently the choice probability is calculated as:

$$\Pr(j|od) = \frac{\exp(V_j)}{\sum_{k=1}^{31}\exp(V_k)} \tag{18}$$

where V_j is a deterministic utility of booking day j of origin o destination d. V_k is a deterministic utility of booking day k of origin o destination d.

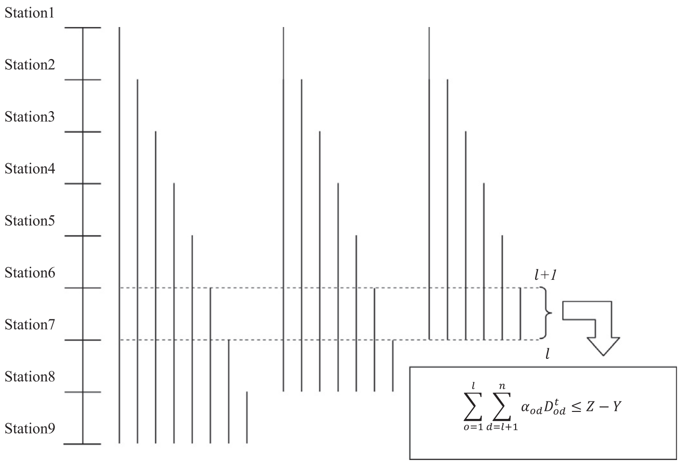

Fig. 5. Capacity constraint.

6.3.2.1. Objective function. The first two summations accounts for the passenger demand departing from origin $o \in \{1,\ldots,n-1\}$ to destination $d \in \{2,\ldots,n\}$ within the same train trip. Note that this index has different value from the station number. The value $o = 1$ represents the south end station (station9) and increases up to $o = 9$ for the north end station (station1). The index t is used to represent the corresponding departure time for each station which loads passenger into this train. The third summation computes the expected ticket revenue for each origin destination pair and is a weighted average of fare ($fare_{od}^{j}$) and probability of being purchased ($\Pr(j|od)$) over the entire sale horizon. The fraction of demand which can be accepted for each origin destination pair is denoted as α_{od}.

6.3.2.2. Constraints. The capacity constraint restricts the number of accepted passenger in each segment to be within the allowable seat capacity which is equals the total seat capacity (Z) subtracted by the seats currently occupied on the excluded stations and seats currently empty (Y). The decision variable α_{od} is an acceptance ratio which is used to control the number of accepted passengers to be within the allowable seat capacity. Fig. 5 represents the capacity constraint, where the line connecting each origin destination pair represents the passenger demand. Fare policy constraint restricts the fare to be within the bound ($fare_{od}^{-}$ and $fare_{od}^{+}$ as lower bound and upper bound respectively). The fare bound is obtained from the dataset based on the maximum and minimum average fare for a particular time of day, and day of week for each departure. This fare bound is also adjusted to ensure that fare for shorter distances does not exceed the fare for longer distances.

6.3.3. LC fare and seat allocation optimization

The application of the latent class (LC) choice model in the fare optimization allows for a discrete segmentation of passenger taste heterogeneity. Given the choice probability of LC model as:

$$\Pr(j|X_M, X_C) = \sum_{s=1}^{S} \Pr(s|X_M)\Pr(j|X_C, S); \quad \forall j \in C \tag{19}$$

where s is class index; $\{1,\ldots,S\}$; X_M is class membership explanatory variable; X_C is class specific choice models explanatory variable.

The corresponding optimization problem can be expressed as:

$$\max_{fare_{od}^{j},\alpha_{od}} revenue = \sum_{o=1}^{n-1}\sum_{d=o+1}^{n}\left[\alpha_{od}D_{od}^{t}\sum_{j=1}^{31}\left\{\sum_{s=1}^{S}\Pr(s|X_M)\Pr(d|X_C, S, od)fare_{od}^{j}\right\}\right] \tag{20}$$

The latent class optimization has the same constraints as the MNL optimization.

7. Optimization results

The optimization problem is solved as a nonlinear programming problem with LINGO 12.0, the optimization software by Lindo System Inc. (Lingo System Inc, 2010). The nonlinearity of this problem is due to the exponential term in the logit choice probability function (MNL and LC). Results from the optimization exercise are shown Table 6. First we compare the number of accepted passengers ($\alpha_{od} \times D_{od}^{t}$) by the modeling system to the actual demand (last row of Table 6). Results show that the proposed strategy increases the total number of passengers in all the four trains. Based on our assumption that limits the number of seats to those occupied in the actual situation, we can conclude that the optimal solution suggests accepting more short-haul passengers thus allowing for the same number of seats to serve more passengers. The acceptance ratio (α_{OD}) between the MNL and LC are also compared; the rates obtained with the two models are similar. Finally, in Table 6 we show the revenues per each train trip across the four train departures. Results indicate that the RM strategy can improve significantly revenues from low traffic trains (Train#1, 2, and 3 from 15.80% to 24.96%) and from high traffic markets but in a less remarkable entity (Train#4 from 13.82% to 16.24%). When comparing LC and MNL, it was found that the optimization problem which incorporate LC choice model generally provides less revenues. A possible explanation for this behavior can be derived from the fact that LC models enable passenger in different class to respond to fare differently. Thus, the behavior of the price sensitive passenger can be revealed more realistically with LC than with MNL which assumes homogeneity across populations. Based on the results obtained, two markets are selected and detailed analyses in terms of pricing and seat allocation are provided in the next two sub-Sections.

7.1. Pricing result

The two ODs under analysis are (9,7) and (4,1) which loads passengers into Train#2 and Train#4. OD (9,7) departs from the south end station at 9AM and 4PM respectively, while OD (4,1) departs at 12PM and 7PM.

Results in terms of fare for OD (9,7) for the 9AM departure time are shown in Fig. 6. The results from MNL and LC models are compared to the existing fare, which is the representative fare pattern for a particular departure time of day and day of week obtained from the real data. In general, the optimization using LC model provides a stepwise pattern which responds

Table 6
Revenue comparison ($).

(O,D)	5AM (Train#1)			9AM (Train#2)			1PM (Train#3)			4PM (Train#4)		
	Exist	MNL	LC	Exist	MNL	LC	Exist	MNL	LC	Exist	MNL	LC
(9,8)	0	28	9	396	190	64	0	0	0	171	86	31
(9,7)	0	1013	994	764	6907	6888	1433	7326	7326	7521	13,310	13,310
(9,6)	0	0	0	0	1882	1686	894	0	0	7244	4317	4317
(9,5)	0	0	1	1704	0	0	1808	0	0	4433	1863	1849
(9,4)	1203	0	0	9589	3709	3606	3575	0	0	14,529	19,035	18,810
(9,3)	137	0	0	183	0	0	0	0	0	1959	0	0
(9,2)	0	0	0	332	0	0	396	0	0	222	0	0
(9,1)	149	0	0	694	0	0	0	0	0	422	0	0
(8,7)	0	287	37	244	0	48	690	1123	1085	132	1492	1145
(8,6)	0	0	0	0	0	43	205	655	655	390	871	770
(8,5)	0	0	194	0	99	72	212	733	731	191	975	943
(8,4)	95	0	0	1717	2186	1698	2038	0	0	2693	0	0
(8,3)	0	0	0	212	0	62	0	0	0	827	0	0
(8,2)	0	0	0	0	0	0	0	0	0		0	0
(8,1)	0	0	0	124	0	0	223	0	0	223	0	0
(7,6)	0	0	0	0	0	0	566	658	658	354	584	250
(7,5)	86	456	395	114	357	298	715	688	688	272	611	491
(7,4)	2064	4225	4086	1860	8663	8879	5025	12,461	12,461	2790	9088	9153
(7,3)	273	0	0	152	0	0	0	0	0	760	0	0
(7,2)	0	0	0	107	0	0	0	0	0	386	0	0
(7,1)	274	0	0	1449	0	0	847	0	0	389	0	0
(6,5)	0	0	0	0	125	84	0	199	200	0	0	0
(6,4)	0	0	0	0	334	278	0	0	0	0	0	0
(6,3)	0	0	0	0	290	272	0	0	0	0	0	0
(6,2)	0	0	0	0	0	0	140	0	0	124	0	0
(6,1)	0	0	0	0	0	0	281	0	0	298	0	0
(5,4)	32	336	342	0	83	35	0	0	0	0	0	0
(5,3)	0	488	479	0	313	224	0	0	0	0	0	0
(5,2)	93	0	0	124	0	0	0	0	0	140	0	0
(5,1)	795	0	0	978	0	0	520	0	0	523	0	0
(4,3)	71	220	162	0	259	190	0	0	0	71	246	202
(4,2)	2366	3396	3361	2263	3429	3362	5293	6747	6830	3061	4320	4251
(4,1)	4781	4416	4331	5430	6103	5896	10,629	12,364	12,278	7117	9865	9794
(3,2)	396	0	0	370	0	0	0	0	0	0	0	0
(3,1)	104	978	956	343	1299	1350	0	0	0	119	0	0
(2,1)	70	181	174	25	227	135	90	243	278	75	100	57
Tot. revenue	12,989	16,023	15,520	29,174	36,456	35,169	35,580	43,197	43,191	57,436	66,761	65,372
% Improve	23.36	15.80		24.96	20.55		21.41	21.39		16.24	13.82	
Tot. passenger												
Accepted 131	166	166	238	334	336	255	306	306	358	401	399	

more realistically to passenger behavior; this is consistent with expectations that assume price sensitivity to be different across the sale horizon. The use of LC choice model also enables differentiating passengers into two classes (class 1 and class 2) depending on the train departure time.

Fig. 7 shows the corresponding number of accepted passengers in OD (9,7) on each day over the sale horizon. The response of passengers demand in both MNL and LC are realistic; when the new fare is lower than the existing (from booking day 25 onward), the passenger demand increases from the existing significantly.

A summary of the results obtained for these two markets in terms of accepted passengers and corresponding revenue across different departure times is provided in Figs. 8 and 9 respectively.

The proposed seat allocation strategy also influences capacity distribution for each segment in the network. We observe for example, that the proposed shares of capacity increase slightly for ODs (9,4), (4,2), and (4,1) and significantly for ODs (9,7)

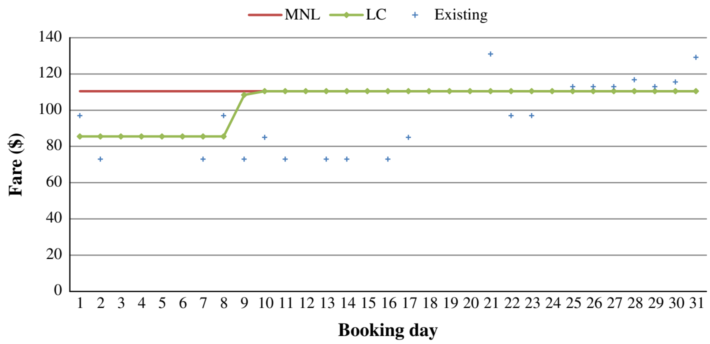

Fig. 6. OD (9,7) 9AM departure fare.

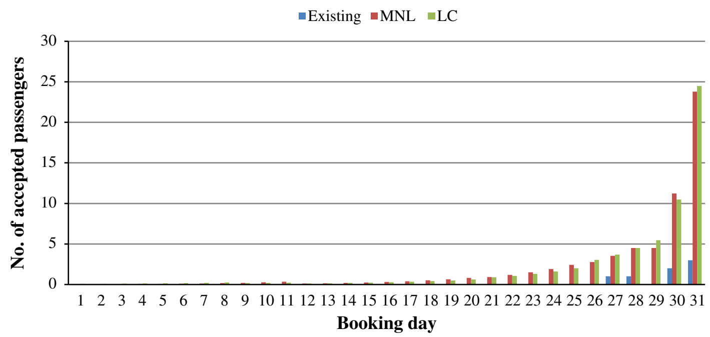

Fig. 7. OD (9,7) 9AM departure demand.

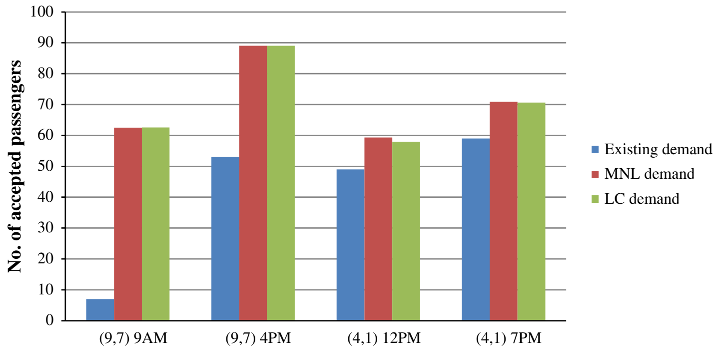

Fig. 8. Total accepted passengers of major markets.

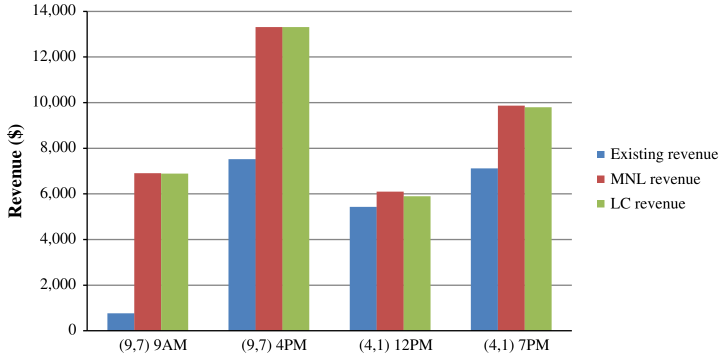

Fig. 9. Revenue ($) of major ODs.

and (7,4). On the other hand, the proposed shares of capacity for ODs (9,6) and (9,5) decrease. This occurs across all the segments utilized by these markets.

8. Conclusions and future research

This paper has proposed an empirical study based on ticket reservation data for intercity passenger railway trips. We have presented a methodological framework that incorporates latent class models for ticket purchase timing decisions in a railway pricing and seat allocation problem. The approach allows RM strategy to explicitly account for passenger taste heterogeneity by classifying passengers into classes based on departure schedules instead of using trip purposes which are not available in the data used for the analysis. Demand response to fare is explicitly represented by an aggregate log-linear function and is incorporated into the optimization problem. The RM revenue optimization considers a joint problem of pricing and seat allocation with the objective of maximizing expected ticket revenue for each train trip. The proposed formulation allows for simultaneous optimization of pricing and seat allocation while accounting for heterogeneous passenger preferences. The capacity constraints are determined on the basis of the railway network characteristics which allow capacity resources to be efficiently utilized across the network.

Results obtained have illustrated the impacts that the strategy derived from the optimization procedure has on the existing conditions in terms of fare, capacity distribution, and revenue. Seat allocation policy results into more short-haul trips acceptance, which contributes to greater revenue than long-haul trip with the same seat capacity. The solution from the proposed framework gives a significant revenue improvement from the actual between 16.24% and 24.96% obtained with MNL and from the actual between 13.82% and 21.39% obtained with LC choice models respectively, depending on the train departure times. Finally, the optimization system also provides indications on how to redistribute capacity efficiently across the markets considered. In conclusions, this paper has illustrated how a railway operator can exploit its existing data sources to better understand the choice behavior of railway passengers and its impact toward RM strategy. In particular, it shows that accounting for passenger taste heterogeneity based on discrete and continuous segmentation approach results in more realistic representation of the passenger behavior in supporting RM strategy.

The following areas indicate possible avenues for future research. The application of methods that allow for continuous class segmentation (i.e. parametric and non-parametric mixed logit) should be investigated and results implemented into the optimization framework. Other choice dimensions should be considered in the railway RM problem; for instance, departure time and departure day choices are important when making ticket purchase decisions. However, for these choice dimensions, additional data are necessary to construct plausible choice set for each passenger. From an optimization perspective, it will be desirable to consider networks with hub and spoke characteristics, which involves the consideration of station transfers and of more complex capacity constraints. It will also be interesting to optimize ticket revenue over multiple departures simultaneously by accounting for demand shift across different departures.

Acknowledgements

The authors would like to thank referees and the editor for very helpful and constructive comments throughout the reviewing process.

Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.tre.2013.10.005.

References

Bastin, F., 2011. Another Mixed Logit Estimation Tool (AMLET) Reference Manual, <http://amlet.slashbin.net/> (accessed 08.07.11).
Bertsimas, D., de Boer, S., 2002. Joint Network Pricing and Resource Allocation.
Bhat, C.R., 1998. Accommodating variations in responsiveness to level-of-service variables in travel mode choice modeling. Transportation Research Part A 32 (7), 495–507.
Carrier, E., 2008. Modeling the Choice of an Airline Itinerary and Fare Product using Booking and Seat Availability Data. Ph.D. Dissertation, Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA.
Chew, E.P. et al, 2008. Joint inventory allocation and pricing decisions for perishable products. International Journal of Production Economics. http://dx.doi.org/10.1016/j.ijpe.2008.07.018.
Cizaire, C., 2011. Optimization Models for Joint Airline Pricing and Seat Inventory Control: Multiple Products, Multiple Periods. Ph.D. Dissertation. Department of Aeronautics and Astronautics. Massachusetts Institute of Technology.
Cote, J., Marcotte, P., Savard, G., 2003. A bilevel modeling approach to pricing and fare optimization in the airline industry. Journal of Revenue and Pricing Management 2 (1), 23.
Erdelyi, A., Topaloglu, H., 2010. Using decomposition methods to solve pricing problems in network revenue management. Journal of Revenue Pricing Management 10 (4), 325–343.
Garrow, L.A., 2010. Discrete Choice Modeling and Air Travel Demand: Theory and Applications. Ashgate Publishing, U.K.
Greene, W.H., Hensher, D.A., 2003. A latent class model for discrete choice analysis: contrasts with mixed logit. Transportation Research Part B 37, 681–698.
Hetrakul, P., Cirillo, C., 2013. Accommodating taste heterogeneity in railway passenger choice models based on internet booking data. Journal of Choice Modeling 6, 1–16.
Jin, X., 2007. Toward Time-of-Day Modeling for Long Distance Trips, Doctoral Dissertation. University of Wisconsin, Milwaukee.
Kuyumcu, A., Garcia-Diaz, A., 2000. A polyhedral graph theory approach to revenue management in the airline industry. Computers and Industrial Engineering 38 (3), 375–395.
Lingo User Manual, 2010. Lindo System Inc., Chicago, IL, <http://www.lindo.com/> (accessed 08.07.11).
McGill, J.I., van Ryzin, G.J., 1999. Revenue management: research overview and prospects. Transportation Science 33, 233–256.
Méndez-Díaz, I., Miranda-Bront, J.J., Vulcano, G., Zabala, P., 2012. A branch-and-cut algorithm for the latent-class logit assortment problem. Discrete Applied Mathematics. http://dx.doi.org/10.1016/j.dam.2012.03.003.
Ongprasert, S., 2006. Passenger Behavior on Revenue Management Systems of Inter-city Transportation. Ph.D. Dissertation. Graduate School of Engineering. Kochi University of Technology, Japan.
Railway Technology, 2011. Acela Express Specification, <http://www.railway-technology.com/projects/amtrak/specs.html>.
Rohr, C., Daly, A., Patruni, B., Tsang, F., 2009. The importance of frequency and destination choice effects in long-distance travel behavior: what choice models can tell us. In: Proceedings of the International Choice Modeling Conference, Yorkshire, England, April 2009.
Rusmevichientong, P., Shmoys, D., Topaloglu, H., 2012. Assortment optimization with mixtures of logits. Operation Research 60 (4), 865–882.
Shen, J., 2009. Latent class model or mixed logit model? A comparison by transport mode choice data. Applied Economics 41 (22), 2915–2924.
Teichert, T., Shehu, E., von Wartburg, I., 2008. Customer segmentation revisited: the case of the airline industry. Transportation Research Part A 42, 227–242.
Topaloglu, H., 2009. Using Lagrangian relaxation to compute capacity-dependent bid prices in network revenue management. Operation Research 57 (3), 637–649.
Train, K.E., 2003. Discrete Choice Methods with Simulation. Cambridge University Press, Cambridge.
Vermunt, J.K., Magdison, J., 2005. Technical Guide for Latent Gold Choice 4.0: Basic and Advanced. Statistical Innovations Inc., Belmont, MA.
Wen, C.H., Lai, S.C., 2010. Latent class models of international air carrier choice. Transportation Research Part E 46, 211–221.
Walker, J. 2001. Extended Discrete Choice Models: Integrated Framework, Flexible Error Structures, and Latent Variables. Ph.D. Dissertation, Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA.
Weatherford, L., 1997. Using prices more realistically as decision variables in perishable-asset revenue management problems. Journal of Combinatorial Optimization 1 (3), 277–304.
Zhang, D., Adelman, D., 2009. An approximate dynamic programming approach to network revenue management with customer choice. Transportation Science 43 (3), 381–394.