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Error measures for fuzzy linear regression: Monte Carlo simulation approach

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ABSTRACT

The focus of this study is to use Monte Carlo method in fuzzy linear regression. The purpose of the study is to figure out the appropriate error measures for the estimation of fuzzy linear regression model parameters with Monte Carlo method. Since model parameters are estimated without any mathematical programming or heavy fuzzy arithmetic operations in fuzzy linear regression with Monte Carlo method. In the literature, only two error measures (E_1 and E_2) are available for the estimation of fuzzy linear regression model parameters. Additionally, accuracy of available error measures under the Monte Carlo procedure has not been evaluated. In this article, mean square error, mean percentage error, mean absolute percentage error, and symmetric mean absolute percentage error are proposed for the estimation of fuzzy linear regression model parameters with Monte Carlo method. Moreover, estimation accuracies of existing and proposed error measures are explored. Error measures are compared to each other in terms of estimation accuracy; hence, this study demonstrates that the best error measures to estimate fuzzy linear regression model parameters with Monte Carlo method are proved to be E_1, E_2, and the mean square error. One the other hand, the worst one can be given as the mean percentage error. These results would be useful to enrich the studies that have already focused on fuzzy linear regression models.

1. Introduction

Regression analysis is a statistical tool that is used to figuring out the mathematical relation between two or more quantitative variables. In the literature, most of the available regression modelling approaches are rather restrictive and their applications to real life problems require various assumptions. Therefore, new techniques have been proposed to relax some of these assumptions. Fuzzy regression is one of these techniques that attracts more attention nowadays.

After introduced by Tanaka et al. [1,2], the fuzzy regression analysis has become very popular with the introduction of fuzziness into regression. Many regression models including crisp input and fuzzy parameters, as well as fuzzy input and crisp parameters have been studied. Diamond [3] implemented regression models for crisp input and fuzzy output and fuzzy input-output. In these models, distance between fuzzy numbers was used to measure the goodness-of-fit for models. Furthermore, Näther and Körner [4] extended estimators of Tanaka et al. [1,2] with a least squares approach in the linear regression with crisp and fuzzy input and fuzzy output cases. Hong et al. [5] adopted a regression model with fuzzy input and fuzzy parameters. Additionally, Bardossy et al. [6] defined a new class of distance measures on fuzzy numbers and considered the regression model involving fuzzy input and fuzzy parameters. Peters [7], Luczynski and Matloka [8], Tanaka et al. [9], and Yen et al. [10] are some of the authors who focused on crisp input and fuzzy output regression models. D'Urso [11] carried out fuzzy linear regression analysis for fuzzy/crisp input and fuzzy/crisp output data. Moreover, Roh et al. [12] presented a new estimation approach based on Polynomial Neural Networks for fuzzy linear regression. Recently, a generalized maximum entropy estimation approach to fuzzy regression model is introduced by Ciavolinoa and Calcagni [13].

Application areas of fuzzy linear regression analysis have been considerably improved by different approaches in recent years. For instance, the relationship between dimensions of health related quality of life and health conditions are investigated under fuzzy linear regression [14]. A new fuzzy linear regression approach for dissolved oxygen prediction is suggested by Khan and Valeo [15]. Fuzzy linear regression is also used in electricity demand forecasting by Sarkar et al. [16] and in global solar radiation prediction by Ramedani et al. [17]. Estimation of relationship between forest fires and meteorological conditions are investigated within the framework of fuzzy linear regression by Akdemir and Tiryaki [18].

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Abdalla and Buckley [19–21] are the first practitioners of Monte Carlo (MC) method within the fuzzy linear regression context. In the MC method, a number of regression coefficient vectors are randomly generated; then values of dependent variables are estimated by using each generated vector, and the vector that gives minimum value to an error measure is taken as the best estimate of regression parameters. Abdalla and Buckley [19,21] used two error measures and tackled problems that are defined on the positive side of the real line. However, they did not mention which definition of the absolute value of a fuzzy number is used in the calculations for the error measures.

In this study, MC method for fuzzy linear regression analysis introduced by Abdalla and Buckley [19,21] is taken as a focal point. The definition of AbuAarqob et al. [22] is used for the absolute value of a fuzzy number and the MC method is applied to the data set of Abdalla and Buckley [19,21]. Six error terms are used in this study. Two of them have already been used by Abdalla and Buckley [19,21] and the other four different error measures that have not been previously calculated for MC method in fuzzy linear regression are used. These error measures are mean square error, mean percentage error, mean absolute percentage error, and symmetric mean absolute percentage error. In order to evaluate estimation accuracy of the new and existing error measures in fuzzy linear regression modeling, an extended simulation study is conducted over the whole real line. In the design of simulation study, two cases that fuzzy input-fuzzy output and crisp input-fuzzy output are taken into consideration. The performance of four error measures along with those used by Abdalla and Buckley [19,21] are evaluated and compared with each other. The best error measure and the one that should not be used for the estimation of fuzzy linear regression parameters are identified by MC method without using any mathematical programming or heavy fuzzy arithmetic operations.

The rest of the paper is organized as follows: some preliminaries for fuzzy numbers, random crisp vectors, and random fuzzy vectors are presented in Section 2. Brief information about fuzzy linear regression models with MC method is given in Section 3.1. Error measures proposed for the MC method are given in Section 3.2. The simulation study that compares the performances of error measures is conducted in Section 4. After the decision of the best and the worst error measures in MC method for fuzzy linear regression models, values of the error measures are calculated for the real data sets used by Abdalla and Buckley [19,21] in Section 5. Concluding remarks and some possible future perspectives are addressed Section 6.

2. Preliminaries

This section contains various definitions of fuzzy numbers and random fuzzy vectors that are defined by Abdalla and Buckley [19,21], Dubois and Prade [23], and AbuAarqob et al. [22].

Definition 2.1. A fuzzy number $\overline{A}$ is a fuzzy subset of the real line $\mathfrak{R}$. Its membership function $\mu_A(x)$ satisfies the following criteria [23]:

- α-cut set of $\mu_A(x)$ is a closed interval,
- $\exists$x such that $\mu_A(x)$=1, and
- convexity such that

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \quad \mu_A(x_2)) \text{ for} \lambda \in [0, 1],$$

where α-cut set contains all x elements that have a membership grade $\mu_A(x) \geq \alpha$.

Definition 2.2. A triangular shaped fuzzy number $\overline{A}$ is a fuzzy number whose membership function defined by three values, $a_1 < a_2 < a_3$, where the base of triangular is the interval $[a_1, a_3]$ and the vertex is $x = a_2$ [23].

Definition 2.3. The α-cut of a fuzzy number $\overline{A}$ is a non-fuzzy set defined as $\overline{A}(\alpha) = \{x \in \mathfrak{R}, \mu_A(x) \geq \alpha\}$. Hence $\overline{A}(\alpha) = [A^L(\alpha), A^U(\alpha)]$ where $A^L(\alpha) = \inf\{x \in \mathfrak{R}, \mu_A(x) \geq \alpha\}$ and $A^U(\alpha) = \sup\{x \in \mathfrak{R}, \mu_A(x) \geq \alpha\}$ [23].

Definition 2.4. The absolute value of a fuzzy number $\overline{A} \in \mathfrak{R}_F$ is a function $F : \mathfrak{R}_F \to \mathfrak{R}_F$ denoted by $F(\overline{A}) := |\overline{A}|$ with α-cut $\overline{A}(\alpha)$. From the interval analysis [24], it is known that if $I = [I^-, I^+]$, then $|I| = [\max(I^-, -I^+, 0), \max(-I^-, I^+)]$; and hence, the α-cut of $|\overline{A}|$ is given by

$$(|\overline{A}|)_\alpha = [\max(\overline{A}^-(\alpha), -\overline{A}^+(\alpha), 0), \max(-\overline{A}^-(\alpha), \overline{A}^+(\alpha))]. \quad (1)$$

From Eq. (1), the absolute value of a triangular fuzzy number is given as follows (for more information see [22,24]):

$$(|\overline{A}|)_\alpha = \begin{cases} \overline{A}(\alpha) & \text{if } \overline{A} \geq 0 \\ -\overline{A}(\alpha) & \text{if } \overline{A} \leq 0 \\ \{0, \max(-\overline{A}^-(\alpha), \overline{A}^+(\alpha))\} & \text{if } x \in (\overline{A}^-(0), \overline{A}^+(0)). \end{cases} \quad (2)$$

Definition 2.5. Random crisp vectors are defined as $\mathbf{v}_k = (v_{0k}, \ldots, v_{mk})$, elements of which are all real numbers in intervals I_i, $i = 0, 1, \ldots, m$. To obtain $\mathbf{v}_k$, firstly randomly crisp vectors $v_k = (x_{1k}, x_{2k}, \ldots, x_{mk})$ with all x_{ik} in [0, 1], $k = 1, 2, \ldots, N$ are needed to be generated. Since each x_{ik} starts out in [0,1], it is possible to put them into $I_i = [c_i, d_i]$ by $v_{ik} = c_i + (d_i - c_i)x_{ik}$, $i = 0, 1, \ldots, m$ [19,25].

Definition 2.6. Random fuzzy vectors are defined as $\overline{\mathbf{V}}_k = (\overline{V}_{0k}, \ldots, \overline{V}_{mk})$, $k = 1, 2, \ldots N$, where each $\overline{V}_{ik}$ is a triangular fuzzy number. Firstly crisp vectors $v_k = (x_{1k}, \ldots, x_{3m+3,k})$ with $x_{ik} \in$ [0, 1], $k = 1, \ldots, N$ are needed to be generated. Then first three numbers in v_k are chosen and ordered from smallest to largest. If it is assumed that $x_{3k} < x_{1k} < x_{2k}$, the first triangular fuzzy number is $\overline{V}_{0k} = (x_{3k}/x_{1k}/x_{2k})$. It is possible to continue with the next three numbers in v_k to form $\overline{V}_{ik}$, $i = 1, 2, \ldots, m$. In order to obtain $\overline{V}_{ik}$ within certain intervals, it is supposed to be in interval $I_i = [a_i, b_i]$, $i = 0, 1, 2, \ldots, m$. Since each $\overline{V}_{ik}$ starts out in [0, 1], it is possible to put it into $[a_i, b_i]$ by computing $a_i + (b_i - a_i)x_{ik}$, $i = 1, 2, \ldots, m$ (for more information see [21,25]).

3. Fuzzy linear regression

Fuzzy regression model is classified into three cases according to the type of independent and dependent variables by Choi and Buckley [26] as the following:

(I) Input and output data are both crisp.
(II) Input data is crisp and output data is fuzzy.
(III) Input and output data are both fuzzy.

The first category is considered as an ordinary regression model. Hence, Case-I is not taken into consideration in this paper. Fuzzy regression model for the second (Case-II) and third (Case-III) cases are considered.

The fuzzy linear regression model for Case-II is given as follows:

$$\overline{Y}_l = \overline{A}_0 + \overline{A}_1 x_{1l} + \overline{A}_2 x_{2l} + \cdots + \overline{A}_m x_{ml} \quad (3)$$

where $x_{1l}, \ldots, x_{ml}$ for $l = 1, \ldots, n$ are crisp numbers and $\overline{A}_0, \ldots, \overline{A}_m$ and $\overline{Y}_l$ are all triangular fuzzy numbers. Given the data, the objective is to find a combination of $\overline{A}_j, j = 1, \ldots, m$ values that makes the overall difference between estimated and observed values of dependent variable (error) minimum. The fuzzy regression model for Case-III is given as follows:

$$\overline{Y}_l = a_0 + a_1\overline{X}_{1l} + a_2\overline{X}_{2l} + \cdots + a_m\overline{X}_{ml}, \quad (4)$$

where $\overline{X}_{il}$ and $\overline{Y}_l$ for $i=1, \ldots, m; l=1, \ldots, n$ are triangular fuzzy numbers and a_i is a crisp number. Here, the objective is to find a combination of $a_j, j=1, \ldots, m$ values that makes the overall error minimum.

Characterization of the above cases is developed from different perspectives; and hence, several conceptual and methodological approaches exist in fuzzy regression. It should also be mentioned that several cases are considered simultaneously to estimate fuzzy regression model parameters [27].

3.1. Fuzzy linear regression with Monte Carlo method

In this section, brief information about the use of Monte Carlo method in fuzzy linear regression is given. Abdalla and Buckley [19,21] introduced the MC method for estimating fuzzy regression model parameters. Random fuzzy vectors $\overline{\mathbf{V}}_k = (\overline{V}_{0k}, \ldots, \overline{V}_{mk})$ for Case-II or random crisp vectors $\mathbf{v}_k = (v_{0k}, \ldots, v_{mk})$ for Case-III are generated in this method.

Arbitrary intervals which include model parameters are chosen for the generation of these vectors. Various optimization methods can be used to find an appropriate interval. Then, values of dependent variable are calculated over each candidate vector by using Eqs. (5) and (6) for Case-II and Case-III, respectively.

$$\overline{Y}^*_{lk} = \overline{V}_{0k} + \overline{V}_{1k}x_{1l} + \cdots + \overline{V}_{mk}x_{ml}, \quad (5)$$

$$\overline{Y}^*_{lk} = v_{0k} + v_{1k}\overline{X}_{1l} + \cdots + v_{mk}\overline{X}_{ml}. \quad (6)$$

Accuracy of each candidate $\overline{\mathbf{V}}_k$ or $\mathbf{v}_k$ vector depends on the used error measure. Abdalla and Buckley [19,21] use error measures E_1 in Eq. (7) and E_2 in Eq. (8) to quantify the error between the given ($\overline{Y}_l$) and the estimated ($\overline{Y}^*_{lk}$) values for both Case-II and Case-III. The first error measure is

$$E_{1k} = \sum_{l=1}^{n}[\int_{-\infty}^{\infty}|\overline{Y}_l(x) - \overline{Y}^*_{lk}(x)|dx]/[\int_{-\infty}^{\infty}\overline{Y}_l(x)dx], \quad (7)$$

where the integrals are only calculated over interval(s) containing the support of the triangular fuzzy numbers $\overline{Y}_l = (y_{l1}/y_{l2}/y_{l3})$ and $\overline{Y}^*_{lk} = (y_{lk1}/y_{lk2}/y_{lk3})$. The second error measure is

$$E_{2k} = \sum_{l=1}^{n}[|y_{l1} - y_{lk1}| + |y_{l2} - y_{lk2}| + |y_{l3} - y_{lk3}|]. \quad (8)$$

This process is repeated for N times, which is usually 10^4 or 10^5. For each case, two solution vectors that give minimum values for E_1 and E_2 are recorded and two best solutions for Case-II (or Case-III) are obtained. The first one is the vector $\overline{\mathbf{V}}_k$ (or $\mathbf{v}_k$) that corresponds to minimum E_{1k} and the second is the one that gives the minimum E_{2k} over all k vectors. If several candidate intervals are chosen, the calculations described above are done for each interval. Best solutions obtained for each interval are used as the most accurate parameter estimates of fuzzy linear regression models.

3.2. New error measures in fuzzy linear regression with Monte Carlo method

In the literature, E_1 and E_2 (given in Section 3.1) are the only error measures that are used for obtaining best parameter estimates of fuzzy linear regression models with the MC method. Moreover, there is not any available approach proposed for the comparison of the values of E_1 and E_2 with each other in order to choose a unique best solution for the fuzzy linear regression models.

The main contribution of this study is the application of new error measures in fuzzy linear regression with the MC method. Within this context, mean square error (MSE_e), mean percentage error (MPE_e), mean absolute percentage error ($MAPE_e$) and symmetric mean absolute percentage error ($SMAPE_e$) are all proposed as the error measures. Furthermore, estimation accuracy of the error measures used by Abdalla and Buckley [19,21] and the proposed error measures are evaluated and compared under a simulation study. In so doing, the aim is to determine the best error measure in order to achieve the best parameter estimate for the fuzzy linear regression models with the MC method.

The error measures proposed for fuzzy linear regression with Monte Carlo method are explained as follows:

- MSE_e describes the average squared difference between estimated and the corresponding true value and is given as

$$MSE_e = \frac{1}{n}\sum_{i=1}^{n}[(y_{l1} - y_{lk1})^2 + (y_{l2} - y_{lk2})^2 + (y_{l3} - y_{lk3})^2]. \quad (9)$$

- MPE_e is computed as average of percentage errors between predicted and the corresponding true value and is given as

$$MPE_e = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{y_{lk1} - y_{l1}}{y_{l1}} + \frac{y_{lk2} - y_{l2}}{y_{l2}} + \frac{y_{lk3} - y_{l3}}{y_{l3}}\right]. \quad (10)$$

- $MAPE_e$ expresses the accuracy as a percentage and is defined by

$$MAPE_e = \frac{100}{n}\sum_{i=1}^{n}\left[|\frac{y_{lk1} - y_{l1}}{y_{l1}}| + |\frac{y_{lk2} - y_{l2}}{y_{l2}}| + |\frac{y_{lk3} - y_{l3}}{y_{l3}}|\right]. \quad (11)$$

- $SMAPE_e$ is an accuracy measure based on percentage errors and is given as

$$SMAPE_e = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{|y_{lk1} - y_{l1}|}{(y_{lk1} - y_{l1})/2} + \frac{|y_{lk2} - y_{l2}|}{(y_{lk2} - y_{l2})/2} + \frac{|y_{lk3} - y_{l3}|}{(y_{lk3} - y_{l3})/2}\right]. \quad (12)$$

A simulation study is conducted by taking into account of the error measures E_1 and E_2, and those given from Eqs. (9)–(12).

4. The simulation study

In this section, estimation accuracy of the error measures mentioned in Sections 3.1 and 3.2 is evaluated within the simulation study. In our simulations, generation of random data for dependent and independent variables for Case-II and Case-III is described in Sections 4.1 and 4.2 respectively.

Simulation scenarios depend on the intervals, which are used for generating candidate solution vectors in the MC method given in Table 1. Both sides of the real line and interval widths are considered in the determination of these intervals.

In Table 1, I_0 is a short interval excluding negative numbers, I_1 is a short interval including negative numbers, I_2 is a long interval excluding negative numbers, I_3 is a long interval including both negative and positive numbers, I_4 is a short interval excluding positive numbers, and I_5 is a long interval including only negative numbers.

The intervals I_0 to I_5 are used for estimating regression parameters $\overline{A}_0, \overline{A}_1, \overline{A}_2$, and $\overline{A}_3$ for Case-II. Then, the same intervals are used for estimating regression parameters a_0, a_1, and a_2 for Case-III. In each scenario, 10^5 candidate solution vectors are generated.

Table 1
Intervals for Case-II and Case-III.

	I_0	I_1	I_2	I_3	I_4	I_5
$\bar{A}_0$ (or a_0)	[0,3]	[−2,1]	[2,15]	[−12,15]	[−3,−2]	[−22,−4.2]
$\bar{A}_1$ (or a_1)	[0,2]	[−1,1]	[10,22]	[−3,27]	[−1.756,0]	[−28,−3.5]
$\bar{A}_2$ (or a_2)	[3,4.5]	[−2.5,1.5]	[4,30]	[−45,18]	[−4.8,−3.75	[−18,−1]
$\bar{A}_3$	[1.2,2.4]	[−1.2,1.4]	[17,35]	[−24,28]	[−1.02,0]	[−27,−7]

Parameter estimates of fuzzy linear regression models are obtained by using the MC method for both Case-II and Case-III for each interval given in Table 1 for 10^3 times.

We need to have an approach for comparing the estimation accuracy of used error measures. Although error measures acquired from different approaches are not comparable, deviations between estimated and observed values of dependent variable obtained by using different error measures are comparable. In both Case-II and Case-III, dependent variable is a fuzzy number. The value of a deviation measure can be calculated as a fuzzy or crisp number. If it is calculated as a fuzzy number, there are several approaches for comparing sizes of fuzzy numbers. However, we can separately compare left, center, and right values of estimated and observed values of dependent variable, calculate deviation for each of them, and obtain average deviation of left, center, and right values. This is a hybrid of fuzzyness and crispness. On this point, mean absolute error (MAE_c) and mean square error (MSE_c) measures are used as a measure of the deviation. These measures will be called as "comparison measures ($_c$)" hereafter and are given in Eqs. (13) and (14):

$$MAE_c = \frac{1}{3}\sum_{j=1}^{3}|y_{lj} - y_{lkj}|, \tag{13}$$

$$MSE_c = \frac{1}{3}\sum_{j=1}^{3}(y_{lj} - y_{lkj})^2. \tag{14}$$

4.1. Simulation study for Case-II

In each of 10^3 replications, data sets of size 12 are randomly generated from Normal (0, 4) distribution for the first (x_1), second (x_2), and third (x_3) independent variables. Fuzzy numbers for the values of each parameter are randomly generated from normal distribution with mean 1 and standard deviation 0.02 for $\bar{A}_0$, from normal distribution with mean 2 and standard deviation 0.02 for $\bar{A}_1$, from normal distribution with mean −1 and standard deviation 0.04 for $\bar{A}_2$, and from normal distribution with mean −2 and standard deviation 0.1 for $\bar{A}_3$. The corresponding values of dependent variable are obtained under the model given in Eq. (3).

Each scenario given in Table 1 is applied for the simulation study, 10^5 vectors are generated, and the described MC method is applied in order to obtain estimates of parameters of the fuzzy linear regression model. Based on the minimum errors, the accuracy between estimated and true values of regression parameters is evaluated by using both MAE_c and MSE_c.

Tables 2 and 3 present simulation results of Case-II for the error measure MAE_c.

Based on Tables 2 and 3, the worst and the best cases for each interval are presented below:

- The same error values for the left, center, and right sides of each fuzzy parameter are obtained by considering error measures E_1 and E_2 for each interval.
- According to interval I_0, minimum error values are given by $MAPE_e$, $SMAPE_e$, and MSE_e for $\bar{A}_0$ and $\bar{A}_2$, $\bar{A}_1$, and $\bar{A}_3$, respectively. On the other hand, maximum error values are observed with MPE_e for each $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_1, the minimum error value is observed with MSE_e for each $\bar{A}_i$, i = 0, 1, 2, 3. Maximum error values are obtained with MPE_e for each $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_2, the minimum error value is produced by MSE_e for each $\bar{A}_i$, i = 0, 2, 3 and by $MAPE_e$ for $\bar{A}_1$. However, maximum error values are obtained with $SMAPE_e$ for $\bar{A}_0$, $\bar{A}_1$ and with MPE_e for $\bar{A}_2$ and $\bar{A}_3$.
- According to interval I_3, the minimum error value is observed with $SMAPE_e$ for $\bar{A}_0$ and with E_1 and E_2 for $\bar{A}_1$, and with $SMAPE_e$ and MSE_e for $\bar{A}_2$ and $\bar{A}_3$, respectively. On the other hand, maximum error values are observed with MPE_e for each $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_4, the minimum error value is obtained with MSE_e for each $\bar{A}_i$, i = 0, 1, 2 and with $SMAPE$ for $\bar{A}_3$. However, maximum error values are obtained with MPE_e for each $\bar{A}_i$, i = 0, 1, 3 and with $SMAPE_e$ for $\bar{A}_2$.
- According to interval I_5, minimum error values are observed with E_1 and E_2 for $\bar{A}_0$, with $MAPE_e$ for $\bar{A}_1$, and with MSE_e for $\bar{A}_2$ and $\bar{A}_3$. Maximum error values are obtained with MPE_e for $\bar{A}_0$, $\bar{A}_1$ and with $SMAPE_e$ observed for $\bar{A}_2$ and $\bar{A}_3$.

Simulation results of Case-II for the error measure MSE_c are given in Tables 4 and 5.

Based on Tables 4 and 5, the worst and the best cases for each interval are presented below:

- The same error values are obtained for the left, center, and right sides of each fuzzy parameter with considering error measures E_1 and E_2 for each interval.
- According to interval I_0, the minimum error value is obtained with $MAPE_e$ for $\bar{A}_0$ and $\bar{A}_2$, with $SMAPE_e$ and MSE_e for $\bar{A}_1$ and $\bar{A}_3$, respectively. On the other hand, maximum error values are obtained by MPE_e for $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_1, the minimum error value is observed with MSE_e for $\bar{A}_i$, i = 0, 2, 3 and with E_1 and E_2 for $\bar{A}_1$. However, maximum error values are observed by MPE_e for each $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_2, the minimum error value is seen with MSE_e for $\bar{A}_i$, i = 0, 2, 3 and with $MAPE_e$ for $\bar{A}_1$. On the other hand, maximum error values are obtained by $SMAPE_e$ for $\bar{A}_0$ and $\bar{A}_1$ and by MPE_e for $\bar{A}_2$ and $\bar{A}_3$.
- According to interval I_3, the minimum error value is observed with $SMAPE$ for $\bar{A}_0$, with E_1 and E_2 for $\bar{A}_1$ and $\bar{A}_2$, respectively, and with MSE_e for $\bar{A}_3$. Maximum values are obtained by MPE_e for each $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_4, the minimum error value is seen with MSE_e for $\bar{A}_0$, $\bar{A}_1$, and $\bar{A}_2$, with $SMAPE_e$ for $\bar{A}_3$. However, maximum values are obtained by MPE_e for each $\bar{A}_i$, i = 0, 1, 2, 3.
- According to interval I_5, the minimum error value is obtained with E_1 and E_2 for $\bar{A}_0$ and $\bar{A}_1$, respectively, and with MSE_e for $\bar{A}_2$ and $\bar{A}_3$. However, maximum values are obtained with MPE_e for $\bar{A}_0$, $\bar{A}_1$ and with $SMAPE_e$ for $\bar{A}_2$ and $\bar{A}_3$.

Table 2
Simulation results of Case-II for MAE_c.

Error	Coef.	I_0			I_1			I_2		
E_1	$\overline{A}_0$	0.851	0.934	1.036	1.348	0.946	0.776	4.286	5.689	8.359
	$\overline{A}_1$	1.039	0.818	0.631	1.332	1.229	1.138	9.278	10.018	11.103
	$\overline{A}_2$	4.120	4.161	4.278	0.634	0.499	0.536	7.059	7.908	10.618
	$\overline{A}_3$	3.369	3.393	3.482	0.926	0.924	1.073	19.681	20.285	22.491
Error	Coef.	I_3			I_4			I_5		
E_1	$\overline{A}_0$	5.756	5.756	6.038	3.386	3.301	3.105	**15.581**	**8.988**	**7.693**
	$\overline{A}_1$	**2.131**	**1.671**	**4.034**	2.336	2.181	2.094	10.873	8.669	6.406
	$\overline{A}_2$	2.934	3.140	4.187	2.951	2.864	2.829	8.210	4.038	1.898
	$\overline{A}_3$	4.084	1.419	3.713	1.154	1.342	1.443	10.417	7.689	5.987
Error	Coef.	I_0			I_1			I_2		
E_2	$\overline{A}_0$	0.851	0.934	1.036	1.348	0.946	0.776	4.286	5.689	8.359
	$\overline{A}_1$	1.039	0.818	0.631	1.332	1.229	1.138	9.278	10.018	11.103
	$\overline{A}_2$	4.120	4.161	4.278	0.634	0.499	0.536	7.059	7.908	10.618
	$\overline{A}_3$	3.369	3.393	3.482	0.926	0.924	1.073	19.681	20.284	22.491
Error	Coef.	I_3			I_4			I_5		
E_2	$\overline{A}_0$	5.757	5.756	6.038	3.386	3.301	3.105	**15.581**	**8.988**	**7.693**
	$\overline{A}_1$	**2.131**	**1.671**	**4.034**	2.336	2.181	2.094	10.873	8.669	6.406
	$\overline{A}_2$	2.934	3.140	4.187	2.951	2.864	2.829	8.210	4.038	1.898
	$\overline{A}_3$	4.084	1.419	3.713	1.154	1.342	1.443	10.417	7.689	5.987
Error	Coef.	I_0			I_1			I_2		
MSE_e	$\overline{A}_0$	0.803	0.898	1.042	**1.306**	**0.939**	**0.779**	**4.050**	**5.171**	**7.726**
	$\overline{A}_1$	1.043	0.833	0.590	**1.321**	**1.233**	**1.142**	9.264	9.862	10.792
	$\overline{A}_2$	4.118	4.154	4.262	**0.619**	**0.485**	**0.473**	**6.773**	**7.508**	**10.162**
	$\overline{A}_3$	**3.364**	**3.385**	**3.454**	**0.911**	**0.900**	**1.042**	**19.669**	**20.267**	**22.528**
Error	Coef.	I_3			I_4			I_5		
MSE_e	$\overline{A}_0$	5.353	6.379	7.052	**3.359**	**3.296**	**3.086**	14.871	9.322	7.928
	$\overline{A}_1$	2.741	2.217	3.063	**2.300**	**2.142**	**2.070**	10.740	9.234	6.755
	$\overline{A}_2$	3.030	3.549	4.046	**2.929**	**2.851**	**2.824**	**7.322**	**3.856**	**1.964**
	$\overline{A}_3$	**3.070**	**1.311**	**3.828**	1.138	1.352	1.459	**10.062**	**7.531**	**6.122**

4.2. Simulation study for Case-III

In each of 10^3 replications, 10 triangular fuzzy numbers that have normal distribution with mean 0 and standard deviation 2 are randomly generated for each independent variable ($\overline{X}_1$, $\overline{X}_2$). Also three crisp numbers for each parameter are randomly generated from normal distribution with mean 1 and standard deviation 0.02 for a_0, from normal distribution with mean 2 and standard deviation 0.02 for a_1, and from normal distribution with mean -1 and standard deviation 0.04 for a_2. The corresponding values of dependent variable are obtained under the model given in Eq. (4).

Each scenario given in Table 1 is applied for the simulation study, 10^5 vectors are generated, and the described MC method is applied in order to obtain estimates of parameters in the fuzzy linear regression model. Afterwards, taking minimum errors into consideration, the accuracy between estimated and true values of regression parameters is evaluated by using MAE_c and MSE_c. Since our parameters are crisp numbers, left, center, and right values are taken into account as a_i for $i = 0, 1, 2$.

Table 6 gives simulation results of Case-III for the error measure MAE_c.

The following inferences are drawn from Table 6:

- The same error values are observed for each parameter by considering error measures E_1 and E_2 for each interval.
- According to interval I_0, the minimum error value is generated by MSE_e for a_0 and by E_1 and E_2 for a_1 and a_2, respectively. On the other hand, maximum error values are obtained with MPE_e for each a_i 0, 1, 2.
- According to interval I_1, the minimum error value is obtained with MSE_e for a_0 and with $MAPE_e$ for a_1 and a_2. However, maximum error values are obtained with MPE_e for each a_i, $i = 0, 1, 2$.
- According to interval I_2, the minimum error value is produced by MSE_e for a_0 and a_1, and by $MAPE_e$ for a_2. Maximum error values are obtained with MPE_e for a_0 and a_2 and with $SMAPE_e$ for a_1.
- According to interval I_3, the minimum value is obtained with MSE_e for a_0, with E_1 and E_2 for a_1, and with $MAPE_e$ for a_2. Maximum error values are obtained with MPE_e for each a_i, $i = 0, 1, 2$.
- According to interval I_4, the minimum error value is given by MSE_e for each a_i, $i = 0, 1, 2$ and maximum error values are obtained with MPE_e for a_0 and a_1 and with $SMAPE_e$ for a_2.
- According to interval I_5, the minimum error value is obtained with MSE_e for a_0, with E_1 and E_2 for a_1, and with MSE_e for a_2. Maximum error values are obtained with MPE_e for a_0 and a_1 and with $SMAPE_e$ for a_2.

Table 7 presents simulation results of Case-III for the error measure MSE_c.

The following inferences are drawn from Table 7:

Table 3
Simulation results of Case-II for MAE_c (cont.).

Error	Coef.	I_0			I_1			I_2		
MPE_e	$\overline{A}_0$	**0.922**	**1.089**	**1.160**	**2.146**	**1.556**	**1.032**	4.465	7.279	9.744
	$\overline{A}_1$	**1.648**	**1.379**	**1.021**	**2.635**	**2.327**	**1.903**	10.124	12.078	14.680
	$\overline{A}_2$	**4.631**	**4.853**	**5.044**	**1.491**	**1.571**	**1.702**	**16.537**	**20.074**	**23.419**
	$\overline{A}_3$	**3.853**	**4.035**	**4.120**	**2.217**	**2.599**	**2.848**	**28.592**	**31.570**	**33.516**
Error	Coef.	I_3			I_4			I_5		
MPE_e	$\overline{A}_0$	**7.744**	**7.270**	**8.515**	**3.708**	**3.517**	**3.360**	**18.090**	**14.424**	**11.295**
	$\overline{A}_1$	**5.247**	**7.617**	**12.219**	**3.449**	**3.238**	**2.949**	**25.943**	**22.565**	**18.028**
	$\overline{A}_2$	**25.307**	**21.951**	**19.343**	3.346	3.197	3.078	10.216	7.264	4.817
	$\overline{A}_3$	**19.449**	**21.508**	**23.517**	**1.572**	**1.700**	**1.743**	14.469	10.900	8.524
Error	Coef.	I_0			I_1			I_2		
$MAPE_e$	$\overline{A}_0$	**0.788**	**0.856**	**0.981**	1.308	0.961	0.648	4.428	5.870	8.551
	$\overline{A}_1$	1.561	1.350	1.145	1.623	1.361	1.223	**9.063**	**9.800**	**11.240**
	$\overline{A}_2$	**4.106**	**4.138**	**4.264**	0.498	0.592	0.817	8.957	10.375	13.032
	$\overline{A}_3$	3.484	3.523	3.629	1.143	1.231	1.393	19.808	20.655	23.022
Error	Coef.	I_3			I_4			I_5		
$MAPE_e$	$\overline{A}_0$	5.902	5.815	6.503	3.454	3.333	3.202	15.270	10.213	8.363
	$\overline{A}_1$	2.646	2.579	4.541	2.582	2.406	2.228	**11.503**	**8.578**	**6.475**
	$\overline{A}_2$	4.493	3.783	4.359	2.952	2.857	2.821	8.870	4.749	2.360
	$\overline{A}_3$	5.018	2.389	4.174	1.226	1.404	1.516	11.336	7.915	6.310
Error	Coef.	I_0			I_1			I_2		
$SMAPE_e$	$\overline{A}_0$	0.913	1.058	1.339	1.419	0.939	0.631	**5.469**	**7.663**	**9.895**
	$\overline{A}_1$	**0.694**	**0.418**	**0.222**	1.568	1.302	1.175	**12.620**	**14.507**	**16.509**
	$\overline{A}_2$	4.517	4.754	5.067	0.617	0.566	0.784	14.614	17.740	21.796
	$\overline{A}_3$	3.506	3.622	3.839	1.064	1.163	1.304	25.040	28.222	31.373
Error	Coef.	I_3			I_4			I_5		
$SMAPE_e$	$\overline{A}_0$	**6.311**	**3.597**	**7.250**	3.643	3.462	3.293	16.571	12.775	9.781
	$\overline{A}_1$	2.538	4.116	11.868	2.854	2.514	2.297	17.077	12.487	9.821
	$\overline{A}_2$	**3.031**	**3.094**	**5.363**	**3.514**	**3.314**	**3.138**	**10.999**	**8.009**	**5.561**
	$\overline{A}_3$	10.425	3.871	5.468	**1.255**	**1.308**	**1.408**	**18.952**	**14.890**	**11.485**

- The same error values are obtained for each parameter by considering error measures E_1 and E_2 for each interval.
- According to interval I_0, minimum error values are generated by MSE_e, $SMAPE_e$, and MSE_e for a_0, a_1, and a_2, respectively. On the other hand, maximum error values are obtained with MPE_e for each a_0, a_1, and a_2.
- According to interval I_1, minimum error values are obtained with MSE_e, $SMAPE_e$, and MSE_e for a_0, a_1, and a_2, respectively. However, maximum error values are obtained with MPE_e for each a_i, $i = 0, 1, 2$.
- According to interval I_2, the minimum error value is given by MSE_e for a_0 and a_1, and by E_1 and E_2 for a_2. Maximum error values are obtained with MPE_e for a_0 and a_2 and with $SMAPE_e$ for a_1.
- According to interval I_3, the minimum error value is obtained with MSE_e for a_0 and a_2 and with E_1 and E_2 for a_1. Maximum error values are obtained with MPE_e for each a_i, $i = 0, 1, 2$.
- According to interval I_4, the minimum error value is observed with MSE_e for each a_i, $i = 0, 1, 2$. Maximum error values are obtained with MPE_e for a_0 and a_1 and with $SMAPE_e$ for a_2.
- According to interval I_5, the minimum error value is obtained with MSE for each a_i, $i = 0, 1, 2$, and maximum error values are obtained with MPE_e for a_0 and a_1 and with $SMAPE_e$ for a_2.

Considering the simulation results, minimum error values are produced by the error measures E_1, E_2, and MSE_e and nearly all maximum error values are observed with the error measure MPE_e for both MAE_c and MSE_c in both Case-II and Case-III. Hence, it is possible to conclude that the best error measures to estimate fuzy/crisp parameters of fuzzy linear regression models are not only E_1 and E_2 but also usage of MSE_e gives precise results. Furthermore, the worst error measure is proved to be MPE_e for estimating the parameters of fuzzy linear regression models with MC method.

5. Application

This section contains two applications of fuzzy linear regression models to Case-II and Case-III. We apply the MC method to estimate regression parameters and calculate error measures E_1, E_2, MSE_e, and MPE_e. The reason behind the selection of these error measures is because the best estimates are obtained with E_1, E_2, and MSE_e, whereas the worst estimates are calculated by MPE_e, as demonstrated in the simulation study.

5.1. Application for the Case-II

The popular data set analysed by several authors including Abdalla and Buckley [21] in the literature is revisited for this application. It contains the cognitive response times of the control room crew in a nuclear power plant to an unusual event as the dependent variable. Three independent variables which are inside control room experience, outside control room experience, and education are shown by x_1, x_2, and x_3, respectively. The dependent and

Table 4
Simulation results of Case-II for MSE_c.

Error	Coef.	I_0			I_1			I_2		
E_1	$\bar{A}_0$	0.915	1.156	1.601	2.041	1.084	0.770	28.992	46.899	93.709
	$\bar{A}_1$	1.540	1.080	0.817	**1.800**	**1.518**	**1.298**	87.442	102.562	128.575
	$\bar{A}_2$	16.988	17.331	18.328	0.492	0.322	0.562	54.759	72.757	129.412
	$\bar{A}_3$	11.365	11.548	12.172	0.871	0.892	1.232	387.422	412.036	507.389
Error	Coef.	I_3			I_4			I_5		
E_1	$\bar{A}_0$	40.414	36.512	44.080	11.505	10.923	9.649	**260.225**	**89.469**	**63.315**
	$\bar{A}_1$	**7.311**	**4.999**	**27.971**	5.536	4.812	4.407	**126.779**	**80.511**	**42.360**
	$\bar{A}_2$	**15.628**	**13.166**	**19.415**	8.740	8.213	8.009	82.772	21.568	6.159
	$\bar{A}_3$	25.974	2.967	18.701	1.350	1.842	2.139	121.613	62.725	36.864
Error	Coef.	I_0			I_1			I_2		
E_2	$\bar{A}_0$	0.915	1.156	1.601	2.041	1.084	0.770	28.992	46.899	93.709
	$\bar{A}_1$	1.540	1.080	0.817	**1.800**	**1.518**	**1.298**	87.442	102.562	128.575
	$\bar{A}_2$	16.988	17.331	18.328	0.492	0.322	0.562	54.759	72.757	129.412
	$\bar{A}_3$	11.365	11.548	12.172	0.871	0.892	1.232	387.421	412.036	507.389
Error	Coef.	I_3			I_4			I_5		
E_2	$\bar{A}_0$	40.414	36.512	44.080	11.505	10.923	9.649	**260.225**	**89.469**	**63.315**
	$\bar{A}_1$	**7.311**	**4.999**	**27.971**	5.536	4.812	4.407	**126.779**	**80.511**	**42.360**
	$\bar{A}_2$	**15.628**	**13.166**	**19.415**	8.740	8.213	8.009	82.772	21.568	6.159
	$\bar{A}_3$	25.974	2.967	18.701	1.350	1.842	2.139	121.613	62.725	36.864
Error	Coef.	I_0			I_1			I_2		
MSE_e	$\bar{A}_0$	0.839	1.067	1.606	**1.864**	**1.007**	**0.733**	**26.043**	**39.383**	**83.920**
	$\bar{A}_1$	1.520	1.086	0.740	1.759	1.523	1.306	86.787	98.699	120.272
	$\bar{A}_2$	16.972	17.268	18.184	**0.454**	**0.296**	**0.457**	**48.139**	**62.361**	**114.398**
	$\bar{A}_3$	**11.330**	**11.490**	**11.975**	**0.839**	**0.841**	**1.159**	**386.928**	**411.306**	**508.555**
Error	Coef.	I_3			I_4			I_5		
MSE_e	$\bar{A}_0$	35.701	43.290	57.721	**11.315**	**10.885**	**9.527**	238.789	98.453	67.920
	$\bar{A}_1$	10.439	7.122	19.404	**5.340**	**4.616**	**4.292**	121.004	91.898	47.290
	$\bar{A}_2$	17.602	15.308	17.534	**8.600**	**8.136**	**7.979**	**64.678**	**19.681**	**5.863**
	$\bar{A}_3$	**17.643**	**2.475**	**21.396**	1.304	1.857	2.175	**111.155**	**59.617**	**38.222**

independent variables are recorded for eight ($n = 8$) teams in the data set (complete data set can be seen in Kim and Bishu [28]).

First, the intervals I_i, $i = 0, 1, 2, 3$ should be obtained to find $\bar{A}_i$ coefficients as explained in Definition 2.6. The solutions for I_i, $i = 0, 1, 2, 3$ are taken from Abdalla and Buckley [21] as the results of our approach should be comparable with the previous approaches. They study under four separate intervals for each $\bar{A}_i$, as outlined in Table 8.

Using the data set in Kim and Bishu [28] with $n = 8$ and $N = 10^5$, $\mathbf{v}_k = (v_{1k}, v_{2k}, \ldots, v_{12k})$ vectors, which define the $\bar{A}_i$, $i = 0, 1, 2, 3$ as described in Section 2, are generated. Table 9 demonstrates the values of $\bar{A}_i$ that give minimum values to E_1, E_2, MSE_e and MPE_e under each I_i setting.

The solutions for the $\bar{A}_i$ are obtained and predicted values are determined. Abdalla and Buckley [21] give optimal solutions for $\bar{A}_i$ as $\bar{A}_0 = (-0.710/-0.539/-0.524)$, $\bar{A}_1 = (-0.610/-0.473/-0.472)$, $\bar{A}_2 = (-1.090/-1.089/-1.088)$, and $\bar{A}_3 = (0.459/0.487/0.68)$. The values of error measures E_1, E_2, MSE_e, and MPE_e are computed by using Eqs. (7)–(10). The results are shown in Table 10, where the results of previous studies are given under the relevant citation and our results are given under "MC" column.

Values of error measures obtained by our algorithm are considerably smaller than the results of Savic and Peryzc [29], Tanaka [30], and Kim and Bishu [28], better than those of Abdalla and Buckley [21] for *MCIII* and *MCVI* for E_1, and nearly same for E_2. Abdalla and Buckley [21] achieved the smallest error value under *MCII*. On the other hand, we get the smallest error value under *MCII* with MSE_e. In addition to the error measures E_1 and E_2, other error measures are also calculated for this application. According to the simulation results, it is obtained that MPE_e cannot be used because it gives negative error values.

5.2. Application for the Case-III

The data set used for this application was also studied by Choi and Buckley [26], Diamond and Körner [31], and Abdalla and Buckley [19]. The data set consists of two independent variables which are shown by $\bar{X}_1$ and $\bar{X}_2$ and 10 observations. The intervals I_i, $i = 0, 1, 2, 3$ should be calculated to obtain a_i for $i = 0, 1, 2$ as explained in Definition 2.5. The solutions for I_i, $i = 0, 1, 2$ are taken from Abdalla and Buckley [19]. They study under four separate intervals for I_i, which are given in Table 11.

Table 12 demonstrates the values of a_i that give minimum values to E_1, E_2, MSE_e, and MPE_e under each I_i setting.

The optimal solutions for a_i are given as $a_0 = 4.19$, $a_1 = 4.97$, and $a_2 = 3.11$ by Abdalla and Buckley [19]. Regarding the obtained estimates according to each error measure and each interval, closest values to the true values are obtained according to *MCI* with E_2.

After the calculation of error measures E_1, E_2, MSE_e, and MPE_e, the obtained results are shown in Table 13, where the results of previous studies are given under the relevant citation.

In Table 13, our results are very close to those obtained by Abdalla and Buckley [19], however our method does not provide

Table 5
Simulation results of Case-II for MSE_c (cont.).

Error	Coef.	I_0			I_1			I_2		
MPE_e	$\overline{A}_0$	**1.027**	**1.492**	**1.851**	**5.332**	**3.480**	**1.975**	30.615	70.288	109.625
	$\overline{A}_1$	**2.966**	**2.343**	**1.529**	**7.143**	**5.769**	**4.015**	110.021	158.699	230.036
	$\overline{A}_2$	**21.699**	**23.793**	**25.660**	**2.517**	**2.987**	**3.576**	**379.315**	**503.296**	**627.930**
	$\overline{A}_3$	**14.996**	**16.414**	**17.057**	**5.753**	**7.460**	**8.572**	**860.502**	**1031.404**	**1147.233**
Error	Coef.	I_3			I_4			I_5		
MPE_e	$\overline{A}_0$	**76.561**	**67.400**	**87.248**	**13.826**	**12.474**	**11.377**	**351.420**	**241.841**	**155.800**
	$\overline{A}_1$	**45.128**	**107.093**	**228.295**	**12.106**	**10.833**	**9.101**	**710.537**	**574.032**	**399.856**
	$\overline{A}_2$	**874.030**	**636.206**	**461.368**	**11.326**	**10.343**	**9.577**	138.653	85.045	50.251
	$\overline{A}_3$	**415.632**	**517.890**	**618.216**	**2.601**	**2.993**	**3.105**	259.960	159.906	98.891
Error	Coef.	I_0			I_1			I_2		
$MAPE_e$	$\overline{A}_0$	**0.757**	**0.961**	**1.403**	2.009	1.158	0.581	29.651	49.680	95.049
	$\overline{A}_1$	2.658	2.113	1.682	2.756	1.905	1.529	**82.906**	**97.548**	**131.604**
	$\overline{A}_2$	**16.869**	**17.139**	**18.231**	0.334	0.451	0.872	106.546	149.771	220.593
	$\overline{A}_3$	12.208	12.525	13.284	1.373	1.616	2.126	392.967	429.015	534.298
Error	Coef.	I_3			I_4			I_5		
$MAPE_e$	$\overline{A}_0$	43.918	39.397	51.861	12.010	11.174	10.293	252.837	123.990	80.857
	$\overline{A}_1$	11.279	13.875	40.021	6.941	6.026	5.088	145.541	79.114	43.186
	$\overline{A}_2$	36.686	19.687	25.012	8.744	8.177	7.963	100.968	35.562	14.196
	$\overline{A}_3$	43.116	12.747	29.355	1.547	2.043	2.375	147.821	69.402	41.943
Error	Coef.	I_0			I_1			I_2		
$SMAPE_e$	$\overline{A}_0$	1.044	1.479	2.175	2.422	1.157	0.557	**42.615**	**73.014**	**109.601**
	$\overline{A}_1$	**0.775**	**0.359**	**0.147**	2.604	1.727	1.396	**171.104**	**223.411**	**283.245**
	$\overline{A}_2$	20.600	22.808	25.839	0.534	0.444	0.892	269.422	379.326	532.499
	$\overline{A}_3$	12.370	13.236	14.881	1.181	1.438	1.831	652.037	827.301	1010.590
Error	Coef.	I_3			I_4			I_5		
$SMAPE_e$	$\overline{A}_0$	**52.589**	**20.186**	**67.238**	13.351	12.077	10.924	297.198	188.286	113.476
	$\overline{A}_1$	10.761	28.492	178.353	8.522	6.599	5.445	346.732	199.206	124.402
	$\overline{A}_2$	286.051	17.092	52.909	12.421	11.076	9.944	**143.533**	**87.925**	**50.159**
	$\overline{A}_3$	136.576	24.055	66.821	**1.642**	**1.809**	**2.096**	**386.544**	**251.038**	**156.263**

Table 6
Simulation results of Case-III for MAE_c.

Error	Coef.	I_0	I_1	I_2	I_3	I_4	I_5
$E1$	a_0	0.883	0.812	3.162	0.719	1.175	5.533
	a_1	**1.098**	1.252	8.144	**0.501**	2.019	**5.657**
	a_2	**4.003**	1.387	5.126	0.825	2.757	0.483
$E2$	a_0	0.883	0.812	3.162	0.719	1.175	5.533
	a_1	**1.098**	1.252	8.144	**0.501**	2.019	**5.657**
	a_2	**4.003**	1.387	5.126	0.825	2.757	0.483
MSE_e	a_0	**0.841**	**0.774**	**2.980**	**0.689**	**1.037**	**5.532**
	a_1	1.139	1.229	**8.137**	0.508	**2.015**	5.658
	a_2	4.002	1.420	5.136	0.872	**2.756**	**0.451**
MPE_e	a_0	**1.478**	**1.429**	**7.903**	**13.036**	**2.372**	**13.708**
	a_1	**1.795**	**2.276**	9.255	**8.530**	**2.657**	**14.616**
	a_2	**4.516**	**1.601**	**13.837**	**17.634**	2.973	3.509
$MAPE_e$	a_0	1.082	0.949	3.764	0.978	1.609	5.815
	a_1	1.609	**1.130**	8.397	0.531	2.056	5.707
	a_2	4.006	**0.823**	**5.117**	**0.692**	2.764	0.593
$SMAPE_e$	a_0	1.017	0.926	5.447	0.938	1.680	11.497
	a_1	0.145	1.263	**9.759**	0.865	2.453	9.805
	a_2	4.315	1.172	6.162	0.915	**3.008**	**9.135**

Table 7
Simulation results of Case-III for MSE_c.

Error	Coef.	I_0	I_1	I_2	I_3	I_4	I_5
	a_0	1.002	1.005	13.280	59.308	2.292	30.557
$E1$	a_1	1.503	1.506	66.337	**26.324**	4.079	32.006
	a_2	16.024	16.025	**26.286**	292.011	7.601	0.354
	a_0	1.002	1.005	13.280	59.308	2.292	30.557
$E2$	a_1	1.503	1.506	66.337	**26.324**	4.079	32.006
	a_2	16.024	16.025	**26.286**	292.011	7.601	0.354
	a_0	**0.889**	**0.891**	**10.950**	**43.481**	**1.806**	**30.531**
MSE_e	a_1	1.513	1.517	**66.221**	26.414	**4.060**	**32.014**
	a_2	**16.017**	**16.018**	26.386	**291.914**	**7.596**	**0.276**
	a_0	**2.446**	**2.449**	**100.537**	**170.782**	**7.879**	**259.572**
MPE_e	a_1	**3.530**	**3.530**	96.317	**318.457**	**7.729**	**346.371**
	a_2	**20.881**	**20.882**	**338.231**	**615.259**	9.001	55.171
	a_0	1.470	1.473	26.526	98.067	4.168	35.910
$MAPE_e$	a_1	2.909	2.913	71.604	28.301	4.265	32.599
	a_2	16.052	16.052	26.332	293.124	7.641	1.978
	a_0	1.389	1.385	45.644	81.992	4.366	170.156
$SMAPE_e$	a_1	**0.118**	**0.118**	**104.065**	104.777	6.314	134.537
	a_2	18.865	18.865	43.149	411.546	**9.203**	**111.423**

Table 8
The intervals for I_i, $i = 0, 1, 2, 3$ for Case-II.

	MCI	*MCII*	*MCIII*	*MCIV*
I_0	[−1,0]	[0,1]	[−18.174,−18.174]	[28.000,47.916]
I_1	[−1,0]	[−1,0]	[−1.083,−1.083]	[−2.542,−2.542]
I_2	[−1.5,−0.5]	[−1.5,−0.5]	[−1.500,−1.500]	[−2.333,−2.333]
I_3	[0,1]	[0,1]	[1.733,2.149]	[−1.354,−1.354]

Table 9
Estimates of coefficients under *MCI–MCII–MCIII–MCIV* setting for Case-II.

		$\bar{A}_0$			$\bar{A}_1$			$\bar{A}_2$			$\bar{A}_3$		
	E_1	−0.654	−0.163	−0.139	−0.285	−0.228	−0.133	−0.643	−0.555	−0.543	0.304	0.317	0.321
MCI	E_2	−0.754	−0.548	−0.421	−0.802	−0.786	−0.684	−1.323	−1.265	−1.251	0.548	0.566	0.661
	MSE_e	−0.712	−0.672	−0.611	−0.934	−0.928	−0.887	−1.202	−1.139	−1.035	0.613	0.758	0.801
	MPE_e	−0.938	−0.836	−0.251	−0.950	−0.810	−0.492	−1.426	−1.349	−1.016	0.010	0.042	0.066
	E_1	0.061	0.316	0.341	−0.271	−0.268	−0.129	−0.822	−0.727	−0.721	0.259	0.294	0.336
MCII	E_2	0.767	0.901	0.923	−0.604	−0.430	−0.145	−1.096	−1.083	−1.015	0.355	0.367	0.517
	MSE_e	0.210	0.262	0.937	−0.970	−0.882	−0.771	−1.285	−1.245	−0.998	0.530	0.629	0.686
	MPE_e	0.062	0.164	0.749	−0.950	−0.891	−0.492	−1.426	−1.349	−1.016	0.010	0.042	0.066
	E_1	−18.174	−18.174	−18.174	−1.083	−1.083	−1.083	−1.500	−1.500	−1.500	1.875	1.876	1.879
MCIII	E_2	−18.174	−18.174	−18.174	−1.083	−1.083	−1.083	−1.500	−1.500	−1.500	1.823	1.888	1.960
	MSE_e	−18.174	−18.174	−18.174	−1.083	−1.083	−1.083	−1.500	−1.500	−1.500	1.904	2.015	2.119
	MPE_e	−18.174	−18.174	−18.174	−1.083	−1.083	−1.083	−1.500	−1.500	−1.500	1.736	1.739	1.741
	E_1	30.645	30.645	30.658	−2.542	−2.542	−2.542	−2.333	−2.333	−2.333	−1.354	−1.354	−1.354
MCIV	E_2	31.102	35.335	36.042	−2.542	−2.542	−2.542	−2.333	−2.333	−2.333	−1.354	−1.354	−1.354
	MSE_e	31.013	35.597	36.814	−2.542	−2.542	−2.542	−2.333	−2.333	−2.333	−1.354	−1.354	−1.354
	MPE_e	28.168	28.168	28.664	−2.542	−2.542	−2.542	−2.333	−2.333	−2.333	−1.354	−1.354	−1.354

Table 10
Comparison of error measures in the application (Case-II).

Error	[29]	[30]	[28]	*MCI*		*MCII*		*MCIII*		*MCIV*	
				[21]	MC	[21]	MC	[21]	MC	[21]	MC
E_1	53.82	48.79	16.98	6.17	9.00	5.81	9.49	7.13	6.83	8.20	7.34
E_2	143.45	131.83	70.99	64.89	63.26	63.59	64.06	66.46	66.42	94.09	94.26
MSE_e	NA	NA	NA	NA	27.18	NA	1.78	NA	26.23	NA	41.03
MPE_e	NA	NA	NA	NA	-3.30	NA	-2.88	NA	-0.81	NA	-1.70

Table 11
The intervals for I_i, $i = 0, 1, 2$ for Case-III.

	MCI	*MCII*	*MCIII*	*MCIV*
I_0	[0,5]	[0,37]	[16.528,16.528]	[33.808,36.601]
I_1	[0,6]	[0,6]	[3.558,3.982]	[1.294,3.765]
I_2	[0,4]	[0,6]	[2.575,2.575]	[0.473,0.473]

Table 12
Estimates of coefficients under *MCI–MCII–MCIII–MCIV* setting for Case-III.

		a_0	a_1	a_2
MCI	E_1	2.657	0.013	0.006
	E_2	4.849	4.882	3.198
	MSE_e	4.919	4.642	3.544
	MPE_e	0.379	0.027	0.055
MCII	E_1	19.661	0.013	0.010
	E_2	9.970	4.458	2.850
	MSE_e	14.540	4.009	2.699
	MPE_e	0.312	0.051	0.200
MCIII	E_1	16.528	3.558	2.575
	E_2	16.528	3.807	2.575
	MSE_e	16.528	3.809	2.575
	MPE_e	16.528	3.558	2.575
MCIV	E_1	36.519	1.295	0.473
	E_2	33.822	3.294	0.473
	MSE_e	33.810	3.053	0.473
	MPE_e	33.835	1.296	0.473

Table 13
Comparison of error measures in the application (Case-III).

Error	[31]	[26]	*MCI*		*MCII*		*MCIII*		*MCIV*	
			[19]	MC	[19]	MC	[19]	MC	[19]	MC
E_1	13.58	11.11	10.02	10.03	9.39	10.03	12.73	15.90	9.59	11.75
E_2	141.63	137.85	133.11	130.16	133.12	129.71	146.53	137.83	170.12	161.07
MSE_e	NA	NA	NA	76.72	NA	72.15	NA	72.72	NA	98.68
MPE_e	NA	NA	NA	−0.99	NA	−0.97	NA	−0.02	NA	−0.20

the smallest error measures for each interval for E_1. Our algorithm gives the smallest E_2 value under each interval whereas algorithm of Abdalla and Buckley [19] produces the smallest E_1 value under each interval.

6. Conclusions

6.1. General findings

In this study, we use different error measures to find parameter estimates of fuzzy linear regression models with MC method. Regression models are applied to two cases. In the first one, input data is crisp and output data is fuzzy (Case-II) and in the second case, input and output data are both fuzzy (Case-III). We utilize a more general definition of absolute value of a triangular shaped fuzzy number given by AbuAraqub et al. [22] in our MC method.

A simulation study is conducted to compare estimation performances of the selected error measures. It is demonstrated that only two error measures (E_1 and E_2) are not sufficient to estimate parameters of fuzzy linear regression models. Additionally, parameter estimates that make each error measure minimum are calculated. The differences between the true and estimated values of the parameters are evaluated by using MAE_c and MSE_c. These error measures are preferred because of the fact that MSE_c at least has two advantages over other distance measures: First, it is analytically tractable and second, it has an interpretation. Besides, MAE_c is a reasonable alternative of MSE_c. Also, the parameters are estimated by considering five different intervals that possibly include their true values. Each interval (I_i, $i = 0, 1, 2, 3, 4$) is arbitrarily determined according to its length and status of involving any negative number.

The results of simulation study are generalized as follows. Considering Case-II, minimum error values are observed in E_1, E_2, and MSE_e. On the other hand, maximum errors are often produced by MPE_e regarding both MAE_c and MSE_c. Considering Case-III, minimum error values are obtained by E_1, E_2, and MSE_e. However, maximum errors are often observed by MPE_e in terms of both MAE_c and MSE_c. As a result, it is possible to conclude that best error measures to estimate fuzzy/crisp parameters of fuzzy linear regression models are not only E_1 and E_2 but also MSE_e. Furthermore, the worst error measure is proved to be MPE_e for estimating the parameters of fuzzy linear regression models.

After obtaining simulation results, the performance of proposed MC method is compared with other MC studies from the literature according to previously studied data sets. In the application for Case-II, the smallest error measure is obtained with E_1 in the previous studies [28–30] and Buckley and Abdalla's study [21] under *MCIII* and *MCIV* settings. Furthermore, the smallest error measure is observed with E_2 in the previous studies [28–30] and it is very close to the values of the previous study of Buckley and Abdalla [21] under each interval. From the overall results of Case-II, the minimum error measure is obtained by MPE_e. However, it can be seen that this error measure is unsuitable to be used for the parameter estimation of fuzzy linear regression model with the simulation study. The maximum error measure is observed with E_2 under *MCIV* setting. In the application for Case-III, the smallest error measure is observed with E_2 in the previous studies [31,26,19] for each interval. The smallest error measure is seen with E_1 in the previous studies [31,26] and it is very close to the values of [19] for each interval. When we look at the overall results of Case-III, the minimum error measure is obtained with MPE_e which is unsuitable to use for fuzzy linear regression model and maximum error measure is observed with E_2 under *MCIV* setting. Finally, it is concluded that the best error measures for fuzzy linear regression with MC method are E_1, E_2, and MSE_e. Nevertheless, MPE_e is the worst one for estimating the fuzzy/crisp parameters of fuzzy linear regression models by MC method.

6.2. Future extension

Obtained results can and will be used to enrich the studies that have already focused on fuzzy linear regression models. For example, investigation on the spreads of estimated fuzzy dependent variables in fuzzy linear regression model [32] can be enhanced by the help of fuzzy linear regression with MC methods according to the best error measures determined in this study. Moreover, Multi-Level fuzzy linear regression model parameters studied by Kazemi et al. [33] can be estimated with the MC method using the error measure *MSE*. Taking into consideration of the best error measures determined in this study, fuzzy linear regression model with MC method can facilitate the fuzzy applications in renewable energy systems [34] such as solar, wind, or space heating due to its ease of applicability. Benchmarking process with fuzzy linear regression model [35] can be regenerated with MC method based on the suitable error measures determined in this study. Outlier detection in the view of linguistic variable [36] is also a potential study area of fuzzy linear regression with Monte Carlo method within the framework of the error measures determined in this study.

It is observed that common error measures such as MAE_c and MSE_c can be successfully used to assess accuracy of parameter estimates of fuzzy regression models. Consequently, another future study can and will focus on the problem that occurs in the assessment of the quality of the fuzzy regression models by using these error measures.

Considering other possible ways to get the absolute value of the triangular fuzzy number, it is possible to apply different methods in MC method in fuzzy linear regression analysis. Therefore, future research may be conducted to apply different definitions of absolute value of triangular fuzzy numbers. Extension of the proposed method for different types of fuzzy regression models, such as non-parametric fuzzy regression or fuzzy nonlinear regression, is, also, a potential area for the future works.

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