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Deriving the priority weights from incomplete hesitant fuzzy preference relations based on multiplicative consistency

Zhiming Zhang

College of Mathematics and Information Science, Hebei University, Baoding, Hebei 071002, China

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ABSTRACT

In this paper, we define the concept of incomplete hesitant fuzzy preference relations to deal with the cases where the decision makers express their judgments by using hesitant fuzzy preference relations with incomplete information, and investigate the consistency of the incomplete hesitant fuzzy preference relations and obtain the reliable priority weights. We first establish a goal programming model for deriving the priority weights from incomplete hesitant fuzzy preference relations based the α -normalization. Then, we give the definition of multiplicative consistent incomplete hesitant fuzzy preference relations based on the β-normalization, and develop a method for complementing the acceptable incomplete hesitant fuzzy preference relations by using the multiplicative consistency property. Furthermore, utilizing a convex combination method, a new algorithm for obtaining the priority weights from complete or incomplete hesitant fuzzy preference relations is presented on the basis of the β-normalization. Finally, several numerical examples are provided to illustrate the validity and practicality of the proposed methods.

1. Introduction

As an extension of Zadeh's fuzzy sets [46], Torra [26] introduced the concept of hesitant fuzzy sets (HFSs) to enhance the modeling abilities of Zadeh's fuzzy sets. Although the memberships of the elements in a HFS could be any subset of [0, 1], practical works dealing with hesitant fuzzy sets frequently restrict to finite sets [30,43]. To address this issue, Bedregal et al. [8] introduced the notion of typical hesitant fuzzy sets (THFSs). The core of a typical hesitant fuzzy set is typical hesitant fuzzy element (THFE) [8,9], which is composed of several possible values for the membership. THFEs are a very useful tool to express a decision maker (DM)'s hesitancy in providing the preference information over objects in the process of decision making. For example, suppose that a group of decision makers (DMs) are hesitant about some possible values as 0.2, 0.3, and 0.4 to assess the membership of an element x to the set A, and the group of DMs cannot persuade one another to change their own opinions. In such cases, the membership of x to A can be modeled by a THFE represented by $h = \left\{0.2, 0.3, 0.4\right\}$, which is significantly different from the situations of using Zadeh's fuzzy sets and its extensions, including interval-valued fuzzy sets [47], intuitionistic fuzzy sets [6], interval-valued intuitionistic fuzzy sets [7], type-2 fuzzy sets [12,44], and fuzzy multisets [20]. Owing to the advantages of handing imprecision whereby two or more sources of vagueness appear simultaneously [50,51].

Decision making is one of the most common activities in the real world. In the process of decision making, an expert (or decision maker) is usually asked to give his/her preferences by comparing the relation of each pair of the considered objects (or alternatives) [39]. Preference relations (or called pairwise comparison matrices, judgment matrices) are very efficient and common tools to express decision makers' preference information on alternatives or criteria [39]. Over the last few decades, a number of studies have focused on the use of preference relations, and various types of preference relations have been developed, including multiplicative preference relation [23], incomplete multiplicative preference relation [14], interval multiplicative preference relation [24], incomplete interval multiplicative preference relation [37], triangular fuzzy multiplicative preference relation [29], incomplete triangular fuzzy multiplicative preference relation [37], fuzzy preference relation [21], incomplete fuzzy preference relation [2], interval fuzzy preference relation [34], incomplete interval fuzzy preference relation [37], triangular fuzzy preference relation [33], incomplete triangular fuzzy preference relation [37], linguistic preference relation [15,16], incomplete linguistic preference relation [35,36], intuitionistic preference relation [38], incomplete intuitionistic preference relation [40], intuitionistic multiplicative preference relation [31], etc. Two important research topics on preference relations are to check their

E-mail addresses: zhimingzhang@ymail.com, zhangzm_1978@126.com

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consistency and to generate weights from them. Porcel and Herrera-Viedma [22] dealt with incomplete information in a fuzzy linguistic recommender system to disseminate information in university digital libraries. Alonso et al. [1] investigated group decision-making with incomplete fuzzy linguistic preference relations. Herrera-Viedma et al. [17] presented a group decision-making model with incomplete fuzzy preference relations based on additive consistency. Alonso et al. [3] proposed a consistency-based procedure to estimate missing pairwise preference values. Alonso et al. [5] developed a web based consensus support system for group decision making problems and incomplete preferences.

However, the DM may not estimate his/her preference with single exact numerical values, interval numbers, or intuitionistic fuzzy numbers, but with THFEs due to the fact that the DMs are hesitant about some possible values for the preference degrees over paired comparisons of alternatives. In such situations, a hesitant fuzzy preference relation (HFPR), initially proposed by Zhu and Xu [49] on the basis of HFSs, may be more suitable for expressing the DM's hesitant preference information than all the aforementioned preference relations, which do not consider the hesitant fuzzy information and cannot provide all the possible evaluation values of the decision makers when comparing paired alternatives (or criteria). Zhu and Xu [49] proposed a regression method to transform hesitant fuzzy preference relations (HFPRs) into fuzzy preference relations (FPRs). Moreover, Zhu et al. [51] explored the ranking methods with HFPRs in the group decision making environments. Liao et al. [19] investigated the multiplicative consistency of HFPRs and its application in group decision making.

However, it is noted that the aforesaid researches [19,49,51] focused on HFPRs with complete information. A complete HFPR of order n necessitates the completion of all $n(n-1)/2$ judgments in its entire top triangular portion. Sometimes, however, a DM may develop an incomplete HFPR in which some of the elements cannot be provided due to a variety of reasons such as time pressure, lack of knowledge, and the DM's limited expertise related with the problem domain. Consider that it is an interesting and important issue to investigate the consistency of the preference relations and obtain the reliable priority weights, and up to now, no investigation has been devoted to the issue on the approach of deriving the priority vector of incomplete HFPR in the existing literatures. Therefore, it is necessary and significant to pay attention to this issue. In addition, many decision making processes, in the real world, take place in multi-person settings because the increasing complexity and uncertainty of the socio-economic environment makes it less and less possible for single decision maker to consider all relevant aspects of a decision making problem, and the existing researches [19,49,51] only consider an individual HFPR and do not consider group decision making situations. Up to now, considering that no technique has been investigated for dealing with group decision making with incomplete HFPRs. Therefore, this paper introduces the use of a new type of incomplete preference relations, which was pointed out in Ref. [28], as a new challenge to study. We shall in this paper define the concept of incomplete HFPRs, and then construct a goal programming model for deriving the priority weights from incomplete HFPRs under group decision making based on the α-normalization. Then, on the basis of the β-normalization, we define the multiplicative consistent HFPRs and multiplicative consistent incomplete HFPRs, and from which, we propose a method to obtain priority interval weights from a complete or incomplete HFPR. Moreover, a new algorithm is also developed to obtain the collective priority weight vector of several complete or incomplete HFPRs under group decision making situations, and finally, we give several numerical examples to illustrate the proposed model and algorithms.

This paper is structured as follows. Section 2 introduces some known results of FPRs HFSs, and HFPRs. In Section 3, we develop a goal programming model for deriving the priority weights from incomplete HFPRs under group decision making based on the α-normalization. On the basis of the β-normalization, Section 4 defines the multiplicative consistent HFPRs and the multiplicative consistent incomplete HFPRs, based on which, an algorithm is shown to complement an acceptable incomplete HFPR, and a novel procedure is further given to obtain a priority vector from an complete or incomplete HFPR. Moreover, Section 4 also addresses a multiplicative consistency analysis of the collective HFPRs. Then, a practical procedure for obtaining a solution of a GDM problem with several complete or incomplete HFPRs is presented. Finally, the main conclusions are given in Section 5.

2. Preliminaries

In this section, we will briefly recall the concepts of fuzzy preference relation, hesitant fuzzy set, and hesitant fuzzy preference relation.

2.1. Fuzzy preference relation

Definition 2.1 *([21])*. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives, then $R = (r_{ij})_{n\times n}$ is called a fuzzy preference relation (FPR) on $X \times X$ with the following conditions:

$$r_{ij} \geq 0, \quad r_{ij} + r_{ji} = 1, \quad i, j = 1, 2, \ldots, n, \tag{1}$$

where r_{ij} denotes the degree that the alternative x_i is prior to the alternative x_j provided by the decision maker. Especially, $r_{ij} = 0.5$ indicates indifference between x_i and x_j; $r_{ij} > 0.5$ indicates x_i is preferred to x_j, the larger the r_{ij}, the greater the preference degree of the alternative x_i over x_j, $r_{ij} = 1$ indicates that x_i is absolutely prior to x_j; $r_{ij} < 0.5$ indicates x_j is preferred to x_i; the smaller the r_{ij}, the greater the preference degree of the alternative x_j over x_i, $r_{ij} = 0$ indicates that x_j is absolutely prior to x_i.

Definition 2.2 *([25])*. Let $R = (r_{ij})_{n\times n}$ be a FPR, then $R = (r_{ij})_{n\times n}$ is called a multiplicative consistent FPR if it satisfies the multiplicative transitivity property:

$$r_{ik}r_{kj}r_{ji} = r_{ki}r_{jk}r_{ij}, \quad i, j, k = 1, 2, \ldots, n, \tag{2}$$

By the simple algebraic manipulation, Eq. (2) can be expressed as [10,11]

$$r_{ij} = \frac{r_{ik}r_{kj}}{r_{ik}r_{kj} + (1-r_{ik})(1-r_{kj})}, \quad i, j, k = 1, 2, \ldots, n \tag{3}$$

where $r_{ik} > 0$, $i, k = 1, 2, \ldots, n$.

In addition, a multiplicative consistent FPR can also be given by

$$r_{ij} = \frac{w_i}{w_i + w_j}, \quad i, j = 1, 2, \cdots, n \tag{4}$$

where $w = (w_1, w_2, \cdots, w_n)$ is the priority vector of R, satisfying $\sum_{i=1}^{n} w_i = 1$, and $w_i > 0, i = 1, 2, \cdots, n$.

Based on Eq. (4), Fedrizzi and Brunelli [13] proposed a method to derive the priority vector of a multiplicative consistent FPR.

Theorem 2.1 ([13]). For a multiplicative consistent FPR $R = (r_{ij})_{n\times n}$, the vector of priority weights can be written as

$$w_i = \frac{\sqrt[n]{\prod_{j=1}^{n} r_{ij}}}{\sqrt[n]{\prod_{j=1}^{n} (1 - r_{ij})}}, \quad i = 1, 2, \cdots, n \tag{5}$$

where $\prod_{i=1}^{n} w_i = 1$ and $w_i > 0, i = 1, 2, \cdots, n$.

2.2. Hesitant fuzzy set

Torra [26] originally proposed the concept of hesitant fuzzy sets to manage the situations in which several values are possible for the definition of the membership of an element.

Definition 2.3 *([8,26])*. Let $\mathcal{P}([0, 1])$ be the set of all subsets of the unitary interval and X be a nonempty set. Let $h_A : X \to \mathcal{P}([0, 1])$, then a hesitant fuzzy set (HFS) A defined over $X$is given by

$$A = \left\{ \langle x, h_A(x)\rangle | x \in X \right\} \tag{6}$$

Although the memberships of the HFSs could be any subset of [0, 1], most of works on HFSs assume explicitly or implicitly that the memberships of the HFSs are finite and nonempty subsets of [0, 1] [30,43]. Therefore, Bedregal et al. [8,9] introduced the concept of typical hesitant fuzzy sets.

Definition 2.4 *([8,9])*. Let $\mathbb{H} \subseteq (\mathcal{P}[0, 1])$ be the set of all finite non-empty subsets of the interval [0, 1], and let X be a non-empty set. A typical hesitant fuzzy set (THFS) A over X is given by Eq. (6), where $h_A : X \to \mathbb{H}$.

Each $h \in \mathbb{H}$ is called a typical hesitant fuzzy element of $\mathbb{H}$ (THFE) and the number of its elements, i.e. the cardinality of h, is referred to as l_h.

Remark 2.1. A type-2 fuzzy set enables us to define the membership of a given element in terms of a fuzzy set, and its definition is introduced as follows:

Let X be a finite and nonempty set, which is referred to as the universe. A type-2 fuzzy set over X, denoted as $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}} : X \times [0, 1] \to [0, 1]$, i.e.,

$$\tilde{A} = \left\{ ((x, t), \mu_{\tilde{A}}(x, t)) | x \in X, t \in [0, 1] \right\}$$

where $0 \le \mu_{\tilde{A}}(x, t) \le 1$.

We define a mapping $\mu_{\tilde{A}}(x) : [0, 1] \to [0, 1]$ such that $\mu_{\tilde{A}}(x)(t) = \mu_{\tilde{A}}(x, t)$ for any $x \in X$ and $t \in [0, 1]$. $\mu_{\tilde{A}}(x)$ is called the secondary membership function at x. We define $J_{\tilde{A}}^x = \left\{ t \in [0, 1] | \mu_{\tilde{A}}(x, t) > 0 \right\}$, which is referred to as the primary membership of x.

Let $\tilde{A}$ be a type-2 fuzzy set in X. If for $x \in X$ and $t \in J_{\tilde{A}}^x$, $\mu_{\tilde{A}}(x, t) = 1$, then $\tilde{A}$ is defined as an interval type-2 fuzzy set.

Though a HFS $A = \left\{ \langle x, h_A(x)\rangle | x \in X \right\}$ can be represented as an interval type 2 fuzzy set by defining $\mu_A(x, t) = 1$ if and only if $t \in h_A(x)$, there are some difference between type 2 fuzzy sets and hesitant fuzzy sets. The main difference between interval type 2 fuzzy sets and HFSs is the representation of membership degrees. In an interval type 2 fuzzy set, each membership degree estimated cannot be precisely specified with a crisp number but can be expressed with an interval number within [0, 1]. But in a hesitant fuzzy set, each membership degree estimated can be represented by a THFE that may include several possible preferences in [0, 1]. Clearly, an interval number contains more elements and uncertainty than a THFE. HFSs are motivated for the common difficulty that often appears when the membership degree of an element must be established and the difficulty is not because of an error margin (as in intuitionistic fuzzy sets) or due to some possibility distribution on the possible values (as in type 2 fuzzy sets), but rather because there are some possible values that make to hesitate about which one would be the right one. This situation is very usual in decision making when the experts provide different membership degrees, and it is not always possible to obtain a consensus to unify these values. In this case, it is useful to deal with all the possible values instead of considering just an aggregation operator. Such a membership set naturally arises if it is considered as a description of the different opinions of several experts. Additionally, a THFE is usually provided by a decision group, while an interval number is provided by a decision maker. Therefore, the hesitant fuzzy set can handle imprecision and hesitation more objectively and precisely than the interval type 2 fuzzy set. For example, in order to obtain a reasonable decision result, a decision organization constructed by a large group of experts is asked to provide the estimation of the membership degree of the element x into the set A. Suppose there are three cases, some experts provide 0.5, some provide 0.6, and the others provide 0.7; additionally, these experts cannot persuade one another to change their opinions. In such a case, the membership degree of the element x into the set A can be represented by a THFE $\left\{0.5, 0.6, 0.7\right\}$, rather than an interval $[0.5, 0.7]$, because the membership degree of the element x into the set A is not the convex of 0.5 and 0.7 or the interval between 0.5 and 0.7 and instead is three possible preference values, 0.5, 0.6, and 0.7. Because the THFE $\left\{0.5, 0.6, 0.7\right\}$ includes all possible preference values that are provided by the experts for the membership degree of the element x into the set A, $\left\{0.5, 0.6, 0.7\right\}$ is much closer to the true preference degree of x_1 over x_2. From the THFE $\left\{0.5, 0.6, 0.7\right\}$, we can approximately estimate the true membership degree of the element x into the set A as $(0.5+0.6+0.7)/3=0.6$. If the membership degree of the element x into the set A is represented by an interval number $[0.5, 0.7]$, then it is difficult for us to approximately estimate the true membership degree of the element x into the set A. As a consequence, the hesitant fuzzy set cannot be replaced by the type 2 fuzzy set. The advantage of using the hesitant fuzzy set is that all preferences provided by the decision group can be taken into account, which is helpful to make better decisions. Let us consider another example. If an expert cannot exactly quantify the membership degree of the element x into the set A by a crisp number but provides his/her estimation by an interval preference $[0.1, 0.3]$, then the true membership degree is within $[0.1, 0.3]$. Suppose that the true membership degree is 0.25. In this case, if the expert provides his/her preference by a THFE $\left\{0.1, 0.2, 0.3\right\}$, then the true membership degree is not within $\left\{0.1, 0.2, 0.3\right\}$. Therefore, the type 2 fuzzy set cannot by replaced by the HFS. The above numerical illustration and analysis reveal the differences between the type 2 fuzzy set and the hesitant fuzzy set, and also verify the validity and effectiveness of the HFS.

2.3. Hesitant fuzzy preference relation

Let $X=\{x_1, x_2, \cdots, x_n\}$ be a fixed set of alternatives. Assume that the DMs hesitate between some possible preferences to provide paired comparison judgments of alternatives, and these preferences are collected into hesitant fuzzy elements (THFEs). On the basis of THFEs and fuzzy preference relations (FPRs), Zhu and Xu [49] developed a concept of hesitant fuzzy preference relations (HFPRs) as follows:

Definition 2.5 *([49])*. Let $X=\{x_1, x_2, \cdots, x_n\}$ be a fixed set. A hesitant fuzzy preference relation (HFPR) H on X is denoted by a matrix $H=(h_{ij})_{n\times n}\subset X\times X$, where $h_{ij}=\left\{h_{ij}^{\sigma(s)}|s=1, 2, \cdots, l_{h_{ij}}\right\}$ is a THFE, indicating hesitant degrees to which x_i is preferred to x_j. For all $i, j=1, 2, \cdots, n$, h_{ij} should satisfy the following conditions:

$$\begin{cases} h_{ij}^{\sigma(s)}+h_{ji}^{\sigma(l_{h_{ij}}-s+1)}=1, \\ h_{ii}=\left\{0.5\right\}, \\ l_{h_{ij}}=l_{h_{ji}}. \end{cases} \tag{7}$$

where $h_{ij}^{\sigma(s)}$ is the sth largest element in h_{ij}.

Example 2.1. Let H be a HFPR shown as follows:

$$H=\left\{\begin{matrix} \left\{0.5\right\} & \left\{0.4, 0.6, 0.7\right\} & \left\{0.2, 0.3\right\} & \left\{0.5, 0.7\right\} \\ \left\{0.6, 0.4, 0.3\right\} & \left\{0.5\right\} & \left\{0.1\right\} & \left\{0.8, 0.9\right\} \\ \left\{0.8, 0.7\right\} & \left\{0.9\right\} & \left\{0.5\right\} & \left\{0.3, 0.4\right\} \\ \left\{0.5, 0.3\right\} & \left\{0.2, 0.1\right\} & \left\{0.7, 0.6\right\} & \left\{0.5\right\} \end{matrix}\right\}$$

In most situations, it is noted that the numbers of elements in two THFEs $h_i\,(i=1, 2)$ are different. In order to more accurately operation on h_1 and h_2, the precondition is to make sure that h_1 and h_2 have the same number of elements [43]. For two THFEs $h_i\ (i=1, 2)$ with limited number of elements, Zhu et al. [51] proposed two opposite principles for normalization:

(1) α-normalization: Remove some elements of h_i which has more elements than the other one;
(2) β-normalization: Add some elements to h_i which has less elements than the other one.

Based on the above two principles, we next concentrate on deriving the priority weights from incomplete hesitant fuzzy preference relations (HFPRs) under group decision making environments based on multiplicative consistency. If we aim to select the optimal preference from all possible ones, the α-normalization is reasonable; if we want to consider all possible preferences provided by the decision group, we should use the β-normalization.

3. Deriving the priority weights from incomplete HFPRs under group decision making with α-normalization

As a new emerging tool for representing preferences from a group of DMs, HFPRs can efficiently deal with the hesitant situation in which the decision makers are hesitant about some possible values for the preference degrees over paired comparisons of alternatives [51]. In practical applications, due to DM' not possessing a precise or sufficient level of knowledge of part of the problem, a DM may provide his/her judgments over some pairs of alternatives, and is unable or unwilling to give his/her judgments over the other pairs of alternatives, especially when the number of the considered alternatives is very large, in such cases, the DM usually constructs an incomplete HFPR, in which some elements are missing. Consequently, we next introduce the concept of an incomplete HFPR:

Definition 3.1. Let $H=(h_{ij})_{n\times n}$ be a HFPR, where $h_{ij}=\left\{h_{ij}^{\sigma(s)}|s=1, 2, \cdots, l_{h_{ij}}\right\}$ $(i, j=1, 2, \cdots, n)$, then H is called an incomplete HFPR, if some of its elements cannot be given by the decision maker, which we denote by the unknown variable "x", and the others can be provided by the decision maker, which satisfy

Table 1
DMs' pair-wise judgments.

Pair of the four alternatives		Pair-wise judgment
x_1 versus	x_2	0.7, 0.8, none
	x_3	None, none, none
	x_4	0.3, 0.5, 0.5
x_2 versus	x_1	0.3, 0.2, none
	x_3	0.7, 0.8, 0.9
	x_4	None, none, none
x_3 versus	x_1	None, none, none
	x_2	0.3, 0.2, 0.1
	x_4	0.2, 0.3, 0.4
x_4 versus	x_1	0.7, 0.5, 0.5
	x_2	None, none, none
	x_3	0.8, 0.7, 0.6

$$\begin{cases} h_{ij}^{\sigma(s)} + h_{ji}^{\sigma(l_{h_{ij}}-s+1)} = 1, \\ h_{ii} = \left\{0.5\right\}, \\ l_{h_{ij}} = l_{h_{ji}}. \end{cases} \tag{8}$$

for all $h_{ij} \in \Omega$, where Ω is the set of all the known elements in H.

In this case, the DM is unable to provide preference values for all pairs of alternatives, and thus some of them are missing.

Incomplete HFPRs can deal with the problems where the DMs are hesitant about several possible values for the preference degrees over some pairs of alternatives and they are unable or unwilling to furnish their judgments over other pairs of alternatives for some reason. In the following, we employ an example to illustrate the necessity of introducing incomplete HFPRs and clarify the meaning of the concept of incomplete HFPRs.

Example 3.1. Consider a decision problem consisting of four alternatives $X = \{x_1, x_2, x_3, x_4\}$. A decision organization (group) including three decision makers is invited to provide their preference degrees over paired comparisons of alternatives. The decision organization compares each pair of the alternatives and furnishes pair-wise judgments as shown in Table 1.

From Table 1, the pair-wise judgments provided by the decision organization are divided into four cases:

Case 1. The decision organization provides three mutually different numerical preference values for one alternative over another. For example, we receive three different preference values 0.7, 0.8 and 0.9 for the alternative x_2 over x_3.

In order to get a more reasonable decision result, forming a group, such as the board of directors of a company, whose members come from different fields is a common way used in realistic application [18]. However, disagreements may arise from the difference in the decision makers subjective evaluations of the decision problem [45]. Once they cannot persuade each other, the THFE can be applied to maintain all of those original assessments provided by the decision makers, while all the other extended forms of fuzzy set, such as interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, type-2 fuzzy sets, even including fuzzy multisets, cannot be used to depict this situation exactly. Moreover, in practical setting of group decision making, anonymity is needed in order to protect the privacy of the decision makers or ensure non interference opinions accumulated. Thus, it is natural to consider all the assessments so as to get more reasonable decision results. This also can only be represented by THFEs. Based on the above analysis, the preference intensity for the alternative x_2 over x_3 is expressed as a THFE $h_{23} = \left\{0.7, 0.8, 0.9\right\}$. Similarly we can denote the symmetric element of h_{23}, i.e., h_{32} by a THFE $h_{32} = \left\{0.3, 0.2, 0.1\right\}$.

Case 2. The decision organization provides three numerical preference values for one alternative over another, and any two or three of these preference values are the same. For example, we receive three preference values 0.3, 0.5 and 0.5 for the alternative x_1 over x_4. It means that there are two decision makers which provide the same preference value 0.5.

For the preference intensity for one alternative over another, although all of the DMs provide their evaluation values, some of these original assessments may be repeated. It is noted that a value repeated more times does not indicate that it has more importance than other values repeated less times. Because the value repeated one time may be provided by a DM who is an expert at this area, while the value repeated twice may be provided by two DMs who are not familiar with this area. In such cases, the value repeated one time may be more important than the one repeated twice. To get a more reasonable result, it is better that the DMs give their evaluations anonymously. We only collect all of the possible preference values for one alternative over another, and each preference value provided only means that it is a possible value, but its importance is unknown. Thus the times that the values repeated are unimportant, and it is reasonable to allow these values repeated many times appear only once [48]. The THFE is just a tool to deal with such cases, and all possible original evaluations for one alternative over another can be considered as a THFE. As per the above analysis, the preference intensity for the alternative x_1 over x_4 can be described as a THFE $h_{14} = \left\{0.3, 0.5\right\}$. Analogously, we can denote the symmetric element of h_{14}, i.e., h_{41} by a THFE $h_{41} = \left\{0.7, 0.5\right\}$.

Case 3. For one alternative over another, some DMs in the decision organization provide their numerical preference values, while the other DMs are unable or unwilling to furnish their judgments for some reason. For example, we receive two preference values 0.7 and 0.8 for the alternative x_1 over x_2. It means that there are two decision makers which provide two preference values 0.7 and 0.8 for the alternative x_1 over x_2 and one decision maker which is unable or unwilling to provide his/her opinion for the alternative x_1 over x_2 due to insufficient information or limited expertise. In this case, based on the above discussions in Cases 1 and 2, the preference intensity for the alternative x_1 over x_2 can be described as a THFE $h_{12} = \left\{0.7, 0.8\right\}$. Similarly we can denote the symmetric element of h_{12}, i.e., h_{21} by a THFE $h_{21} = \left\{0.3, 0.2\right\}$.

Case 4. All of the three DMs in the decision organization are unable or unwilling to furnish their judgments for some reason, such as insufficient information or limited expertise. For instance, we do not receive any preference values from the decision organization for the alternative x_1 over x_3. In this case, the preference information for the alternative x_1 over x_3 is totally unknown and is denoted by "–". Accordingly, we also do not receive any preference values from the decision organization for the alternative x_3 over x_1. In this case, the preference information for the alternative x_3 over x_1 is totally unknown and is denoted by " ".

According to the aforesaid transform method and principle, the DMs' pair-wise judgments listed in Table 1 are included in the following incomplete HFPR.

$$H = (h_{ij})_{4\times 4} = \begin{Bmatrix} \{0.5\} & \{0.7, 0.8\} & - & \{0.3, 0.5\} \\ \{0.3, 0.2\} & \{0.5\} & \{0.7, 0.8, 0.9\} & - \\ - & \{0.3, 0.2, 0.1\} & \{0.5\} & \{0.2, 0.3, 0.4\} \\ \{0.7, 0.5\} & - & \{0.8, 0.7, 0.6\} & \{0.5\} \end{Bmatrix}$$

In some practical decision making problems, anonymity is required in order to protect the decision makers' privacy or avoid influencing each other, for example, the presidential election or the blind peer review of thesis, in which we do not know what preference values that each decision maker provides for an alternative over another, and thus, leading us to consider all the situations in order to get more reasonable decision results. Therefore, an incomplete HFPR is not preferring to be split into several fuzzy preference relations or incomplete fuzzy preference relations. Previous approaches, based on fuzzy preference relations, treated a GDM problem through aggregating the individual fuzzy preference relation into the group's one, causing the loss of information. These fuzzy preference structures mentioned above are unable to directly incorporate the differences of opinions of different DMs and, hence, they are not applied to GDM problems under hesitant fuzzy environments. The difficulty can be overcome by introducing an incomplete hesitant fuzzy preference relation, whose allows the DMs to provide several possible fuzzy preference values when they compare two alternatives, and thus keeps all of the preference values proposed by the DMs; that is, potentially, it keeps more information about the DM's opinion, avoids performing information aggregation and directly reflects the differences of preference information of different DMs.

In the following example, we use incomplete FPRs instead of incomplete HFPRs to express the DMs' pair-wise judgments in Table 1, and show the advantages of incomplete HFPRs compared to incomplete FPRs with hesitance and uncertainty.

Example 3.2 *((Continued with Example 3.1)).* The common methods which are not based on the use of HFSs, is to aggregate first, for each pair of alternatives, the different opinions of the DMs in the framework of group decision making. HFSs can be used to eliminate this step. In contrast with the methods which do not use HFSs, the preference degrees for each pair of alternatives can be defined as the HFS defined in terms of the opinions of the DMs.

Generally speaking, in order to model and analyze the group decision making problems in which all the DMs' preferences may not be of equal importance, a weight vector is used to characterize these differences between the importance of all the DMs' preferences. However, in a group decision making within the hesitant fuzzy environment, in order to avoid influencing each other, the DMs are required to provide their preferences in anonymity. Because of anonymity, we do not know what preferences that each DM provides. Therefore, we employ the arithmetic average (AA) operator to aggregate all of possible values $h_{ij}^{\sigma(s)}$ $(s = 1, 2, \cdots, l_{h_{ij}})$ in each element h_{ij} of H, and obtain an averaged incomplete FPR $R = (r_{ij})_{m\times n}$ as follows:

$$r_{ij} = \frac{1}{l_{h_{ij}}} \sum_{s=1}^{l_{h_{ij}}} h_{ij}^{\sigma(s)}, \quad i, j = 1, 2, \cdots, n, \quad h_{ij} \in \Omega$$

Therefore, if we use the method which is not based on the use of hesitant fuzzy sets to handle Example 3.1, then preference information given in Table 1 can be expressed as the following incomplete FPR $R = (r_{ij})_{4\times 4}$:

$$R = (r_{ij})_{4\times 4} = \begin{Bmatrix} 0.5 & 0.75 & - & 0.4 \\ 0.25 & 0.5 & 0.8 & - \\ - & 0.2 & 0.5 & 0.3 \\ 0.6 & - & 0.7 & 0.5 \end{Bmatrix}$$

It can be observed from $R = (r_{ij})_{4\times 4}$ that the averaged incomplete FPR, obtained by the previous methods which do not use HFSs, uses the single numerical values to represent the preference information and neglects the hesitance experienced by the decision group, thereby causing the loss of information about the DMs' opinions. In contrast, the incomplete HFPR $H = (h_{ij})_{4\times 4}$ proposed in Example 3.1 uses THFEs to represent the preference information and takes fully the hesitance and differences of opinions of the DMs into account, thereby preserving the original information about the DMs' opinions as much as possible. For example, suppose that the true preference degrees provided by the decision group are included in the following incomplete FPR $R' = (r'_{ij})_{4\times 4}$:

$$R' = (r'_{ij})_{4\times 4} = \begin{Bmatrix} 0.5 & 0.8 & - & 0.3 \\ 0.2 & 0.5 & 0.7 & - \\ - & 0.3 & 0.5 & 0.2 \\ 0.7 & - & 0.8 & 0.5 \end{Bmatrix}$$

It is easy to verify that $R' = (r'_{ij})_{4\times 4}$ is remarkably different from $R = (r_{ij})_{4\times 4}$, indicating that the method which does not use HFSs results in significant loss of information in original DMs' judgments and do not properly reflect the information of DMs' judgments, which makes the final decision results unreasonable and unconvincing.

In contrast, the proposed incomplete HFPR $H=(h_{ij})_{4\times4}$ contains the true incomplete FPR $R'=(r'_{ij})_{4\times4}$, implying that the information in original judgments can be properly reflected by the proposed incomplete HFPR $H=(h_{ij})_{4\times4}$. Because an incomplete HFPR could be any of the possible fuzzy preference values with any of the possible membership values, $H=(h_{ij})_{4\times4}$ contains any true preference information.

In fact, in a usual fuzzy logic, it is reasonable to average these fuzzy preference degrees or take the smallest preference that contains all these fuzzy preference degrees, i.e., a preference between the smallest and the largest of the DMs' preference degrees. The idea of hesitant fuzzy logic is that instead, it keeps all the fuzzy preference values proposed by the DMs; that is, potentially, it keeps more information about the DMs' opinion, the information that is normally dismissed. This may give a better and credible result.

Owing to the increasing complexity of socio-economic environment, it is less and less possible for a single DM to consider all relevant aspects of the problem. Therefore, many organizations employ groups to make decision, which is called as group decision making (GDM). Moreover, due to the complexity and uncertainty involved in real decision problems, time pressure, lack of knowledge and the DMs' limited expertise related to the problem domain, there are also many GDM problems with incomplete information. Here we consider the GDM problem in which the DMs give their judgments in terms of incomplete HFPRs. We will establish a goal programming models for deriving the priority weights from incomplete HFPRs under GDM environments based on α-normalization.

For a group decision making problem, in which there are a finite set of alternatives, $X=\{x_1,x_2,\cdots,x_n\}$, and a group of DMs $D=\left\{d_1,d_2,\cdots,d_m\right\}$, each DM has his/her importance weight λ_t in the process of group decision making, where $\lambda_t\in[0,1], t=1,2,\cdots,m$, and $\sum_{t=1}^{m}\lambda_t=1$. Suppose that the DM $d_t\in D$ compares each pair of the alternatives x_i $(i=1,2,\cdots,n)$ and constructs an incomplete HFPR $H_t=(h_{ij,t})_{n\times n}=\left(\left\{h_{ij,t}^{\sigma(s)}|s=1,2,\cdots,l_{h_{ij,t}}\right\}\right)_{n\times n}$ in which some of the elements are missing, where $h_{ij,t}\in\Omega_t$ (Ω_t is the set of all the known elements in H_t) is a THFE, which indicates hesitant degrees to which x_i is preferred to x_j, and satisfies

$$\begin{cases} h_{ij,t}^{\sigma(s)}+h_{ji,t}^{\sigma(l_{h_{ij,t}}-s+1)}=1,\\ h_{ii,t}=\left\{0.5\right\},\\ l_{h_{ij,t}}=l_{h_{ji,t}}. \end{cases} \tag{9}$$

For convenience, let w_i be the importance weight of the alternative x_i, and all the weights w_i $(i=1,2,\cdots,n)$ satisfy $w_i\geq0, i=1,2,\cdots,n$, and $\sum_{i=1}^{n}w_i=1$. Since each element in $h_{ij,t}$ is a possible preference degree for the comparison of the alternative x_i over x_j, then by Eq. (4), if $H_t=(h_{ij,t})_{n\times n}=\left(\left\{h_{ij,t}^{\sigma(s)}|s=1,2,\cdots,l_{h_{ij,t}}\right\}\right)_{n\times n}$ is an incomplete multiplicative consistent HFPR, then such a preference relation can be obtained by

$$\delta_{ij,t}\frac{w_i}{w_i+w_j}=\delta_{ij,t}\left(h_{ij,t}^{\sigma(1)}\,\mathrm{or}\,h_{ij,t}^{\sigma(2)}\,\mathrm{or}\cdots\mathrm{or}\,h_{ij,t}^{\sigma(l_{h_{ij,t}})}\right),\quad i,j=1,2,\cdots,n,\quad t=1,2,\cdots,m \tag{10}$$

where $h_{ij,t}^{\sigma(s)}$ $(s=1,2,\cdots,l_{h_{ij,t}})$ is the sth largest element in $h_{ij,t}$, and $\delta_{ij,t}$ is a zero or one integer variable defined as

$$\delta_{ij,t}=\begin{cases}1,\ \mathrm{if}\,h_{ij,t}\,\mathrm{is\ known},\\ 0,\ \mathrm{if}\,h_{ij,t}\,\mathrm{is\ unknown},\end{cases}\quad i,j=1,2,\cdots,n\quad t=1,2,\cdots,m \tag{11}$$

Let $S(h_{ij,t})=h_{ij,t}^{\sigma(1)}\,\mathrm{or}\,h_{ij,t}^{\sigma(2)}\,\mathrm{or}\cdots\mathrm{or}\,h_{ij,t}^{\sigma(l_{h_{ij,t}})}$, then by Eq. (10), we can obtain

$$\begin{aligned} &\delta_{ij,t}\frac{w_i}{w_i+w_j}=\delta_{ij,t}\left(h_{ij,t}^{\sigma(1)}\,\mathrm{or}\,h_{ij,t}^{\sigma(2)}\,\mathrm{or}\cdots\mathrm{or}\,h_{ij,t}^{\sigma\left(l_{h_{ij,t}}\right)}\right)\\ &\Leftrightarrow\delta_{ij,t}\frac{w_i}{w_i+w_j}=\delta_{ij,t}S\left(h_{ij,t}\right)\\ &\Leftrightarrow\delta_{ij,t}w_i=\delta_{ij,t}\left(w_i+w_j\right)S\left(h_{ij,t}\right)\\ &\Leftrightarrow\delta_{ij,t}w_i=\delta_{ij,t}w_iS\left(h_{ij,t}\right)+\delta_{ij,t}w_jS\left(h_{ij,t}\right)\\ &\Leftrightarrow\delta_{ij,t}w_i-\delta_{ij,t}w_iS\left(h_{ij,t}\right)=\delta_{ij,t}w_jS\left(h_{ij,t}\right)\\ &\Leftrightarrow\delta_{ij,t}w_i\left(1-S\left(h_{ij,t}\right)\right)=\delta_{ij,t}w_jS\left(h_{ij,t}\right) \end{aligned} \tag{12}$$

Let

$$\begin{aligned} 1-S\left(h_{ij,t}\right)&=\left(1-h_{ij,t}^{\sigma(1)}\right)\ \mathrm{or}\ \left(1-h_{ij,t}^{\sigma(2)}\right)\ \mathrm{or}\cdots\mathrm{or}\ \left(1-h_{ij,t}^{\sigma\left(l_{h_{ij,t}}\right)}\right)\\ &=h_{ji,t}^{\sigma\left(l_{h_{ij,t}}\right)}\,\mathrm{or}\,h_{ji,t}^{\sigma\left(l_{h_{ij,t}}-1\right)}\ \mathrm{or}\cdots\mathrm{or}\ \left(h_{ji,t}^{\sigma(1)}\right)=S\left(h_{ji,t}\right) \end{aligned}$$

then, Eq. (12) can be rewritten as

$$\begin{aligned} &\delta_{ij,t}\frac{w_i}{w_i+w_j}=\delta_{ij,t}\left(h_{ij,t}^{\sigma(1)}\,\mathrm{or}\,h_{ij,t}^{\sigma(2)}\,\mathrm{or}\cdots\mathrm{or}\,h_{ij,t}^{\sigma\left(l_{h_{ij,t}}\right)}\right)\\ &\Leftrightarrow \delta_{ij,t}w_i\left(1-S\left(h_{ij,t}\right)\right)=\delta_{ij,t}w_jS\left(h_{ij,t}\right)\\ &\Leftrightarrow \delta_{ij,t}w_iS\left(h_{ji,t}\right)=\delta_{ij,t}w_jS\left(h_{ij,t}\right) \end{aligned} \tag{13}$$

Nevertheless, Eq. (13) does not always hold in the actual applications. Here, we shall relax Eq. (13) to obtain consistent preferences as much as possible by minimizing the error $\varepsilon_{ij,t}$, where

$$\varepsilon_{ij,t}=|\delta_{ij,t}w_iS(h_{ji,t})-\delta_{ij,t}w_jS(h_{ij,t})|=\delta_{ij,t}|w_iS(h_{ji,t})-w_jS(h_{ij,t})|,\quad i,j=1,2,\cdots,n,\quad t=1,2,\cdots,m \tag{14}$$

In order to derive the weights w_i $(i=1,2,\cdots,n)$ of the alternatives x_i $(i=1,2,\cdots,n)$, all H_t $(t=1,2,\cdots,m)$ should be integrated, and thus, based on Eq. (14), we establish the optimization problem:

$$\begin{aligned} &\min\sum_{t=1}^{m}\lambda_t\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}\varepsilon_{ij,t}=\sum_{t=1}^{m}\lambda_t\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}\delta_{ij,t}|w_iS(h_{ji,t})-w_jS(h_{ij,t})|\\ &\mathrm{s.t.}\,w_i\geq 0,\ i=1,2,\cdots,n,\ \sum_{i=1}^{n}w_i=1 \end{aligned} \tag{15}$$

Solution to the above minimization problem is found by solving the following goal programming model:

$$\begin{aligned} &\min J=\sum_{t=1}^{m}\lambda_t\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}(s_{ij,t}d_{ij,t}^{+}+t_{ij,t}d_{ij,t}^{-})\\ &\mathrm{s.t.}\begin{cases}\delta_{ij,t}(w_iS(h_{ji,t})-w_jS(h_{ij,t}))-s_{ij,t}d_{ij,t}^{+}+t_{ij,t}d_{ij,t}^{-}=0,\\ i,j=1,2,\cdots,n,\ i\neq j,\ t=1,2,\cdots,m,\\ w_i\geq 0,\ i=1,2,\cdots,n,\ \sum_{i=1}^{n}w_i=1,\\ d_{ij,t}^{+},d_{ij,t}^{-}\geq 0,\ i,j=1,2,\cdots,n,\ i\neq j,\ t=1,2,\cdots,m\end{cases} \end{aligned} \tag{16}$$

where $d_{ij,t}^{+}$ is the positive deviation from the target of the goal $\varepsilon_{ij,t}$, defined as

$$d_{ij,t}^{+}=\max\left\{\delta_{ij,t}(w_iS(h_{ji,t})-w_jS(h_{ij,t})),0\right\}$$

$d_{ij,t}^{-}$ is the negative deviation from the target of the goal $\varepsilon_{ij,t}$, defined as

$$d_{ij,t}^{-}=\max\left\{\delta_{ij,t}(w_jS(h_{ij,t})-w_iS(h_{ji,t})),0\right\}$$

$s_{ij,t}$ is the weighting factor corresponding to positive deviation $d_{ij,t}^{+}$, and $t_{ij,t}$ is the weighting factor corresponding to the negative deviation $d_{ij,t}^{-}$.

Consider that all the goal functions $\varepsilon_{ij,t}$ $(i,j=1,2,\cdots,n,\ t=1,2,\cdots,m)$ are fair, thus we can set $s_{ij,t}=t_{ij,t}=1$ $(i,j=1,2,\cdots,n,\ t=1,2,\cdots,m)$, and then rewrite Eq. (16) as

$$\begin{aligned} &\min J=\sum_{t=1}^{m}\lambda_t\sum_{i=1}^{n}\sum_{j=1,i\neq j}^{n}(d_{ij,t}^{+}+d_{ij,t}^{-})\\ &\mathrm{s.t.}\begin{cases}\delta_{ij,t}(w_iS(h_{ji,t})-w_jS(h_{ij,t}))-d_{ij,t}^{+}+d_{ij,t}^{-}=0,\\ i,j=1,2,\cdots,n,\ i\neq j,\ t=1,2,\cdots,m,\\ w_i\geq 0,\ i=1,2,\cdots,n,\ \sum_{i=1}^{n}w_i=1,\\ d_{ij,t}^{+},d_{ij,t}^{-}\geq 0,\ i,j=1,2,\cdots,n,\ i\neq j,\ t=1,2,\cdots,m\end{cases} \end{aligned} \tag{17}$$

Solving this model is equivalent to a selection process selecting the optimal preference value from all possible ones for each paired comparison of alternatives. Through this selection process, we can obtain an incomplete FPR consisting of all the optimal preferences, which we call a reduced incomplete HFPR.

Example 3.3. For a GDM problem, suppose that there are four decision alternatives x_i $(i=1,2,3,4)$ and three DMs d_t $(t=1,2,3)$, whose weight vector $\lambda=(0.5,0.2,0.3)$. The DMs provide their hesitant preferences over paired comparisons of these four decision alternatives, and give three incomplete HFPRs as follows, respectively:

$$H_1 = \begin{Bmatrix} \{0.5\} & \{0.3, 0.5\} & - & - \\ \{0.7, 0.5\} & \{0.5\} & \{0.5, 0.6, 0.7\} & - \\ - & \{0.5, 0.4, 0.3\} & \{0.5\} & \{0.7, 0.8\} \\ - & - & \{0.3, 0.2\} & \{0.5\} \end{Bmatrix}$$

$$H_2 = \begin{Bmatrix} \{0.5\} & \{0.7, 0.8\} & - & \{0.3, 0.5\} \\ \{0.3, 0.2\} & \{0.5\} & \{0.7, 0.8, 0.9\} & - \\ - & \{0.3, 0.2, 0.1\} & \{0.5\} & \{0.2, 0.3, 0.4\} \\ \{0.7, 0.5\} & - & \{0.8, 0.7, 0.6\} & \{0.5\} \end{Bmatrix}$$

$$H_3 = \begin{Bmatrix} \{0.5\} & \{0.3, 0.4\} & \{0.4, 0.5, 0.6\} & \{0.5, 0.7\} \\ \{0.7, 0.6\} & \{0.5\} & - & \{0.8, 0.9\} \\ \{0.6, 0.5, 0.4\} & - & \{0.5\} & \{0.6, 0.7, 0.8\} \\ \{0.5, 0.3\} & \{0.2, 0.1\} & \{0.4, 0.3, 0.2\} & \{0.5\} \end{Bmatrix}$$

To derive the priority weights of alternatives, according to Eq. (17), we can build the following programming model:

$$\min J = 0.5 \times \sum_{i=1}^{4} \sum_{j=1, i \neq j}^{4} (d_{ij,1}^{+} + d_{ij,1}^{-}) + 0.2 \times \sum_{i=1}^{4} \sum_{j=1, i \neq j}^{4} (d_{ij,2}^{+} + d_{ij,2}^{-}) + 0.3 \times \sum_{i=1}^{4} \sum_{j=1, i \neq j}^{4} (d_{ij,3}^{+} + d_{ij,3}^{-})$$

$$\text{s.t.} \begin{cases} w_1(0.7 or 0.5) - w_2(0.3 or 0.5) - d_{12,1}^{+} + d_{12,1}^{-} = 0, \\ w_2(0.5 or 0.4 or 0.3) - w_3(0.5 or 0.6 or 0.7) - d_{23,1}^{+} + d_{23,1}^{-} = 0, \\ w_3(0.3 or 0.2) - w_4(0.7 or 0.8) - d_{34,1}^{+} + d_{34,1}^{-} = 0, \\ w_1(0.3 or 0.2) - w_2(0.7 or 0.8) - d_{12,2}^{+} + d_{12,2}^{-} = 0, \\ w_1(0.7 or 0.5) - w_4(0.3 or 0.5) - d_{14,2}^{+} + d_{14,2}^{-} = 0, \\ w_2(0.3 or 0.2 or 0.1) - w_3(0.7 or 0.8 or 0.9) - d_{23,2}^{+} + d_{23,2}^{-} = 0, \\ w_3(0.8 or 0.7 or 0.6) - w_4(0.2 or 0.3 or 0.4) - d_{34,2}^{+} + d_{34,2}^{-} = 0, \\ w_1(0.7 or 0.6) - w_2(0.3 or 0.4) - d_{12,3}^{+} + d_{12,3}^{-} = 0, \\ w_1(0.6 or 0.5 or 0.4) - w_3(0.4 or 0.5 or 0.6) - d_{13,3}^{+} + d_{13,3}^{-} = 0, \\ w_1(0.5 or 0.3) - w_4(0.5 or 0.7) - d_{14,3}^{+} + d_{14,3}^{-} = 0, \\ w_2(0.2 or 0.1) - w_4(0.8 or 0.9) - d_{24,3}^{+} + d_{24,3}^{-} = 0, \\ w_3(0.4 or 0.3 or 0.2) - w_4(0.6 or 0.7 or 0.8) - d_{34,3}^{+} + d_{34,3}^{-} = 0, \\ w_i \geq 0, i = 1, 2, 3, 4, \sum_{i=1}^{4} w_i = 1, \\ d_{ij,t}^{+}, d_{ij,t}^{-} \geq 0, i, j = 1, 2, 3, 4, i \neq j, t = 1, 2, 3 \end{cases} \tag{18}$$

Using the MATLAB Optimization Tool, we can obtain the results as follows:

$J = 0.0938$, $w_1 = 0.3022$, $w_2 = 0.4532$, $w_3 = 0.1942$, $w_4 = 0.0504$, $d_{12,1}^{+} = 0$, $d_{12,1}^{-} = 0.0755$, $d_{13,1}^{+} = 0$, $d_{13,1}^{-} = 0$, $d_{14,1}^{+} = 0$, $d_{14,1}^{-} = 0$, $d_{21,1}^{+} = 0$, $d_{21,1}^{-} = 0$, $d_{23,1}^{+} = 0$, $d_{23,1}^{-} = 0$, $d_{24,1}^{+} = 0$, $d_{24,1}^{-} = 0$,$d_{31,1}^{+} = 0$, $d_{31,1}^{-} = 0$, $d_{32,1}^{+} = 0$, $d_{32,1}^{-} = 0$, $d_{34,1}^{+} = 0$, $d_{34,1}^{-} = 0.0014$, $d_{41,1}^{+} = 0$, $d_{41,1}^{-} = 0$, $d_{42,1}^{+} = 0$, $d_{42,1}^{-} = 0$, $d_{43,1}^{+} = 0$, $d_{43,1}^{-} = 0$, $d_{12,2}^{+} = 0$, $d_{12,2}^{-} = 0.3022$, $d_{13,2}^{+} = 0$, $d_{13,2}^{-} = 0$, $d_{14,2}^{+} = 0.1259$, $d_{14,2}^{-} = 0$, $d_{21,2}^{+} = 0$, $d_{21,2}^{-} = 0$, $d_{23,2}^{+} = 0$, $d_{23,2}^{-} = 0.1295$, $d_{24,2}^{+} = 0$, $d_{24,2}^{-} = 0$, $d_{31,2}^{+} = 0$, $d_{31,2}^{-} = 0$, $d_{32,2}^{+} = 0$, $d_{32,2}^{-} = 0$, $d_{34,2}^{+} = 0.0964$, $d_{34,2}^{-} = 0$, $d_{41,2}^{+} = 0$, $d_{41,2}^{-} = 0$, $d_{42,2}^{+} = 0$, $d_{42,2}^{-} = 0$, $d_{43,2}^{+} = 0$, $d_{43,2}^{-} = 0$, $d_{12,3}^{+} = 0$, $d_{12,3}^{-} = 0$, $d_{13,3}^{+} = 0.0043$, $d_{13,3}^{-} = 0$, $d_{14,3}^{+} = 0.1259$, $d_{14,3}^{-} = 0$, $d_{21,3}^{+} = 0$, $d_{21,3}^{-} = 0$, $d_{23,3}^{+} = 0$, $d_{23,3}^{-} = 0$, $d_{24,3}^{+} = 0$, $d_{24,3}^{-} = 0$, $d_{31,3}^{+} = 0$, $d_{31,3}^{-} = 0$, $d_{32,3}^{+} = 0$, $d_{32,3}^{-} = 0$, $d_{34,3}^{+} = 0$, $d_{34,3}^{-} = 0.0014$, $d_{41,3}^{+} = 0$, $d_{41,3}^{-} = 0$, $d_{42,3}^{+} = 0$, $d_{42,3}^{-} = 0$, $d_{43,3}^{+} = 0$, $d_{43,3}^{-} = 0$.

Since $w_2 > w_1 > w_3 > w_4$, then the ranking of alternatives is $x_2 \succ x_1 \succ x_3 \succ x_4$, and the optimal alternative is x_2.

Furthermore, we can obtain three corresponding incomplete FPRs R'_1, R'_2, and R'_3 consisting of the optimal preference values as follows:

$$R'_1 = \begin{Bmatrix} 0.5 & 0.3 & - & - \\ 0.7 & 0.5 & 0.7 & - \\ - & 0.3 & 0.5 & 0.7 \\ - & - & 0.3 & 0.5 \end{Bmatrix}, \quad R'_2 = \begin{Bmatrix} 0.5 & 0.7 & - & 0.5 \\ 0.3 & 0.5 & 0.7 & - \\ - & 0.3 & 0.5 & 0.4 \\ 0.5 & - & 0.6 & 0.5 \end{Bmatrix}, \quad R'_3 = \begin{Bmatrix} 0.5 & 0.3 & 0.5 & 0.7 \\ 0.7 & 0.5 & - & 0.8 \\ 0.5 & - & 0.5 & 0.7 \\ 0.3 & 0.2 & 0.3 & 0.5 \end{Bmatrix}$$

which can be considered as three reduced incomplete HFPRs of H_1, H_2, and H_3, respectively.

To facilitate to compare with the ranking results of alternatives obtained by the model (17) based on HFSs, Example 3.3 is further employed here to present the results of ranking by the GDM methods based on fuzzy sets.

Example 3.4. First, fuse all of the possible preference opinions in each incomplete HFPR $H_t = (h_{ij,t})_{4\times 4}$ $(t = 1, 2, 3)$ and obtain the averaged incomplete FPR $R_t = (r_{ij,t})_{4\times 4}$ $(t = 1, 2, 3)$, which are shown as follows:

$$R_1 = \begin{Bmatrix} 0.5 & 0.4 & - & - \\ 0.6 & 0.5 & 0.6 & - \\ - & 0.4 & 0.5 & 0.75 \\ - & - & 0.25 & 0.5 \end{Bmatrix}, \quad R_2 = \begin{Bmatrix} 0.5 & 0.75 & - & 0.4 \\ 0.25 & 0.5 & 0.8 & - \\ - & 0.2 & 0.5 & 0.3 \\ 0.6 & - & 0.7 & 0.5 \end{Bmatrix}, \quad R_3 = \begin{Bmatrix} 0.5 & 0.35 & 0.5 & 0.6 \\ 0.65 & 0.5 & - & 0.85 \\ 0.5 & - & 0.5 & 0.7 \\ 0.4 & 0.15 & 0.3 & 0.5 \end{Bmatrix}$$

For example, for the DM d_1, $r_{12,1} = \frac{1}{2}(0.3 + 0.5) = 0.4$ and $r_{23,1} = \frac{1}{3}(0.5 + 0.6 + 0.7) = 0.6$.

Subsequently, plug $R_t = (r_{ij,t})_{4\times 4}$ $(t = 1, 2, 3)$ into the hesitant fuzzy goal programming model (17) in which $S(h_{ji,t}) = r_{ij,t}$ for all $i, j = 1, 2, 3, 4$, $t = 1, 2, 3$, and obtain the following goal programming model:

$$\min J = 0.5 \times \sum_{i=1}^{4} \sum_{j=1, i\neq j}^{4} (d_{ij,1}^{+} + d_{ij,1}^{-}) + 0.2 \times \sum_{i=1}^{4} \sum_{j=1, i\neq j}^{4} (d_{ij,2}^{+} + d_{ij,2}^{-}) + 0.3 \times \sum_{i=1}^{4} \sum_{j=1, i\neq j}^{4} (d_{ij,3}^{+} + d_{ij,3}^{-})$$

$$\text{s.t.} \begin{cases} 0.6w_1 - 0.4w_2 - d_{12,1}^{+} + d_{12,1}^{-} = 0, \\ 0.4w_2 - 0.6w_3 - d_{23,1}^{+} + d_{23,1}^{-} = 0, \\ 0.25w_3 - 0.75w_4 - d_{34,1}^{+} + d_{34,1}^{-} = 0, \\ 0.25w_1 - 0.75w_2 - d_{12,2}^{+} + d_{12,2}^{-} = 0, \\ 0.6w_1 - 0.4w_4 - d_{14,2}^{+} + d_{14,2}^{-} = 0, \\ 0.2w_2 - 0.8w_3 - d_{23,2}^{+} + d_{23,2}^{-} = 0, \\ 0.7w_3 - 0.3w_4 - d_{34,2}^{+} + d_{34,2}^{-} = 0, \\ 0.65w_1 - 0.35w_2 - d_{12,3}^{+} + d_{12,3}^{-} = 0, \\ 0.5w_1 - 0.5w_3 - d_{13,3}^{+} + d_{13,3}^{-} = 0, \\ 0.4w_1 - 0.6w_4 - d_{14,3}^{+} + d_{14,3}^{-} = 0, \\ 0.15w_2 - 0.85w_4 - d_{24,3}^{+} + d_{24,3}^{-} = 0, \\ 0.3w_3 - 0.7w_4 - d_{34,3}^{+} + d_{34,3}^{-} = 0, \\ w_i \geq 0, i = 1, 2, 3, 4, \sum_{i=1}^{4} w_i = 1, \\ d_{ij,t}^{+}, d_{ij,t}^{-} \geq 0, i, j = 1, 2, 3, 4, i \neq j, t = 1, 2, 3 \end{cases}$$

Finally, solving the above model by means of the MATLAB Optimization Tool, we obtain the optimal priority weight vector $w = (w_1, w_2, w_3, w_4)^T = (0.2545, 0.3818, 0.2545, 0.1091)^T$, indicating that the ranking order of four alternatives is $x_2 \succ x_1 \sim x_3 \succ x_4$.

Although both the above two approaches in Example 3.3 and 3.4 give the same priority to x_2, the method that does not use HFSs cannot distinguish the preference order between the alternatives x_1 and x_3; whereas the proposed model (17) that uses HFSs can distinguish the preference order among the alternatives x_1, x_2, x_3 and x_4, which is more reasonable and credible.

The common approach of fuzzy sets for GDM first aggregates the opinions of the DMs based on a pair-wise comparison of alternatives and only obtains the average preference information. Such an aggregation actually amounts to perform a transformation of THFEs into fuzzy numbers. The obtained average preference information may not be the optimal and true one. As a result, it leads to the result that the method that does not use HFSs cannot distinguish the preference order between the alternatives x_1 and x_3. However, introducing HFSs does not need to perform such an aggregation and, hence, provides a more comprehensive description of the opinions of these DMs. An incomplete HFPR could be any of the possible fuzzy preference values with any of the possible membership values. Just because of this, an incomplete HFPR contains the true fuzzy preference values and can be used to represent the DMs' hesitancy and uncertainty regarding several possible values for the preference degrees over the pairs of alternatives. The proposed hesitant fuzzy goal programming model (17) based on the α-normalization can not only derive the priority weights from incomplete HFPRs but also select the optimal preference value from all possible ones for each paired comparison of alternatives, and then obtain corresponding FPRs consisting of these optimal preferences as the reduction of the original incomplete HFPR. Hence, our method is more useful and efficient than the method which is not based on the use of HFSs.

In the present section, we have reduced HFPRs to FPRs by utilizing the α-normalization to remove some preference values in HFPRs. In the following, we use the opposite β-normalization to add some elements to HFPRs.

4. Deriving the priority weights from incomplete HFPRs under group decision making with β-normalization

4.1. Multiplicative consistency of HFPRs

Based on the β-normalization, Zhu et al. [51] introduced a method to add some elements to THFEs.

Definition 4.1 *([51])*. Assume a THFE, $h = \{h^{\sigma(s)} | s = 1, 2, \cdots, l_h\}$, let h^+ and h^- be the maximum and minimum elements in h respectively, and ς $(0 \le \varsigma \le 1)$ be an optimized parameter, which can be chosen by the decision makers according to their own risk preferences, then we call $\bar{h} = \varsigma h^+ + (1-\varsigma)h^-$ an added element.

Especially, $\bar{h} = h^+$ and $\bar{h} = h^-$ can be respectively derived from the conditions that $\varsigma = 1$ and $\varsigma = 0$, which correspond with the optimism and pessimism rules introduced by Xu and Xia [43], respectively.

Based on Definition 3.2, Zhu et al. [51] used the optimized parameter ς to add some elements to an HFPR, and obtained a normalized hesitant fuzzy preference relation (NHFPR) defined as follows.

Definition 4.2. Assume a HFPR, $H = (h_{ij})_{n\times n}$, and an optimized parameter ς $(0 \le \varsigma \le 1)$, where ς is used to add some elements to h_{ij} $(i<j)$, and $1-\varsigma$ is used to add some elements to h_{ji} $(i<j)$ to obtain a HFPR $^-H = (^-h_{ij})_{n\times n}$. And for all $i, j = 1, 2, \cdots, n$, this preference relation should satisfy the following conditions:

$$\begin{cases} l_{\bar{h}_{ij}} = \max\{l_{h_{ij}} | i, j = 1, 2, \cdots, n, i \ne j\} \\ \bar{h}_{ij}^{\sigma(s)} + \bar{h}_{ji}^{\sigma(s)} = 1, \bar{h}_{ii} = \{0.5\} \\ \bar{h}_{ij}^{\sigma(s)} \le \bar{h}_{ij}^{\sigma(s+1)}, \bar{h}_{ji}^{\sigma(s+1)} \le \bar{h}_{ji}^{\sigma(s)} \end{cases} \tag{19}$$

where $\bar{h}_{ij}^{\sigma(s)}$ and $\bar{h}_{ji}^{\sigma(s)}$ are the sth elements in $\bar{h}_{ij}$ and $\bar{h}_{ji}$, respectively. Then, we call $\bar{H} = (\bar{h}_{ij})_{n\times n}$ a normalized hesitant fuzzy preference relation (NHFPR) with the optimized parameter ς, and $\bar{h}_{ij}$ is a normalized hesitant fuzzy element (NTHFE).

Example 4.1. Let H be a HFPR given in Example 2.1. According to Definition 4.1, and let $\varsigma = 1$, then we can get the NHFPR $\bar{H}$ as follows:

$$\bar{H} = \begin{Bmatrix} \{0.5\} & \{0.4, 0.6, 0.7\} & \{0.2, 0.3, 0.3\} & \{0.5, 0.7, 0.7\} \\ \{0.6, 0.4, 0.3\} & \{0.5\} & \{0.1, 0.1, 0.1\} & \{0.8, 0.9, 0.9\} \\ \{0.8, 0.7, 0.7\} & \{0.9, 0.9, 0.9\} & \{0.5\} & \{0.3, 0.4, 0.4\} \\ \{0.5, 0.3, 0.3\} & \{0.2, 0.1, 0.1\} & \{0.7, 0.6, 0.6\} & \{0.5\} \end{Bmatrix}$$

Based on NHFPRs, Zhu et al. [51] defined the multiplicative consistent HFPR.

Definition 4.3 *([51])*. Given a HFPR $H = (h_{ij})_{n\times n}$ and its NHFPR $\bar{H} = (\bar{h}_{ij})_{n\times n}$ with ς, if for any $i, j, k = 1, 2, \cdots, n, i \ne j \ne k$,

$$\bar{h}_{ik}^{\sigma(s)} \bar{h}_{kj}^{\sigma(s)} \bar{h}_{ji}^{\sigma(s)} = \bar{h}_{ki}^{\sigma(s)} \bar{h}_{jk}^{\sigma(s)} \bar{h}_{ij}^{\sigma(s)} \tag{20}$$

where $\bar{h}_{ij}^{\sigma(s)}$ is the sth element in $\bar{h}_{ij}$, then H is called a multiplicative consistent HFPR with ς.

Theorem 4.1. Given a HFPR $H = (h_{ij})_{n\times n}$ and its NHFPR $\bar{H} = (\bar{h}_{ij})_{n\times n}$ with ς, the following statements are equivalent:

(1) H is multiplicative consistent;

(2) $$\bar{h}_{ij}^{\sigma(s)} = \frac{\bar{h}_{ik}^{\sigma(s)} \bar{h}_{kj}^{\sigma(s)}}{\bar{h}_{ik}^{\sigma(s)} \bar{h}_{kj}^{\sigma(s)} + (1-\bar{h}_{ik}^{\sigma(s)})(1-\bar{h}_{kj}^{\sigma(s)})}, \quad i, j, k = 1, 2, \cdots, n, \quad s = 1, 2, \cdots, l; \tag{21}$$

(3) $$\bar{h}_{ij}^{\sigma(s)} = \frac{\sqrt[n]{\prod_{k=1}^{n} \bar{h}_{ik}^{\sigma(s)} \bar{h}_{kj}^{\sigma(s)}}}{\sqrt[n]{\prod_{k=1}^{n} \bar{h}_{ik}^{\sigma(s)} \bar{h}_{kj}^{\sigma(s)}} + \sqrt[n]{\prod_{k=1}^{n} (1-\bar{h}_{ik}^{\sigma(s)})(1-\bar{h}_{kj}^{\sigma(s)})}}, \quad i, j = 1, 2, \cdots, n, \quad s = 1, 2, \cdots, l. \tag{22}$$

Proof. (1)⇔(2) This proof has been finished by Zhu et al. [51].

(1)⇒(3) By Definition 4.3, we have

$$\bar{h}_{ik}^{\sigma(s)} \bar{h}_{kj}^{\sigma(s)} \bar{h}_{ji}^{\sigma(s)} = \bar{h}_{ki}^{\sigma(s)} \bar{h}_{jk}^{\sigma(s)} \bar{h}_{ij}^{\sigma(s)}, \quad i, j, k = 1, 2, \cdots, n$$

that is

$$\bar{h}_{i1}^{\sigma(s)} \bar{h}_{1j}^{\sigma(s)} \bar{h}_{ji}^{\sigma(s)} = \bar{h}_{1i}^{\sigma(s)} \bar{h}_{j1}^{\sigma(s)} \bar{h}_{ij}^{\sigma(s)}, \quad i, j = 1, 2, \cdots, n \tag{23}$$

$$\bar{h}_{i2}^{\sigma(s)} \bar{h}_{2j}^{\sigma(s)} \bar{h}_{ji}^{\sigma(s)} = \bar{h}_{2i}^{\sigma(s)} \bar{h}_{j2}^{\sigma(s)} \bar{h}_{ij}^{\sigma(s)}, \quad i, j = 1, 2, \cdots, n \tag{24}$$

$$\vdots$$

$$\bar{h}_{in}^{\sigma(s)}\bar{h}_{nj}^{\sigma(s)}\bar{h}_{ji}^{\sigma(s)}=\bar{h}_{ni}^{\sigma(s)}\bar{h}_{jn}^{\sigma(s)}\bar{h}_{ij}^{\sigma(s)},\quad i,j=1,2,\cdots,n \tag{25}$$

Multiplying Eqs. (23)–(25) together, we have

$$\begin{aligned}&(\bar{h}_{ji}^{\sigma(s)})^n\prod_{k=1}^{n}\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}=(\bar{h}_{ij}^{\sigma(s)})^n\prod_{k=1}^{n}\bar{h}_{ki}^{\sigma(s)}\bar{h}_{jk}^{\sigma(s)},\ i,j=1,2,\cdots,n\\ &\Rightarrow(1-\bar{h}_{ij}^{\sigma(s)})\sqrt[n]{\prod_{k=1}^{n}\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}=\bar{h}_{ij}^{\sigma(s)}\sqrt[n]{\prod_{k=1}^{n}\bar{h}_{ki}^{\sigma(s)}\bar{h}_{jk}^{\sigma(s)}},\ i,j=1,2,\cdots,n\\ &\Rightarrow\bar{h}_{ij}^{\sigma(s)}=\frac{\sqrt[n]{\prod_{k=1}^{n}\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}}{\sqrt[n]{\prod_{k=1}^{n}\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}+\sqrt[n]{\prod_{k=1}^{n}(1-\bar{h}_{ik}^{\sigma(s)})(1-\bar{h}_{kj}^{\sigma(s)})}},\ i,j=1,2,\cdots,n\end{aligned} \tag{26}$$

(3)⇒(1) On the conversion, Eq. (26) can be written as

$$\left(\prod_{t=1}^{n}\bar{h}_{it}^{\sigma(s)}\bar{h}_{tj}^{\sigma(s)}\right)\left(\bar{h}_{ji}^{\sigma(s)}\right)^n=\left(\prod_{t=1}^{n}\bar{h}_{ti}^{\sigma(s)}\bar{h}_{jt}^{\sigma(s)}\right)\left(\bar{h}_{ij}^{\sigma(s)}\right)^n,\ i,j=1,2,\cdots,n \tag{27}$$

and we have the following equations:

$$\left(\prod_{t=1}^{n}\bar{h}_{it}^{\sigma(s)}\bar{h}_{tk}^{\sigma(s)}\right)\left(\bar{h}_{ki}^{\sigma(s)}\right)^n=\left(\prod_{t=1}^{n}\bar{h}_{ti}^{\sigma(s)}\bar{h}_{kt}^{\sigma(s)}\right)\left(\bar{h}_{ik}^{\sigma(s)}\right)^n,\ i,k=1,2,\cdots,n \tag{28}$$

$$\left(\prod_{t=1}^{n}\bar{h}_{kt}^{\sigma(s)}\bar{h}_{tj}^{\sigma(s)}\right)\left(\bar{h}_{jk}^{\sigma(s)}\right)^n=\left(\prod_{t=1}^{n}\bar{h}_{tk}^{\sigma(s)}\bar{h}_{jt}^{\sigma(s)}\right)\left(\bar{h}_{kj}^{\sigma(s)}\right)^n,\ k,j=1,2,\cdots,n \tag{29}$$

Multiplying Eqs. (28) and (29):

$$\left(\prod_{t=1}^{n}\bar{h}_{it}^{\sigma(s)}\bar{h}_{tj}^{\sigma(s)}\right)\left(\bar{h}_{ki}^{\sigma(s)}\right)^n\left(\bar{h}_{jk}^{\sigma(s)}\right)^n=\left(\prod_{t=1}^{n}\bar{h}_{ti}^{\sigma(s)}\bar{h}_{jt}^{\sigma(s)}\right)\left(\bar{h}_{ik}^{\sigma(s)}\right)^n\left(\bar{h}_{kj}^{\sigma(s)}\right)^n,\quad i,j=1,2,\cdots,n \tag{30}$$

Dividing Eq. (30) by Eq. (27), we have

$$\frac{\left(\bar{h}_{ki}^{\sigma(s)}\right)^n\left(\bar{h}_{jk}^{\sigma(s)}\right)^n}{\left(\bar{h}_{ji}^{\sigma(s)}\right)^n}=\frac{\left(\bar{h}_{ik}^{\sigma(s)}\right)^n\left(\bar{h}_{kj}^{\sigma(s)}\right)^n}{\left(\bar{h}_{ij}^{\sigma(s)}\right)^n},\quad i,j,k=1,2,\cdots,n \tag{31}$$

Namely,

$$\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}\bar{h}_{ji}^{\sigma(s)}=\bar{h}_{ki}^{\sigma(s)}\bar{h}_{jk}^{\sigma(s)}\bar{h}_{ij}^{\sigma(s)}\quad \mathrm{i,j,k}=1,2,\cdots,n \tag{32}$$

which indicates that H is multiplicative consistent, which completes the proof of Theorem 4.1.

Theorem 4.2 ([51]). Assume a HFPR $H=(h_{ij})_{n\times n}$ and its NHFPR $\bar{H}=(\bar{h}_{ij})_{n\times n}$ with ς, for $i,j=1,2,\cdots,n$, $i\neq j\neq k$, $s=1,2,\cdots,l$, let

$$\tilde{h}_{ij}^{\sigma(s)}=\frac{\sqrt[n]{\prod_{k=1}^{n}\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}}{\sqrt[n]{\prod_{k=1}^{n}\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}+\sqrt[n]{\prod_{k=1}^{n}\left(1-\bar{h}_{ik}^{\sigma(s)}\right)\left(1-\bar{h}_{kj}^{\sigma(s)}\right)}} \tag{33}$$

then, $\tilde{H}=(\tilde{h}_{ij})_{n\times n}$ is a multiplicative consistent HFPR with ς.

Theorem 4.2 provides us a method of constructing the multiplicative consistent HFPR from an inconsistent one.

Algorithm 1. ***Step 1:*** Assume a HFPR $H=(h_{ij})_{n\times n}$, and an optimized parameter ς $(0\leq\varsigma\leq 1)$, use Definition 4.2 to obtain a NHFPR $\bar{H}=(\bar{h}_{ij})_{n\times n}$.

Step 2: Utilize Theorem 4.2 to obtain the multiplicative consistent HFPR $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$.

Example 4.2. Let H be a HFPR shown in Example 2.1. Using Algorithm 1, the multiplicative consistent HFPR $\tilde{H} = (\tilde{h}_{ij})_{n\times n}$ of H is obtained as follows:

$$\tilde{H} = \begin{Bmatrix} \{0.5\} & \{0.4142, 0.5505, 0.6044\} & \{0.2438, 0.3639, 0.3899\} & \{0.4223, 0.6816, 0.7051\} \\ \{0.5858, 0.4495, 0.3956\} & \{0.5\} & \{0.3132, 0.3184, 0.2949\} & \{0.5083, 0.6361, 0.6101\} \\ \{0.7562, 0.6361, 0.6101\} & \{0.6868, 0.6816, 0.7051\} & \{0.5\} & \{0.6940, 0.7891, 0.7891\} \\ \{0.5777, 0.3184, 0.2949\} & \{0.4917, 0.3639, 0.3899\} & \{0.3060, 0.2109, 0.2109\} & \{0.5\} \end{Bmatrix}$$

4.2. An estimation procedure for acceptable incomplete HFPRs

In Section 3, we have defined the incomplete HFPRs. In the following, based on Definition 4.1, we can use the optimized parameter ς to add some elements to an incomplete HFPR, and obtained an incomplete NHFPR defined as follows.

Definition 4.4. Assume an incomplete HFPR, $H = (h_{ij})_{n\times n}$, and an optimized parameter ς $(0 \le \varsigma \le 1)$, where ς is used to add some elements to $h_{ij} \in \Omega\,(i < j)$, and $1 - \varsigma$ is used to add some elements to $h_{ji} \in \Omega\,(i < j)$ to obtain an incomplete HFPR ${}^{-}H = ({}^{-}h_{ij})_{n\times n}$. And for any $i, j = 1, 2, \cdots, n$, $\bar{h}_{ij} \in \Omega$, this preference relation should satisfy the following conditions:

$$\begin{cases} l_{\bar{h}_{ij}} = \max\left\{l_{h_{ij}} | i, j = 1, 2, \cdots, n, i \ne j, h_{ij} \in \Omega\right\} \\ \bar{h}_{ij}^{\sigma(s)} + \bar{h}_{ji}^{\sigma(s)} = 1, \bar{h}_{ii} = \{0.5\} \\ \bar{h}_{ij}^{\sigma(s)} \le \bar{h}_{ij}^{\sigma(s+1)}, \bar{h}_{ji}^{\sigma(s+1)} \le \bar{h}_{ji}^{\sigma(s)}, i < j \end{cases} \tag{34}$$

Then, we call $\bar{H} = (\bar{h}_{ij})_{n\times n}$ an incomplete NHFPR with the optimized parameter ς.

Example 4.3. Let H_1 be an incomplete HFPR given in Example 3.2. According to Definition 4.4, and let $\varsigma = 1$, then we can get the incomplete NHFPR $\bar{H}_1$ as follows:

$$\bar{H}_1 = \begin{Bmatrix} \{0.5\} & \{0.3, 0.5, 0.5\} & - & - \\ \{0.7, 0.5, 0.5\} & \{0.5\} & \{0.5, 0.6, 0.7\} & - \\ - & \{0.5, 0.4, 0.3\} & \{0.5\} & \{0.7, 0.8, 0.8\} \\ - & - & \{0.3, 0.2, 0.2\} & \{0.5\} \end{Bmatrix}$$

Similar to Definition 4.3, we define the following:

Definition 4.5. Assume an incomplete HFPR $H = (h_{ij})_{n\times n}$ and its incomplete NHFPR $\bar{H} = (\bar{h}_{ij})_{n\times n}$ with ς, if for any $i, j, k = 1, 2, \cdots, n$, $i \ne j \ne k$, $\bar{h}_{ij}, \bar{h}_{ik}, \bar{h}_{kj} \in \Omega$,

$$\bar{h}_{ij}^{\sigma(s)} = \frac{\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}{\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)} + (1 - \bar{h}_{ik}^{\sigma(s)})(1 - \bar{h}_{kj}^{\sigma(s)})} \tag{35}$$

where $\bar{h}_{ij}^{\sigma(s)}$ is the sth element in $\bar{h}_{ij}$, then H is called a multiplicative consistent incomplete HFPR with ς.

Let $H = (h_{ij})_{n\times n}$ be an incomplete HFPR, if $(i, j) \cap (k, l) \ne \emptyset$, then the elements h_{ij} and h_{kl} in H are called adjoining. For the missing element h_{ij}, if there exist two adjoining known elements h_{ik} and h_{kj}, then h_{ij} is called available. In such a case, the element h_{ij} can be obtained indirectly according to the known elements h_{ik} and h_{kj}. If each missing element can be derived from its adjoining known elements, then the incomplete HFPR H is called acceptable; otherwise, H is called unacceptable.

For an acceptable incomplete HFPR H, there exists at least one known element (except diagonal elements) in each line or each column of H, i.e., there exist at least $n - 1$ judgments provided by the expert (that is to say, each one of the alternatives is compared at least once). In what follows, we develop an approach to estimate the missing values from the known values and get the complete HFPR $\hat{H}$ of H.

Algorithm 2. ***Step 1:*** Assume an acceptable incomplete HFPR, $H = (h_{ij})_{n\times n}$, and an optimized parameter ς $(0 \le \varsigma \le 1)$, use Definition 4.4 to obtain an incomplete NHFPR, $\bar{H} = (\bar{h}_{ij})_{n\times n}$, with the optimized parameter ς.

Step 2: Each missing element $\bar{h}_{ij}$ in $\bar{H} = (\bar{h}_{ij})_{n\times n}$ can be estimated as $\hat{h}_{ij} = \left\{\hat{h}_{ij}^{\sigma(s)} | s = 1, 2, \cdots, l\right\}$, where

$$\hat{h}_{ij}^{\sigma(s)} = \frac{1}{k_{ij}} \sum_{k \in K_{ij}} \frac{\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)}}{\bar{h}_{ik}^{\sigma(s)}\bar{h}_{kj}^{\sigma(s)} + (1 - \bar{h}_{ik}^{\sigma(s)})(1 - \bar{h}_{kj}^{\sigma(s)})},$$

$$\text{for all} \quad \bar{h}_{ik}, \bar{h}_{kj} \in \Omega, \quad \text{and} \quad s = 1, 2, \cdots, l \tag{36}$$

where $l = \max\left\{l_{h_{ij}} | i, j = 1, 2, \cdots, n, i \neq j, h_{ij} \in \Omega\right\}$, $\bar{h}_{ik} = \left\{\bar{h}_{ik}^{\sigma(s)} | s = 1, 2, \cdots, l\right\}$, $\bar{h}_{kj} = \left\{\bar{h}_{kj}^{\sigma(s)} | s = 1, 2, \cdots, l\right\}$, and Ω is the set of all the known elements in $\bar{H} = (\bar{h}_{ij})_{n\times n}$, $K_{ij} = \left\{k | \bar{h}_{ik}, \bar{h}_{kj} \in \Omega\right\}$, and k_{ij} is the number of the elements in K_{ij}. After doing so, we get a complete HFPR $\hat{H} = (\hat{h}_{ij})_{n\times n}$ with the optimized parameter ς, where (1) $\hat{h}_{ij} = \bar{h}_{ij}$, if $\bar{h}_{ij} \in \Omega$; and (2) $\hat{h}_{ij} = \hat{h}_{ij}$, if $\bar{h}_{ij} \notin \Omega$.

Example 4.4. Let H_t $(t = 1, 2, 3)$ be three incomplete HFPR shown in Example 3.2, and $\varsigma = 1$. Using Algorithm 2, the complete HFPRs $\hat{H}_t$ $(t = 1, 2, 3)$ of H_t $(t = 1, 2, 3)$ are obtained as follows, respectively:

$$\hat{H}_1 = \begin{Bmatrix} \{0.5\} & \{0.3, 0.5, 0.5\} & \{0.3, 0.6, 0.7\} & \{0.5, 0.8571, 0.9032\} \\ \{0.7, 0.5, 0.5\} & \{0.5\} & \{0.5, 0.6, 0.7\} & \{0.7, 0.8571, 0.9032\} \\ \{0.7, 0.4, 0.3\} & \{0.5, 0.4, 0.3\} & \{0.5\} & \{0.7, 0.8, 0.8\} \\ \{0.5, 0.1429, 0.0968\} & \{0.3, 0.1429, 0.0968\} & \{0.3, 0.2, 0.2\} & \{0.5\} \end{Bmatrix}$$

$$\hat{H}_2 = \begin{Bmatrix} \{0.5\} & \{0.7, 0.8, 0.8\} & \{0.7382, 0.8206, 0.7865\} & \{0.3, 0.5, 0.5\} \\ \{0.3, 0.2, 0.2\} & \{0.5\} & \{0.7, 0.8, 0.9\} & \{0.2618, 0.4158, 0.5286\} \\ \{0.2618, 0.1794, 0.2135\} & \{0.3, 0.2, 0.1\} & \{0.5\} & \{0.2, 0.3, 0.4\} \\ \{0.7, 0.5, 0.5\} & \{0.7382, 0.5842, 0.4714\} & \{0.8, 0.7, 0.6\} & \{0.5\} \end{Bmatrix}$$

$$\hat{H}_3 = \begin{Bmatrix} \{0.5\} & \{0.3, 0.4, 0.4\} & \{0.4, 0.5, 0.6\} & \{0.5, 0.7, 0.7\} \\ \{0.7, 0.6, 0.6\} & \{0.5\} & \{0.6680, 0.6971, 0.6923\} & \{0.8, 0.9, 0.9\} \\ \{0.6, 0.5, 0.4\} & \{0.3320, 0.3029, 0.3077\} & \{0.5\} & \{0.6, 0.7, 0.8\} \\ \{0.5, 0.3, 0.3\} & \{0.2, 0.1, 0.1\} & \{0.4, 0.3, 0.2\} & \{0.5\} \end{Bmatrix}$$

4.3. Deriving the priority weights from a HFPR based on multiplicative consistency

Let $H = (h_{ij})_{n\times n}$ be a HFPR and $\bar{H} = (\bar{h}_{ij})_{n\times n} = \left(\left\{\bar{h}_{ij}^{\sigma(s)} | s = 1, 2, \cdots, l\right\}\right)_{n\times n}$ ($l = \max\left\{l_{h_{ij}} | i, j = 1, 2, \cdots, n, i \neq j\right\}$) be its NHFPR. $\bar{H}$ can be divided into l fuzzy preference relations (FPRs) denoted by $R^{(s)} = (r_{ij}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$, where

$$r_{ij}^{(s)} = \bar{h}_{ij}^{\sigma(s)}, \quad i, j = 1, 2, \cdots, n, \quad s = 1, 2, \cdots, l \tag{37}$$

From Eqs. (19) and (37), it follows that $r_{ij}^{(s)} + r_{ji}^{(s)} = \bar{h}_{ij}^{\sigma(s)} + \bar{h}_{ji}^{\sigma(s)} = 1$, for $i \neq j$; $r_{ij}^{(s)} + r_{ij}^{(s)} = 0..5 + 0.5 = 1$, for $i = j$. Thus, $R^{(s)} = (r_{ij}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$ are l FRRs constructed by the possible values of $\bar{H}$.

Based on Definition 4.3, we can easily obtain the following theorem:

Theorem 4.3. Assume a HFPR $H = (h_{ij})_{n\times n}$ and its NHFPR $\bar{H} = (\bar{h}_{ij})_{n\times n}$ with ς, H is a multiplicative consistent HFPR if and only if $R^{(s)} = (r_{ij}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$ defined by Eq. (37) are all multiplicative consistent.

From Theorem 4.3, in order to check whether H is multiplicative consistent or not, it is sufficient to check the multiplicative consistency of $R^{(s)} = (r_{ij}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$. When one of $R^{(s)} = (r_{ij}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$ does not possess multiplicative consistency, H is not multiplicative consistent.

In the following, we introduce a convex combination method for generating the priority weights from a HFPR H.

Definition 4.6. Given a HFPR $H = (h_{ij})_{n\times n}$ and its NHFPR $\bar{H} = (\bar{h}_{ij})_{n\times n}$ with ς, $R^{(s)} = (r_{ij}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$ are defined by Eq. (37), $\alpha \in L = \left\{(\alpha_1, \alpha_2, \cdots, \alpha_l)^T | \alpha_s \geq 0, s = 1, 2, \cdots, l, \sum_{s=1}^{l} \alpha_s = 1\right\}$, then we call $R(\alpha) = (r_{ij}(\alpha))_{n\times n}$ a $\alpha-$ reciprocal relation of H, where

$$r_{ij}(\alpha) = \frac{\prod_{s=1}^{l} (r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l} (r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l} (1 - r_{ij}^{(s)})^{\alpha_s}} \tag{38}$$

From Eq. (38), one can find that every element of $R(\alpha)$ is a convex combination of the corresponding elements in $R^{(s)}$ $(s = 1, 2, \cdots, l)$. Making use of Eq. (38), it is easy to obtain the following property theorem:

Theorem 4.4. Let $H = (h_{ij})_{n\times n}$ be a HFPR and $\bar{H} = (\bar{h}_{ij})_{n\times n}$ be its NHFPR with ς, then we have

(1) For $\alpha = \left(\underbrace{0,\cdots,0}_{s-1}, 1, \underbrace{0,\cdots,0}_{l-s}\right)$, $R(\alpha) = R^{(s)}$.

(2) $r_{ij}(\alpha) \in \left[\min\limits_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}, \max\limits_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}\right]$, where $\forall\alpha \in L$.

(3) $R(\alpha)$ is a FPR, where $\forall\alpha \in L$.

Proof. (1) If $\alpha = \left(\underbrace{0,\cdots,0}_{s-1}, 1, \underbrace{0,\cdots,0}_{l-s}\right)$, then $r_{ij}(\alpha) = \frac{r_{ij}^{(s)}}{r_{ij}^{(s)}+(1-r_{ij}^{(s)})} = r_{ij}^{(s)}$, therefore, we have $R(\alpha) = R^{(s)}$.

(2) From Eq. (38), it follows that

$$r_{ij}(\alpha) = \frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}} = \frac{1}{1+\prod_{s=1}^{l}(\frac{1}{r_{ij}^{(s)}}-1)^{\alpha_s}}$$

Since

$$\prod_{s=1}^{l}(\frac{1}{\max\limits_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}}-1)^{\alpha_s} \le \prod_{s=1}^{l}(\frac{1}{r_{ij}^{(s)}}-1)^{\alpha_s} \le \prod_{s=1}^{l}(\frac{1}{\min\limits_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}}-1)^{\alpha_s}$$

then

$$\min_{1\le s\le l}\left\{r_{ij}^{(s)}\right\} = \frac{1}{1+\prod_{s=1}^{l}(\frac{1}{\min\limits_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}}-1)^{\alpha_s}} \le \frac{1}{1+\prod_{s=1}^{l}(\frac{1}{r_{ij}^{(s)}}-1)^{\alpha_s}} \le \frac{1}{1+\prod_{s=1}^{l}(\frac{1}{\max\limits_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}}-1)^{\alpha_s}} = \max_{1\le s\le l}\left\{r_{ij}^{(s)}\right\}$$

(3) For any $i, j = 1, 2, \cdots, n$,

$$\begin{aligned} r_{ij}(\alpha)+r_{ji}(\alpha) &= \frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}} + \frac{\prod_{s=1}^{l}(r_{ji}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ji}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(1-r_{ji}^{(s)})^{\alpha_s}} \\ &= \frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}} + \frac{\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}} \\ &= \frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s} + \prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}} = 1 \end{aligned}$$

Therefore, $R(\alpha)$ is a FPR.

Theorem 4.5. *H is multiplicative consistent if and only if $R(\alpha)$ is multiplicative consistent for any $\alpha \in L$.*

Proof. Suppose that $R(\alpha)$ is multiplicative consistent, then $R(\alpha) = R^{(s)}$ for $\alpha = \left(\underbrace{0,\cdots,0}_{s-1}, 1, \underbrace{0,\cdots,0}_{l-s}\right)$ and $s = 1, 2, \cdots, l$, thus, $R^{(s)}$ ($s = 1, 2, \cdots, l$) are all multiplicative consistent. By Theorem 4.3, we can obtain that H is multiplicative consistent.

Conversely, assume that H is multiplicative consistent, then $R^{(s)}$ $(s=1,2,\cdots,l)$ are multiplicative consistent, and we have

$$
\begin{aligned}
& r_{ik}(\alpha)r_{kj}(\alpha)r_{ji}(\alpha) \\
&= \frac{\prod_{s=1}^{l}(r_{ik}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ik}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ik}^{(s)})^{\alpha_s}} \times \frac{\prod_{s=1}^{l}(r_{kj}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{kj}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{kj}^{(s)})^{\alpha_s}} \times \frac{\prod_{s=1}^{l}(r_{ji}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ji}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ji}^{(s)})^{\alpha_s}} \\
&= \frac{\prod_{s=1}^{l}(r_{ik}^{(s)}r_{kj}^{(s)}r_{ji}^{(s)})^{\alpha_s}}{(\prod_{s=1}^{l}(r_{ki}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ki}^{(s)})^{\alpha_s})\times(\prod_{s=1}^{l}(r_{jk}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{jk}^{(s)})^{\alpha_s})\times(\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s})} \\
&= \frac{\prod_{s=1}^{l}(r_{ki}^{(s)}r_{jk}^{(s)}r_{ij}^{(s)})^{\alpha_s}}{(\prod_{s=1}^{l}(r_{ki}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ki}^{(s)})^{\alpha_s})\times(\prod_{s=1}^{l}(r_{jk}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{jk}^{(s)})^{\alpha_s})\times(\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s})} \\
&= \frac{\prod_{s=1}^{l}(r_{ki}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ki}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ki}^{(s)})^{\alpha_s}} \times \frac{\prod_{s=1}^{l}(r_{jk}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{jk}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{jk}^{(s)})^{\alpha_s}} \times \frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}} \\
&= r_{ki}(\alpha)r_{jk}(\alpha)r_{ij}(\alpha)
\end{aligned}
$$

By Definition 2.2, we can obtain that $R(\alpha)$ is multiplicative consistent, which completes the proof of Theorem 4.5.

As observed in Theorem 4.5, when H is a multiplicative consistent HFPR, a family of FPRs $R(\alpha)$ $(\alpha \in L)$ with multiplicative consistency are obtained.

When a HFPR is multiplicative consistent by Theorem 4.3, the derived weights are consistent. In what follows, the weights generated from acceptably consistent HFPRs are only considered. In the case of HFPRs, it is considered that interval weights should be more logical and natural than an exact priority vector to represent part or complete imprecision of the DM's hesitant judgments. Consequently, interval weights derived from acceptably consistent HFPRs will be determined.

To achieve the interval weights, it is convenient to investigate the weights of crisp FPRs $R(\alpha)$ $(\alpha \in L)$ with multiplicative consistency by considering Theorem 4.5. Let us suppose that the weights $w_i(\alpha)$ of $R(\alpha)$ satisfy $\prod_{i=1}^{n} w_i(\alpha)=1$, and the weights $w_i(R^{(s)})$ of $R^{(s)}$ satisfy $\prod_{i=1}^{n} w_i(R^{(s)})=1$. Then, the relationships between $w_i(\alpha)$ and $w_i(R^{(s)})$ are shown in the following theorem:

Theorem 4.6. If $w(\alpha)$ and $w(R^{(s)})$ $(s=1,2,\cdots,l)$ are the weight vectors of $R(\alpha)$ and $R^{(s)}$ $(s=1,2,\cdots,l)$, respectively, defined by Eq. (5), then

$$
w_i(\alpha)=\prod_{s=1}^{l}(w_i(R^{(s)}))^{\alpha_s}, \quad i=1,2,\cdots,n, \tag{39}
$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_l)^T$, $\sum_{s=1}^{l}\alpha_s=1$, $w(\alpha)=(w_1(\alpha),w_2(\alpha),\cdots,w_n(\alpha))^T$, $\prod_{i=1}^{n}w_i(\alpha)=1$, $w(R^{(s)})=(w_1(R^{(s)}),w_2(R^{(s)}),\cdots,w_n(R^{(s)}))$, and $\prod_{i=1}^{n}w_i(R^{(s)})=1$, $s=1,2,\cdots,l$.

Proof. By Eqs. (5) and (38), we have

$$w_i(\alpha)=\frac{\sqrt[n]{\prod_{j=1}^{n} r_{ij}(\alpha)}}{\sqrt[n]{\prod_{j=1}^{n}(1-r_{ij}(\alpha))}}=\frac{\sqrt[n]{\prod_{j=1}^{n}\frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}}}}{\sqrt[n]{\prod_{j=1}^{n}(1-\frac{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}{\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}+\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}})}}=\frac{\sqrt[n]{\prod_{j=1}^{n}\prod_{s=1}^{l}(r_{ij}^{(s)})^{\alpha_s}}}{\sqrt[n]{\prod_{j=1}^{n}\prod_{s=1}^{l}(1-r_{ij}^{(s)})^{\alpha_s}}}$$

$$=\prod_{s=1}^{l}\left(\frac{\sqrt[n]{\prod_{j=1}^{n} r_{ij}^{(s)}}}{\sqrt[n]{\prod_{j=1}^{n}(1-r_{ij}^{(s)})}}\right)^{\alpha_s}=\prod_{s=1}^{l}(w_i(R^{(s)}))^{\alpha_s}$$

Thus, the proof is completed.

Furthermore, in order to obtain the synthetic priority of $H=(h_{ij})_{n\times n}$, the weight functions $w_i(\alpha)$ $(\alpha\in L)$ derived from $R(\alpha)$ are further aggregated as the following interval numbers:

$$w_i=[\min\left\{w_i(\alpha)|\alpha\in L\right\},\max\left\{w_i(\alpha)|\alpha\in L\right\}],\quad i=1,2,\cdots,n \tag{40}$$

Lemma 4.1 ([27,32]). Let $z_k>0$, $\upsilon_k>0$, $k=1,2,\cdots,m$, and $\sum_{k=1}^{m}\upsilon_k=1$. Then,

$$\prod_{k=1}^{m} z_k^{\upsilon_k}\le\sum_{k=1}^{m}\upsilon_k z_k \tag{41}$$

with equality if and only if $z_1=z_2=\cdots=z_m$.

Theorem 4.7. Let w_i $(i=1,2,\cdots,n)$ are given by Eq. (40), then Eq. (40) can be equivalently rewritten as

$$w_i=[\min\left\{w_i(R^{(s)})|s=1,2,\cdots,l\right\},\max\left\{w_i(R^{(s)})|s=1,2,\cdots,l\right\}],\quad i=1,2,\cdots,n \tag{42}$$

Proof. Let $\max\left\{w_i(R^{(s)})|s=1,2,\cdots,l\right\}=w_i(R^{(p)})$, then by Eqs. (39) and (41), we have

$$w_i(\alpha)=\prod_{s=1}^{l}(w_i(R^{(s)}))^{\alpha_s}\le\sum_{s=1}^{l}(\alpha_s\cdot w_i(R^{(s)}))\le(\sum_{s=1}^{l}\alpha_s)\cdot w_i(R^{(p)})=w_i(R^{(p)}).$$

Let $\alpha=\left(\underbrace{0,\cdots,0}_{p-1},1,\underbrace{0,\cdots,0}_{l-p}\right)$, then $w_i(\alpha)=\prod_{s=1}^{l}(w_i(R^{(s)}))^{\alpha_s}=w_i(R^{(p)})$, thus we can obtain

$$\max\left\{w_i(\alpha)|\alpha\in L\right\}=w_i(R^{(p)})=\max\left\{w_i(R^{(s)})|s=1,2,\cdots,l\right\}$$

In addition, let $\min\left\{w_i(R^{(s)})|s=1,2,\cdots,l\right\}=w_i(R^{(q)})$, then by Eq. (39), we have

$$w_i(\alpha)=\prod_{s=1}^{l}(w_i(R^{(s)}))^{\alpha_s}\ge\prod_{s=1}^{l}(w_i(R^{(q)}))^{\alpha_s}=(w_i(R^{(q)}))^{\sum_{s=1}^{l}\alpha_s}=w_i(R^{(q)})$$

Let $\alpha=\left(\underbrace{0,\cdots,0}_{q-1},1,\underbrace{0,\cdots,0}_{l-q}\right)$, then $w_i(\alpha)=\prod_{s=1}^{l}(w_i(R^{(s)}))^{\alpha_s}=w_i(R^{(q)})$, thus we can obtain

$$\min\left\{w_i(\alpha)|\alpha\in L\right\}=w_i(R^{(q)})=\min\left\{w_i(R^{(s)})|s=1,2,\cdots,l\right\}$$

Based on the above analysis, we can conclude that

$$w_i = \left[\min\left\{w_i(\alpha)|\alpha \in L\right\}, \max\left\{w_i(\alpha)|\alpha \in L\right\}\right]$$
$$= \left[\min\left\{w_i(R^{(s)})|s = 1, 2, \cdots, l\right\}, \max\left\{w_i(R^{(s)})|s = 1, 2, \cdots, l\right\}\right]$$

Thus, the proof is completed.

4.4. Deriving the priority weights from HFPRs based on multiplicative consistency under group decision making environments

In real world, however, many decision making processes take place in multi-person settings because the increasing complexity and uncertainty of the socio-economic environment makes it less and less possible for single decision maker to consider all relevant aspects of a decision making problem. Next, a procedure is developed for deriving the priority weights from HFPRs under GDM based on β-normalization.

A GDM can be described as follows: let $X = \{x_1, x_2, \cdots, x_n\}$ be the set of alternatives, and $D = \left\{d_1, d_2, \cdots, d_m\right\}$ be the set of DMs. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T$ be the weight vector of DMs, where $\lambda_t \in [0, 1]$, $t = 1, 2, \cdots, m$, and $\sum_{t=1}^{m}\lambda_t = 1$. Suppose that the DM $d_t \in D$ compares each pair of the alternatives x_i $(i = 1, 2, \cdots, n)$ and constructs a HFPR $H_t = (h_{ij,t})_{n\times n} = \left(\left\{h_{ij,t}^{\sigma(s)}|s = 1, 2, \cdots, l_{h_{ij,t}}\right\}\right)_{n\times n}$, where $h_{ij,t}$ $(i < j)$ is a THFE, which indicates hesitant degrees to which x_i is preferred to x_j, and satisfies

$$\begin{cases} h_{ij,t}^{\sigma(s)} + h_{ji,t}^{\sigma(l_{h_{ij,t}}-s+1)} = 1, \\ h_{ii,t} = \left\{0.5\right\}, \\ l_{h_{ij,t}} = l_{h_{ji,t}} \end{cases} \tag{43}$$

where $h_{ij,t}^{\sigma(s)}$ and $h_{ji,t}^{\sigma(s)}$ are the sth elements in $h_{ij,t}$ and $h_{ji,t}$, respectively.

Utilizing Definition 4.1, we can transform all the individual HFPRs $H_t = (h_{ij,t})_{n\times n} = \left(\left\{h_{ij,t}^{\sigma(s)}|s = 1, 2, \cdots, l_{h_{ij,t}}\right\}\right)_{n\times n}$ $(t = 1, 2, \cdots, m)$ into their normalized HFPRs (NHFPRs) $\bar{H}_t = (\bar{h}_{ij,t})_{n\times n} = \left(\left\{\bar{h}_{ij,t}^{\sigma(s)}|s = 1, 2, \cdots, l\right\}\right)_{n\times n}$ with ς, which satisfy the following conditions:

$$\begin{cases} l_{\bar{h}_{ij,t}} = \max\left\{l_{h_{ij,t}}|i, j = 1, 2, \cdots, n, i \neq j, t = 1, 2, \cdots, m\right\}, \\ \bar{h}_{ij,t}^{\sigma(s)} + \bar{h}_{ji,t}^{\sigma(s)} = 1, \\ \bar{h}_{ii,t} = \left\{0.5\right\}, \\ \bar{b}_{ij,k}^{\sigma(s)} \leq \bar{b}_{ij,k}^{\sigma(s+1)}, \bar{b}_{ji,k}^{\sigma(s+1)} \leq \bar{b}_{ji,k}^{\sigma(s)}, i < j \end{cases} \tag{44}$$

where $\bar{h}_{ij,t}^{\sigma(s)}$ and $\bar{h}_{ji,t}^{\sigma(s)}$ are the sth elements in $\bar{h}_{ij,t}$ and $\bar{h}_{ji,t}$, respectively.

A collective HFPR $H_c = (h_{ij,c})_{n\times n}$ can be derived from all the individual HFPRs $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2, \cdots, m)$ by the following fusion method:

$$h_{ij,c}^{\sigma(s)} = \frac{\prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}} \quad (i, j = 1, 2, \cdots, n, s = 1, 2, \cdots, l) \tag{45}$$

Theorem 4.8. Let $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ be m HFPRs, $\bar{H}_t = (\bar{h}_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ be their NHFPRs with ς, and $\lambda_t \in [0, 1]$, $t = 1, 2, \cdots, m$, $\sum_{t=1}^{m}\lambda_t = 1$. If $H_c = (h_{ij,c})_{n\times n}$ is constructed through Eq. (45), then H_c is a HFPR.

Proof. According to Eq. (45), for all $i, j = 1, 2, \cdots, n$, we have

$$h_{ij,c}^{\sigma(s)} + h_{ji,c}^{\sigma(s)} = \frac{\prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}} + \frac{\prod_{t=1}^{m}(1-\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(1-\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}}$$
$$= \frac{\prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-\bar{h}_{ij,t}^{\sigma(s)})^{\lambda_t}} = 1$$

According to Definition 2.5, H_c is a HFPR, which completes the proof of Theorem 4.8.

Theorem 4.9. If all individual HFPRs $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ are multiplicative consistent, then their collective HFPR $H_c = (h_{ij,c})_{n\times n}$ is also multiplicative consistent.

Proof. Assume that $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ are multiplicative consistent, then for any $i, j, k = 1, 2, \cdots, n$, $i \neq j \neq k$, $s = 1, 2, \cdots, l$, we have

$$h_{ik,c}^{\sigma(s)} \times h_{kj,c}^{\sigma(s)} \times h_{ji,c}^{\sigma(s)}$$
$$= \frac{\prod_{t=1}^{m}(h_{ik,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(h_{ik,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-h_{ik,t}^{\sigma(s)})^{\lambda_t}} \times \frac{\prod_{t=1}^{m}(h_{kj,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(h_{kj,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-h_{kj,t}^{\sigma(s)})^{\lambda_t}} \times \frac{\prod_{t=1}^{m}(h_{ji,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(h_{ji,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-h_{ji,t}^{\sigma(s)})^{\lambda_t}}$$
$$= \frac{\prod_{t=1}^{m}(h_{ik,t}^{\sigma(s)} \times h_{kj,t}^{\sigma(s)} \times h_{ji,t}^{\sigma(s)})^{\lambda_t}}{(\prod_{t=1}^{m}(1-h_{ki,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(h_{ki,t}^{\sigma(s)})^{\lambda_t}) \times (\prod_{t=1}^{m}(1-h_{jk,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(h_{jk,t}^{\sigma(s)})^{\lambda_t}) \times (\prod_{t=1}^{m}(1-h_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(h_{ij,t}^{\sigma(s)})^{\lambda_t})}$$
$$= \frac{\prod_{t=1}^{m}(h_{ki,t}^{\sigma(s)} \times h_{jk,t}^{\sigma(s)} \times h_{ij,t}^{\sigma(s)})^{\lambda_t}}{(\prod_{t=1}^{m}(1-h_{ki,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(h_{ki,t}^{\sigma(s)})^{\lambda_t}) \times (\prod_{t=1}^{m}(1-h_{jk,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(h_{jk,t}^{\sigma(s)})^{\lambda_t}) \times (\prod_{t=1}^{m}(1-h_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(h_{ij,t}^{\sigma(s)})^{\lambda_t})}$$
$$= \frac{\prod_{t=1}^{m}(h_{ki,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(h_{ki,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-h_{ki,t}^{\sigma(s)})^{\lambda_t}} \times \frac{\prod_{t=1}^{m}(h_{jk,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(h_{jk,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-h_{jk,t}^{\sigma(s)})^{\lambda_t}} \times \frac{\prod_{t=1}^{m}(h_{ij,t}^{\sigma(s)})^{\lambda_t}}{\prod_{t=1}^{m}(h_{ij,t}^{\sigma(s)})^{\lambda_t} + \prod_{t=1}^{m}(1-h_{ij,t}^{\sigma(s)})^{\lambda_t}}$$
$$= h_{ki,c}^{\sigma(s)} \times h_{jk,c}^{\sigma(s)} \times h_{ij,c}^{\sigma(s)}$$

which implies that $H_c = (h_{ij,c})_{n\times n}$ is multiplicative consistent.

From the above analysis, we develop an algorithm for obtaining the priority weight vector from HFPRs under GDM situations based on multiplicative consistency, which consists of the following steps:

Algorithm 3. ***Step 1.*** If the HFPRs $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ given by the decision makers d_t $(t = 1, 2\cdots, m)$ are complete, then go to step 2; otherwise, construct the complete HFPR $\hat{H}_t = (\hat{h}_{ij,t})_{n\times n}$ from H_t by Algorithm 2, and let $H_t = \hat{H}_t$, go to Step 2.

Step 2. Construct multiplicative consistent HFPRs $\tilde{H}_t = (\tilde{h}_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ from $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ by Algorithm 1.

Step 3. Aggregate the individual HFPRs $\tilde{H}_t = (\tilde{h}_{ij,t})_{n\times n}$ $(t = 1, 2\cdots, m)$ into a collective HFPR $H_c = (h_{ij,c})_{n\times n}$ by Eq. (45).

Step 4. Derive s FPRs $R_c^{(s)} = (r_{ij,c}^{(s)})_{n\times n}$ $(s = 1, 2, \cdots, l)$ from $H_c = (h_{ij,c})_{n\times n}$ by Eq. (37), and then compute the priority weight vector $w(R_c^{(s)})$ $(s = 1, 2, \cdots, l)$ of $R_c^{(s)}$ $(s = 1, 2, \cdots, l)$ by Eq. (5).

Step 5. Compute the priority weight vector $w = (w_1, w_2, \cdots, w_n)$ of H_c by Eq. (42).

Step 6. To rank these interval weights $w_i = [w_i^-, w_i^+]$ $(i = 1, 2, \cdots, n)$, we first compare each w_i with all the w_j $(j = 1, 2, \cdots, n)$, then compute the degree of possibility of $w_i \geq w_j$ as follows [41,42]:

$$p(w_i \geq w_j) = \max\left\{1 - \max\left\{\frac{w_j^+ - w_i^-}{w_i^+ - w_i^- + w_j^+ - w_j^-}, 0\right\}, 0\right\} \quad (46)$$

that is, w_i is superior to w_j to the degree of $p(w_i \geq w_j)$, denoted by $w_i \overset{p(w_i \geq w_j)}{\succ} w_j$.

For simplicity, we let $p_{ij} = p(w_i \geq w_j)$, then we develop a complementary matrix as $P = (p_{ij})_{m\times m}$ as below:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \cdots, n$.

Summing all elements in each line of the matrix P, we have

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad i = 1, 2, \cdots, n \tag{47}$$

Then we rank these interval weights $w_i = [w_i^-, w_i^+]$ $(i = 1, 2, \cdots, n)$ in descending order in accordance with the values of p_i $(i = 1, 2, \cdots, n)$.

Step 7. End.

Example 4.5 *((Continued with Example 3.3))*. In the following, we use Algorithm 3 to derive a collective priority weight vector from three incomplete HFPRs $H_t = (h_{ij,t})_{n\times n}$ $(t = 1, 2, 3)$ given in Example 3.2.

Step 1. Since H_t $(t = 1, 2\cdots, m)$ are incomplete, we construct the complete HFPRs $\hat{H}_t$ $(t = 1, 2, 3)$ from H_t $(t = 1, 2, 3)$ by Algorithm 2. $\hat{H}_t$ $(t = 1, 2, 3)$ have been obtained in Example 4.4.

Step 2. Construct multiplicative consistent HFPRs $\tilde{H}_t$ $(t = 1, 2\cdots, m)$ from $\hat{H}_t$ $(t = 1, 2\cdots, m)$ by Algorithm 1.

$$\tilde{H}_1 = \begin{Bmatrix} \{0.5\} & \{0.3, 0.5, 0.5\} & \{0.3, 0.6, 0.7\} & \{0.5, 0.8571, 0.9032\} \\ \{0.7, 0.5, 0.5\} & \{0.5\} & \{0.5, 0.6, 0.7\} & \{0.7, 0.8571, 0.9032\} \\ \{0.7, 0.4, 0.3\} & \{0.5, 0.4, 0.3\} & \{0.5\} & \{0.7, 0.8, 0.8\} \\ \{0.5, 0.1429, 0.0968\} & \{0.3, 0.1429, 0.0968\} & \{0.3, 0.2, 0.2\} & \{0.5\} \end{Bmatrix}$$

$$\tilde{H}_2 = \begin{Bmatrix} \{0.5\} & \{0.6268, 0.6925, 0.6085\} & \{0.7459, 0.8409, 0.8388\} & \{0.3639, 0.6058, 0.6456\} \\ \{0.3732, 0.3075, 0.3915\} & \{0.5\} & \{0.6361, 0.7013, 0.7699\} & \{0.2541, 0.4057, 0.5396\} \\ \{0.2541, 0.1591, 0.1612\} & \{0.3639, 0.2987, 0.2301\} & \{0.5\} & \{0.1631, 0.2253, 0.2593\} \\ \{0.6361, 0.3942, 0.3544\} & \{0.7459, 0.5943, 0.4604\} & \{0.8369, 0.7747, 0.7407\} & \{0.5\} \end{Bmatrix}$$

$$\tilde{H}_3 = \begin{Bmatrix} \{0.5\} & \{0.2599, 0.3211, 0.3449\} & \{0.4155, 0.5267, 0.5422\} & \{0.5336, 0.7471, 0.7891\} \\ \{0.7401, 0.6789, 0.6551\} & \{0.5\} & \{0.6693, 0.7017, 0.6923\} & \{0.7651, 0.8620, 0.8767\} \\ \{0.5845, 0.4733, 0.4578\} & \{0.3307, 0.2983, 0.3077\} & \{0.5\} & \{0.6168, 0.7264, 0.7595\} \\ \{0.4664, 0.2529, 0.2109\} & \{0.2349, 0.1380, 0.1233\} & \{0.3832, 0.2736, 0.2405\} & \{0.5\} \end{Bmatrix}$$

Step 3. Aggregate the individual multiplicative consistent HFPRs $\tilde{H}_t$ $(t = 1, 2, 3)$ into a collective multiplicative consistent HFPR H_c by Eq. (45).

$$H_c = \begin{Bmatrix} \{0.5\} & \{0.3466, 0.4844, 0.4740\} & \{0.4230, 0.6383, 0.6909\} & \{0.4822, 0.7870, 0.8365\} \\ \{0.6534, 0.5156, 0.5260\} & \{0.5\} & \{0.5801, 0.6525, 0.7127\} & \{0.6371, 0.7972, 0.8503\} \\ \{0.5770, 0.3617, 0.3091\} & \{0.4199, 0.3475, 0.2873\} & \{0.5\} & \{0.5596, 0.6768, 0.6960\} \\ \{0.5178, 0.2130, 0.1635\} & \{0.3629, 0.2028, 0.1497\} & \{0.4404, 0.3232, 0.3040\} & \{0.5\} \end{Bmatrix}$$

Step 4. Derive s FPRs $R_c^{(s)} = (r_{ij,c}^{(s)})_{n\times n}$ $(s = 1, 2, 3)$ from $H_c = (h_{ij,c})_{n\times n}$ by Eq. (37), and compute the priority weight vector $w(R_c^{(s)})$ $(s = 1, 2, 3)$ of $R_c^{(s)}$ $(s = 1, 2, 3)$ by Eq. (5), where

$w(R_c^{(1)}) = (0.7757, 1.4622, 1.0583, 0.8330)$, $w(R_c^{(2)}) = (1.5732, 1.6742, 0.8916, 0.4258)$, $w(R_c^{(3)}) = (1.7917, 1.9885, 0.8016, 0.3501)$

Step 5. Compute the priority weight vector $w = (w_1, w_2, w_3, w_4)$ of H_c by Eq. (42) as follows:

$$w = ([0.7757, 1.7917], [1.4622, 1.9885], [0.8016, 1.0583], [0.3501, 0.8330])^T$$

Step 6. By using Eq. (46) to compare each w_i with each w_j $(j = 1, 2, 3, 4)$, we develop a complementary matrix $P = (p_{ij})_{4\times 4}$ as

$$P = \begin{bmatrix} 0.5000 & 0.2136 & 0.7779 & 0.9618 \\ 0.7864 & 0.5000 & 1.0000 & 1.0000 \\ 0.2221 & 0 & 0.5000 & 0.9575 \\ 0.0382 & 0 & 0.0425 & 0.5000 \end{bmatrix}$$

Summing all elements in each line of the matrix P, we have,

$$p_1=\sum_{j=1}^{4}p_{1j}=2.4534,\quad p_2=\sum_{j=1}^{4}p_{2j}=3.2864,\quad p_3=\sum_{j=1}^{4}p_{3j}=1.6796,\quad p_4=\sum_{j=1}^{4}p_{4j}=0.5807$$

thus, the ranking of four alternatives is as below:

$$w_2 \overset{0.7864}{\succ} w_1 \overset{0.7779}{\succ} w_3 \overset{0.9575}{\succ} w_4$$

which indicates that w_2 is superior to w_1 to the degree of 78.64%, w_1 is superior to w_3 to the degree of 77.79%, and w_3 is superior to w_4 to the degree of 95.75%.

Example 4.6. In the following, Example 4.5 is further used to compare the results of ranking obtained with hesitant fuzzy preference information and with fuzzy preference information. We use a method which does not use HFSs to revisit Example 4.5 and clarify the advantages of the proposed Algorithm 3 compared to the method which are not based on HFSs.

Step 1: Average all of the possible preference opinions in each incomplete HFPR $H_t=(h_{ij,t})_{4\times 4}$ $(t=1,2,3)$ and obtain the averaged incomplete FPR $R_t=(r_{ij,t})_{4\times 4}$ $(t=1,2,3)$, which have been completed in Example 3.4.

Step 2: Substituting $R_t=(r_{ij,t})_{4\times 4}$ $(t=1,2,3)$ into Eq. (36) in which $\bar{h}_{ij,t}=r_{ij,t}$ for all $i,j=1,2,3,4$, $t=1,2,3$, we obtain the complete FPRs $\hat{R}_t=(\hat{r}_{ij,t})_{4\times 4}$ $(t=1,2,3)$ as follows:

$$\hat{R}_1=\begin{Bmatrix}0.5 & 0.4 & 0.5 & 0.75\\ 0.6 & 0.5 & 0.6 & 0.8182\\ 0.5 & 0.4 & 0.5 & 0.75\\ 0.25 & 0.1818 & 0.25 & 0.5\end{Bmatrix},\quad \hat{R}_2=\begin{Bmatrix}0.5 & 0.75 & 0.7659 & 0.4\\ 0.25 & 0.5 & 0.8 & 0.4067\\ 0.2341 & 0.2 & 0.5 & 0.3\\ 0.6 & 0.5933 & 0.7 & 0.5\end{Bmatrix},\quad \hat{R}_3=\begin{Bmatrix}0.5 & 0.35 & 0.5 & 0.6\\ 0.65 & 0.5 & 0.6792 & 0.85\\ 0.5 & 0.3208 & 0.5 & 0.7\\ 0.4 & 0.15 & 0.3 & 0.5\end{Bmatrix}$$

Step 3: Compute the averaged preference value $\hat{r}_{i,t}$ of the alternative x_i over all the other alternatives for the DM d_t $(t=1,2,3)$ by using the arithmetic average (AA) operator:

$$\hat{r}_{i,t}=\mathrm{AA}(\hat{r}_{i1,t},\hat{r}_{i2,t},\hat{r}_{i3,t},\hat{r}_{i4,t})=\frac{1}{n}\sum_{j=1}^{n}\hat{r}_{ij,t}$$

The aggregation results of the DM d_t $(t=1,2,3)$ are as follows

$$d_1:\quad \hat{r}_{1,1}=0.5375,\quad \hat{r}_{2,1}=0.6296,\quad \hat{r}_{3,1}=0.5375,\quad \hat{r}_{4,1}=0.2954$$

$$d_2:\quad \hat{r}_{1,2}=0.6040,\quad \hat{r}_{2,2}=0.4892,\quad \hat{r}_{3,2}=0.3085,\quad \hat{r}_{4,2}=0.5983$$

$$d_3:\quad \hat{r}_{1,3}=0.4875,\quad \hat{r}_{2,3}=0.6698,\quad \hat{r}_{3,3}=0.5052,\quad \hat{r}_{4,3}=0.3375$$

Step 4. Aggregate all $\hat{r}_{i,t}$ $(t=1,2,3)$ into a collective preference value $\hat{r}_{i,c}$ of the alternative x_i over all the other alternatives by using the weighted average (WA) operator:

$$\hat{r}_{i,c}=\mathrm{WA}(\hat{r}_{i,1},\hat{r}_{i,2},\hat{r}_{i,3})=\sum_{t=1}^{3}\lambda_t\hat{r}_{i,t}$$

In order to be consistent with Example 4.5, the same weight vector for the DMs, i.e., $\lambda=(0.5,0.2,0.3)$ are adopted here. Then, the calculated results are as follows:

$$\hat{r}_{1,c}=0.5358,\quad \hat{r}_{2,c}=0.6136,\quad \hat{r}_{3,c}=0.4820,\quad \hat{r}_{4,c}=0.3686$$

Step 5. Rank all of the alternatives in accordance with $\hat{r}_{i,c}$ $(i=1,2,3,4)$ as: $x_2\succ x_1\succ x_3\succ x_4$, which is the same as that derived by our Algorithm 3 in Example 4.5, thereby validating the effectiveness, feasibility and applicability of our method.

From the above example, we can see that compared with the method which does not use HFSs, our proposed method has the following advantages:

(1) According to Example 3.4 and 4.6, the ranking orders of alternatives obtained by two methods which do not use HFSs are slightly different. However, according to Examples 3.3 and 4.5, the ranking orders of alternatives obtained by our two methods based on HFSs are the same. This result reveals that our two methods are more reliable and stable than the methods which do not use HFSs.

(2) The method in Example 4.6 first transfers the original incomplete HFPR given by the DM into its corresponding average incomplete FPR by using the AA operator, which seems to be an indirect computation process and easily leads to the information losing. Moreover, as mentioned in Example 3.4, the transferred average incomplete FPR may not be the optimal and true one, which makes the produced results unreliable and doubtful. However, our method in Example 4.5 is carried out on the original incomplete HFPR directly, which keeps all possible opinions of the group members through HFSs, incorporates more information of different DMs' opinions and ensures the obtained decision results to be convincing and reasonable.

(3) The method in Example 4.6 derives the preference values from each individual completed average FPR by the AA aggregation operator, and then utilizes the WA aggregation operator to fuse them into the collective one, from which we can obtain the ranking order of alternatives. It is worth noting that the use of these two aggregation operators may make the information losing take place, which

affects the final ranking results. In contrast, our method in Example 4.5 is based on the β-normalization. With our method, a family of crisp FPRs with multiplicative consistency can be obtained from a multiplicative consistent HFPR by using a convex combination method, whose weights are found to be a form of convex combination. An interval weight generation method is also used to obtain interval weights from multiplicative consistent HFPRs. The obtained results show that the proposed methods are simple, effective and applicable, without solving any mathematical programming and complicated algebraic aggregation operations, thereby avoiding the information loss.

5. Conclusions

In the process of group decision making, the decision makers may uses incomplete hesitant fuzzy preference relations to express their preference information due to the complexity, uncertainty, and hesitancy involved in real decision problems. In order to correctly operation on incomplete HFPRs, we have two principles to normalize them, i.e., the α-normalization and the β-normalization. The α-normalization reduces a HFPR to a crisp FPR by removing some preference values of some elements of the HFPR, while the β-normalization adds some preference values to some elements which have less lengths than the other ones until all the elements of the HFPR have the same length. Based on the α-normalization, we have developed a goal programming model to derive the priority weights from incomplete HFPRs under GDM situations. Then, on the basis of the β-normalization, a multiplicative consistent hesitant fuzzy preference relation, an acceptable incomplete hesitant fuzzy preference relation, and a multiplicative consistent incomplete hesitant fuzzy preference relation have been defined. An algorithm for estimating the missing preference values in an acceptable incomplete HFPR has been proposed. Furthermore, a practical algorithm for obtaining the collective priority weight vector of the GDM problem with several complete or incomplete HFPRs has been proposed. Several numerical examples have been given to verify the proposed goal programming model and these algorithms.

The main advantages of the proposed methods are summarized as follows: (1) The proposed methods are applicable to the GDM problem in which all the DMs use the incomplete HFPRs to express their preferences. (2) According to the α-normalization, a new hesitant fuzzy goal programming model is developed to derive the priority weights from incomplete HFPRs based on multiplicative consistency. Meanwhile, by solving this model we can select the optimal preference value from all possible ones for each paired comparison of alternatives, and then obtain a corresponding FPR consisting of these optimal preferences as the reduction of the original HFPR. (3) According to the β-normalization, a family of crisp FPRs with acceptable consistency can be obtained from an acceptably consistent HFPR by using a convex combination method, whose weights are found to be a form of convex combination. An interval weight generation method is also used to obtain interval weights from acceptably consistent HFPR. The obtained results show that the proposed methods are simple, effective and applicable to both consistent HFPRs and inconsistent ones generated from concrete decision-making problems, without solving any mathematical programming and complicated algebraic operation.

According to β-normalization and Definition 4.1, we add some elements to the original HFPR and obtain its corresponding normalized HFPR, from which we derive the interval weights and further determine the ranking of the alternatives. Whether and how adding different elements affect the final decision result have been clarified in this paper, which is a drawback of our approaches and is worthy to be further studied in the future.

The proposed estimation procedure for acceptable incomplete HFPRs in Algorithm 2 usually needs at least a piece of information about every alternative in the problem, but it fails to address situations in which an expert does not provide any information about a particular alternative. Motivated by the idea in [4], in the future we will deal with these situations by employing two strategies: (i) individual strategies that can be applied to each individual preference relation without taking into account any information from the rest of experts and (ii) social strategies, that is, strategies that make use of the information available from the group of experts. Both individual and social strategies use extra assumptions or knowledge, which could not be directly instantiated in the experts' preference relations.

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References

[1] S. Alonso, F.J. Cabrerizo, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group decision-making with incomplete fuzzy linguistic preference relations, Int. J. Intell. Syst. 24 (2) (2009) 201–222.
[2] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, A procedure for learning missing values in fuzzy preference relations based on additive consistency, Lect. Notes Artif. Intell. 3131 (2004) 227–238.
[3] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, J. Alcalá-Fdez, C. Porcel, A consistency-based procedure to estimate missing pairwise preference values, Int. J. Intell. Syst. 23 (2) (2008) 155–175.
[4] S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, Individual and social strategies to deal with ignorance situations in multi-person decision making, Int. J. Inf. Technol. Decis. Mak. 8 (2) (2009) 313–333.
[5] S. Alonso, E. Herrera-Viedma, F. Chiclana, F. Herrera, A web based consensus support system for group decision making problems and incomplete preferences, Inf. Sci. 180 (23) (2010) 4477–4495.
[6] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1) (1986) 87–96.
[7] K. Atanassov, G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets Syst. 31 (3) (1989) 343–349.
[8] B. Bedregal, R. Reiser, H. Bustince, C. Lopez-Molina, V. Torra, Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms, Inf. Sci. 255 (2014) 82–99.
[9] B. Bedregal, R.H.N. Santiago, H. Bustince, D. Paternain, R. Reiser, Typical hesitant fuzzy negations, Int. J. Intell. Syst. 29 (6) (2014) 525–543.
[10] F. Chiclana, E. Herrera-Viedma, S. Alonso, F. Herrera, A note on the estimation of missing pair wise preference values: a U-consistency based method, Int. J. Uncertain. Fuzziness Knowl. Based Syst. 16 (2008) 19–32.
[11] F. Chiclana, E. Herrera-Viedma, S. Alonso, F. Herrera, Cardinal consistency of reciprocal preference relation: a characterization of multiplicative transitivity, IEEE Trans. Fuzzy Syst. 17 (2009) 14–23.
[12] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[13] M. Fedrizzi, M. Brunelli, On the priority vector associated with a reciprocal relation and with a pairwise comparison matrix, Soft Comput. 14 (6) (2010) 639–645.
[14] P.T. Harker, Alternative modes of questioning in the analytic hierarchy process, Math. Model. 9 (1987) 353–360.

[15] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, A model of consensus in group decision making under linguistic assessments, Fuzzy Sets Syst. 78 (1996) 73–87.
[16] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, Direct approach processes in group decision making using linguistic OWA operators, Fuzzy Sets Syst. 79 (1996) 175–190.
[17] E. Herrera-Viedma, F. Chiclana, F. Herrera, S. Alonso, Group decision-making model with incomplete fuzzy preference relations based on additive consistency, IEEE Trans. Syst. Man Cybern. B Cybern. 37 (1) (2007) 176–189.
[18] H.C. Liao, Z.S. Xu, Some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment, J. Intell. Fuzzy Syst. 26 (4) (2014) 1601–1617.
[19] H.C. Liao, Z.S. Xu, M.M. Xia, Multiplicative consistency of hesitant fuzzy preference relation and its application in group decision making, Int. J. Inf. Technol. Decis. Mak. 13 (1) (2014) 47–76.
[20] S. Miyamoto, Multisets and fuzzy multisets, in: Z.-Q. Liu, S. Miyamoto (Eds.), Soft Computing and Human-Centered Machines, Springer, Berlin, 2000, pp. 9–33.
[21] S.A. Orlovsky, Decision-making with a fuzzy preference relation, Fuzzy Sets Syst. 1 (3) (1978) 155–167.
[22] C. Porcel, E. Herrera-Viedma, Dealing with incomplete information in a fuzzy linguistic recommender system to disseminate information in university digital libraries, Knowl. Based Syst. 23 (2010) 32–39.
[23] T.L. Saaty, The Analytical Hierarchy Process, McGraw-Hill, New York, 1980.
[24] T.L. Saaty, L.G. Vargas, Uncertainty and rank order in the analytic hierarchy process, Eur. J. Oper. Res. 32 (1987) 107–117.
[25] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets Syst. 12 (2) (1984) 117–131.
[26] Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (6) (2010) 529–539.
[27] Y. Narukawa, Torra, Modeling Decisions: Information Fusion and Aggregation Operators, Springer, 2007.
[28] M.R. Ureña, F. Chiclana, J.A. Morente-Molinera, E. Herrera-Viedma, Managing incomplete preference relations in decision making: a review and future trends, Inf. Sci. 302 (1) (2015) 14–32.
[29] P.J.M. Van Laarhoven, W. Pedrycz, A fuzzy extension of Saaty's priority theory, Fuzzy Sets Syst. 11 (1983) 199–227.
[30] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, Int. J. Approx. Reason. 52 (3) (2011) 395–407.
[31] M.M. Xia, Z.S. Xu, H.C. Liao, Preference relations based on intuitionistic multiplicative information, IEEE Trans. Fuzzy Syst. 21 (1) (2013) 113–133.
[32] Z.S. Xu, On consistency of the weighted geometric mean complex judgment matrix in AHP, Eur. J. Oper. Res. 126 (3) (2000) 683–687.
[33] Z.S. Xu, A method for priorities of triangular fuzzy number complementary judgment matrices, Fuzzy Syst. Math. 16 (2002) 47–50.
[34] Z.S. Xu, On compatibility of interval fuzzy preference matrices, Fuzzy Optim. Decis. Mak. 3 (2004) 217–225.
[35] Z.S. Xu, An approach to group decision making based on incomplete linguistic preference relations, Int. J. Inf. Technol. Decis. Mak. 4 (2005) 153–160.
[36] Z.S. Xu, Incomplete linguistic preference relations and their fusion, Inf. Fusion 7 (2006) 331–337.
[37] Z.S. Xu, Method for group decision making with various types of incomplete judgment matrices, Control Decis. 21 (2006) 28–33.
[38] Z.S. Xu, Intuitionistic preference relations and their application in group decision making, Inf. Sci. 177 (11) (2007) 2363–2379.
[39] Z.S. Xu, Intuitionistic Fuzzy Preference Modeling and Interactive Decision Making Berlin, Springer-Verlag, Heidelberg, 2013.
[40] Z.S. Xu, X.Q. Cai, E. Szmidt, Algorithms for estimating missing elements of incomplete intuitionistic preference relations, Int. J. Intell. Syst. 26 (9) (2011) 787–813.
[41] Z.S. Xu, J. Chen, Some models for deriving the priority weights from interval reciprocal relations, Eur. J. Oper. Res. 184 (2008) 266–280.
[42] Z.S. Xu, Q.L. Da, The uncertain OWA operator, Int. J. Intell. Syst. 17 (6) (2002) 569–575.
[43] Z.S. Xu, M.M. Xia, Distance and similarity measures for hesitant fuzzy sets, Inf. Sci. 181 (11) (2011) 2128–2138.
[44] R.R. Yager, On the theory of bags, Int. J. Gen. Syst. 13 (1986) 23–37.
[45] P.L. Yu, A class of solutions for group decision problems, Manag. Sci. 19 (1973) 936–946.
[46] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (3) (1965) 338–353.
[47] L. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part 1, Inf. Sci. 8 (3) (1975) 199–249.
[48] L. Zhang, Z.S. Xu, Hesitant fuzzy QUALIFLEX approach with a signed distance-based comparison method for multiple criteria decision analysis, Expert Syst. Appl. 42 (2) (2015) 873–884.
[49] B. Zhu, Z.S. Xu, Regression methods for hesitant fuzzy preference relations, Technol. Econ. Dev. Econ. 19 (Suppl. 1) (2013) S214–S227.
[50] B. Zhu, Z.S. Xu, Consistency measures for hesitant fuzzy linguistic preference relations, IEEE Trans. Fuzzy Syst. 22 (1) (2014) 35–45.
[51] B. Zhu, Z.S. Xu, J.P. Xu, Deriving a ranking from hesitant fuzzy preference relations under group decision making, IEEE Trans. Cybern. 44 (8) (2014) 1328–1337.