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Relations between complex vague soft sets

Q1 Ganeshsree Selvachandran [a,*], P.K. Maji [b], Israa E. Abed [c], Abdul Razak Salleh [c]

[a] Department of Actuarial Science and Applied Statistics, Faculty of Business & Information Science UCSI University, Jalan Menara Gading, 56000, Cheras, Kuala Lumpur, Malaysia
[b] Department of Mathematics, B.C. College, Asansol, West Bengal, India
[c] School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600, UKM Bangi, Selangor DE, Malaysia

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ABSTRACT

Complex vague soft sets are essentially vague soft sets characterized by an additional parameter called the phase term which is defined over the set of complex numbers. In this study, we introduce and discuss the relations between complex vague soft sets. We present the definitions of the Cartesian product of complex vague soft sets and subsequently that of complex vague soft relations. The definition of the composition of complex vague soft sets is also provided. The notions of symmetric, transitive, reflexive and equivalence complex vague soft relations are then proposed and the algebraic properties of these concepts are verified. The relation between complex vague soft sets is then discussed in the context of a real-life problem: the relation between the financial indicators of the Chinese economy which are characterized by their degrees of influence on the financial indicators of the Malaysian economy, and the time required for the former to affect the latter. Interpretations of the results obtained from this example are then proposed by relating them to recent significant real-life events in the Chinese and Malaysian economies.

1. Introduction

The complex number set allows us to solve many problems that traditionally cannot be solved by using the real number set, such as improper integrals that represent electrical resistance in the field of engineering. Thus applying soft, fuzzy and hybrid sets to complex numbers is an essential step to incorporate the advantages of complex numbers into the notion of soft sets, fuzzy sets, and their generalizations. This work initiated by Ramot et al. [1], who introduced the concept of complex fuzzy sets which extend the notion of fuzzy sets, are made possible by adding a phase term that describes the periodicity of the elements with respect to time. The range of Ramot's complex fuzzy sets is not limited to [0,1] but rather extends to the unit circle in the complex plane. The theory behind the development of complex fuzzy sets says that in many instances a second dimension must be added to the expression of the membership value of an element or object which is particularly important in situations wherein the elements of the set vary with time. Examples include meteorological time series such as sunspot cycles which describe the number of sunspots that appear on the surface of the sun as a function of time and economic time series such as the fluctuation of stock prices on a daily or hourly basis or the effects of certain financial factors on the economy of a country or a region. Among other notable research in this relatively unexplored area are the application of complex fuzzy sets to traditional fuzzy logic via the introduction of complex fuzzy logic by Ramot et al. [2] and Dick [3] who further expanded the theory of complex fuzzy logic by introducing several new operators pertaining to this theory. Q3 Q4

The study of uncertainties and nonlinear problems and by extension, the modelling of uncertainties and nonlinear problems has always been a major area in the study of mathematics. Over the years, many techniques and methods have been proposed as tools to be used to find the solutions of problems that are nonlinear or vague in nature, with every method introduced superior to its predecessors. The study of nonlinear problems is of particular interest to engineers, physicists, mathematicians and other scientists as most systems in the real world are inherently nonlinear in nature and often appear as chaotic, unpredictable and sometimes even counterintuitive. Applied mathematical techniques and artificial intelligence are the most commonly used methodologies used to handle perturbation and chaotic

* Corresponding author.
E-mail addresses: ganeshsree86@yahoo.com (G. Selvachandran), pabitra_maji@yahoo.com (P.K. Maji).

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behavior of nonlinear problems and find accurate solutions to these problems. Among the notable research done in this area are due to Argyros & George [4] and Argyros & Gonzalez [5]. In Ref. [4], the authors presented a local convergence analysis for a family of Steffenson-type fourth order methods in order to obtain approximate solutions for nonlinear equations and proved that the method proposed here has the ability to be extended under weaker hypotheses. A local convergence analysis was also presented in Ref. [5], but for an improved Jarrett-type method of at least five to obtain approximate solutions for nonlinear equations in a Banach space setting. It was proven that the convergence ball and error estimates introduced here which was derived using the hypotheses up to the first Frechet derivative is superior and more widely applicable in finding approximate solutions to nonlinear problems as opposed to methods introduced in previous studies, many of which uses hypotheses up to the third Frechet derivative. Besides that, one of the most recent research done in the study of complex dynamics pertaining to nonlinear problems is due to Amat et al. [6] who utilized the dynamics of a two-step Newton-type method for solving nonlinear equations and systems and went on to prove that there exists significant characteristics of attraction, chaos and perturbation for some choices of the damping factor.

Another commonly used method in handling uncertainties and representing incomplete and unreliable data is soft computing, which was born as a direct result of the establishment of soft set theory by Molodtsov [7]. Soft computing is in fact, a collection of methodologies and techniques that aim to push the limits of tolerance for imprecision, uncertainty and partial truths, with the objective of obtaining robust and accurate solutions with a reasonable amount of computation. These soft computing techniques are widely used in the modelling of imprecision, vagueness and subjective data, which is another major area of study in both pure and applied mathematics. Since its inception, soft computing methods have been combined with many of the classical mathematical theories that are used to handle uncertainties with the most prominent tools being fuzzy sets, soft sets and their generalizations.

In recent years, there has been a significant increase in the application of artificial intelligence or applied mathematics techniques which are combined with mathematical models constructed using fuzzy sets, soft sets and/or hybrid models of these two sets. It is also worth noting that there have been an exponential increase in the utilization of applied mathematical techniques to aid with and improve the efficiency of computation in several prominent areas in the study of fuzzy sets and systems. This is particularly true in the study of fuzzy differential equations and fuzzy partial differential equations, where the recent trend in this area include among others, using fuzzy differential transform methods and Laplace-Fourier transforms in finding the solutions to fuzzy Volterra integro differential equations (see Refs. [8–10]). In the same vein, in this paper, we apply soft computing techniques to develop a better model to evaluate the degree of interaction between two hybrid models which are used to represent two-dimensional data that is subjective, humanistic, vague and subject to perceptional differences.

Selvachandran et al. [11] introduced the concept of complex vague soft sets (CVSSs) which combine the key features of soft and complex fuzzy sets. The CVSS model is an improved hybrid model of complex fuzzy sets that can (1) represent two-dimensional information i.e. information about a parameter and its periodicity by virtue of a phase term; (2) be more expressive in capturing the vagueness and unreliability of data by virtue of the vague soft set used in this model, which allows interval-based membership values to overcome the problem of assigning suitable membership values to the elements in a set compared with fuzzy or intuitionistic fuzzy sets and better corresponds to the intuition of representing vague data and (3) provide a more adequate parametrization tool that can represent the parameters of a problem in a more comprehensive manner. The CVSS model possesses all of these abilities because it embodies all of the features of complex fuzzy sets with the added advantage of vague sets and the adequate parameterization characteristic of soft sets. We refer the readers to Ref. [11] for further information on CVSSs and their properties.

In the present work, we introduce and study the various types of relations between CVSSs. The relations between fuzzy sets, soft sets and their many generalizations are a topic that has been extensively studied. However, the most significant shortcoming of these relations is their inability to capture information pertaining to the time frame of the intersection between the parameters, which is a very important component of most real-life situations. Improving this deficiency in our understanding of the relations between various hybrids of fuzzy and soft sets formed the motivation for this work. As such, we herein introduce the complex vague soft relation between CVSSs, which represents the degree of presence or absence of interaction and the phase of the interaction between the elements of the CVSSs. We then provide verifications of some fundamental properties of this concept, present the application of this concept to an economics example and support it with real-life events in the Malaysian and Chinese economies.

2. Preliminaries

In this section, we recapitulate some of the important concepts pertaining to soft sets and the hybrid structures of soft sets such as vague soft sets and complex fuzzy sets that are relevant to this paper.

Definition 2.1 *([12])*. Let X be a space of points (objects) with elements of denoted by x. A *vague set* V in X is characterized by a truth-membership function $t_V : X \to [0, 1]$ and a false-membership function $f_V : X \to [0, 1]$. The function $t_V(x)$ gives the lower bound of the grade of membership of x derived from the evidence for x and $f_V(x)$ gives the lower bound of the negation of x derived from the evidence against x. The functions $t_V(x)$ and $f_V(x)$ each associate a real number in the interval [0, 1] with each point in X, where $0 \leq t_V(x) + f_V(x) \leq 1$. This approach bounds the grade of membership of x to a closed subinterval $[t_V(x), 1 - f_V(x)]$ of [0, 1].

Q5 When X is continuous, the vague set V can be written as:

$$V = \frac{\int [t_V(x), 1 - f_V(x)]}{x}, \quad \text{where} x \in X.$$

When X is discrete, V can be written as:

$$V = \sum_{i=1}^{n} \frac{[t_V(x_i), 1 - f_V(x_i)]}{x_i}, \quad \text{where} x_i \in X.$$

By using vague sets, the accuracy of our knowledge of x is immediately clear through the uncertainty, which is characterized by the difference between $1 - f_V(x)$ and $t_V(x)$. If this difference is small, then our knowledge of x is accurate. Conversely, if this difference is large, then our knowledge of x is inaccurate. If $1 - f_V(x) = t_V(x)$, then our knowledge of x is exact and the vague set then degenerates into a fuzzy set.

Soft sets were initially introduced in Ref. [7] as a parametrized family of subsets of the universe of discourse U. This well-known basic definition of soft sets has undergone a lot of changes over the years, as a result of being hybridized with other types of fuzzy sets to produce new and improved hybrid models, with each one being superior to its predecessors. One such instance in recent years, is the introduction of the taxonomy of soft sets by Saraf [13], who proposed several new categories of soft sets which are based on the concept of a neighbourhood system. Among the variations of soft sets introduced by Saraf in Ref. [13] are weighted soft sets, finite-multi soft sets, partitioned soft sets, basic neighbourhood soft sets, covering soft sets and double fuzzy soft sets. We refer the readers to Ref. [13] for more information pertaining to these new types of soft sets.

The notion of vague soft sets was introduced by Xu et al. [14] as a generalization of the notion of soft sets. It is an improvement in the theory of soft sets and provides a means to deal with the vagueness of problems involving complex data with a high level of uncertainty and imprecision.

Definition 2.2 *([14])*. A pair $(\hat{F}, A)$ is called a *vague soft set* over U, where $\hat{F}$ is a mapping given by $\hat{F}: A \to V(U)$ and $V(U)$ is the set of all vague sets of U. In other words, a vague soft set over U is a parametrized family of vague sets of the universe U. For all $a \in A$, $\mu_{\hat{F}_a} : U \to [0, 1] \times [0, 1]$ is regarded as the set of ε-approximate elements of the vague soft set $(\hat{F}, A)$. Therefore a vague soft set is a collection of approximations of the form:

$$(\hat{F}, A) = \left\{\hat{F}(a_i) : i = 1, 2, 3, \ldots\right\} = \left\{ \frac{\left[t_{\hat{F}(a_i)}(x_i),\, 1 - f_{\hat{F}(a_i)}(x_i)\right]}{x_i} : i = 1, 2, 3, \ldots \right\}$$

for all $a_i \in A$ and for all $x_i \in U$.

Alhazaymeh and Hassan [15] on the other hand, furthered the study of vague soft set theory by introducing the notion of relations, partition and functions in this context. They also derived some of the fundamental theories pertaining to the relations and functions of vague soft sets. This vague soft relation is later compared to the relation between complex vague soft sets introduced in this paper.

Definition 2.3 *([15])*. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft sets over U. Then the relation from $(\hat{F}, A)$ to $(\hat{G}, B)$ is a vague soft subset of the Cartesian product of $(\hat{F}, A)$ and $(\hat{G}, B)$, denoted by $(\hat{F}, A) \times (\hat{G}, B)$. A *vague soft set relation* from $(\hat{F}, A)$ to $(\hat{G}, B)$ is of the form (H_1, S), where $S \subset A \times B$ and $H_1(a, b) \forall a, b \in S$. This relationship is denoted by $(\hat{F}, A)\Re(\hat{G}, B) = (H_1, S)$. Any subset of $(\hat{F}, A) \times (\hat{F}, A)$ is called a relation on $(\hat{F}, A)$ in the parametrized form as follows:

If $(\hat{F}, A) = \{\hat{F}(a), \hat{F}(b), \ldots\}$, then $\hat{F}(a)\Re\hat{F}(b)$ if and only if $\hat{F}(a) \times \hat{F}(b) \in \Re$, where $\Re$ is a vague soft set relation.

The novelty of the concept of complex fuzzy sets introduced by Ramot et al. [1] lies in the inclusion of the phase term in the membership function. The phase term enables a complex fuzzy set to capture the periodicity of the interactions between its parameters, which cannot be done by using a traditional fuzzy set.

Definition 2.4 *([1])*. A *complex fuzzy set* A defined over a universe of discourse U is characterized by a membership function $\mu_A(x)$, which assigns to any element $x \in U$, a complex-valued grade of membership in A. By definition, the values that $\mu_A(x)$ may receive all lie within the unit circle in the complex plane and are of the form $\mu_A(x) = r_A(x) e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, both $r_A(x)$ and $\omega_A(x)$ are real-valued and $r_A(x) \in [0, 1]$. The complex fuzzy set A may be represented as the set of ordered pairs:

$$A = \left\{(x, \mu_A(x)) : x \in U\right\} = \left\{(x, r_A(x) e^{i\omega_A(x)}) : x \in U\right\}.$$

The CVSS model is a hybrid of the soft and complex fuzzy set models and is defined by extending the range of truth and false membership functions of the corresponding vague soft set from the original interval of [0,1] to a unit circle in the complex plane. The formal definition of this concept is as follows:

Definition 2.5 *([11])*. Let U be an initial universe, E be the set of parameters under consideration and $A \subset E$. Furthermore, let $P(U)$ denote the complex vague power set of U. A pair (F, A) is called a *complex vague soft set* (CVSS) over U, where F is a mapping given by $F : A \to P(U)$ such that

$$F\left(x_j\right) = \left\{ \left(x, \left[r_{t_{F_a}}(x), 1 - k_{t_{F_a}}(x)\right] \times e^{i\left[{w^r}_{t_{F_a}}(x), 2\pi - {w^k}_{f_{F_a}}(x)\right]} \right) : x \in U \right\}$$

where $j = 1, 2, 3, \ldots$ is the number of parameters, $\left[r_{t_{F_a}}(x), 1 - k_{t_{F_a}}(x)\right]$ are real-valued in the closed interval $[0, 1]$, the phase terms $\left[{w^r}_{t_{F_a}}(x), 2\pi - {w^k}_{f_{F_a}}(x)\right]$ are real-valued in the interval $(0, 2\pi]$, $0 \le r_{t_{F_a}}(x) + k_{F_a}(x) \le 1$ and $i = \sqrt{-1}$.

The complex truth-membership function $t_{F_a}(x)$ is defined as

$$t_{F_a}(x) : U \to \left\{a : a \in \mathbb{C}, |a| \le 1\right\}, \quad t_{F_a}(x) = r_{t_{F_a}}(x) \times e^{i{w^r}_{t_{F_a}}(x)},$$

and the complex false-membership function $f_{F_a}(x)$ is defined as

$$f_{F_a}(x) : U \to \left\{a : a \in \mathbb{C}, |a| \le 1\right\}, 1 - f_{F_a}(x) = \left(1 - k_{f_{F_a}}(x)\right) \times e^{i\left(2\pi - {w^k}_{f_{F_a}}(x)\right)},$$

where $t_{F_a}(x)$ is the lower bound of the complex grade of membership of x derived from the evidence for x and $f_{F_a}(x)$ is the lower bound on the negation of x derived from the evidence against x. By definition, the values $t_{F_a}(x)$ and $f_{F_a}(x)$ and their sum all lie within a unit circle in the complex plane. In other words, a CVSS over a universe U is a parametrized family of complex vague sets of the universe U.

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Definition 2.6 *([11])*. Let (F, A) and (G, B) be complex vague soft sets over U. Then:

(a) (F, A) is said to be a *complex vague soft subset* of (G, B) if and only if the following conditions are satisfied for all $x \in U$:
  (i) $r_{t_{F_a}}(x) \le r_{t_{G_b}}(x)$ and $k_{f_{G_b}}(x) \le k_{f_{F_a}}(x)$ for the amplitude terms;
  (ii) $w^r_{t_{F_a}}(x) \le w^r_{t_{G_b}}(x)$ and $w^k_{f_{G_b}}(x) \le w^k_{f_{F_a}}(x)$ for the phase terms.

This relationship is denoted as $(F, A) \subset (G, B)$.

(b) (F, A) is said to be *complex vague soft* equal to (G, B) if and only if the following conditions are satisfied for all $x \in U$:
  (i) $r_{t_{F_a}}(x) \le r_{t_{G_b}}(x)$ and $k_{f_{G_b}}(x) \le k_{f_{F_a}}(x)$ for the amplitude terms;
  (ii) $w^r_{t_{F_a}}(x) \le w^r_{t_{G_b}}(x)$ and $w^k_{f_{G_b}}(x) \le w^k_{f_{F_a}}(x)$ for the phase terms.

This relationship is denoted as $(F, A) \equiv (G, B)$.

3. Relations between complex vague soft sets

In this section, we consider the Cartesian product between CVSSs and the relations between CVSSs. For the sake of simplicity, the following discussion is limited to the relations between two sets. However, all of the results presented in this paper can easily be extended to include any number of sets.

Definition 3.1. If U is an initial universal set, E is a set of parameters, $A, B \subseteq E$ and (F, A) and (G, B) are CVSSs over a universe (U, E), then the *Cartesian product* of (F, A) and (G, B), denoted by $(F, A) \times (G, B)$, is a CVSS (H, C), where $C = A \times B$ and $H : C \to P(U)$, and is defined as:

$$(H, C) = (F, A) \times (G, B) = \left\{ \left(x, \left[\min\left(r_{t_{F_a}}(x), r_{t_{G_b}}(x) \right), \min\left(1 - k_{f_{F_a}}(x), 1 - k_{f_{G_b}}(x) \right) \right] \right.\right.$$
$$\left.\left. \times e^{i\left[\min\left(w^r{}_{t_{F_a}}(x), w^r{}_{t_{G_b}}(x) \right), \min\left(2\pi - w^r{}_{t_{F_a}}(x), 2\pi - w^r{}_{t_{G_b}}(x) \right) \right]} \right) : x \in U, (a, b) \in A \times B \right\}.$$

This relationship can also be written as $H(a, b) = F(a) \times G(b)$, for all $(a, b) \in C$.

Definition 3.2. If U is an initial universal set, E is a set of parameters, $A, B \subseteq E$, (U, E) is a soft universe and (F, A) and (G, B) are CVSSs over (U, E) then a *complex vague soft (CVS) relation* from (F, A) to (G, B) is a complex vague soft subset of $(F, A) \times (G, B)$, and is of the form (R, C) where $C \subseteq A \times B$, $R(a, b) \subseteq (F, A) \times (G, B)$, $\forall (a, b) \in C \subseteq A \times B$ and $(F, A) \times (G, B)$ is the Cartesian product of (F, A) and (G, B).

We now present an example of two CVSSs and the relation between them. The interpretations of the upper and lower bounds of the amplitude terms and the phase terms in the context of this example are also presented.

Example 3.3. Consider two CVSSs (F, A) and (G, B) over a universe U, where $U = \{x_1 = \text{minimal influence}, x_2 = \text{average influence}, x_3 = \text{strong influence}\}$ is a set that describes the degrees of influence of the parameters related to the Chinese and Malaysian economies. Let A and B be the sets of financial indicators that describe the Chinese and Malaysian economies respectively, as defined below:

$$A = \left\{ a_1 = \text{exchange rate of the Chinese Yuan}, a_2 = \text{volume of Chinese exports}, a_3 = \text{China's GDP growth rate}, a_4 = \text{interest rate} \right\},$$

$$B = \left\{ b_1 = \text{exchange rate of the Malaysian Ringgit}, b_2 = \text{prices of Malaysian commodities}, b_3 = \text{volume of Malaysian exports}, \right.$$
$$\left. b_4 = \text{Malaysia's GDP growth rate} \right\}.$$

The amplitude and phase terms given in the CVSSs (F, A) and (G, B) can be obtained from or determined by data obtained from various agencies such as the Central Bank of Malaysia, the Kuala Lumpur Stock Exchange, the International Monetary Fund (IMF), the Asian Development Bank and/or international brokerage firms such as Credit Suisse as well as from experts such as economists, financial analysts, investment bankers and researchers. The CVSSs (F, A) and (G, B) are defined as

$$(F, A)$$

$$= \left\{ F(a_1) = \left\{ \left(x_1, [0, 0.1] e^{i\left[\frac{2\pi}{3}, \pi\right]} \right), \left(x_2, [0.5, 0.6] e^{i\left[\frac{2\pi}{3}, \pi\right]} \right), \left(x_3, [0.9, 1] e^{i\left[\frac{2\pi}{3}, \pi\right]} \right) \right\}, \right.$$

$$= \left\{ F(a_2) = \left\{ \left(x_1, [0.1, 0.2] e^{i\left[\frac{5\pi}{6}, \pi\right]} \right), \left(x_2, [0.8, 0.9] e^{i\left[\frac{5\pi}{6}, \pi\right]} \right), \left(x_3, [0.6, 0.7] e^{i\left[\frac{5\pi}{6}, \pi\right]} \right) \right\}, \right.$$

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$$= \left\{ F(a_3) = \left\{ \left(x_1, [0.1, 0.3]\, e^{i\left[\frac{3\pi}{2}, 2\pi\right]} \right), \left(x_{2,}\, [0.7, 0.8]\, e^{i\left[\frac{3\pi}{2}, 2\pi\right]} \right), \left(x_3, [0.8, 0.9]\, e^{i\left[\frac{3\pi}{2}, 2\pi\right]} \right) \right\}, \right.$$

$$= \left\{ F(a_4) = \left\{ \left(x_1, [0.3, 0.45]\, e^{i\left[\pi, \frac{7\pi}{6}\right]} \right), \left(x_{2,}\, [0.9, 1]\, e^{i\left[\pi, \frac{7\pi}{6}\right]} \right), \left(x_3, [0.6, 0.75]\, e^{i\left[\pi, \frac{7\pi}{6}\right]} \right) \right\}, \right.$$

and

$$(G, B)$$

$$= \left\{ G(b_1) = \left\{ \left(x_1, [0, 0.1]\, e^{i\left[0, \frac{\pi}{6}\right]} \right), \left(x_{2,}\, [0.5, 0.7]\, e^{i\left[0, \frac{\pi}{6}\right]} \right), \left(x_3, [0.9, 1]\, e^{i\left[0, \frac{\pi}{6}\right]} \right) \right\}, \right.$$

$$= \left\{ G(b_2) = \left\{ \left(x_1, [0.2, 0.4]\, e^{i\left[\frac{2\pi}{3}, \pi\right]} \right), \left(x_{2,}\, [0.5, 0.6]\, e^{i\left[\frac{2\pi}{3}, \pi\right]} \right), \left(x_3, [0.8, 0.9]\, e^{i\left[\frac{2\pi}{3}, \pi\right]} \right) \right\}, \right.$$

$$= \left\{ G(b_3) = \left\{ \left(x_1, [0.1, 0.3]\, e^{i\left[\frac{\pi}{24}, \frac{\pi}{6}\right]} \right), \left(x_{2,}\, [0.6, 0.7]\, e^{i\left[\frac{\pi}{24}, \frac{\pi}{6}\right]} \right), \left(x_3, [0.8, 1]\, e^{i\left[\frac{\pi}{24}, \frac{\pi}{6}\right]} \right) \right\}, \right.$$

$$= \left\{ G(b_4) = \left\{ \left(x_1, [0.2, 0.3]\, e^{i\left[\frac{4\pi}{3}, 2\pi\right]} \right), \left(x_{2,}\, [0.75, 0.85]\, e^{i\left[\frac{4\pi}{3}, 2\pi\right]} \right), \left(x_3, [0.8, 0.9]\, e^{i\left[\frac{4\pi}{3}, 2\pi\right]} \right) \right\}. \right.$$

Suppose that the interactions between these CVSSs are measured over the limited time frame of 12 months (1 year). This interaction is determined by computing the relations between (F, A) and (G, B). Let the CVS relation be denoted by (S, C), where $C \subseteq A \times B$ and $S(a, b) \subseteq (F, A) \times (G, B), \forall (a, b) \in C$. Then (S, C) is a CVSS that measures the degree of association or influence between the Chinese and Malaysian financial indicators over the limited time span of 12 months.

In the context of this example, the lower and upper bounds of the amplitude terms of (S, C) are measures of the degree of influence of a Chinese financial indicator on a Malaysian financial indicator, whereas the lower and upper bounds of the phase term represent the "phase" of influence, which in this case represents the amount of time that elapses before the influence of (or a particular event related to) a Chinese financial indicator becomes evident in the Malaysian economy. As both the amplitude and phase terms lie in the closed interval [0, 1], an amplitude term with value close to zero implies that a particular Chinese financial index has very little or no influence on the corresponding Malaysian financial index whereas an amplitude term close to one implies that a particular Chinese financial index strongly influences the corresponding Malaysian financial index. Similarly, a phase term with value close to zero (one) implies that a very short (long) time elapses before the influence of a Chinese financial index becomes evident on a Malaysian financial index.

Continuing in this direction, we present the computation the relations between (F, A) and (G, B) and provide possible interpretations of the results by considering possible scenarios. Given

$$(S, C) = \left\{ S(a_1, b_1), S(a_1, b_2), S(a_1, b_3), S(a_1, b_4), \ldots, S(a_4, b_3), S(a_4, b_4) \right\},$$

then we have:

$$\begin{aligned}
(S, C) = \Big\{ S(a_1, b_1) &= \left\{ \left(x_1, [0, 0.1]\, e^{i\left[0, \frac{\pi}{6}\right]} \right), \left(x_{2,}\, [0.5, 0.6]\, e^{i\left[0, \frac{\pi}{6}\right]} \right), \left(x_3, [0.9, 1]\, e^{i\left[0, \frac{\pi}{6}\right]} \right) \right\}, \\
S(a_1, b_2) &= \left\{ \left(x_1, [0, 0.1]\, e^{i\left[\frac{\pi}{24}, \frac{\pi}{6}\right]} \right), \left(x_{2,}\, [0.5, 0.6]\, e^{i\left[\frac{\pi}{24}, \frac{\pi}{6}\right]} \right), \left(x_3, [0.8, 1]\, e^{i\left[\frac{\pi}{24}, \frac{\pi}{6}\right]} \right) \right\}, \\
S(a_1, b_3) &= \left\{ \left(x_1, [0, 0.1]\, e^{i\left[\frac{2\pi}{3}, \pi\right]} \right), \left(x_{2,}\, [0.5, 0.6]\, e^{i\left[\frac{2\pi}{3}, \pi\right]} \right), \left(x_3, [0.8, 0.9]\, e^{i\left[\frac{2\pi}{3}, \pi\right]} \right) \right\}, \ldots \\
S(a_4, b_3) &= \left\{ \left(x_1, [0.1, 0.3]\, e^{i\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]} \right), \left(x_{2,}\, [0.6, 0.7]\, e^{i\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]} \right), \left(x_3, [0.6, 0.75]\, e^{i\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]} \right) \right\}, \\
S(a_4, b_4) &= \left\{ \left(x_1, [0.2, 0.3]\, e^{i[\pi, 2\pi]} \right), \left(x_{2,}\, [0.75, 0.85]\, e^{i[\pi, 2\pi]} \right), \left(x_3, [0.6, 0.75]\, e^{i[\pi, 2\pi]} \right) \right\} \Big\}.
\end{aligned}$$

The term $S(a_1, b_1) = (x_3, [0.9, 1]\, e^{i[0, \pi/6]})$ indicates the existence of a strong relationship between the Chinese Yuan (Yuan) exchange rate and the Malaysian Ringgit (MYR), which implies that the Yuan exchange rate strongly influences the MYR and that this effect very quickly becomes evident in the Malaysian economy (i.e. within two months or less). The term $S(a_1, b_2) = (x_3, [0.8, 1]\, e^{i[\pi/24, \pi/6]})$ indicates that the Yuan exchange rate strongly influences the prices of Malaysian commodities and that this effect very quickly becomes evident in the Malaysian economy (i.e. between a week and a month). Similarly, the term $S(a_1, b_3) = (x_3, [0.8, 0.9]\, e^{i[2\pi/3, \pi]})$ indicates a very strong relationship between the Yuan exchange rate and the volume of Malaysian exports. However, in this case, the time required for this influence on the Malaysian economy to become evident is four to six months, unlike the previous two financial indicators for which the effects very quickly manifest. This temporal difference is present because, although the fluctuations in the Yuan exchange rate would affect the price of exports from Malaysia and thereby make them more or less attractive (depending on which way the exchange rate fluctuates), a few months would pass before this change would become evident, because the volume of exports from one country to another is usually decided on a quarterly or semi-annual basis and sales contracts normally fix the prices for similar periods of time. Thus, the change in the volume of exports would not be immediate, but rather would occur only after four to six months. All of these findings are supported by recent events in the Chinese and Malaysian economies.

China's recent move to devalue the Yuan has sent shockwaves through global markets, raising fears of a fresh round of currency wars amid heightened fears and uncertainties in an already volatile global environment. In August 2015, about a month after China's stock market crashed and caused increased anxiety in global financial markets, the People's Bank of China devalued the Yuan for a record three times in a row in a single week. This action adversely affected equity markets worldwide and most Asian currencies. The MYR was one of the currencies that declined drastically against the US Dollar (USD). The MYR quickly weakened beyond the critical psychological level of 4.00 MYR = 1.00 USD over the week and closed at MYR 4.0805 against the greenback on August 14, 2015, losing 3.5% of its value in just one week.

Furthermore, because China is the world's largest consumer of commodities, the move by the People's Bank of China to devalue the Yuan caused declines in global prices for everything from oil to industrial metals. The devaluation was interpreted as a signal that the Chinese economy was weakening faster than expected, thus fuelling fear worldwide because a slowdown in the Chinese economy would significantly lower Chinese demand for imports, which would negatively impact economies that rely on exports to China. A weaker Yuan could lower China's domestic spending because the costs of imports would increase, which would drive consumers toward locally produced goods. Devaluing the Yuan would also affect China's trading partners and exporting countries that compete with China because a weaker Yuan would make China's exports cheaper and thus more attractive, giving Chinese exports a competitive edge against other exporters.

Malaysia is perceived to be among the countries that stand to lose the most from the decline of the Yuan against the USD. Many international financial institutions such as the IMF, the Asian Development Bank and the international brokerage firm Credit Suisse are of the opinion that Malaysia is in a particularly vulnerable situation, given China's position as Malaysia's largest trading partner for the seventh consecutive year in 2015 and Malaysia's position as China's largest trading partner within the 10-member ASEAN community with a total trade of USD 6.07 billion. Furthermore, China is Malaysia's largest export market, accounting for 12.6% of Malaysia's total shipments in the first six months of 2015 alone. Although Malaysian exports do not compete directly with Chinese exports, Malaysia still stands to be affected because a weaker Yuan could potentially lower China's domestic spending on imported goods which would make locally produced goods more attractively priced, thus lowering the demand for imported goods. This situation directly affects Malaysia because a large portion of Malaysia's exports and commodities is sold to China.

Thus, weakening of the Yuan causes a decline in the MYR exchange rate and a drop in the Chinese demand for Malaysian exports and commodities, thus resulting in a drop in the volume of Malaysian exports to China and a drop in the prices of global commodities, such as minerals, rubber, palm oil and petroleum that are produced in Malaysia and vice-versa. These factors explain why fluctuations in the Yuan exchange rate strongly influence the MYR exchange rate, prices of Malaysian commodities and volume of Malaysian exports.

Thus, Example 3.3 illustrates how to use the relations between CVSSs to represent and interpret interrelated real-life scenarios.

3.1. Comparison with existing methods in literature

In Example 3.3, we discussed the relations between the Chinese and Malaysian financial indicators and how the Chinese indicators influence the Malaysian indicators. To evaluate these relationships, we calculated the degrees of association between these indicators and, as with most other economic-related activities in which timing is of utmost importance, we discussed the time required for changes in the Chinese financial indicators to affect the Malaysian financial indicator. Although a wide variety of hybrid models of fuzzy and soft sets exists to describe the relations between such indicators, the models that are the closest in structure to the CVS relation are the complex fuzzy relations and vague soft relations. Therefore, we now compare these two relations with our proposed CVS relation.

One method to determine the degrees of association and the phase described above is to use vague soft sets to represent the information and then to find the relations between these vague soft sets. In this case, we would have to define two vague soft sets with parameters such as "minimal influence", "average influence" and "strong influence" for the first set and "1–3 months", "4–6 months" and "6–12 months" for the second set, so that the degrees of influence of the Chinese financial indicators on the Malaysian financial indicators and the time required for the former to affect the latter were characterized by their truth and false membership grades in two separate vague soft sets. Although determining the degrees of association or interaction between these vague soft sets is definitely possible, the outcome would be highly ambiguous because it would be rather difficult to explain the relations between the information in both vague soft sets accurately to see the complete picture without causing both aspects of the information to lose significance or meaning. Alternatively, one could find the link between the information through composition operations which are undoubtedly cumbersome and would result in additional computations that do not in any way increase the accuracy with which the information is represented.

An alternative to using the vague soft set model is to determine the truth and false membership grades of an indicator by accounting for both the degree of influence of a Chinese financial indicator on a Malaysian financial indicator and the phase or time required for the former to influence the latter and then to combine these values to obtain the truth and false membership values. However, were this the case, a Chinese financial indicator that strongly and rapidly (weakly and slowly) influences a Malaysian financial indicator would receive a rather small (large) membership grade in the vague soft set. Both of these methods are inaccurate and can prove very misleading to those who use this information, especially for the purpose of research and development.

Complex fuzzy relations can overcome the problems inherent in using vague soft sets. However, the complex fuzzy model has two major problems: (1) It lacks of a parameterization tool that would enable the problem parameters to be defined more comprehensively, unlike CVS relations which have the added advantage of soft sets; (2) the single-valued grade of membership that is assigned to each element lacks the accuracy of the interval-based grade of membership provided by the CVS model. As explained previously, assigning a single value to describe the degree of influence of one indicator on another and the time required for the effect to become evident is extremely difficult because these values would normally be obtained from different sources that may have different opinions. For example, the IMF may opine that the volume of Chinese exports strongly influences the volume of Malaysian exports and that only one to two months are required for any change in the former to affect the latter. Thus, it may assign 0.95 to the amplitude term and $\pi/8$ to the phase term. In contrast, the international brokerage firm Credit Suisse may opine that, although the volume of Chinese exports substantially affects the volume of Malaysian exports, anywhere from five to six months would usually be required for this effect to become evident in the economy. As such, Credit Suisse may choose to assign values of 0.6 and π to the amplitude and phase terms, respectively of this element. The CVS relation overcomes this problem by providing interval based membership values for the amplitude and phase terms of the elements which is more practical for describing real-life situations which are often subjective in nature. These features succinctly describe the advantages of using the CVS relation structure to model situations that contain vague, uncertain and subjective data.

Definition 3.4. If (F, A) and (G, B) are CVSSs over a soft universe (U, E) and (R, C) is a CVS relation from (F, A) to (G, B), then (R^{-1}, C^{-1}) is a *CVS relation* and is defined as:

$$R^{-1}(a, b) = R(b, a), \forall (a, b) \in C \subseteq A \times B.$$

We now present the proofs of some of the fundamental properties of complex vague soft relations. For the sake of similarity and to avoid repetition, proofs are given only for the lower bound of the amplitude term; the proofs for the upper bound of the amplitude term and the lower and upper bounds of the phase term can be obtained in a similar manner.

Proposition 3.5. *If (F, A) and (G, B) are CVSSs over a soft universe (U, E) and (R, C) is a CVS relation from (F, A) to (G, B), then R^{-1} is a CVS relation from (G, B) to (F, A).*

Proof. Since (R, C) is a CVS relation from (F, A) to (G, B), then $R(a, b) = F(a) \cap G(b), \forall (a, b) \in C \subseteq A \times B$. Thus, it follows that

$$r_{t_{(R^{-1})}}(a, b) = r_{t_R}(b, a) = \min\left(r_{t_{G_b}}(x), r_{t_{F_a}}(x)\right)$$

and

$$1 - k_{f_{(R^{-1})}}(a, b) = 1 - k_{f_R}(b, a) = \min\left(1 - k_{f_{G_b}}(x), 1 - k_{f_{F_a}}(x)\right)$$

and similarly for the phase term. This completes the proof. □

Proposition 3.6. *If (F, A) and (G, B) are CVSSs over a soft universe (U, E) and if (R_1, C) and (R_2, C) are CVS relations from (F, A) to (G, B), where , then the following results hold:*

(i) $\left(R_1^{-1}\right)^{-1} = R_1$
(ii) $R_1 \subseteq R_2 \Rightarrow R_1^{-1} \subseteq R_2^{-1}$.

Proof. The proof is straightforward. □

4. Composition of complex vague soft relations

There are many types of fuzzy compositions, with the most common being the max–min and the max-product compositions. The composition of CVS relations introduced here is based on the fuzzy max–min composition which was made compatible with the base concept of CVSSs.

Definition 4.1. If (F, A), (G, B) and (H, C) are CVSSs over a soft universe (U, E) and (R_1, D_1) and (R_2, D_2) are CVS relations from (F, A) to (G, B) and from (G, B) to (H, C) respectively, where $D_1 \subseteq A \times B$ and $D_2 \subseteq B \times C$, then the *composition* of the CVS relations (R_1, D_1) and (R_2, D_2), denoted by $(R_1 \circ R_2)(a, c) = R_1(a, b) \cap R_2(b, c)$, is defined as:

$$(R_1 \circ R_2)(a, c) = \left\{ x, \left[r_{t_{(R_1 \circ R_2)}}(a, c), 1 - k_{f_{(R_1 \circ R_2)}}(a, c)\right] \times e^{i\left[w^r{}_{t_{(R_1 \circ R_2)}}(a,c), 2\pi - w^k{}_{f_{(R_1 \circ R_2)}}(a,c)\right]} : x \in U \right\},$$

where

$$r_{t_{(R_1 \circ R_2)}}(a, c) = \max\left(r_{t_{R_1}}(a, b), r_{t_{R_2}}(b, c)\right) = \max\left(\min\left(r_{t_{(Fa)}}(x), r_{t_{(G_b)}}(x)\right), \min\left(r_{t_{(G_b)}}(x), r_{t_{(Hc)}}(x)\right)\right),$$

$$1 - k_{f_{(R_1 \circ R_2)}}(a, c) = \max\left(1 - k_{f_{R_1}}(a, b), 1 - k_{f_{R_2}}(b, c)\right) = \max\left(\min\left(1 - k_{f_{(Fa)}}(x), 1 - k_{f_{(G_b)}}(x)\right), \min\left(1 - k_{f_{(G_b)}}(x), 1 - k_{f_{(Hc)}}(x)\right)\right),$$

$$w^r{}_{t_{(R_1 \circ R_2)}}(a, c) = \max\left(w^r{}_{t_{R_1}}(a, b), w^r{}_{t_{R_2}}(b, c)\right) = \max\left(\min\left(w^r{}_{t_{(Fa)}}(x), w^r{}_{t_{(G_b)}}(x)\right), \min\left(w^r{}_{t_{(G_b)}}(x), w^r{}_{t_{(Hc)}}(x)\right)\right),$$

$$2\pi - w^k{}_{f_{(R_1 \circ R_2)}}(a, c) = \max\left(2\pi - w^k{}_{f_{R_1}}(a, b), 2\pi - w^k{}_{f_{R_2}}(b, c)\right) = \max\begin{pmatrix} \min\left(2\pi - w^k{}_{f_{Fa}}(x), 2\pi - w^k{}_{f_{G_b}}(x)\right), \\ \min\left(2\pi - w^k{}_{f_{G_b}}(x), 2\pi - w^k{}_{f_{Hc}}(x)\right) \end{pmatrix},$$

for all $(a, b) \in D_1 \subseteq A \times B$ and $(b, c) \in D_2 \subseteq B \times C$.

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Example 4.2. Let $U = \{m_1, m_2, m_3, m_4\}$ be a set of medicines under consideration, E be a set of parameters and $A, B, C \subseteq E$ be defined as:

$$A = \left\{a_1 = \text{pneumonia}, a_2 = \text{influenza}, a_3 = \text{tuberculosis}, a_4 = \text{asthma}\right\},$$

$$B = \left\{b_1 = \text{leptospirosis}, b_2 = \text{influenza}, b_3 = \text{diabetes}\right\},$$

$$C = \left\{c_1 = \text{encephalitis}, c_2 = \text{hypertension}, c_3 = \text{chronic obstructive pulmonary disorder (COPD)}\right\}.$$

Now suppose that (F, A), (G, B) and (H, C) are CVSSs defined as:

$$(F, A) =$$

$$F(a_1) = \left\{\left(m_1, [0.2, 0.3]\, e^{i[0.2\pi, 0.8\pi]}\right), \left(m_2, [0.5, 0.8]\, e^{i[0.2\pi, 1.2\pi]}\right), \left(m_3, [0.4, 0.7]\, e^{i[0.8\pi, 1.8\pi]}\right), \left(m_4, [0.3, 0.5]\, e^{i[0.3\pi, \pi]}\right)\right\},$$

$$F(a_2) = \left\{\left(m_1, [0.2, 0.4]\, e^{i[0.4\pi, 0.6\pi]}\right), \left(m_2, [0.9, 1]\, e^{i[0, \pi]}\right), \left(m_3, [0.8, 0.95]\, e^{i[0.6\pi, 0.6\pi]}\right), \left(m_4, [0.5, 0.5]\, e^{i[0.3\pi, 1.5\pi]}\right)\right\},$$

$$F(a_3) = \left\{\left(m_1, [0.6, 0.7]\, e^{i[0, 0.7\pi]}\right), \left(m_2, [0.6, 0.8]\, e^{i[0.4\pi, 1.4\pi]}\right), \left(m_3, [0.4, 0.8]\, e^{i[0.2\pi, 0.6\pi]}\right), \left(m_4, [0.2, 0.5]\, e^{i[0.1\pi, 0.5\pi]}\right)\right\},$$

$$F(a_4) = \left\{\left(m_1, [0.4, 0.6]\, e^{i[0.4\pi, 0.8\pi]}\right), \left(m_2, [0.1, 0.4]\, e^{i[0.1\pi, 0.8\pi]}\right), \left(m_3, [0.3, 0.9]\, e^{i[0.2\pi, 1.4\pi]}\right), \left(m_4, [0.4, 0.9]\, e^{i[\pi, 2\pi]}\right)\right\}.$$

$$(G, B) =$$

$$G(b_1) = \left\{\left(m_1, [0.8, 0.9]\, e^{i[0.1\pi, 0.6\pi]}\right), \left(m_2, [0.15, 0.4]\, e^{i[0, \pi]}\right), \left(m_3, [0.2, 0.5]\, e^{i[0.6\pi, 1.2\pi]}\right), \left(m_4, [0.3, 0.8]\, e^{i[0.4\pi, 1.1\pi]}\right)\right\},$$

$$G(b_2) = \left\{\left(m_1, [0.1, 0.9]\, e^{i[0.4\pi, 1.2\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0.6\pi, 1.8\pi]}\right), \left(m_3, [0.5, 0.75]\, e^{i[\pi, 1.3\pi]}\right), \left(m_4, [0.6, 0.9]\, e^{i[0.2\pi, \pi]}\right)\right\},$$

$$G(b_3) = \left\{\left(m_1, [0.2, 0.95]\, e^{i[0.7\pi, 1.3\pi]}\right), \left(m_2, [0.1, 0.8]\, e^{i[0, 0.9\pi]}\right), \left(m_3, [0.4, 0.75]\, e^{i[0.5\pi, 0.8\pi]}\right), \left(m_4, [0.7, 1]\, e^{i[1.1\pi, 1.3\pi]}\right)\right\}.$$

$$(H, C) =$$

$$H(c_1) = \left\{\left(m_1, [0.3, 0.5]\, e^{i[0, 0.5\pi]}\right), \left(m_2, [0.6, 0.7]\, e^{i[\pi, 2\pi]}\right), \left(m_3, [0.9, 1]\, e^{i[0, \pi]}\right), \left(m_4, [0.1, 0.3]\, e^{i[1.2\pi, 1.5\pi]}\right)\right\},$$

$$H(c_2) = \left\{\left(m_1, [0.2, 0.6]\, e^{i[\pi, 1.8\pi]}\right), \left(m_2, [0.1, 0.15]\, e^{i[0, 0]}\right), \left(m_3, [0.95, 1]\, e^{i[\pi, 1.4\pi]}\right), \left(m_4, [0.7, 0.9]\, e^{i[1.4\pi, 1.8\pi]}\right)\right\},$$

$$H(c_3) = \left\{\left(m_1, [0.65, 0.85]\, e^{i[0.6\pi, 1.2\pi]}\right), \left(m_2, [0, 0]\, e^{i[0, 0]}\right), \left(m_3, [0.25, 0.6]\, e^{i[0.4\pi, 0.9\pi]}\right), \left(m_4, [0.75, 0.99]\, e^{i[0.8\pi, 1.4\pi]}\right)\right\}.$$

The CVS relations (R_1, D_1) from (F, A) to (G, B) and (R_2, D_2) from (G, B) to (H, C) are defined below for all $(a, b) \in D_1 \subseteq A \times B$ and $(b, c) \in D_2 \subseteq B \times C$ respectively if and only if a, b and c are respiratory system related diseases.

Q6 In this case, the CVS relations (R_1, D_1) and (R_2, D_2) are given by:

$$(R_1, D_1) =$$

$$\left\{\left\{F(a_1) \times G(b_2) = \left\{\left(m_1, [0.1, 0.3]\, e^{i[0.2\pi, 0.8\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0.2\pi, 1.2\pi]}\right), \left(m_3, [0.4, 0.7]\, e^{i[0.8\pi, 1.3\pi]}\right), \left(m_4, [0.3, 0.5]\, e^{i[0.2\pi, \pi]}\right)\right\},\right.\right.$$

$$\left\{F(a_2) \times G(b_2) = \left\{\left(m_1, [0.1, 0.4]\, e^{i[0.4\pi, 0.6\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0, \pi]}\right), \left(m_3, [0.5, 0.75]\, e^{i[0.6\pi, 0.6\pi]}\right), \left(m_4, [0.5, 0.5]\, e^{i[0.2\pi, \pi]}\right)\right\},\right.$$

$$\left\{F(a_3) \times G(b_2) = \left\{\left(m_1, [0.1, 0.7]\, e^{i[0, 0.7\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0.4\pi, 1.4\pi]}\right), \left(m_3, [0.4, 0.75]\, e^{i[0.2\pi, 0.6\pi]}\right), \left(m_4, [0.2, 0.5]\, e^{i[0.1\pi, 0.5\pi]}\right)\right\}\right\},$$

$$\left\{F(a_4) \times G(b_2) = \left\{\left(m_1, [0.1, 0.6]\, e^{i[0.4\pi, 0.8\pi]}\right), \left(m_2, [0.1, 0.4]\, e^{i[0.1\pi, 0.8\pi]}\right), \left(m_3, [0.3, 0.75]\, e^{i[0.2\pi, 1.3\pi]}\right), \left(m_4, [0.4, 0.9]\, e^{i[0.2\pi, \pi]}\right)\right\}\right\}.$$

$$(R_2, D_2) =$$

$$\left\{\left\{G(b_2) \times H(c_3) = \left\{\left(m_1, [0.1, 0.85]\, e^{i[0.4\pi, 1.2\pi]}\right), \left(m_2, [0, 0]\, e^{i[0, 0]}\right), \left(m_3, [0.25, 0.6]\, e^{i[0.4\pi, 0.9\pi]}\right), \left(m_4, [0.6, 0.9]\, e^{i[0.2\pi, \pi]}\right)\right\}.\right.\right.$$

Then the composition between the CVS relations (R_1, D_1) and (R_2, D_2) is

$$(R_1, D_1) \circ (R_2, D_2)$$

$$= (R_1 \circ R_2)(a, c)$$

$$= R_1(a, b) \cap R_2(b, c)$$

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$$= \Big\{ \big\{ F(a_1) \times H(c_3) = \big\{ \left(m_1, [0.1, 0.85]\, e^{i[0.4\pi, 1.2\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0.2\pi, 1.2\pi]}\right), \left(m_3, [0.4, 0.7]\, e^{i[0.8\pi, 1.3\pi]}\right), \left(m_4, [0.6, 0.9]\, e^{i[0.2\pi, \pi]}\right) \big\},$$

$$\big\{ F(a_2) \times H(c_3) = \big\{ \left(m_1, [0.1, 0.85]\, e^{i[0.4\pi, 1.2\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0, \pi]}\right), \left(m_3, [0.5, 0.75]\, e^{i[0.6\pi, 0.9\pi]}\right), \left(m_4, [0.6, 0.9]\, e^{i[0.2\pi, \pi]}\right) \big\},$$

$$\big\{ F(a_3) \times H(c_3) = \big\{ \left(m_1, [0.1, 0.85]\, e^{i[0.4\pi, 1.2\pi]}\right), \left(m_2, [0.4, 0.5]\, e^{i[0.4\pi, 1.4\pi]}\right), \left(m_3, [0.4, 0.75]\, e^{i[0.4\pi, 0.9\pi]}\right), \left(m_4, [0.6, 0.9]\, e^{i[0.2\pi, \pi]}\right) \big\},$$

$$\big\{ F(a_4) \times H(c_3) = \big\{ \left(m_1, [0.1, 0.85]\, e^{i[0.4\pi, 1.2\pi]}\right), \left(m_2, [0.1, 0.4]\, e^{i[0.1\pi, 0.8\pi]}\right), \left(m_3, [0.3, 0.75]\, e^{i[0.4\pi, 1.3\pi]}\right), \left(m_4, [0.6, 0.9]\, e^{i[0.2\pi, \pi]}\right) \big\} \Big\}.$$

Theorem 4.3. *If $(F, A), (G, B), (H, C)$ and (J, D) are CVSSs over a soft universe (U, E) and $(R_1, K_1), (R_2, K_2)$ and (R_3, K_3) are CVS relations from (F, A) to (G, B), (G, B) to (H, C) and (H, C) to (J, D) respectively, where $K_1 \subseteq A \times B$, $K_2 \subseteq B \times C$ and $K_3 \subseteq C \times D$, then*

$$((R_1, K_1) \circ (R_2, K_2)) \circ (R_3, K_3) = (R_1, K_1) \circ ((R_2, K_2) \circ (R_3, K_3)).$$

Proof. If $(R_1, K_1), (R_2, K_2)$ and (R_3, K_3) are CVS relations from (F, A) to (G, B), (G, B) to (H, C) and (H, C) to (J, D) respectively, where $K_1 \subseteq A \times B$, $K_2 \subseteq B \times C$ and $K_3 \subseteq C \times D$, then $R_1(a, b) = F(a) \cap G(b)$, $R_2(b, c) = G(b) \cap H(c)$ and $R_3(c, d) = H(c) \cap J(d)$ for all $(a, b) \in K_1 \subseteq A \times B$, $(b, c) \in K_2 \subseteq B \times C$ and $(c, d) \in K_3 \subseteq C \times D$ respectively, then $(R_1, K_1) \circ (R_2, K_2) = (R_1 \circ R_2)(a, c)$ and $(R_2, K_2) \circ (R_3, K_3) = (R_2 \circ R_3)(b, d)$. Therefore, it follows that:

$$r_{t_{(R_1 \circ R_2)}}(a, c) \circ r_{t_{R_3}}(c, d)$$

$$= \max\left(r_{t_{R_1}}(a, b), r_{t_{R_2}}(b, c)\right) \circ r_{t_{R_3}}(c, d)$$

$$= \max\left[\min\left(r_{t_{F_a}}(x), r_{t_{G_b}}(x)\right), \min\left(r_{t_{G_b}}(x), r_{t_{H_c}}(x)\right)\right] \circ \min\left(r_{t_{H_c}}(x), r_{t_{J_d}}(x)\right)$$

$$= \max\left[\max\left[\min\left(r_{t_{F_a}}(x), r_{t_{G_b}}(x)\right), \min\left(r_{t_{G_b}}(x), r_{t_{H_c}}(x)\right)\right], \min\left(r_{t_{H_c}}(x), r_{t_{J_d}}(x)\right)\right]$$

$$= \max\left[\min\left(r_{t_{F_a}}(x), r_{t_{G_b}}(x)\right), \max\left[\min\left(r_{t_{G_b}}(x), r_{t_{H_c}}(x)\right), \min\left(r_{t_{H_c}}(x), r_{t_{J_d}}(x)\right)\right]\right]$$

$$= \max\left(r_{t_{R_1}}(a, b), r_{t_{(R_2 \circ R_3)}}(b, d)\right)$$

$$= r_{t_{R_1}}(a, b) \circ r_{t_{(R_2 \circ R_3)}}(b, d),$$

proving that $r_{t_{(R_1 \circ R_2)}}(a, c) \circ r_{t_{R_3}}(c, d) = r_{t_{R_1}}(a, b) \circ r_{t_{(R_2 \circ R_3)}}(b, d)$. The proofs for other terms follow similarly. This completes the proof. □

Theorem 4.4. *If (F, A) and (G, B) are CVSSs over a soft universe (U, E), and (R_1, D_1) and (R_2, D_2) are CVS relations from (F, A) to (G, B) and (G, B) to (H, C) respectively, where $D_1 \subseteq A \times B$ and $D_2 \subseteq B \times C$, then $(R_1 \circ R_2)^{-1} = {R_2}^{-1} \circ {R_1}^{-1}$.*

Proof. If (R_1, D_1) and (R_2, D_2) are CVS relations from (F, A) to (G, B) and (G, B) to (H, C) respectively, then $R_1(a, b) = F(a) \cap G(b)$ and $R_2(b, c) = G(b) \cap H(c)$, for all $(a, b) \in D_1 \subseteq A \times B$ and $(b, c) \in D_2 \subseteq B \times C$ respectively. Thus, it follows that:

$$r_{t_{(R_1 \circ R_2)^{-1}}}(a, c)$$

$$= r_{t_{(R_1 \circ R_2)}}(c, a)$$

$$= \max\left(r_{t_{R_1}}(c, b), r_{t_{R_2}}(b, a)\right)$$

$$= \max\left(r_{t_{R_2}}(b, a), r_{t_{R_1}}(c, b)\right)$$

$$= \max\left[\min\left(r_{t_{G_b}}(x), r_{t_{F_a}}(x)\right), \min\left(r_{t_{H_c}}(x), r_{t_{G_b}}(x)\right)\right]$$

$$= \max\left[\min\left(r_{t_{F_a}}(x), r_{t_{G_b}}(x)\right), \min\left(r_{t_{G_b}}(x), r_{t_{H_c}}(x)\right)\right]$$

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$$= \max\left(r_{t_{R_2{}^{-1}}}(a,b), r_{t_{R_1{}^{-1}}}(b,c)\right)$$

$$= r_{t_{\left(R_2{}^{-1} \circ R_1{}^{-1}\right)}}(a,c).$$

Similar results follow for the rest of the terms. This completes the proof. □

Next, we introduce the notion of symmetric, transitive and reflexive relations between CVSSs.

Definition 4.5. If (R, C) is a CVS relation from (F, A) to (G, B), then

(i) (R, C) is a *CVS symmetric relation* from (F, A) to (G, B) if $R(a, b) = R(b, a)$, for all $(a, b) \in C \subseteq A \times B$.
(ii) (R, C) is a *CVS transitive relation* from (F, A) to (G, B) if $R(a, b) \circ R(b, a) \subseteq R(a, b)$, for all $(a, b) \in C \subseteq A \times B$.
(iii) (R, C) is a *CVS reflexive relation* from (F, A) to (G, B) if $R(a, b) \subseteq R(a, a)$ and $R(b, a) \subseteq R(a, a)$ for all $(a, b) \in C \subseteq A \times B$.

Definition 4.6. If (R, C) is a CVS relation from (F, A) to (G, B), then (R, C) is a *CVS equivalence relation* from (F, A) to (G, B) if it is a CVS relation that is symmetric, transitive and reflexive for all $(a, b) \in C \subseteq A \times B$.

Proposition 4.7. *If (R, C) is a CVS relation from (F, A) to (G, B), then*

(i) *(R, C) is a CVS symmetric relation if and only if (R^{-1}, C^{-1}) is a CVS symmetric relation.*
(ii) *(R, C) is a CVS symmetric relation if and only if $(R, C) = (R^{-1}, C^{-1})$.*
(iii) *If (R, C) is a CVS transitive relation, then (R^{-1}, C^{-1}) is also a CVS transitive relation.*
(iv) *If (R, C) is a CVS transitive relation, then $(R, C) \circ (R, C)$ is also a CVS transitive relation.*
(v) *If (R, C) is a CVS reflexive relation, then (R^{-1}, C^{-1}) is also a CVS reflexive relation.*
(vi) *If (R, C) is a CVS relation that is symmetric and transitive, then (R, C) is a CVS reflexive relation.*

Proof. (i) ($\Rightarrow$) If (R, C) is a CVS symmetric relation, then $R(a, b) = R(b, a)$. Thus it follows that:

$$r_{t_{\left(R^{-1}\right)}}(a,b) = r_{t_R}(b,a) = r_{t_R}(a,b) = r_{t_{\left(R^{-1}\right)}}(b,a)$$

and

$$1 - k_{f_{\left(R^{-1}\right)}}(a,b) = 1 - k_{f_R}(b,a) = 1 - k_{f_R}(a,b) = 1 - k_{f_{\left(R^{-1}\right)}}(b,a).$$

($\Leftarrow$) If (R^{-1}, C^{-1}) is a CVS symmetric relation, then $R^{-1}(a, b) = R^{-1}(b, a)$. Thus, it follows that:

$$r_{t_R}(a,b) = r_{t_{\left(R^{-1}\right)^{-1}}}(a,b) = r_{t_{\left(R^{-1}\right)}}(b,a) = r_{t_{\left(R^{-1}\right)}}(a,b) = r_{t_R}(b,a)$$

and

$$1 - k_{f_R}(a,b) = 1 - k_{f_{\left(R^{-1}\right)^{-1}}}(a,b) = 1 - k_{f_{\left(R^{-1}\right)}}(b,a) = 1 - k_{f_{\left(R^{-1}\right)}}(a,b) = 1 - k_{f_R}(b,a).$$

Similar results can be proven for both the lower and upper bounds of the phase term for both sides of the proof.

This completes the proof. □

(ii)–(vi) The proofs are straightforward using Definition 4.5. □

Proposition 4.8. *If (R_1, D_1) and (R_2, D_2) are CVS symmetric relations from (F, A) to (G, B) and (G, B) to (H, C) respectively, then $(R_1, D_1) \circ (R_2, D_2)$ is a CVS symmetric relation if and only if $(R_1, D_1) \circ (R_2, D_2) = (R_2, D_2) \circ (R_1, D_1)$.*

Proof. Let (R_1, D_1) and (R_2, D_2) be complex vague soft symmetric relations. Then $(R_1, D_1) = \left(R_1^{-1}, D_1^{-1}\right)$ and $(R_2, D_2) = \left(R_2^{-1}, D_2^{-1}\right)$ and we have $(R_1 \circ R_2)^{-1} = \left(R_2^{-1} \circ R_1^{-1}\right)$.

($\Rightarrow$) Let $R_1 \circ R_2$ be a complex vague soft symmetric relation. Then $(R_1 \circ R_2)(a, b) = (R_1 \circ R_2)^{-1}(a, b)$. It is sufficient to prove that $(R_1 \circ R_2)(a, b) = (R_2 \circ R_1)(a, b)$ and this will be proven for the amplitude terms only. The proof for the phase terms can be obtained in a similar manner. Thus we have,

$$r_{t_{(R_1 \circ R_2)}}(a,b) = r_{t_{(R_1 \circ R_2)^{-1}}}(a,b) = r_{t_{\left(R_2^{-1} \circ R_1^{-1}\right)}}(a,b) = r_{t_{(R_2 \circ R_1)}}(a,b)$$

and

$$1 - k_{f_{(R_1 \circ R_2)}}(a,b) = 1 - k_{f_{(R_1 \circ R_2)^{-1}}}(a,b) = 1 - k_{f_{\left(R_2^{-1} \circ R_1^{-1}\right)}}(a,b) = 1 - k_{f_{(R_2 \circ R_1)}}(a,b).$$

This proves that $(R_1, D_1) \circ (R_2, D_2) = (R_2, D_2) \circ (R_1, D_1)$.

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($\Leftarrow$) Now let $(R_1, D_1) \circ (R_2, D_2) = (R_2, D_2) \circ (R_1, D_1)$, which implies that $(R_1 \circ R_2)(a, b) = (R_2 \circ R_1)(a, b)$. To prove that $R_1 \circ R_2$ is a complex vague soft symmetric relation, it is sufficient to prove that $(R_1 \circ R_2)(a, b) = (R_1 \circ R_2)^{-1}(a, b)$. This result is proven below for the amplitude terms and the proof for the phase terms follow in a similar manner. Hence we have,

$$r_{t_{(R_1 \circ R_2)^{-1}}}(a, b) = r_{t_{\left(R_2^{-1} \circ R_1^{-1}\right)}}(a, b) = r_{t_{(R_2 \circ R_1)}}(a, b) = r_{t_{(R_1 \circ R_2)}}(a, b)$$

and

$$1 - k_{f_{(R_1 \circ R_2)^{-1}}}(a, b) = 1 - k_{f_{\left(R_2^{-1} \circ R_1^{-1}\right)}}(a, b) = 1 - k_{f_{(R_2 \circ R_1)}}(a, b) = 1 - k_{f_{(R_1 \circ R_2)}}(a, b).$$

This completes the proof. □

5. Conclusion

This work establishes the notion of relations between CVSSs and the composition of these relations. It improves on the concept of relations between fuzzy sets, soft sets and hybrid models of these two types of sets (such as complex fuzzy sets) because the relations are constructed based on the CVSS model which allows for an interval-based membership function that better corresponds to the intuition and subjectivity factors that exist in assigning membership values and provide an adequate parameterization tool to deal with the aspects of uncertainties and vagueness which are features that are pervasive in most real-life problems. The amplitude term of this CVS relation functions similarly to that of an ordinary membership function and describes the degree of presence or the absence of interaction between the parameters, albeit in the more accurate form of a closed interval. The phase term of this relation on the other hand, accurately captures and deals with the element of periodicity that exists in most (if not all) complex data consisting of two dimensional information. Verification of some of the fundamental properties of these concepts was also presented.

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