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# Approaches to decision making with linguistic preference relations based on additive consistency

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ABSTRACT

The linguistic preference relation (LPR) is introduced to efficiently deal with situations in which the decision makers (DMs) provide their preference information by using linguistic labels over paired comparisons of alternatives. However, the lack of consistency in decision making with LPRs can lead to inconsistent conclusions. In this paper, two new decision making methods are developed to improve the additive consistency of LPRs until they are acceptable, and eventually obtain the reliable decision making results. First, the new concepts of order consistency and additive consistency of LPRs are introduced, and followed by a discussion of the characterization about additive consistent LPRs. Then, a consistency index is defined to measure whether an LPR is of acceptable additive consistency. For an unacceptable additive consistent LPR, two automatic iterative algorithms are further proposed to help DMs improve additive consistency level until it is acceptable. In addition, the proposed algorithms can derive the priority weight vector from LPRs and obtain the ranking of the alternatives. Finally, the proposed methods are applied to an emergency operating center (EOC) selection problem. The comparative analysis demonstrates the applicability and effectiveness of the proposed methods.



## 1. Introduction

In a decision making situation, the DMs are faced with the problem of which one to be selected among a set of alternatives that fit with DMs' desired goal. To model this problem, in the process of decision making, DMs are usually required to provide their exact preference over a set of alternatives and construct preference relation judgement matrices using their expressed pairwise comparison information [1–3].

Preference relations are popular techniques used to model DMs' knowledge and preferences regarding decision problems. However, it may be difficult for DMs to express their preference information with a crisp number in many situations due to that (1) the DM may not possess a precise or sufficient level of knowledge of the problem; (2) the DM is unable to discriminate explicitly the degree to which one alternative are better than others [4,5,20]. In these situations, various types of preference relations have been investigated in the existing literatures, including multiplicative preference relation (MPR) [6–9], fuzzy preference relation (FPR) [10–14], fuzzy interval preference relation (FIPR) [15,16], triangular fuzzy preference relation (TFPR) [17], trapezoid fuzzy preference relation (TDFPR) [18], intuitionistic fuzzy preference relation (IFPR) [19–21].

For the reason that in practice the preference relation for any two alternatives given by an individual is the perception obtained from an appropriate semantic scale, the pairwise comparison values are provided with qualitative description rather than numerical values [22,23], and then the DMs usually provide the linguistic preference relations (LPRs) [24–27] in decision making. For example, when evaluating the design of a house, the linguistic labels like good, fair, poor can be used. During the recent decades, many researchers pay attention to decision making using LPRs [28–37]. Xu [28] presented a comprehensive survey of LPRs, and briefly discussed their properties and introduced some new preference relations. Based on the ordered weighted averaging operator [29], Herrera et al. [30] introduced the linguistic ordered weighted averaging (LOWA) operator to aggregate LPRs. Xu [31] proposed the linguistic ordered weighted geometric (LOWG) operator, and then he developed an approach to group decision making (GDM). Xu [32–34] also investigated some new linguistic aggregation operators, such as extended order weighted averaging (EOWA) operator,

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extended order weighted geometric (EOWG) operator, uncertain LOWA operator, uncertain LOWG operator, induced uncertain LOWA operator and induced uncertain LOWG operator. Under the uncertain environment, Wang [35] proposed incomplete LPRs to ensure comparison consistency. Based on the induced linguistic ordered weighted geometric (ILOWG) operator [33] and the linguistic continuous ordered weighted geometric (LCOWG) operator [36], Zhou and Chen [37] developed the induced linguistic continuous ordered weighted geometric (ILCOWG) operator and studied some properties of the ILCOWG operator.

Due to an inconsistent preference relation may lead to inconsistent conclusions, then for various types of preference relations, an important research topic is to check their consistency. Based on the additive consistency and the order consistency, Lee [38] presented a method for GDM with incomplete FPRs, and then established the consistent matrix which satisfies the additive consistency and the order consistency. To overcome the drawbacks of Lee's method [38], Chen et al. [39] constructed a modified consistent matrix, and proposed a new method for GDM using incomplete FPRs. Wu et al. [40] developed a trust based estimation method and a visual consensus aggregation model for multi-criteria GDM with incomplete linguistic information. Ureña et al. [41] presented a review of the foundations, developments and applications in estimating missing preferences in GDM problems, and then outlined some of the current trends and potential future research areas. In order to deal with GDM problems with uncertain multiplicative LPRs, Zhou and Chen [42] developed an optimal model based on the criterion of minimizing the compatibility index to determine the optimal weights of DMs. Zhang et al. [43] studied the consistency and consensus measures for GDM based on distribution LPRs. Chen et al. [44] developed a new approach for the uncertain additive LPRs and utilized it to determine the optimal weights of experts in the GDM. Cabrerizo et al. [45] proposed the additive transitivity for unbalanced LPRs and introduced a consistency index to measure the degree of consistency of unbalanced LPRs. Dong et al. [46] presented an optimization-based approach to improve the consistency level of unbalanced LPRs.

From above analysis, we can see that LPR, as a new tool used for expressing preference information in GDM with linguistic terms, is a very useful tool to cope with uncertainty and vagueness. More and more decision making methods and theories have been developed on the basis of LPRs. In GDM with LPRs, just as the numerical preference relations, studying the additive consistency of LPRs, deriving the priority weight vector from LPRs and exploring the ranking methods with LPRs are the important issues.

Inspired by the concept of deviation measure between two LPRs, Dong et al. [47] proposed two techniques to deal with the inconsistency in LPRs and discussed the consistency properties of collective LPRs. However, due to the fact that the iterative algorithm proposed by Dong et al. [47] is not based on LPRs directly, we need to transfer the original LPR into its corresponding LPR and FIPR, thus, it seems to be an indirect computation process and the information losing may take place when the number of alternatives is too large. To circumvent this issue, it is natural and logical to expect that the decision results should be directly derived by the original LPRs. Moreover, it is necessary to develop new simple yet effective models for GDM with LPRs to improve the additive consistency of LPRs and derive the priority weight vector of the LPRs directly. At present, there are few techniques about these issues.

In this paper, we introduce the new concepts of LPRs, order consistent LPRs and additive consistent LPRs, and then the characterization about the additive consistency of LPRs is discussed. A consistency index of LPR is defined to measure whether a LPR is of acceptable additive consistency. For the unacceptable additive consistent LPRs, two automatic iterative algorithms are proposed to improve additive consistency until they are acceptable. To do this, the rest of the paper is organized as follows. In Section 2, we review some related works of the linguistic term sets and LPRs. Section 3 introduces the new concepts of LPRs, order consistent LPRs and additive consistent LPRs, and the characterization about the additive consistency of LPRs is discussed. In Section 4, two automatic iterative algorithms are established to improve additive consistency until the adjusted LPRs are accepTable Section 5 presents a numerical example to verify the proposed methods and provides the comprehensive comparative analysis. Finally, some conclusions and future research possibilities are given in Section 6.

## 2. Preliminaries

In this section, we furnish a brief review on some basic concepts of linguistic term sets.

Let $S=\{s_0, s_1, \cdots, s_g\}$ be a finite and totally ordered discrete linguistic term set [22], where $s_i$ represents a possible value of a linguistic variable, $g$ is an even number. Note that a feasible cardinality of the linguistic term set is the number among 7–9 [48]. The linguistic term set $S$ should satisfy the following characteristics:

(1) If $\alpha \geq \beta$, then $s_\alpha \geq s_\beta$. Therefore, there exist a maximization operator and a minimization operator;
(2) The negation operator: $neg(s_\alpha)=s_{g-\alpha}$, especially $neg(s_{g/2})=s_{g/2}$.

**Example 1**. A set of nine terms $S$ can be defined as:

$$S=\begin{pmatrix} s_0: extremelypoor, s_1: verypoor, s_2: poor, \\ s_3: sightlypoor, s_4: fair, s_5: sightlygood, \\ s_6: good, s_7: verygood, s_8: extremelygood \end{pmatrix}.$$

For convenience, let $g=2\tau$, then $S=\{s_0, s_1, \cdots, s_g\}$ can be denoted by $S=\{s_0, s_1, \cdots, s_{2\tau}\}$.

In order to preserve all of the given information, Xu [31] further extended the discrete linguistic term set $S$ to a continuous linguistic term set $\tilde{S}=\{s_\alpha | s_0 \leq s_\alpha \leq s_{2\tau}, \alpha \in [0, 2\tau]\}$, where $\tau$ is a sufficiently large positive integer. If $s_\alpha \in S$, then we call $s_\alpha$ the original linguistic term; otherwise, we call $s_\alpha$ the virtual linguistic term. In general, the original linguistic terms come from the DMs when they evaluate the considered alternatives, and the virtual linguistic terms only appear in operations.

Suppose that $s_\alpha, s_\beta \in \tilde{S}, \lambda \in [0, 1]$, then the operational laws are defined as follows [45]:

$$s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha = s_{\alpha+\beta}\ ; \tag{1}$$

$$s_\alpha \otimes s_\beta = s_\beta \otimes s_\alpha = s_{\alpha\beta}\ ; \tag{2}$$

$$\lambda s_\alpha = s_{\lambda\alpha}\ ; \tag{3}$$

$$(s_\alpha)^\lambda = s_{\alpha^\lambda}. \tag{4}$$

As can be seen from the operational laws, the virtual linguistic term computes the lower indices of linguistic terms directly, then we introduce a function $I(\cdot): \tilde{S} \to [0, 2\tau]$ to obtain the lower index of linguistic term $s_\alpha \in \tilde{S}$, such that $I(s_\alpha)=\alpha$. Obviously, there exists an inverse function $I^{-1}(\cdot): [0, 2\tau] \to \tilde{S}$, such that $I^{-1}(\alpha)=s_\alpha$ for any $\alpha \in [0, 2\tau]$.

## 3. Order consistency and additive consistency of LPRs

In this section, several new concepts are introduced, including LPRs, order consistent LPRs and additive consistent LPRs, and then some properties of additive consistent LPRs are discussed.

### 3.1. New definition of LPRs and order consistency of LPRs

Linguistic expressions are a natural way of human's thinking and reasoning. For the convenience of the following discussions, we now develop a new concept of LPRs, which is based on the linguistic term set $S = \{s_0, s_1, \cdots, s_{2\tau}\}$.

**Definition 1.** A LPR $A$ on $X = \{x_1, x_2, \cdots, x_n\}$ is characterized by a compassion matrix $A = (a_{ij})_{n\times n}$, where $a_{ij} \in S$, and

$$a_{ij} + a_{ji} = s_{2\tau}, a_{ii} = s_\tau, \forall i, j \in N \quad , \tag{1}$$

where $a_{ij}$ represents a linguistic preference degree of the alternative $x_i$ over $x_j$. If $a_{ij} = s_\tau$, then it denotes that there is no difference between alternative $x_i$ and $x_j$; if $a_{ij} > s_\tau$, then it denotes that alternative $x_i$ is preferred to $x_j$; the larger $r_{ij}$, the greater the preference degree of the alternative $x_i$ over $x_j$; if $a_{ij} = s_{2\tau}$, then it denotes that alternative $x_i$ is absolutely preferred to $x_j$.

In what follows, we present the concept of order consistent LPR, it helps DMs to distinguish the priority relations for a set of alternatives.

**Definition 2.** A LPR $A = (a_{ij})_{n\times n}$ with $a_{ij} \in S$ is called an order consistent LPR, if there exists a permutation $\sigma : N \to N$, such that $a_{\sigma(1),j} \geq a_{\sigma(2),j} \geq \cdots \geq a_{\sigma(n),j}$ for all $j \in N$.

Definition 2 shows that $\sigma(i)$-th row is greater than $\sigma(i+1)$-th row in an order consistent LPR, $i = 1, 2, \cdots, n-1$.

**Example 2.** Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives, suppose that $B = (b_{ij})_{4\times 4}$ is a LPR on $X$, which is shown as follows:

$$B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} s_4 & s_5 & s_6 & s_8 \\ s_3 & s_4 & s_5 & s_7 \\ s_2 & s_3 & s_4 & s_6 \\ s_0 & s_1 & s_2 & s_4 \end{bmatrix} \end{array} .$$

From the LPR $B = (b_{ij})_{4\times 4}$, one can obtain that $b_{1j} \geq b_{2j}$, $b_{2j} \geq b_{3j}$ and $b_{3j} \geq b_{4j}$ for all $j \in \{1, 2, 3, 4\}$, which denotes that the alternative $x_1$ is preferred to $x_2$, the alternative $x_2$ is preferred to $x_3$ and the alternative $x_3$ is preferred to $x_4$. Therefore, the ranking order among the alternatives $x_1, x_2, x_3$ and $x_4$ is that $x_1 \succ x_2 \succ x_3 \succ x_4$.

### 3.2. Additive consistency of LPRs

Inspired by the additive consistency of FPRs, we introduce the definitions of consistent LPRs as follows:

**Definition 3.** A LPR $A = (a_{ij})_{n\times n}$ is called an additive consistent LPR, if it satisfies the following additive transitivity:

$$I(a_{ij}) + I(a_{jk}) + I(a_{ki}) = I(a_{ik}) + I(a_{kj}) + I(a_{ji}), \forall i, j, k \in N \quad . \tag{2}$$

**Remark 1.** According to Eq. (2), we have

$$\frac{I(a_{ij}) + I(a_{jk}) + I(a_{ki})}{I(s_{2\tau})} = \frac{I(a_{ik}) + I(a_{kj}) + I(a_{ji})}{I(s_{2\tau})},$$

$$\forall i, j, k \in N \text{i.e}, \frac{I(a_{ij})}{2\tau} + \frac{I(a_{jk})}{2\tau} + \frac{I(a_{ki})}{2\tau}$$

$$= \frac{I(a_{ik})}{2\tau} + \frac{I(a_{kj})}{2\tau} + \frac{I(a_{ji})}{2\tau}, \forall \text{i, j, k} \in \text{N} \tag{3}$$

Let $p_{ij} = \frac{I(a_{ij})}{2\tau}$, $\forall i, j \in N$, then Eq. (3) reduces to $p_{ij} + p_{jk} + p_{ki} = p_{ik} + p_{kj} + p_{ji}$ and $p_{ij} \in [0, 1]$, $\forall i, j, k \in N$, which is equivalent to the condition of additive transitivity of FPR proposed by Tanino [12].

**Definition 4.** For a LPR $A = (a_{ij})_{n\times n}$, if there exists a normalized crisp weight vector $w = (w_1, w_2, \cdots, w_n)^T$ with $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$, $i \in N$, such that

$$I(a_{ij}) = \tau(w_i - w_j + 1), \forall i, j \in N \quad , \tag{4}$$

then $A = (a_{ij})_{n\times n}$ is an additive consistent LPR.

$$\frac{I(a_{ij})}{I(s_{2\tau})} = \frac{\tau(w_i - w_j + 1)}{I(s_{2\tau})}, \forall i, j \in N \quad ,$$

that is

$$\frac{I(a_{ij})}{2\tau} = 0.5(w_i - w_j + 1), \forall i, j \in N \quad . \tag{5}$$

Let $p_{ij} = \frac{I(a_{ij})}{2\tau}$, $\forall i, j \in N$, then Eq. (5) is reduced to $p_{ij} = 0.5(w_i - w_j + 1)$ and $p_{ij} \in [0, 1]$, for all $i, j \in N$. In this case, Eq. (4) is able to be transformed as Xu's [13] transformation relation between additive consistent FPR and normalized crisp priority weights.

**Theorem 1.** Let $A = (a_{ij})_{n\times n}$ be a LPR with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$, $\forall i, j \in N$, then the following statements are equivalent:

(i) $A$ isadditive consistent;

$$I(a_{ij}) = I(a_{ik}) + I(a_{kj}) - \tau, \forall i, j, k \in N \quad ; \tag{ii}$$

$$I(a_{ij}) = \frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik}) + I(a_{kj})\right) - \tau, \forall i, j \in N \quad . \tag{iii}$$

**Proof.** (i)⇔(ii) According Eq. (1), we have $I(a_{ij}) + I(a_{ji}) = 2\tau$, $I(a_{ii}) = \tau$, $\forall i, j \in N$. If $A$ is additive consistent, by Definition 3, it follows that

$$I(a_{ij}) + I(a_{jk}) + I(a_{ki}) = I(a_{ik}) + I(a_{kj}) + I(a_{ji}), \quad \forall i, j, k \in N$$

$$\Leftrightarrow I(a_{ij}) + (2\tau - I(a_{kj})) + (2\tau - I(a_{ik}))$$
$$= I(a_{ik}) + I(a_{kj}) + (2\tau - I(a_{ij})), \quad \forall i, j, k \in N$$

$$\Leftrightarrow 2I(a_{ij}) = 2I(a_{ik}) + 2I(a_{kj}) - 2\tau, \quad \forall i, j, k \in N$$

$$\Leftrightarrow I(a_{ij}) = I(a_{ik}) + I(a_{kj}) - \tau, \forall i, j, k \in N.$$

(i) ⇒ (iii) According to Definition 3, we have

$$I(a_{ij}) + I(a_{jk}) + I(a_{ki}) = I(a_{ik}) + I(a_{kj}) + I(a_{ji}), \forall i, j, k \in N \quad ,$$

then

$$I(a_{ij}) + I(a_{j1}) + I(a_{1i}) = I(a_{i1}) + I(a_{1j}) + I(a_{ji}), \forall i, j \in N \quad , \tag{6}$$

$$I(a_{ij}) + I(a_{j2}) + I(a_{2i}) = I(a_{i2}) + I(a_{2j}) + I(a_{ji}), \forall i, j \in N \quad , \tag{7}$$

$$\vdots$$

$$I(a_{ij}) + I(a_{jn}) + I(a_{ni}) = I(a_{in}) + I(a_{nj}) + I(a_{ji}), \forall i, j \in N \quad . \tag{8}$$

Adding Eqs. (6)–(8), one can get

$$nI(a_{ij}) + \sum_{k=1}^{n}\left(I(a_{jk}) + I(a_{ki})\right)$$

$$= \sum_{k=1}^{n}\left(I(a_{ik}) + I(a_{kj})\right) + nI(a_{ji}), \forall i, j \in N \quad ,$$

then we have

$$nI(a_{ij})+\sum_{k=1}^{n}\left(2\tau-I(a_{kj})+2\tau-I(a_{ik})\right)$$

$$=\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)+n\left(2\tau-I(a_{ij})\right),\forall i,j\in N,$$

it follows that

$$2nI(a_{ij})=2\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)-2n\tau,\forall i,j\in N,$$

i.e.,

$$I(a_{ij})=\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)-\tau,\forall i,j\in N. \tag{9}$$

(iii) $\Rightarrow$ (i) Since $I(a_{ij})=\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)-\tau,\quad \forall i,j\in N,$ then we can obtain that

$$nI(a_{ij})+\sum_{t=1}^{n}\left(I(a_{jt})+I(a_{ti})\right)=\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tj})\right)+nI(a_{ji}),\forall i,j\in N. \tag{10}$$

Similarly, we have

$$nI(a_{ik})+\sum_{t=1}^{n}\left(I(a_{kt})+I(a_{ti})\right)$$

$$=\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tk})\right)+nI(a_{ki}),\forall i,k\in N, \tag{11}$$

$$nI(a_{kj})+\sum_{t=1}^{n}\left(I(a_{jt})+I(a_{tk})\right)$$

$$=\sum_{t=1}^{n}\left(I(a_{kt})+I(a_{tj})\right)+nI(a_{jk}),\forall k,j\in N. \tag{12}$$

Adding Eqs. (11) and (12), we have

$$nI(a_{ik})+nI(a_{kj})+\sum_{t=1}^{n}\left(I(a_{jt})+I(a_{ti})\right)$$

$$=\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tj})\right)+nI(a_{ki})+nI(a_{jk}),\forall i,j,k\in N. \tag{13}$$

By subtracting Eq. (10) from Eq. (13), one can obtain

$$nI(a_{ik})+nI(a_{kj})-nI(a_{ij})=nI(a_{ki})+nI(a_{jk})-nI(a_{ji}),\forall i,j,k\in N,$$

i.e.,

$$nI(a_{ik})+nI(a_{kj})+nI(a_{ji})=nI(a_{ij})+nI(a_{ki})+nI(a_{jk}),\forall i,j,k\in N. \tag{14}$$

Therefore, $I(a_{ik})+I(a_{kj})+I(a_{ji})=I(a_{ij})+I(a_{ki})+I(a_{jk})$, $\forall i,j,k\in N$, which indicates that $A$ is additive consistent. □

**Theorem 2**. Suppose that $A=(a_{ij})_{n\times n}$ is a LPR with $a_{ij}\in S=\{s_0,s_1,\cdots,s_{2\tau}\}$, $\quad\forall i,j\in N$, let

$$\tilde{a}_{ij}=I^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)-\tau\right),\forall i,j\in N, \tag{15}$$

then $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ is an additive consistent LPR.

**Proof**. According to Eq. (15), we have

$$I(\tilde{a}_{ij})=\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)-\tau,\forall i,j\in N, \tag{16}$$

$$I(\tilde{a}_{ik})=\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tk})\right)-\tau,\forall i,k\in N, \tag{17}$$

$$I(\tilde{a}_{kj})=\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{kt})+I(a_{tj})\right)-\tau,\forall k,j\in N, \tag{18}$$

then for all $i,j\in N$,

$$\begin{aligned}I(\tilde{a}_{ij})+I(\tilde{a}_{ji})&=\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{kj})\right)-\tau+\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{jk})+I(a_{ki})\right)-\tau\\&=\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik})+I(a_{ki})+I(a_{kj})+I(a_{jk})\right)-2\tau=\frac{1}{n}\sum_{k=1}^{n}(2\tau+2\tau)-2\tau=2\tau\end{aligned} \tag{19}$$

On the other hand, for $i,j,k\in N$,

$$\begin{aligned}I(\tilde{a}_{ik})+I(\tilde{a}_{kj})-\tau&=\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tk})\right)-\tau\\&+\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{kt})+I(a_{tj})\right)-\tau-\tau\\&=\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tk})+I(a_{kt})+I(a_{tj})\right)-3\tau\\&=\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{it})+2\tau+I(a_{tj})\right)-3\tau\\&=\frac{1}{n}\sum_{t=1}^{n}\left(I(a_{it})+I(a_{tj})\right)-\tau\\&=I(\tilde{a}_{ij}).\end{aligned}$$

By Definition 1 and Theorem 1, $\tilde{A}=(\tilde{a}_{ij})_{n\times n}$ is an additive consistent LPR, which completes the proof of Theorem 2. □

**Example 3**. LPRs are widely used for emergency management evaluation. Emergency risk management (ERM) is a process which involves dealing with risks to the community arising from emergency events. Emergency management evaluation as one of the important parts of ERM, aims evaluating and improving social preparedness and organizational ability of an emergency operating center (EOC) in analyzing, identifying and treating emergency risks. Choosing a good EOC will protect people from the community risks arising from emergency events [20,49,50]. Suppose that there are four EOCs $X=\{x_1,x_2,x_3,x_4\}$ to be evaluated, and the evaluator employs the linguistic terms to express his/her assessments over the four EOCs. When all the pairwise comparisons have been done, a LPR $A=(a_{ij})_{4\times 4}$ can be constructed as follows:

$$A=\begin{pmatrix}s_4 & s_6 & s_5 & s_5\\ s_2 & s_4 & s_8 & s_6\\ s_3 & s_0 & s_4 & s_8\\ s_3 & s_2 & s_0 & s_4\end{pmatrix}.$$

Using Theorem 2, we obtain the additive consistent LPR $\tilde{A} = (\tilde{a}_{ij})_{4\times 4}$:

$$\tilde{A} = \begin{pmatrix} s_4 & s_{4.00} & s_{5.25} & s_{6.75} \\ s_{4.00} & s_4 & s_{5.25} & s_{6.75} \\ s_{2.75} & s_{2.75} & s_4 & s_{5.50} \\ s_{1.25} & s_{1.25} & s_{2.50} & s_4 \end{pmatrix}.$$

**Theorem 3**. Suppose that $A = (a_{ij})_{n\times n}$ is a LPR with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$, $\forall i, j \in N$, and its additive consistent LPR is $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, then $A$ is additive consistent if and only if $A = \tilde{A}$.

Due to complexity and uncertainty involved in real decision making problems, it is difficult for the DMs to provide the additive consistent LPR, and then Theorem 3 cannot hold. In such cases, there exist $i, j \in N$, such that $a_{ij} \neq \tilde{a}_{ij}$, then the values of $|I(a_{ij}) - I(\tilde{a}_{ij})|$ can be used to measure the consistency level of the LPR $A = (a_{ij})_{n\times n}$ and should be kept as small as possible. Therefore, we introduce the following definition:

**Definition 5**. Suppose that $A = (a_{ij})_{n\times n}$ is a LPR with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$, $\forall i, j \in N$, and its additive consistent LPR is $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, then the consistency index of $A$ can be defined to measure the deviation between $A$ and $\tilde{A}$, which is denoted as

$$CI(A) = \frac{1}{\tau n(n-1)} \sum_{i<j} |I(a_{ij}) - I(\tilde{a}_{ij})|. \tag{20}$$

**Remark 3**. It is obvious that $CI(A) \in [0, 1]$. Note that $CI(A)$ should be kept as small as possible, and if $CI(A) = 0$, then $A$ is an additive consistent LPR. Given a threshold value $\delta_0$, if $CI(A) \leq \delta_0$, then $A$ is called a LPR with acceptable additive consistency.

## 4. Decision making methods based on additive consistent LPRs

Usually, with complexity and uncertainty involved in real decision making problems, it is difficult for the DMs to provide the acceptable additive consistent LPR, i.e., $CI(A) > \delta_0$. Due to an inconsistent LPR may lead to unreasonable conclusions, then $A$ needs to be optimized to approximate its additive consistent LPR, until it reaches the consistency threshold $\delta_0$. In what follows, we develop two automatic iterative algorithms to improve the additive consistency of LPR $A$.

### 4.1. Automatic iterative Algorithm I

With the analysis in Section 3, we present Algorithm I to improve the additive consistency of LPR $A = (a_{ij})_{n\times n}$ as follows:

**Algorithm I**

**Step 1:** Let $A^{(t)} = (a_{ij}^{(t)})_{n\times n} = A = (a_{ij})_{n\times n}$ and $t = 0$, and give the threshold $\delta_0$ and the adjusted parameter $\theta (0 \leq \theta \leq 1)$, which can reflect the DMs' pReferences

**Step 2:** Calculating the additive consistent LPR $\tilde{A}^{(t)} = (\tilde{a}_{ij}^{(t)})_{n\times n}$, where

$$\tilde{a}_{ij}^{(t)} = I^{-1}\left(\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik}^{(t)}) + I(a_{kj}^{(t)})\right) - \tau\right), \forall i, j \in N. \tag{21}$$

**Step 3:** Computing the consistency index $CI(A^{(t)})$ of $A^{(t)}$, where

$$CI(A^{(t)}) = \frac{1}{\tau n(n-1)} \sum_{i<j} |I(a_{ij}^{(t)}) - I(\tilde{a}_{ij}^{(t)})|. \tag{22}$$

**Step 4:** Acceptable additive consistency of LPR $A^{(t)}$ is checked. If $CI(A^{(t)}) \leq \delta_0$, then go to Step 6. Otherwise, continue with the next step.

**Step 5:** Let

$$a_{ij}^{(t+1)} = I^{-1}\left((1-\theta)I(a_{ij}^{(t)}) + \theta I(\tilde{a}_{ij}^{(t)})\right), \forall i, j \in N, \tag{23}$$

then we obtain the adjusted LPR $A^{(t+1)} = (a_{ij}^{(t+1)})_{n\times n}$. Let $t = t + 1$and return to Step 2.

**Step 6:** Let $A^* = A^{(t)}$. Output the adjusted LPR $A^*$, its additive consistency index $CI(A^*)$ and the number of the iteration $t$.

**Step 7:** Order consistency of $A^*$ is checked, i.e., if the adjusted LPR $A^*$ satisfies the order consistency, then the ranking of the alternatives can be obtained. Otherwise, go to the next step.

**Step 8:** Utilizing the linguistic arithmetic averaging operator [30]:

$$a_j^* = LAA(a_{1j}^*, a_{2j}^*, \cdots, a_{nj}^*) = \frac{1}{n}\overset{n}{\underset{i=1}{\oplus}} a_{ij}^*, j \in N, \tag{24}$$

to aggregate all the linguistic information $a_{ij}^* (j \in N)$ in the $j$th column of the $A^*$.

**Step 9:** Rank all the alternatives $x_j (j \in N)$ and select the best alternative(s) in accordance with the values of $a_j^* (j \in N)$. Since $a_j^*$ is the averaged value of all of the alternatives over the $j$th alternative, therefore, the smaller $a_j^*$, the better alternative $x_j$.

**Step 10:** End.

In the following, we will show that Algorithm I is convergent.

**Theorem 4**. Let $A = (a_{ij})_{n\times n}$ be a LPR with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$, $\theta (0 \leq \theta \leq 1)$ be the adjusted parameter, $\{\tilde{A}^{(t)}\}$ be an additive consistent LPR sequence in Algorithm I, then $\tilde{A}^{(t+1)} = \tilde{A}^{(t)}$ for each $t$.

**Proof**. According to Eqs. (21) and (23), for all $i, j \in N$, we have

$$I(a_{ij}^{(t+1)}) = (1-\theta)I(a_{ij}^{(t)}) + \theta I(\tilde{a}_{ij}^{(t)}), \forall i, j \in N\ ,$$

then

$$I(\tilde{a}_{ij}^{(t+1)}) = \frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik}^{(t+1)}) + I(a_{kj}^{(t+1)})\right) - \tau$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left((1-\theta)I(a_{ik}^{(t)}) + \theta I(\tilde{a}_{ik}^{(t)}) + (1-\theta)I(a_{kj}^{(t)}) + \theta I(\tilde{a}_{kj}^{(t)})\right) - \tau$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left((1-\theta)\left(I(a_{ik}^{(t)}) + I(a_{kj}^{(t)}) - \tau\right) + \theta\left(I(\tilde{a}_{ik}^{(t)}) + I(\tilde{a}_{kj}^{(t)}) - \tau\right)\right)$$

$$= (1-\theta)\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik}^{(t)}) + I(a_{kj}^{(t)})\right) - \tau\right)$$

$$+ \theta\cdot\frac{1}{n}\sum_{k=1}^{n}\left(I(\tilde{a}_{ik}^{(t)}) + I(\tilde{a}_{kj}^{(t)}) - \tau\right) \tag{25}$$

Since $\tilde{A}^{(t)}$ is an additive consistent LPR for each $t$, by using Theorems 1 and 2, one can obtain that

$$I(\tilde{a}_{ij}^{(t)}) = \frac{1}{n}\sum_{k=1}^{n}\left(I(a_{ik}^{(t)}) + I(a_{kj}^{(t)})\right) - \tau, \forall i, j \in N\ ,$$

$$I(\tilde{a}_{ij}^{(t)}) = I(\tilde{a}_{ik}^{(t)}) + I(\tilde{a}_{kj}^{(t)}) - \tau, \forall i, j, k \in N\ .$$

Therefore, for all $i, j \in N$,

$$I(\tilde{a}_{ij}^{(t+1)}) = (1-\theta)\cdot I(\tilde{a}_{ij}^{(t)}) + \theta\cdot\frac{1}{n}\sum_{k=1}^{n} I(\tilde{a}_{ij}^{(t)})$$

$$= (1-\theta)\cdot I(\tilde{a}_{ij}^{(t)}) + \theta\cdot I(\tilde{a}_{ij}^{(t)}) = I(\tilde{a}_{ij}^{(t)}), \quad (26)$$

then,

$$\tilde{a}_{ij}^{(t+1)} = \tilde{a}_{ij}^{(t)}, \forall i, j \in N, \quad (27)$$

i.e. , $\tilde{A}^{(t+1)} = \tilde{A}^{(t)}$ for each $t$, which completes the proof of Theorem 4. □

**Theorem 5**. Assume that $A = (a_{ij})_{n\times n}$ is a LPR with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$, $\theta(0 \le \theta \le 1)$ is the adjusted parameter, $\{A^{(t)}\}$ is a LPR sequence in Algorithm I. Let $CI(A^{(t)})$ be the consistency index of $A^{(t)}$, then $CI(A^{(t+1)}) \le CI(A^{(t)})$ for each $t$.

**Proof**. From Eq. (22) and Theorem 4, for each $t$, we have

$$\begin{aligned} \mathrm{CI}(A^{(t+1)}) &= \frac{2}{n(n-1)}\sum_{i<j}|I(a_{ij}^{(t+1)}) - I(\tilde{a}_{ij}^{(t+1)})| \\ &= \frac{2}{n(n-1)}\sum_{i<j}|I(a_{ij}^{(t+1)}) - I(\tilde{a}_{ij}^{(t)})| \\ &= \frac{2}{n(n-1)}\sum_{i<j}|(1-\theta)I(a_{ij}^{(t)}) + \theta I(\tilde{a}_{ij}^{(t)}) - I(\tilde{a}_{ij}^{(t)})| \\ &= \frac{2}{n(n-1)}\sum_{i<j}|(1-\theta)\cdot\left(I(a_{ij}^{(t)}) - I(\tilde{a}_{ij}^{(t)})\right)| \\ &= (1-\theta)\cdot\frac{2}{n(n-1)}\sum_{i<j}|I(a_{ij}^{(t)}) - I(\tilde{a}_{ij}^{(t)})| \\ &= (1-\theta)\mathrm{CI}(A^{(t)}) \le \mathrm{CI}(A^{(t)}) \end{aligned} \quad (28)$$

This completes the proof of Theorem 5. □

### 4.2. Automatic iterative Algorithm II

From the Definition 4, a LPR $A = (a_{ij})_{n\times n}$ with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$ is additive consistent, if there exists a normalized crisp weight vector $w = (w_1, w_2, \cdots, w_n)^T$ with $\sum_{i=1}^{n} w_i = 1, w_i \ge 0, i \in N$, such that $I(a_{ij})$ can be expressed as Eq. (4).

In many real decision making problems, there usually exists incomplete and uncertain decision information, the LPR $A = (a_{ij})_{n\times n}$ constructed by DM is always with unacceptable additive consistency, and then Eq. (4) cannot hold. In such cases, one can get that

$$I(a_{ij}) \ne \tau(w_i - w_j + 1). \quad (29)$$

Then, the non-negative deviation variables $d_{ij}^-$ and $d_{ij}^+$,$d_{ij}^- \cdot d_{ij}^+ = 0, i, j \in N$ are introduced as follows:

$$I(a_{ij}) + d_{ij}^- - d_{ij}^+ = \tau(w_i - w_j + 1), i, j \in N,$$

i.e.,

$$d_{ij}^- - d_{ij}^+ = \tau(w_i - w_j + 1) - I(a_{ij}), i, j \in N. \quad (30)$$

As the smaller the deviation variables $d_{ij}^-$ and $d_{ij}^+$, the better the additive consistency of LPR. Therefore, in order to find the smallest deviation variables, the following linear optimization model is established to derive a normalized crisp weight vector:

$$(\mathrm{M}-\mathbf{1})\ \min J_1 = \sum_{i=1}^{n}\sum_{j=i}^{n}(d_{ij}^- + d_{ij}^+)$$

$$\text{s.t.}\begin{cases} d_{ij}^- - d_{ij}^+ = \tau(w_i - w_j + 1) - I(a_{ij}), i, j \in N, \\ d_{ij}^- \ge 0, d_{ij}^+ \ge 0, d_{ij}^- \cdot d_{ij}^+ = 0, i, j \in N, \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0, i \in N. \end{cases} \quad (31)$$

According to Definition 1 and Eq. (30), we have

$$d_{ij}^- - d_{ij}^+ = \tau(w_i - w_j + 1) - \mathrm{I}(a_{ij}) = \tau(w_i - w_j + 1) - (2\tau - \mathrm{I}(a_{ji}))$$
$$= -(\tau(w_j - w_i + 1) - \mathrm{I}(a_{ji})) = d_{ji}^- - d_{ji}^+,$$

then, $d_{ij}^- + d_{ji}^- = d_{ij}^+ + d_{ji}^+$.

Since $d_{ij}^-, d_{ij}^+ \ge 0$ and $d_{ij}^- \cdot d_{ij}^+ = 0, i, j \in N$, then $d_{ij}^- = d_{ji}^+, d_{ij}^+ = d_{ji}^-, i, j \in N$. Thus, the model (M-1) can be simplified by considering only the upper diagonal elements as follows:

$$(\mathrm{M}-\mathbf{2})\min J_2 = \sum_{i<j}^{n}(d_{ij}^- + d_{ij}^+)$$

$$\text{s.t.}\begin{cases} d_{ij}^- - d_{ij}^+ = \tau(w_i - w_j + 1) - I(a_{ij}), i < j, \\ d_{ij}^- \ge 0, d_{ij}^+ \ge 0, d_{ij}^- \cdot d_{ij}^+ = 0, i < j, \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0, i \in N. \end{cases} \quad (32)$$

Solving the model (M-2), we determine the optimal normalized crisp weight vector $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_n)^T$ for LPR $A = (a_{ij})_{n\times n}$, and the optimal deviation values $\tilde{d}_{ij}^-$ and $\tilde{d}_{ij}^+$ are obtained. If $\tilde{J}_2 = 0$, i.e., $\tilde{d}_{ij}^- = \tilde{d}_{ij}^+ = 0$, for all $i, j = 1, 2, \cdots, n$, then LPR $A = (a_{ij})_{n\times n}$ is additive consistent. If $\tilde{J}_2 > 0$, by using the optimal nonzero deviation values $\tilde{d}_{ij}^-$ and $\tilde{d}_{ij}^+$, we construct an additive consistent LPR as follows:

Suppose that

$$O = \{(i,j)|(i,j) \in n\times n, \tilde{d}_{ij}^- > 0, \tilde{d}_{ij}^+ = 0\},$$

$$P = \{(i,j)|(i,j) \in n\times n, \tilde{d}_{ij}^- = 0, \tilde{d}_{ij}^+ = 0\},$$

$$Q = \{(i,j)|(i,j) \in n\times n, \tilde{d}_{ij}^- = 0, \tilde{d}_{ij}^+ > 0\}.$$

Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where

$$\tilde{a}_{ij} = \begin{cases} I^{-1}\left(I(a_{ij}) + \tilde{d}_{ij}^-\right), i, j \in O \\ a_{ij}, i, j \in P \\ I^{-1}\left(I(a_{ij}) - \tilde{d}_{ij}^+\right), i, j \in Q \end{cases}, \quad (33)$$

then, according to Definition 4, one can obtain that $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ is an additive consistent LPR.

With the above analysis, the Automatic Iterative Algorithm II can be developed to improve the additive consistency of LPR $A = (a_{ij})_{n\times n}$ as follows:

**Algorithm II**

**Step 1′:** See Step 1 in Algorithm I.

**Step 2′:** Based on the model (M-2), we determine the optimal normalized crisp weight vector $\tilde{w}^{(t)} = (\tilde{w}_1^{(t)}, \tilde{w}_2^{(t)}, \cdots, \tilde{w}_n^{(t)})^T$, and the additive consistent LPR $\tilde{A}^{(t)} = (\tilde{a}_{ij}^{(t)})_{n\times n}$ can be obtained, where

$$\tilde{a}_{ij}^{(t)} = I^{-1}\left(I(a_{ij}^{(t)}) + \tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+}\right), \forall i, j \in N. \quad (34)$$

**Step 3′–5′:** See Step 3–5 in Algorithm I.

**Step 6′:** Let $\tilde{A} = A^{(t)}$, $w^* = \tilde{w}^{(t)}$. Output the adjusted LPR $\tilde{A}$ and optimal weight vector $w^*$, its additive consistency index $CI(\tilde{A})$ and the number of the iteration $t$.

**Step 7′:** Ranking all the priority weights $w_i^*(i \in N)$ in descending order.

**Step 8′:** Rank the alternatives $x_i(i \in N)$ according to the ranking of $w_i^*(i \in N)$, and then select the optimal one(s).

**Step 9′**: End.

The convergence of Algorithm II is shown by the following theorem.

**Theorem 6**. Assume that $A=(a_{ij})_{n\times n}$ is a LPR with $a_{ij} \in S = \{s_0, s_1, \cdots, s_{2\tau}\}$, $\theta(0 \le \theta \le 1)$ is the adjusted parameter, $\{A^{(t)}\}$ is a LPR sequence in Algorithm II. Let $CI(A^{(t)})$ be the consistency index of $A^{(t)}$, then $CI(A^{(t+1)}) \le CI(A^{(t)})$ for each $t$.

**Proof.** According to Algorithm II, let $\tilde{d}_{ij}^{(t)-}, \tilde{d}_{ij}^{(t)+}$ and $\tilde{w}^{(t)} = (\tilde{w}_1^{(t)}, \tilde{w}_2^{(t)}, \cdots, \tilde{w}_n^{(t)})^T$ be the optimal deviation values and the optimal normalized crisp weight vector, respectively, such that $\tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+} = \tau(\tilde{w}_i^{(t)} - \tilde{w}_j^{(t)} + 1) - I(a_{ij}^{(t)}), \forall i,j \in N$ for each $t$.

Since $\tilde{d}_{ij}^{(t)-} \ge 0, \tilde{d}_{ij}^{(t)+} \ge 0$ and $\tilde{d}_{ij}^{(t)-} \cdot \tilde{d}_{ij}^{(t)+} = 0, \forall i,j \in N$ for each $t$, then

$$\tilde{d}_{ij}^{(t)-} + \tilde{d}_{ij}^{(t)+} = |\tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+}|, \forall i,j \in N \quad .$$

From Eq. (34), we have

$$I(\tilde{a}_{ij}^{(t)}) - I(a_{ij}^{(t)}) = \tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+}, \forall i,j \in N \ for \ \ each \ \ t \ . \quad (35)$$

Moreover, it follows from model (M-2) that

$$\sum_{i<j}^{n} \left(\tilde{d}_{ij}^{(t+1)-} + \tilde{d}_{ij}^{(t+1)+}\right) = \sum_{i<j}^{n} |\tilde{d}_{ij}^{(t+1)-} - \tilde{d}_{ij}^{(t+1)+}| = \sum_{i<j}^{n} |\tau(\tilde{w}_i^{(t+1)} - \tilde{w}_j^{(t+1)} + 1) - I(a_{ij}^{(t+1)})| \ , \quad (36)$$

while $\tilde{d}_{ij}^{(t+1)-}$ and $\tilde{d}_{ij}^{(t+1)+}, i,j \in N$ are the optimal objective values in model (M-2), then we have

$$\sum_{i<j}^{n} \left(\tilde{d}_{ij}^{(t+1)-} + \tilde{d}_{ij}^{(t+1)+}\right) \le \sum_{i<j}^{n} \left(d_{ij}^{(t+1)-} + d_{ij}^{(t+1)+}\right) = \sum_{i<j}^{n} |d_{ij}^{(t+1)-} - d_{ij}^{(t+1)+}| = \sum_{i<j}^{n} |\tau(w_i^{(t+1)} - w_j^{(t+1)} + 1) - I(a_{ij}^{(t+1)})|,$$

Let $w_i^{(t+1)} = \tilde{w}_i^{(t)}, i \in N$, then

$$\sum_{i<j}^{n} \left(\tilde{d}_{ij}^{(t+1)-} + \tilde{d}_{ij}^{(t+1)+}\right) \le \sum_{i<j}^{n} |\tau(\tilde{w}_i^{(t)} - \tilde{w}_j^{(t)} + 1) - I(a_{ij}^{(t+1)})| \quad . (37)$$

By Eqs. (35)–(37), we can get

$$\begin{aligned}
\mathrm{CI}(A^{(t+1)}) &= \frac{2}{n(n-1)}\sum_{i<j} |I(a_{ij}^{(t+1)}) - I(\tilde{a}_{ij}^{(t+1)})| = \frac{2}{n(n-1)}\sum_{i<j}^{n} |\tilde{d}_{ij}^{(t+1)+} - \tilde{d}_{ij}^{(t+1)-}| \\
&= \frac{2}{n(n-1)}\sum_{i<j}^{n} |\tilde{d}_{ij}^{(t+1)-} - \tilde{d}_{ij}^{(t+1)+}| = \frac{2}{n(n-1)}\sum_{i<j}^{n} \left(\tilde{d}_{ij}^{(t+1)-} + \tilde{d}_{ij}^{(t+1)+}\right) \\
&\le \frac{2}{n(n-1)}\sum_{i<j}^{n} |\tau(\tilde{w}_i^{(t)} - \tilde{w}_j^{(t)} + 1) - I(a_{ij}^{(t+1)})| \\
&\quad \frac{2}{n(n-1)}\sum_{i<j}^{n} |\tau(\tilde{w}_i^{(t)} - \tilde{w}_j^{(t)} + 1) - \left((1-\theta)I(a_{ij}^{(t)}) + \theta I(\tilde{a}_{ij}^{(t)})\right)| \\
&= \frac{2}{n(n-1)}\sum_{i<j}^{n} |\left(\tau(\tilde{w}_i^{(t)} - \tilde{w}_j^{(t)} + 1) - I(a_{ij}^{(t)})\right) - \theta\left(I(\tilde{a}_{ij}^{(t)}) - I(a_{ij}^{(t)})\right)| \\
&= \frac{2}{n(n-1)}\sum_{i<j}^{n} |\left(\tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+}\right) - \theta\left(\tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+}\right)| \\
&= (1-\theta)\cdot\frac{2}{n(n-1)}\sum_{i<j}^{n} |\tilde{d}_{ij}^{(t)-} - \tilde{d}_{ij}^{(t)+}| \\
&= (1-\theta)\cdot\frac{2}{n(n-1)}\sum_{i<j}^{n} |I(\tilde{a}_{ij}^{(t)}) - I(a_{ij}^{(t)})| \\
&= (1-\theta)\cdot\frac{2}{n(n-1)}\sum_{i<j}^{n} |I(a_{ij}^{(t)}) - I(\tilde{a}_{ij}^{(t)})| \\
&= (1-\theta)\mathrm{CI}(A^{(t)}) \le \mathrm{CI}(A^{(t)})
\end{aligned} \quad (38)$$

This completes the proof of Theorem 6. □

## 5. Numerical example

**Example 4**. (Continued with Example 3). According to the DM's preference information and the decision making situation, the threshold value of consistency index $\delta_0$ can be obtained. The feasible bounds for checking the additive consistency of $A$ can be set at $\delta_0 = 0.1347$[47]. In the following, in order to select the optimal EOC(s), we utilize Algorithm I and Algorithm II to deal with this decision making problem, respectively. The main steps are as follows:

**Algorithm I**

**Step 1–2:** According to the results in Example 3, we have the additive consistent LPR $\tilde{A}$. Let $A^{(0)} = A, \tilde{A}^{(0)} = \tilde{A}$, the adjusted parameter $\theta = 0.5$.

**Step 3:**

By using Eq. (22), we have $CI(A^{(0)}) = 0.4167$.

**Step 4:**

Since $CI(A^{(0)}) > \delta_0$, then continue with the next step.

**Step 5:** By Eq. (23), we can obtain the adjusted LPR $A^{(1)} = (a_{ij}^{(1)})_{4\times 4}$ as follows:

$$A^{(1)} = \begin{pmatrix} s_4 & s_{5.000} & s_{5.125} & s_{5.875} \\ s_{3.000} & s_4 & s_{6.625} & s_{6.375} \\ s_{2.875} & s_{1.375} & s_4 & s_{6.750} \\ s_{2.125} & s_{1.625} & s_{1.250} & s_4 \end{pmatrix} .$$

According to Theorem 4, we have $\tilde{A}^{(1)} = \tilde{A}^{(0)}$. By using Eq. (22), we have $CI(A^{(1)}) = 0.2083$. Since $CI(A^{(1)}) > \delta_0$, then by Eq. (23), the adjusted LPR $A^{(2)} = (a_{ij}^{(2)})_{4\times4}$ can be obtained as follows:

$$A^{(2)} = \begin{pmatrix} s_4 & s_{4.500} & s_{5.1875} & s_{6.3125} \\ s_{3.500} & s_4 & s_{5.9375} & s_{6.5625} \\ s_{2.8125} & s_{2.0625} & s_4 & s_{6.1250} \\ s_{2.6875} & s_{1.4375} & s_{1.8750} & s_4 \end{pmatrix}.$$

According to Theorem 4, we have $\tilde{A}^{(2)} = \tilde{A}^{(1)} = \tilde{A}^{(0)}$. By using Eq. (22), we have $CI(A^{(2)}) = 0.1042 < \delta_0$, then go to Step 6.

**Step 6:** Let $A^* = A^{(2)}$. Output the acceptable additive consistent LPR $A^*$, its additive consistency index $CI(A^*) = 0.1042$ and the number of the iteration $t = 2$.

**Step 7:** As $a_{11}^* > a_{21}^*$ and $a_{13}^* < a_{23}^*$, i.e., LPR $A^*$ is non-order consistent, then go to Step 8;

**Step 8:** By Eq. (24), we calculate the overall linguistic values $a_j^*(j = 1, 2, 3, 4)$ of all of the EOCs over the $j$th EOC:

$a_1^* = s_{3.25}, a_2^* = s_{3.00}, a_3^* = s_{4.25}, a_4^* = s_{5.75}$.

**Step 9:** Since $a_2^* < a_1^* < a_3^* < a_4^*$, then we have $x_2 \succ x_1 \succ x_3 \succ x_4$, and the most desirable EOC is $x_2$.

In what follows, we utilize Algorithm II to improve the additive consistency of LPR $A = (a_{ij})_{n\times n}$ and select the optimal EOC.

**Algorithm II**

**Step 1′:** Let $A^{(0)} = (a_{ij}^{(0)})_{4\times4} = A = (a_{ij})_{4\times4}$ and the adjusted parameter $\theta = 0.5$.

**Step 2′:** Utilizing the model (M-2), we determine the optimal normalized crisp weight vector $\tilde{w}^{(0)} = (0.2976, 0.4291, 0.1367, 0.1366)^T$, the optimal nonzero deviation values $\tilde{d}_{12}^{(0)+} = 2.6667, \tilde{d}_{23}^{(0)+} = 2.0000, \tilde{d}_{34}^{(0)+} = 4.0000$, then the additive consistent LPR $\tilde{A}^{(0)} = (\tilde{a}_{ij}^{(0)})_{4\times4}$ can be obtained by using Eq. (35) as follows:

$$\tilde{A}^{(0)} = \begin{pmatrix} s_4 & s_{3.3333} & s_{5.0000} & s_{5.0000} \\ s_{4.6667} & s_4 & s_{6.0000} & s_{6.0000} \\ s_{3.0000} & s_{2.0000} & s_4 & s_{4.0000} \\ s_{3.0000} & s_{2.0000} & s_{4.0000} & s_4 \end{pmatrix}$$

**Step 3′:**

According to Eq. (22), we have $CI(A^{(0)}) = 0.3611$.

**Step 4′:**

Since $CI(A^{(0)}) > \delta_0$, then continue with the next step.

**Step 5′:** By Eq. (24), we can obtain the adjusted LPR $A^{(1)} = (a_{ij}^{(1)})_{4\times4}$ as follows:

$$A^{(1)} = \begin{pmatrix} s_4 & s_{4.6667} & s_{5.0000} & s_{5.0000} \\ s_{3.3333} & s_4 & s_{7.0000} & s_{6.0000} \\ s_{3.0000} & s_{1.0000} & s_4 & s_{6.0000} \\ s_{3.0000} & s_{2.0000} & s_{2.0000} & s_4 \end{pmatrix}.$$

Utilizing the model (M-2), we obtain the optimal nonzero deviation values $\tilde{d}_{12}^{(1)+} = 1.3334, \tilde{d}_{23}^{(1)+} = 1.0000, \tilde{d}_{34}^{(1)+} = 2.0000$, and the additive consistent LPR $\tilde{A}^{(1)}$ as follows:

$$\tilde{A}^{(1)} = \begin{pmatrix} s_4 & s_{3.3333} & s_{5.0000} & s_{5.0000} \\ s_{4.6667} & s_4 & s_{6.0000} & s_{6.0000} \\ s_{3.0000} & s_{2.0000} & s_4 & s_{4.0000} \\ s_{3.0000} & s_{2.0000} & s_{4.0000} & s_4 \end{pmatrix},$$

then we have $CI(A^{(1)}) = 0.1806 > \delta_0$. By using Eq. (23), we obtain the adjusted LPR $A^{(2)} = (a_{ij}^{(2)})_{4\times4}$ as follows:

$$A^{(2)} = \begin{pmatrix} s_4 & s_{4.0000} & s_{5.0000} & s_{5.0000} \\ s_{4.0000} & s_4 & s_{6.5000} & s_{6.0000} \\ s_{3.0000} & s_{1.5000} & s_4 & s_{5.0000} \\ s_{3.0000} & s_{2.0000} & s_{3.0000} & s_4 \end{pmatrix}.$$

Utilizing the model (M-2), we obtain optimal normalized crisp weight vector $\tilde{w}^{(2)} = (0.3016, 0.4107, 0.1942, 0.0935)^T$ and the optimal nonzero deviation values $\tilde{d}_{12}^{(2)+} = 0.6667, \tilde{d}_{23}^{(2)+} = 0.5000, \tilde{d}_{34}^{(2)+} = 1.0000$, and the additive consistent LPR $\tilde{A}^{(2)}$ as follows:

$$\tilde{A}^{(2)} = \begin{pmatrix} s_4 & s_{3.3333} & s_{5.0000} & s_{5.0000} \\ s_{4.6667} & s_4 & s_{6.0000} & s_{6.0000} \\ s_{3.0000} & s_{2.0000} & s_4 & s_{4.0000} \\ s_{3.0000} & s_{2.0000} & s_{4.0000} & s_4 \end{pmatrix},$$

then we have $CI(A^{(2)}) = 0.0903 < \delta_0$, then go to Step 6′.

**Step 6′:** Let $\tilde{A} = A^{(2)}, w^* = \tilde{w}^{(2)}$. Output the acceptable additive consistent LPR $\tilde{A}$ and optimal weight vector $w^* = (0.3016, 0.4107, 0.1942, 0.0935)^T$, its additive consistency index $CI(\tilde{A}) = 0.0903$ and the number of the iteration $t = 2$.

**Step 7′:** Since $w_2^* > w_1^* > w_3^* > w_4^*$, then the ranking of the EOCs is $x_2 \succ x_1 \succ x_3 \succ x_4$, and the most desirable EOC is $x_2$.

By extended linguistic arithmetical averaging (ELAA) operator, Xu [32] developed a decision technique for addressing decision making problems with linguistic information. If we utilize the method in [32] to deal with the aforementioned problem, then main steps are as follows:

**Step 1.** Based on LPR $A = (a_{ij})_{4\times4}$ in Example 3, we utilize the ELAA operator

$$a_i = ELAA(a_{i1}, a_{i2}, a_{i3}, a_{i4}) = I^{-1}\left(\frac{1}{4}\sum_{j=1}^{4} I(a_{ij})\right), i = 1, 2, 3, 4 \quad (39)$$

to aggregate all the linguistic information $a_{ij}(j = 1, 2, 3, 4)$ into a collective linguistic information $a_i(i = 1, 2, 3, 4)$, then we have $a_1 = s_{5.00}, a_2 = s_{5.00}, a_3 = s_{3.00}, a_4 = s_{2.25}$.

**Step 2.** Ranking all collective linguistic information $a_i(i = 1, 2, 3, 4)$ in descending order: $a_1 = a_2 > a_3 > a_4$.

**Step 3.** Rank all the EOCs $x_i(i = 1, 2, 3, 4)$ in accordance with the ranking of $a_i(i = 1, 2, 3, 4)$ as:

$x_1 \sim x_2 \succ x_3 \succ x_4$,

and the most desirable EOC is $x_1$ or $x_2$.

In what follows, we utilize the iterative algorithm proposed by Dong et al. [47] to cope with the aforementioned problem, and then the following steps are involved:

**Step 1.**

Let $A^{(0)} = (a_{ij}^{(0)})_{4\times4} = A = (a_{ij})_{4\times4}$ and $t = 0$.

**Step 2.** Let $P^{(0)} = (p_{ij}^{(0)})_{4\times4}$, where $p_{ij}^{(t)} = \frac{1}{4}\sum_{k=1}^{4}(a_{ik}^{(t)} + a_{kj}^{(t)}) - 4$, then we have

$$P^{(0)} = \begin{pmatrix} s_4 & s_4 & s_{5.25} & s_{6.75} \\ s_4 & s_4 & s_{5.25} & s_{6.75} \\ s_{2.75} & s_{2.75} & s_4 & s_{5.50} \\ s_{1.25} & s_{1.25} & s_{2.50} & s_4 \end{pmatrix},$$

$$V^{(0)} = \begin{pmatrix} s_4 & [s_4, s_6] & [s_5, s_{5.25}] & [s_5, s_{6.75}] \\ [s_2, s_4] & s_4 & [s_{5.25}, s_8] & [s_6, s_{6.75}] \\ [s_{2.75}, s_3] & [s_0, s_{2.75}] & s_4 & [s_{5.50}, s_8] \\ [s_{1.25}, s_3] & [s_{1.25}, s_2] & [s_0, s_{2.50}] & s_4 \end{pmatrix}.$$

**Step 3.** Calculate the consistency index $CI(A^{(0)})$ of $A^{(0)}$, and we obtain that $CI(A^{(0)}) = d(A^{(0)}, P^{(0)}) = 0.2103 > \delta_0$, then go to next step.

**Step 4.** For convenience, let $V^{(t)} = (v_{ij}^{(t)})_{4\times4}$, where $v_{ij}^{(t)} = [\min\{a_{ij}^{(t)}, p_{ij}^{(t)}\}, \max\{a_{ij}^{(t)}, p_{ij}^{(t)}\}]$. Return $A^{(0)}$ to the DM, and the DM constructs a new LPR $A^{(1)} = (a_{ij}^{(1)})_{4\times4}$ as follows (where $a_{ij}^{(1)} \in v_{ij}^{(0)}$ and $a_{ij}^{(1)} \oplus a_{ji}^{(1)} = s_4$):

$$A^{(1)} = \begin{pmatrix} s_4 & s_6 & s_5 & s_6 \\ s_2 & s_4 & s_7 & s_6 \\ s_3 & s_1 & s_4 & s_8 \\ s_2 & s_2 & s_0 & s_4 \end{pmatrix}.$$

Calculate the $P^{(1)} = (p_{ij}^{(1)})_{4\times4}$ as follows:

$$P^{(1)} = \begin{pmatrix} s_4 & s_{4.5} & s_{5.25} & s_{7.25} \\ s_{3.5} & s_4 & s_{4.75} & s_{6.75} \\ s_{2.75} & s_{3.25} & s_4 & s_6 \\ s_{0.75} & s_{1.25} & s_2 & s_4 \end{pmatrix},$$

$$V^{(1)} = \begin{pmatrix} s_4 & [s_{4.5}, s_6] & [s_5, s_{5.25}] & [s_6, s_{7.25}] \\ [s_2, s_{3.5}] & s_4 & [s_{4.75}, s_7] & [s_6, s_{6.75}] \\ [s_{2.75}, s_3] & [s_1, s_{3.25}] & s_4 & [s_6, s_8] \\ [s_{0.75}, s_2] & [s_{1.25}, s_2] & [s_0, s_2] & s_4 \end{pmatrix}.$$

Since $CI(A^{(1)}) = d(A^{(1)}, P^{(1)}) = 0.1667 > \delta_0$, then we compute $V^{(1)} = (v_{ij}^{(1)})_{4\times4}$ and return $A^{(1)}$ to the DM. The DM constructs a new LPR $A^{(2)} = (a_{ij}^{(2)})_{4\times4}$ as follows (where $a_{ij}^{(2)} \in v_{ij}^{(1)}$ and $a_{ij}^{(2)} \oplus a_{ji}^{(2)} = s_4$):

$$A^{(2)} = \begin{pmatrix} s_4 & s_5 & s_5 & s_6 \\ s_3 & s_4 & s_7 & s_{6.5} \\ s_3 & s_1 & s_4 & s_7 \\ s_2 & s_{1.5} & s_1 & s_4 \end{pmatrix},$$

$$P^{(2)} = \begin{pmatrix} s_4 & s_{3.875} & s_{5.250} & s_{6.875} \\ s_{4.125} & s_4 & s_{5.375} & s_{7.000} \\ s_{2.750} & s_{2.625} & s_4 & s_{5.625} \\ s_{1.125} & s_{1.000} & s_{2.375} & s_4 \end{pmatrix}.$$

Calculating the $P^{(2)} = (p_{ij}^{(2)})_{4\times4}$, then we have $CI(A^{(2)}) = d(A^{(2)}, P^{(2)}) = 0.1189 < \delta_0$.

**Step 5.** Let $\bar{A} = A^{(2)}$. Output the acceptable consistent LPR $\bar{A}$ and its consistency index $CI(\bar{A}) = 0.1189$.

**Table 1**
The decision making results with respect to different methods.

| Method | The ranking of the EOCs | Most desirable EOC |
|---|---|---|
| Algorithm I in this paper | $x_2 \succ x_1 \succ x_3 \succ x_4$ | $x_2$ |
| Algorithm II in this paper | $x_2 \succ x_1 \succ x_3 \succ x_4$ | $x_2$ |
| Xu [32]' method | $x_1 \sim x_2 \succ x_3 \succ x_4$ | $x_1, x_2$ |
| Dong et al. [47]' method | $x_2 \succ x_1 \succ x_3 \succ x_4$ | $x_2$ |

**Step 6.** By Eq. (24), we calculate the overall linguistic preference values $\bar{a}_j (j = 1, 2, 3, 4)$ of all of the EOCs over the $j$th EOC:

$$\bar{a}_1 = s_{3.000}, \bar{a}_2 = s_{2.875}, \bar{a}_3 = s_{4.250}, \bar{a}_4 = s_{5.875}.$$

**Step 7.** Since $\bar{a}_2 < \bar{a}_1 < \bar{a}_3 < \bar{a}_4$, then the ranking of the four EOCs is $x_2 \succ x_1 \succ x_3 \succ x_4$, and the most desirable EOC is $x_2$.

Based on the methods in this paper, Xu [32], and Dong et al. [47], the ranking of the four EOCs and the most desirable EOC are summarized in Table 1.

By observational analysis, we find that different consistency models lead to different decision results. Compared with the methods developed by Xu [32] and Dong et al. [47], our proposed methods have many advantages. More details of advantages of our models are analyzed in the following.

(1) Comparison with Xu [32]: Our methods are more reasonable and reliable. In Algorithm I, we first use an iterative algorithm to modify the consistency of original LPR into the adjusted LPR, which satisfies the acceptable additive consistency. Then, the Algorithm I in this paper uses the linguistic arithmetic averaging operator to aggregate all of the linguistic preference information in each column of the adjusted LPR and get the ranking of all the EOCs. In Algorithm II, we also use a new iterative algorithm to obtain the adjusted LPR with acceptable additive consistency. Then, based the adjusted LPR, we calculate optimal weight vector and get the ranking of all the EOCs. However, with the Xu [32]'s method, we directly utilize the ELAA operator to aggregate all of the linguistic preference information and does not check the acceptable additive consistency of LPR. It is known that the lack of acceptable consistency cannot ensure that the DMs are being neither random nor illogical, and it easily leads to inconsistent conclusions. Therefore, our method is much more reasonable and reliable than Xu [32]'s method.
(2) Comparison with Dong et al. [47]: Our approaches would be more scientific and efficient. In the process of decision making, our algorithms use the original LPR and all the calculations directly using the linguistic preference information to produce results, then it can reflect the original decision making information of the DMs. This further makes the final results of our proposed methods more trustworthy. However, the consistency improvement process used in [47] need to transfer the original LPR into its corresponding LPR $P^{(t)}$ and interval-valued LPR $V^{(t)}$, which makes the computation process seem to be a little bit indirect. Meanwhile, our approaches are also convenient to implement in actual applications.
(3) Based on order consistency, this paper proposes a new approach to derive the ranking among the alternatives from a LPR in some cases. Example 2 shows that we can directly get the ranking among the alternatives $x_1, x_2, x_3$ and $x_4$ is that $x_1 \succ x_2 \succ x_3 \succ x_4$. Therefore, our methods are more simplified than the other methods in some cases.

## 6. Conclusions

In this paper, we first introduce some new concepts, including LPR, order consistent LPR and additive consistent LPR, and the characterization about the additive consistency of LPRs is proposed. A consistency index of LPR is defined to measure whether a LPR is of acceptable additive consistency. Moreover, two automatic iterative algorithms are developed to improve LPR with unacceptable additive consistency until the adjusted LPR is acceptably additive consistent. The corresponding automatic iterative algorithms can help the DMs provide the acceptable consistent preferences so as to guarantee the reasonable and identified decision results. In the end, a numerical example is supplied to illustrate the effectiveness and practicality of the developed methods. Comparative analysis are also provided to discuss the performances of our approaches. On the whole, the methodology and algorithm presented in this paper are very important for the application of LPRs in decision making.

In terms of future work, we will focus on investigating the multiplicative consistency and consensus reaching models of LPRs on the basis of the results in this paper. Besides, we also intend to apply our methods to the fields of decision making, such as pattern recognition and medical diagnosis, etc.

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