ARCHIVES OF CIVIL AND MECHANICAL ENGINEERING 16 (2016) 668–677

Available online at www.sciencedirect.com

ScienceDirect

journal homepage: http://www.elsevier.com/locate/acme

Original Research Article

Analytical solution for beams with multipoint boundary conditions on two-parameter elastic foundations

M. Aslami [a,b,*], P.A. Akimov [a]

[a] Department of Applied Mathematics and Computer Sciences, Moscow State University of Civil Engineering, 26, Yaroslavskoe Sh., Moscow, 129337, Russia

[b] Fasa University, Daneshjou blvd, Fasa, Fars Province, Iran

ARTICLE INFO

Article history:
Received 28 November 2015
Accepted 9 April 2016
Available online

Keywords:
Analytical solution
Continuous beams
Elastic foundation
Jordan form
Fundamental solution

ABSTRACT

An efficient analytical method is presented for the closed form solution of continuous beams on two-parameter elastic foundations. The general form of the governing equation is reduced to a system of first-order differential equations with constant coefficients. The system is then solved using Jordan form decomposition for the coefficient matrix and construction of the fundamental solution. Common types of boundary conditions (pinned and roller support, hinge connection, fixed and free end) can be applied to an arbitrary point on the beam. The method has a completely computer-oriented algorithm, computational stability, and optimal conditionality of the resultant system and is a powerful alternative to the analytical solution of beams with multipoint boundary conditions on one- or two-parameter elastic foundations. Examples with different types of loading, boundary conditions, and foundation are presented to verify the method.

1. Introduction

Recent developments in computer science and mathematics and the need for accurate solutions to problems have resulted in the development of analytical [1] and semi-analytical or discrete-continual methods [2]. For practical problems, it is often more suitable and easier to use an analytical solution than to employ an expensive finite element method (FEM)-based software. Analytical solutions employ a mathematical expression that yields the values of the unknown quantities at any location on a body (the total structure or physical system of interest) and are valid for an infinite number of locations. This property considerably reduces the computational complexity of the problem, an issue that requires special consideration in numerical methods [3].

Akimov and Sidorov [4] proposed an analytical solution to multipoint boundary problems for systems of ordinary differential equations with piecewise constant coefficients. Their method can be applied to continuous beams, which is a leading engineering problem. Beams in different types of bridges, continuous beams of multi-span girders [5], long strip foundations of buildings, and railroad and retaining walls [6] are included in this type of problem. The most important application

* Corresponding author at: Fasa University, Daneshjou blvd, Fasa, Fars Province, Iran. Tel.: +98 71 53340519.
E-mail addresses: aslami@fasau.ac.ir (M. Aslami), akimov@mgsu.ru (P.A. Akimov).

http://dx.doi.org/10.1016/j.acme.2016.04.005

of the proposed method is an analytical solution to continuous beams on elastic foundations. The method can also be applied to beams with structural or foundation discontinuities (changes in the elastic property or cross-section of the beam or stiffness of the soil) for beams without foundations by changing the coefficient matrix at the relevant interval.

Section 3 presents the formulation of the problem. Section 4 describes the analytical solution to the resultant multipoint boundary problem in detail. Various types of boundary conditions are introduced in Section 5. Section 6 uses numerical examples to demonstrate the efficiency, accuracy, and validity of the method. Section 7 presents the concluding remarks.

2. Beam on elastic foundation

A computational model of a beam on an elastic foundation is often used to describe different engineering problems [7]. Much literature has been devoted to evaluation of the behavior of soil-structure interaction (beam, in this case) [8–22]. Some [8–13] present the soil as an idealized Winkler model [23] in which disconnected soil springs are used to represent the compressive resistance of soil. These Winkler springs are characterized by the spring constant k_s, which is often related to the soil subgrade modulus. Lin and Adams [8] and Kaschiev and Mikhajlov [9] investigated the problem of beams on tensionless Winkler foundations. DasGupta [10], Banan et al. [11] and Aköz and Kadioğlu [12] researched finite element approaches for the solution of beams on elastic Winkler foundations.

In the analytical field, Borák and Marcián [13] proposed a modified Betti's theorem for an analytical solution to beams on elastic foundations. The results were acceptable and the ground behavior could be more realistically simulated than the Winkler model. Mechanical resistance in soil arises from both compressive and shear strains; thus, it is more realistic to consider shear interactions between the soil springs, which distributes ground displacement and stress beyond the loaded region. This leads to a simplified-continuum model [14] in which shear interaction is mathematically taken into account by introducing a shear parameter t_s. This parameter represents the shear force at any vertical section of the foundation. Considering the general nature of the governing differential equation describing beam deflection on a two-parameter (k_s and t_s) foundation, t_s can also be interpreted as the tensile force in the membrane connecting the soil springs. In this way, the tensile resistance generated in the ground from the placement of geosynthetics can also be taken into account [15].

Zhaohua et al. [16], Karamanlidis et al. [17], Razaqpur and Shah [18], and Morfidis et al. [19] researched a finite element solution to beams resting on two-parameter elastic foundations. Beams on three-parameter elastic foundation were studied by Avramidis and Morfidis [20] and Morfidis [21]. Dinev [22] proposed an analytical solution to a beam on an elastic foundation using singularity functions and considering two parameters for the soil model. This is applicable only for the solution to problems without special external boundary conditions; however, many practical problems have different external boundary conditions that must be considered, such as continuous strip foundations resting on piles, railroad foundations, and fixed-end foundations. The present paper proposes an exact analytical solution to beams with multiple external boundary conditions. In this method, common types of boundary conditions such as pinned and roller supports, hinge connections, and fixed and free ends at arbitrary points along the beam also allows consideration of an elastic foundation with one or two soil parameters.

3. Formulation of problem

The derivation of a field equation is based on variation in the total potential energy function and employs the following assumptions [22]:

- The beam and the soil materials are linearly elastic, homogeneous and isotropic;
- The displacements are small compared to the beam thickness;
- The axial strains are small compared to unity;
- The transversal normal strains and the shear stresses are negligibly small;
- The cross-sections are plane and perpendicular to the longitudinal axis before and after deformation (Bernoulli hypothesis).

$$\frac{\partial^4 w}{\partial x^4} - \alpha\frac{\partial^2 w}{\partial x^2} + \beta w = F(x) \tag{1}$$

$$\alpha = \frac{2t_s}{EI}, \quad \beta = \frac{k_s}{EI}, \quad F(x) = \frac{P}{EI}\delta(x - x_0) \tag{2}$$

Here, w is vertical displacement of the beam, α and β are soil parameters, and F(x) is the load applied to the structure and is represented by the delta function in distribution theory [24]. A typical beam on an elastic foundation with different types of loading and boundary conditions is shown in Fig. 1. Consider the following relations:

$$\begin{cases} w'(x) = \theta(x); \\ \theta'(x) = \dfrac{M(x)}{EI}; \\ M'(x) = Q(x); \\ Q'(x) = F(x) + \dfrac{\alpha M(x)}{EI} - \beta w(x) \end{cases} \tag{3}$$

The assumed sign convention for moments, shear force, deflection, and cross-section rotation for Eq. (3) are presented in Fig. 2. The governing differential equation of the problem (Eq. (1)) can now be transformed into a system of four differential equations of first order in matrix form as:

$$\bar{y}'(x) = \bar{f}(x) + A\bar{y}(x) \tag{4}$$

$$\bar{y}(x) = [y_1(x) \quad y_2(x) \quad y_3(x) \quad y_4(x)]^T \tag{5}$$

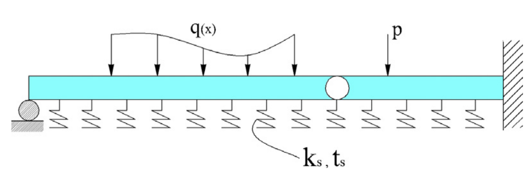

Fig. 1 – Beam on elastic foundation with different types of loading and boundary conditions.

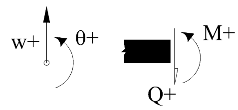

Fig. 2 – Sign convention for moments, shear force, deflection and cross-section rotation.

$$y_1(x) = w(x), \quad y_2(x) = \theta(x), \quad y_3(x) = M(x), \quad y_4(x) = Q(x) \tag{6}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & \alpha & 0 \end{bmatrix}; \quad \bar{f}(x) = [0 \quad 0 \quad 0 \quad F(x)]^T \tag{7}$$

where $\bar{y}(x)$ is the n-dimensional vector of unknowns, A is the matrix of constant coefficients of the n^{th} order, $\bar{f}(x)$ is the n-dimensional vector function of the right side, x is variable for the longitudinal direction, $w(x)$ is transverse displacement of the beam, $\theta(x)$ is the angle of rotation, M(x) is the bending moment, and Q(x) is the shear force.

4. Analytical solution to multipoint boundary problem

Assume the system consists of n equations, n_b boundary conditions, and $k = 1, \ldots, n_b - 1$ individual fragments in which the physical and geometrical parameters (boundary conditions, beam properties, soil parameters, etc.) of the structure are constant. The solution to the problem can be obtained by convolution of fundamental function ($\epsilon(x)$) [24] of the system of ordinary differential equations (Eq. (4)) and the applied forces.

4.1. *Jordan decomposition of coefficient matrix*

The first step for the solution of the problem is the Jordan decomposition [25,26] of the coefficient matrix of this system (A_k), for each individual fragment, as follows:

$$A_k = T_k J_k T_k^{-1}, J_k = \begin{bmatrix} J_{k,1} & & & \\ & J_{k,2} & & \\ & & \ddots & \\ & & & J_{k,u} \end{bmatrix}, \quad J_{k,p} = \begin{bmatrix} \lambda_{k,p} & 1 & 0 & \ldots & 0 \\ 0 & \lambda_{k,p} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_{k,p} \end{bmatrix}$$

where T_k is a non-singular matrix of the nth order whose columns are the eigenvectors of the matrix A_k; J_k the Jordan matrix of the nth order; $J_{k,p}$ the Jordan block corresponding to the eigenvalue $\lambda_{k,p}$ with order of $m_{k,p}$; u the number of different eigenvalues. A typical example of the above Jordan decomposition, for the coefficient matrix, is presented in Appendix A.

4.2. *Construction of the fundamental function*

The essential approach to obtain the solution of the problem is the construction of the fundamental matrix function of the system, which satisfies the following condition:

$$\epsilon_k^{(1)}(x) - A_k\epsilon_k(x) = \delta(x)I \quad \text{or} \quad \begin{cases} \epsilon_k^{(1)}(x) - A_k\epsilon_k(x) = 0, & x \neq 0 \\ \epsilon_k(+0) - \epsilon_k(-0) = I \end{cases} \tag{9}$$

The construction of this function involves calculating the Jordan form of coefficient matrix. However, there is no universal numerically stable method for constructing and numerical implementation of Jordan forms [25,26]. Besides, coefficient matrices in problems of structural mechanics normally have the Jordan cells with non-identity order. Therefore, the following method is proposed for decomposition of the coefficient matrix and construction of fundamental matrix function:

As known, the nonzero vector $\bar{t}_k$ in $A_k\bar{t}_k = \lambda_k\bar{t}_k$ is called the right eigenvector of the matrix A_k corresponding to the eigenvalue λ_k, and the nonzero vector $\bar{\bar{t}}_k$ in $A_k^T\bar{\bar{t}}_k = \lambda_k\bar{\bar{t}}_k$, is called the left eigenvector of matrix A_k, corresponding to the eigenvalue λ_k. Let T_k be a nonsingular nth order matrix, containing eigenvectors and root vectors of the matrix A_k:

$$T_k = [\bar{t}_{k,1} \quad \bar{t}_{k,2} \quad \ldots \quad \bar{t}_{k,n}]^T; A_k\bar{t}_{k,s} = \lambda_{k,p}\bar{t}_{k,s} \quad s = 1 + \sum_{i=1}^{p-1} m_{k,i} \tag{10}$$

Then in accordance with equality of the eigenvalues of matrices A_k and A_k^T, we can write:

$$\tilde{T}_k = T_k^{-1} \tag{11}$$

where $\tilde{T}_k$ is the nonsingular nth order matrix, containing eigenvectors and root vectors of the matrix A_k^T, and

$$\tilde{T}_k = [\bar{\bar{t}}_{k,1} \quad \bar{\bar{t}}_{k,2} \quad \ldots \quad \bar{\bar{t}}_{k,n}]^T; A_k^T\bar{\bar{t}}_{k,s} = \lambda_{k,p}\bar{\bar{t}}_{k,s} \Leftrightarrow \bar{\bar{t}}_{k,s}^T A_k = \lambda_{k,p}\bar{\bar{t}}_{k,s}^T, \quad s = 1 + \sum_{i=1}^{p-1} m_{k,i} \tag{12}$$

Thus, Jordan decomposition of matrix A_k has the following form:

$$A_k = T_k J_k \tilde{T}_k \tag{13}$$

The eigenvalues of the matrices A_k and A_k^T should be reordered according to the following condition:

$$\begin{cases} \forall \lambda_{k,p}, & p = 1, \ldots, l_{k,1}: & \lambda_{k,p} \neq 0, & m_{k,p} = 1, & \tilde{m}_{k,p} = 1 \\ \forall \lambda_{k,p}, & p = l_{k,1} + 1, \ldots, l_{k,2}: & \lambda_{k,p} \neq 0, & m_{k,p} = 1, & \tilde{m}_{k,p} > 1 \\ \forall \lambda_{k,p}, & p = l_{k,2} + 1, \ldots, u_k: & \lambda_{k,p} = 0 \end{cases} \tag{14}$$

where $\tilde{m}_{k,p}, m_{k,p}$ are the multiplicity of the eigenvalue $\lambda_{k,p}$ and dimension of the Jordan block, respectively. This sorting is guarantees the proper correspondence between the eigenvectors of matrices A_k and A_k^T, and allow to use $\tilde{T}_k$ instead of T_k^{-1}. For the construction of the fundamental matrix function, the coefficients matrix A_k of the system is represented by the following formulas:

$$A_k = A_{k,+} + A_{k,-} + A_{k,0}, \quad A_{k,+} = P_{k,+}A_k, \quad A_{k,-} = P_{k,-}A_k, \quad A_{k,0} = P_{k,0}A_k = A_k - A_{k,+} - A_{k,-} \tag{15}$$

where $P_{k,+}$ is projection onto the subspace, corresponding to eigenvectors of nonzero eigenvalues with nonnegative real part; $P_{k,-}$ is projection onto the subspace corresponding to the eigenvectors of nonzero eigenvalues with negative real parts and $P_{k,0}$ is the projection onto the subspace corresponding to eigenvectors of zero eigenvalues. These projection matrices can be obtained as:

$$P_{k,+} = T_{k,+}(\tilde{T}_{k,+}T_{k,+})^{-1}\tilde{T}_{k,+}, \quad P_{k,-} = T_{k,-}(\tilde{T}_{k,-}T_{k,-})^{-1}\tilde{T}_{k,-}, \quad P_{k,0} = I - P_{k,+} - P_{k,-} \tag{16}$$

where $T_{k,+}$ and $\tilde{T}_{k,+}$ are the $n \times n_+$ and $n_+ \times n$ matrices, containing right and left eigenvectors corresponding to nonzero eigenvalues of the matrix A_k with nonnegative real part, respectively; $T_{k,-}$ and $\tilde{T}_{k,-}$ are the $n \times n_-$ and $n_- \times n$ matrices, containing right and left eigenvectors corresponding to nonzero eigenvalues of the matrix A_k with negative real parts, respectively. I is the identity matrix of the appropriate order; n_+ and n_- are the number of nonzero eigenvalues with non-negative and negative real parts, respectively. Finally, the fundamental solution of the problem (3) can be constructed by formulas:

$$\varepsilon_k(x) = T_{k,1}\tilde{\varepsilon}_{k,0}(x)\tilde{T}_{k,1} + \chi(x,0)\left[P_{k,0} + \sum_{j=1}^{m_{k,\max}-1}\frac{x^j}{j!}A^j_{k,0}\right] \tag{17}$$

$$T_{k,1} = [T_{k,+}|T_{k,-}], \quad \tilde{T}_{k,1} = [\tilde{T}_{k,+}|\tilde{T}_{k,-}] \tag{18}$$

$$\chi(x,\lambda_{k,p}) = \begin{cases}\chi(x), & Re(\lambda_{k,p}) \le 0\\ -\chi(-x), & Re(\lambda_{k,p}) > 0\end{cases}; \quad \chi(x) = \begin{cases}1, & x>0\\ 0, & x<0\end{cases}; \tag{19}$$

$$\tilde{\varepsilon}_{k,0}(x) = diag\{\chi(x,\lambda_{k,1})\exp(\lambda_{k,1}x), \ldots, \chi(x,\lambda_{k,l})\exp(\lambda_{k,l}x)\} \tag{20}$$

where $\epsilon_k(x)$ is the fundamental solution of the problem; $m_{k,\max} = \max_{l \le i \le u} m_{k,i}$ ($m_{k,\max}$ has a finite and small value); $m_{k,i}$ is the order of the Jordan block corresponding to the eigenvalue $\lambda_{k,i}$; $l = n_{k,+} + n_{k,-}$ ($n_{k,+} + n_{k,-}$ is the number of non-zero eigenvalues of the matrix A_k).

4.3. *Final solution of the problem*

The solution, which is obtained by the construction of the fundamental matrix function, can be used for the beams on an infinite elastic foundation (the foundation extends beyond the edges of the beam) and without additional internal boundary points. The solution of the problems with the finite foundation (the foundation does not extend beyond the edges of the beam) and other boundary conditions, the following procedure should be considered. Let consider a multipoint boundary problem (i.e. a problem with additional internal boundary points) for the first order ordinary differential equations (4).

$$\bar{y}'_k - A_k\bar{y}_k = \bar{f}_k, \quad x \in \bigcup_{k=1}^{n_b-1}(x^b_k, x^b_{k+1}) \tag{21}$$

$$B^-_i\bar{y}(x^b_i - 0) + B^+_i\bar{y}(x^b_i + 0) = \bar{g}^-_i + \bar{g}^+_i, \quad i = 2, \ldots, n_b - 1 \tag{22}$$

$$B^+_1\bar{y}(x^b_1 + 0) + B^-_{n_b}\bar{y}(x^b_{n_b} - 0) = \bar{g}^+_1 + \bar{g}^-_{n_b} \tag{23}$$

where $B^-_i, B^+_i (i = 2, \ldots, n_b - 1), B^+_1$ and $B^-_{n_b}$ are given $n \times n$ matrices of boundary conditions; $\bar{g}^-_i, \bar{g}^+_i (i = 2, \ldots, n_b - 1)$, $\bar{g}^+_1$ and $\bar{g}^-_{n_b}$ are given n-dimensional vectors of the right parts of the boundary conditions. The solution of the problem (21)–(23) on an arbitrary interval (x^b_k, x^b_{k+1}) is represented by $\bar{y}_k(x)$ and is defined by:

$$\bar{y}_k(x) = (\epsilon(x - x^b_k) - \epsilon(x - x^b_{k+1}))\bar{C}_k + \epsilon(x) * \bar{f}_k(x), \quad x \in (x^b_k, x^b_{k+1}) \tag{24}$$

where $\bar{C}_k$ is the vector of constant coefficients of n order, * denotes the convolution operation and

$$\bar{f}_k(x) \equiv f(x)\theta(x, x^b_k, x^b_{k+1}), \quad \theta(x, x^b_k, x^b_{k+1}) = \begin{cases}1, & x \in (x^b_k, x^b_{k+1})\\ 0, & x \notin (x^b_k, x^b_{k+1})\end{cases} \tag{25}$$

Eq. (24) can be rewritten in the following form

$$\bar{y}_k(x) = E_k(x)\bar{C}_k + \bar{S}_k(x), \quad x \in (x^b_k, x^b_{k+1}) \tag{26}$$

where $E_k(x) = \epsilon(x - x^b_k) - \epsilon(x - x^b_{k+1})$; $\bar{S}_k(x) = \epsilon(x) * \bar{f}_k(x)$.

Substituting (26) in (22) and (23), the resultant system of linear algebraic equations for the coefficients $\bar{C}_k$, can be obtained.

This system can be rewritten in a matrix form:

$$\begin{cases} B^-_iE_{i-1}(x^b_i - 0)\bar{C}_{i-1} + B^+_iE_i(x^b_i + 0)\bar{C}_i = \bar{g}^-_i + \bar{g}^+_i - B^-_iS_{i-1}(x^b_i - 0) - B^+_iS_i(x^b_i + 0), \quad i = 2, \ldots, n_i - 1\\ B^+_1E_1(x^b_1 + 0)\bar{C}_1 + B^-_{n_b}E_{n_b-1}(x^b_{n_b} - 0)\bar{C}_{n_b-1} = \bar{g}^+_1 + \bar{g}^-_{n_b} - B^+_1S_1(x^b_1 + 0) - B^-_{n_b}S_{n_b-1}(x^b_{n_b} - 0)\end{cases} \tag{27}$$

$$K\bar{C} = \bar{G} \tag{28}$$

where

$$K = \begin{bmatrix} K_{1,1} & 0 & 0 & \ldots & 0 & K_{1,n_b-1}\\ K_{2,1} & K_{2,2} & 0 & \ldots & 0 & 0\\ 0 & K_{3,2} & K_{3,3} & \ldots & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & K_{n_b-1,n_b-2} & K_{n_b-1,n_b-1}\end{bmatrix} \tag{29}$$

$$\bar{G} = \left[\bar{G}^T_1 \ \ \bar{G}^T_2 \ \ \ldots \ \ \bar{G}^T_{n_b-1}\right]^T \quad \bar{C} = \left[\bar{C}^T_1 \ \ \bar{C}^T_2 \ \ \ldots \ \ \bar{C}^T_{n_b-1}\right]^T \tag{30}$$

$$K_{i,i-1} = B^-_iE_{i-1}(x^b_i - 0); \quad K_{i,i} = B^+_iE_i(x^b_i + 0); \quad K_{1,1} = B^+_1E_1(x^b_1 + 0); \quad K_{1,n_b-1} = B^-_{n_b}E_{n_b-1}(x^b_{n_b} - 0) \tag{31}$$

$$\begin{cases}\bar{G}_1 = \bar{g}^+_1 + \bar{g}^-_{n_b} - B^+_1S_1(x^b_1 + 0) - B^-_{n_b}S_{n_b-1}(x^b_{n_b} - 0)\\ \bar{G}_i = \bar{g}^-_i + \bar{g}^+_i - B^-_iS_{i-1}(x^b_i - 0) - B^+_iS_i(x^b_i + 0), \quad i = 2, \ldots, n_i - 1\end{cases} \tag{32}$$

It is vital to note that diagonal blocks of the matrix K are practically singular, thereby resulting in several problems to which iterative solution methods cannot be applied in particular [27]. Hence, the Gaussian elimination method with pivoting is required [27]. It is useful to specify ways of eliminating this disadvantage. Therefore, the obtained system of equation (29) is transformed as follows:

1- Each equation of the system, since the first one, is replaced by the sum of this equation and the subsequent one (instead of the initial first equation, we took the sum of the first and second initials, instead of the second initial – the sum of the second and third initials, and so on).

2- Finally, take the sum of the initial last equation with the initial first equation, instead of the initial last equation. Finally, we had:

$$K = \begin{bmatrix} \tilde{K}_{1,1} & \tilde{K}_{1,2} & 0 & \dots & 0 & \tilde{K}_{1,n_b-1} \\ \tilde{K}_{2,1} & \tilde{K}_{2,2} & \tilde{K}_{2,3} & \dots & 0 & 0 \\ 0 & \tilde{K}_{3,2} & \tilde{K}_{3,3} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \tilde{K}_{n_b-1,1} & 0 & 0 & \dots & \tilde{K}_{n_b-1,n_b-2} & \tilde{K}_{n_b-1,n_b-1} \end{bmatrix} \quad (33)$$

5. Boundary conditions

The solution will be complete after the addition of boundary conditions. In this problem, the boundary condition is the relation of transverse displacement of the beam and its derivatives at the left- and right-hand sides of the boundary points. The type of support connection determines the degrees of freedom (displacement or force) that the support can resist. A boundary condition is distinctly defined by four-in-four matrices for the left and right hand sides of the boundary point. This matrix specifies the relation for each $y_i(x)$ to the left and right sides of the support (for the first and last boundary point, only a right-hand condition is needed). Note that, in practice, these types of boundary conditions are ideal representations of special conditions regarding the beam (e.g., connected pile or retaining wall or arrangement of reinforced bars to construct hinge connection). Common types of boundary conditions are explained below.

5.1. *Fixed end*

In this type of boundary condition, transverse deflection w and angle of rotation θ equal zero; therefore, the boundary matrices for this support are:

$$B_1^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{n_b}^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \bar{g}_1^+ = \bar{g}_{n_b}^- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (34)$$

5.2. *Pinned support*

In this type of boundary condition, transverse deflection w and bending moment M equal zero; thus, the boundary matrices for this support are:

$$B_1^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{n_b}^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{g}_1^+ = \bar{g}_{n_b}^- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (35)$$

5.3. *Roller support*

In this type of boundary condition, the relation between the left and right sides of the support and corresponding matrices are:

$$\Delta w = w^+ - w^- = 0, \quad w^- + w^+ = 0, \quad \Delta M = M^+ - M^- = 0, \quad \theta^+ + \theta^- = 0 \quad (36)$$

$$B_1^- = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad B_i^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \bar{g}_i^+ = \bar{g}_i^- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (37)$$

5.4. *Hinge connection*

Here, the relation between the left and right sides of the support and the corresponding matrices of this type of boundary condition are:

$$\Delta w = w^+ - w^- = 0, \quad M^- + M^+ = 0, \quad \Delta M = M^+ - M^- = 0, \quad \Delta Q = Q^+ - Q^- = 0 \quad (38)$$

$$B_i^- = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B_i^+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{g}_i^+ = \bar{g}_i^- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (39)$$

5.5. *Free end*

In this type of boundary condition, bending moment M and shear force Q equal zero; therefore, the matrices for this support are:

$$B_1^+ = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{n_b}^- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{g}_1^+ = \bar{g}_{n_b}^- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (40)$$

6. Examples and comparison with results of previous studies

Numerous examples have been considered to illustrate the efficiency of the method and three are presented below. The software was developed using Intel Parallel Studio XE [28] (FORTRAN programming language [29]) and the results were compared with a FEM solution (ANSYS [30]) and results of previous studies.

6.1. *Beam on Winkler foundation*

In this example, a straight free-ended beam 5.0 m in length with rectangular cross-sections where $b = 0.4$ m, $h = 0.2$ m, $k_s = 1 \times 10^8 \mathrm{N/m^3}$, and $E = 2 \times 10^{11}$ Pa is supported by a Winkler elastic foundation. It was subjected to three types of loads and the results were compared with the results of Borák and Marcián [13] (Fig. 3). The results of the solution for vertical deflection, angle of rotation, and internal forces for a finite elastic foundation (does not extend beyond the edges

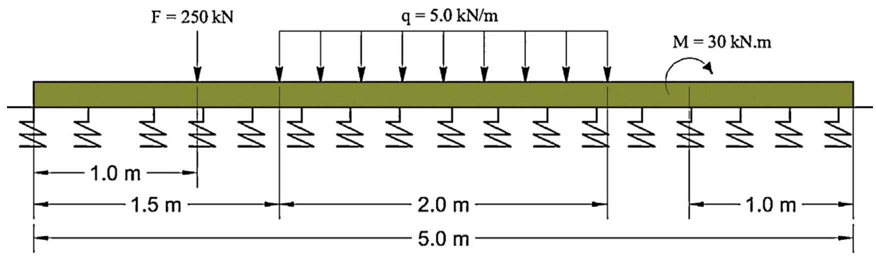

Fig. 3 – Free-ended beam on Winkler foundation.

of the beam) and an infinite elastic foundation (extends beyond the edges of the beam) are shown in Fig. 4 and Table 1.

For a beam on a finite elastic foundation, the results were coincident with the reference; however, when assuming an infinite elastic foundation, the maximum deflection and angle of rotation decreased and the maximum bending moment increased about 50%. Deformation in the part of the foundation that extended beyond the edges of the beam may mean that the shear at the free ends will not equal zero. The advantage of the proposed method is its consideration of an infinite foundation and multiple boundary conditions for the beam, which allow its use for general applications.

6.2. *Beam on elastic foundation with two-parameter for soil behavior*

A continuous beam on a two-parameter elastic foundation [18] was solved using the proposed method and is shown in Fig. 5. The parameters were $k_s = 64.0$ kN/m^2, $2t_s = 800.0$ kN, and $EI = 2000.0$ kN m^2. The solution was presented for a Winkler foundation and a two-parameter foundation and the results compared with those from Razaqpur and Shah [18] (Fig. 6). In

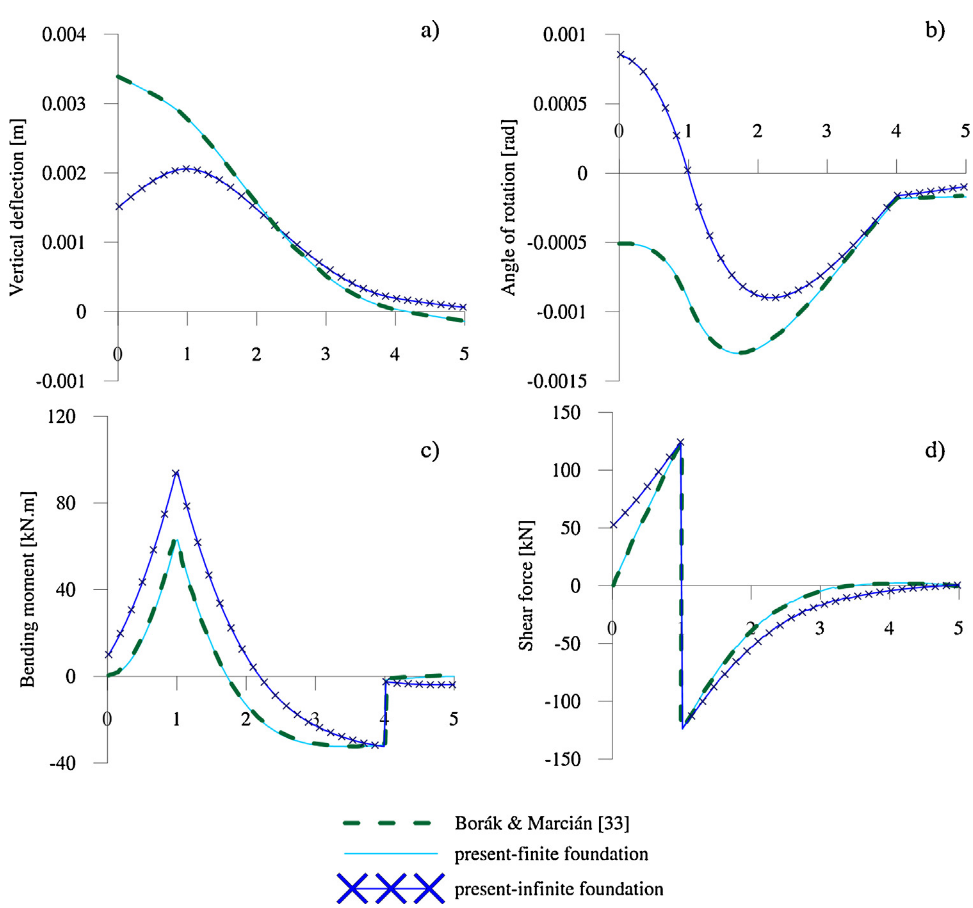

Fig. 4 – Results of the solution: (a) vertical deflection, (b) angle of rotation, (c) bending moment and (d) shear force.

Table 1 – Comparison of vertical deflection, angle of rotation and internal forces for the finite and infinite elastic foundation.

L (m)	w (m)			θ (rad)			Q (kN)			M (kN m)		
	Ref [13]	Present A[a]	Present B[b]	Ref [13]	Present A	Present B	Ref [13]	Present A	Present B	Ref [13]	Present A	Present B
0.00	0.00339	0.00338	0.00152	−0.00051	−0.00051	0.00085	2.8	2.8	52.9	0.1	0.1	10.3
0.50	0.00313	0.00313	0.00188	−0.00056	−0.00056	0.00063	66.7	66.7	87.4	17.3	17.3	43.1
1.00	0.00279	0.00279	0.00206	−0.00092	−0.00092	−0.00001	124.2	124.2	124.7	64.4	64.4	94.2
							−125.2	−125.2	−122.9			
1.50	0.00222	0.00222	0.00187	−0.00127	−0.00127	−0.00065	−73.5	−73.5	−83.9	14.5	14.5	42.7
2.00	0.00157	0.00157	0.00148	−0.00126	−0.00126	−0.00088	−39.1	−39.1	−53.1	−12.9	−12.9	9.0
2.50	0.00099	0.00099	0.00103	−0.00107	−0.00107	−0.00086	−16.8	−16.8	−29.8	−26.7	−26.7	−11.2
3.00	0.00052	0.00052	0.00064	−0.00079	−0.00079	−0.00070	−3.7	−3.7	−16.8	−31.5	−31.5	−23.0
3.50	0.00020	0.00020	0.00035	−0.00049	−0.00049	−0.00045	−0.1	−0.1	−9.4	−32.5	−32.5	−28.9
4.00	0.00004	0.00004	0.00020	−0.00019	−0.00019	−0.00017	1.8	1.8	−3.7	−31.5	−31.5	−32.3
										−1.6	−1.6	−2.8
4.50	−0.00006	−0.00006	0.00012	−0.00017	−0.00017	−0.00013	1.8	1.8	−1.0	−0.6	−0.6	−3.7
5.00	−0.00014	−0.00014	0.00006	−0.00017	−0.00017	−0.00010	−0.1	−0.1	0.8	0.1	0.1	−3.7

[a] A: finite foundation.
[b] B: infinite foundation.

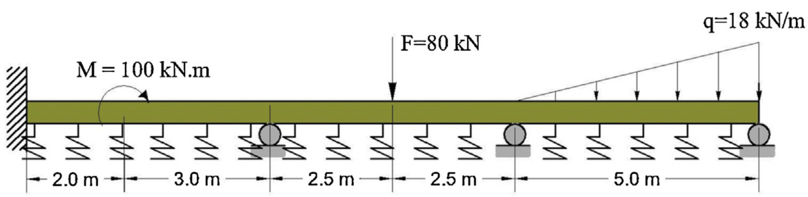

Fig. 5 – Continuous beam on two-parameter elastic foundation.

this example, a beam with multipoint boundary conditions and different types of loading was accurately solved and compared with the finite element solution proposed by Razaqpur and Shah [18]. The proposed analytical approach was very efficient and considerably reduced the computational complexity of the problem, especially for long beams. The finite element model accurately evaluated the behavior of the beam on a two-parameter elastic foundation; however, in accordance with the proposed analytical solution, the high value of the second soil parameter ($2t_s$) affected the shear force diagram with respect to Winkler foundation (maximum 38%), which did not agree with the Razaqpur and Shah model. The extreme values of the results, especially under concentrated force, considerably increased for the Winkler foundation with respect to a beam on a two-parameter elastic foundation. The extreme values are listed in Table 2.

6.3. *Continuous beam on elastic Vlasov foundation*

The continuous beam with concentrated forces shown in Fig. 7 was considered. The soil parameters were obtained using the Vlasov model [14] as $k_s = 6730.77 \text{kN/m}^2$ and $t_s = 7692.31$ kN. The beam was 20 m in length and had a bending stiffness of $EI = 276041.66$ kN m^2. The diagrams for vertical deflection and the internal forces in comparison with the FEM solution (Winkler foundation) are shown in Fig. 8.

The analytical solution obtained by the proposed method was coincident with the FEM solution; however, the accuracy

Table 2 – Comparison of extreme values for Winkler and two-parameter elastic foundation.

Parameter	Minimum		Maximum	
	Winkler foundation	Two-parameter elastic foundation	Winkler foundation	Two-parameter elastic foundation
w (m)	−0.035128	−0.02542	0.00286	0.001424
θ (rad)	−0.01927	−0.014017	0.01923	0.013896
Q (kN)	−40.0	−41.5	40.0	40.0
M (kN m)	−50.0	−51.5	54.8	51.5

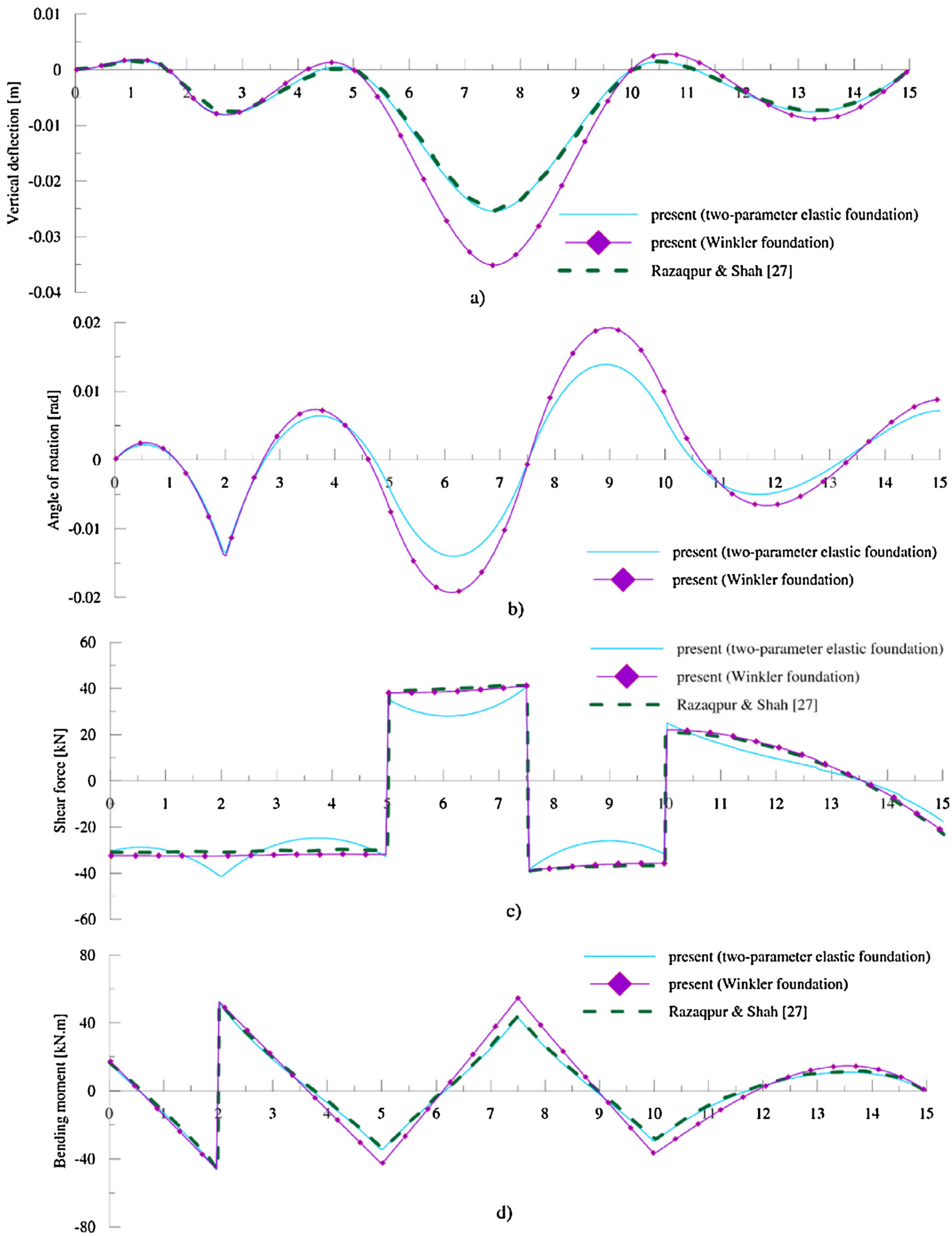

Fig. 6 – Results of the solution: (a) vertical deflection, (b) angle of rotation, (c) shear force and (d) bending moment.

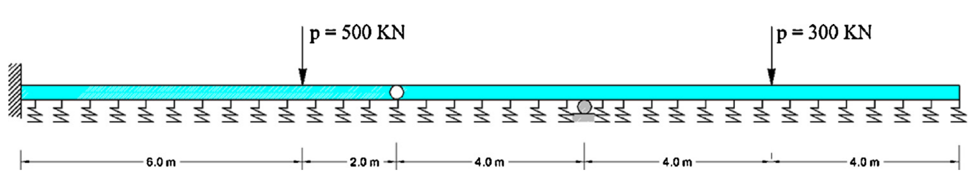

Fig. 7 – Continuous beam on Vlasov elastic foundation.

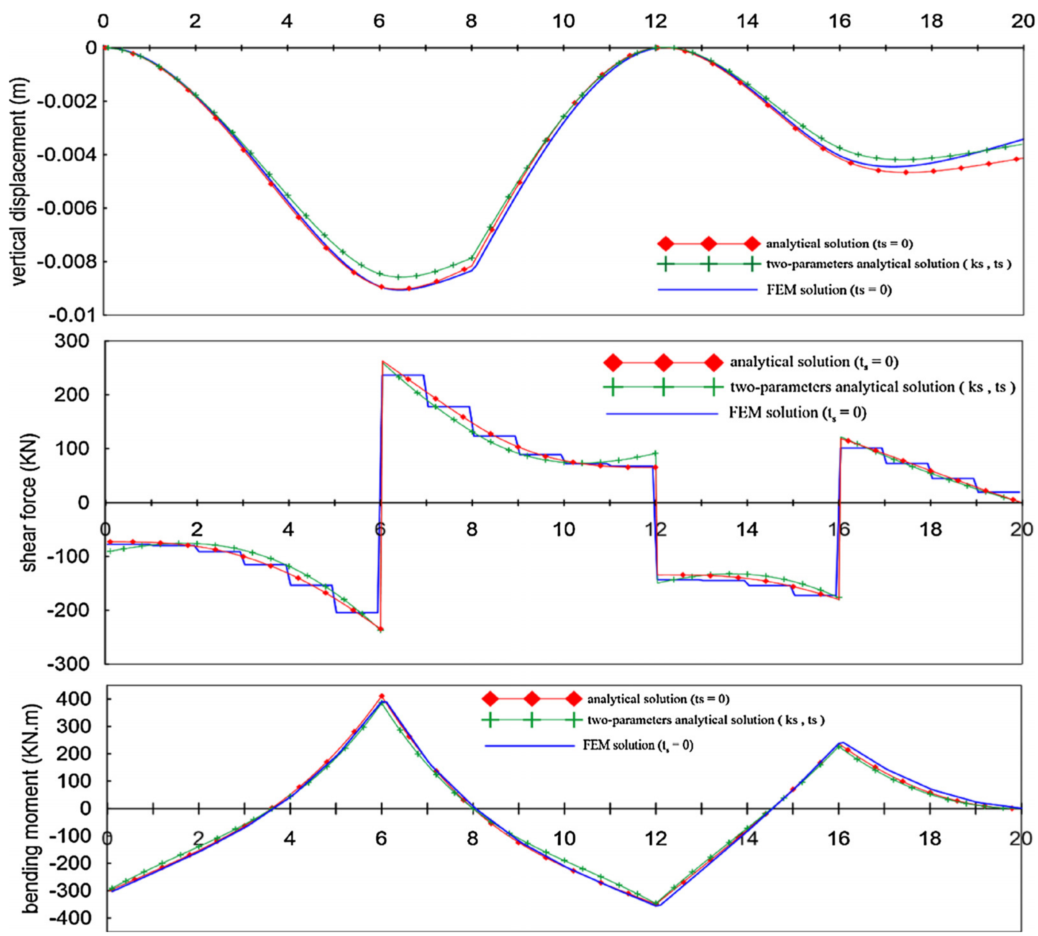

Fig. 8 – Results of the solution; vertical deflection, shear force and bending moment.

and continuity of the shear diagram increased considerably. The small difference between the results of the two soil models can be attributed to the relatively small value for the shear parameter of the foundation (t_s). Consideration of this parameter lowers the vertical deflection of the beam, especially under concentrated forces.

7. Conclusion

The examples given demonstrate the advantages of the proposed approach for an analytical solution to a continuous beam on one or two-parameter elastic foundations. The method has a completely computer-oriented algorithm, computational stability, optimal conditionality of the resultant system, and is applicable for different loads at an arbitrary point or region on the beam. In addition, common boundary conditions such as pinned and roller supports, hinge connections, and fixed and free ends at arbitrary points along the beam, can be considered. Structural or foundation discontinuities (changes in the physical properties of the beam or foundation) can be applied by changing the coefficient matrix at the relevant interval. The method has been shown to be a powerful alternative to the analytical solution of beams with multipoint boundary conditions on one or two-parameter elastic foundations, especially for programming of specialized software packages oriented to analytical solutions. In future research, the analytical solution for curved beams on elastic foundations and beams on infinite elastic foundations with additional internal boundary points can be investigated.

Appendix A

A typical example of Jordan decomposition for the 4×4 coefficient matrix of a system first order differential equations of a beam on Winkler elastic foundation:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix}; \quad J = \begin{bmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & -1+i & 0 \\ 0 & 0 & 0 & -1-i \end{bmatrix};$$

$$T = 0.1826 \begin{bmatrix} -1-i & -1+i & 1-i & 1+i \\ -2i & 2i & 2i & 2i \\ 2-2i & 2+2i & -2-2i & -2+2i \\ 4 & 4 & 4 & 4 \end{bmatrix};$$

$$T^{-1} = 0.3423 \begin{bmatrix} -2+2i & 2i & 1+i & 1 \\ -2-2i & -2i & 1-i & 1 \\ 2+2i & -2i & -1+i & 1 \\ 2-2i & 2i & -1-i & 1 \end{bmatrix}$$

REFERENCES

[1] S.V. Meleshko, Methods for constructing exact solutions of partial differential equations: mathematical and analytical techniques with applications to engineering, Springer Science & Business Media, 2006.

[2] P.A. Akimov, V.N. Sidorov, Correct method of analytical solution of multipoint boundary problems of structural analysis for systems of ordinary differential equations with piecewise constant coefficients, Advanced Materials Research 250 (2011) 3652–3655.

[3] P.A. Akimov, A.M. Belostoskiy, M.L. Mozgaleva, M. Aslami, O. A. Negrozov, Correct multilevel discrete-continual finite element method of structural analysis, Advanced Materials Research 1040 (2014) 664–669.

[4] M. Aslami, P.A. Akimov, Wavelet-based finite element method for multilevel local plate analysis, Thin-Walled Structures 98 (2016) 392–402.

[5] A. Kassimali, Structural analysis, Cengage Learning, 2014.

[6] E.S. Melerski, Design of beams, circular plates and cylindrical tanks on elastic foundation, 2nd edition, Taylor & Francis Group, 2006.

[7] E. Tsudik, Analysis of Structures on Elastic Foundations, J. Ross Publishing (2012).

[8] L. Lin, G.G. Adams, Beam on tensionless elastic foundation, Journal of Engineering Mechanics 113 (4) (1987) 542–553.

[9] M.S. Kaschiev, K. Mikhajlov, A beam resting on a tensionless Winkler foundation, Computers & Structures 55 (2) (1995) 261–264.

[10] S. DasGupta, Axially constrained beams on elastic foundation, International Journal of Mechanical Sciences 16 (5) (1974) 305–310.

[11] M.R. Banan, G. Karami, M. Farshad, Finite element analysis of curved beams on elastic foundations, Computers & Structures 32 (1) (1989) 45–53.

[12] A.Y. Aköz, F. Kadioğlu, The mixed finite element solution of circular beam on elastic foundation, Computers & Structures 60 (4) (1996) 643–651.

[13] L. Borák, P. Marcián, Beams on elastic foundation using modified Betti's theorem, International Journal of Mechanical Sciences 88 (2014) 17–24.

[14] V.Z. Vlasov, U.N. Leont'ev, Beams, Plates and Shells on Elastic Foundations, Israel Program for Scientific Translations Ltd, Jerusalem, Israel, 1966.

[15] D. Basu, N.S.V. Kameswara Rao, Analytical solutions for Euler–Bernoulli beam on visco-elastic foundation subjected to moving load, International Journal for Numerical and Analytical Methods in Geomechanics 37 (8) (2013) 945–960.

[16] F. Zhaohua, R.D. Cook, Beam elements on two-parameter elastic foundations, Journal of Engineering Mechanics 109 (6) (1983) 1390–1402.

[17] D. Karamanlidis, V. Prakash, Exact transfer and stiffness matrices for a beam/column resting on a two-parameter foundation, Computer Methods in Applied Mechanics and Engineering 72 (1) (1989) 77–89.

[18] A.G. Razaqpur, K.R. Shah, Exact analysis of beams on two-parameter elastic foundations, International Journal of Solids and Structures 27 (4) (1991) 435–454.

[19] K. Morfidis, I.E. Avramidis, Formulation of a generalized beam element on a two-parameter elastic foundation with semi-rigid connections and rigid offsets, Computers & Structures 80 (25) (2002) 1919–1934.

[20] I.E. Avramidis, K. Morfidis, Bending of beams on three-parameter elastic foundation, International Journal of Solids and Structures 43 (2) (2006) 357–375.

[21] K. Morfidis, Exact matrices for beams on three-parameter elastic foundation, Computers & Structures 85 (15) (2007) 1243–1256.

[22] D. Dinev, Analytical solution of beam on elastic foundation by singularity functions, Engineering Mechanics 19 (6) (2012) 381–392.

[23] E. Winkler, Die Lehre yon der Elastizitat und Festigkeit, 141, H. Dominicus, Prague, 1867, pp. 182–184.

[24] P. Teodorescu, W.W. Kecs, A. Toma, Distribution Theory with Applications in Engineering and Physics, John Wiley & Sons, 2013.

[25] R. Piziak, P.L. Odell, Matrix theory: From generalized inverses to Jordan form, Vol. 288, CRC Press, 2007.

[26] S.H. Weintraub, Jordan canonical form: theory and practice, Synthesis Lectures on Mathematics and Statistics 2 (1) (2009) 1–108.

[27] Å. Björck, Numerical Methods in Matrix Computations, Springer, 2015.

[28] S. Blair-Chappell, A. Stokes, Parallel Programming with Intel Parallel Studio XE, John Wiley & Sons, 2012.

[29] M. Metcalf, J. Reid, M. Cohen, Modern FORTRAN Explained, Oxford University Press. Inc., 2011.

[30] E. Madenci, I. Guven, The Finite Element Method and Applications in Engineering using ANSYS®, Springer, 2015.