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Group decision making based on incomplete multiplicative and fuzzy preference relations

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ABSTRACT

The main aim of this paper is to investigate the group decision making on incomplete multiplicative and fuzzy preference relations without the requirement of satisfying reciprocity property. This paper introduces a new characterization of the multiplicative consistency condition, based on which a method to estimate unknown preference values in an incomplete multiplicative preference relation is proposed. Apart from the multiplicative consistency property among three known preference values, the method proposed also takes the multiplicative consistency property among more than three values into account. In addition, two models for group decision making with incomplete multiplicative preference relations and incomplete fuzzy preference relations are presented, respectively. Some properties of the collective preference relation are further discussed. Numerical examples are provided to make a discussion and comparison with other similar methods.

1. Introduction

Preference relations (PRs) are commonly used methods to express the preference information of decision makers (DMs) in decision making. In the past decades, numerous studies on this issue have been performed and various types of PRs [11,15,16,33,40–42] have been introduced in succession, among which multiplicative preference relations (MPRs) and fuzzy preference relations (FPRs) received much research attention [8–10,12–14,21,22,26,32,34–38,44,45,52–57]. In MPRs and FPRs based decision making process, DMs should provide their preferences by means of evaluations over each pair of alternatives and construct the corresponding judgement matrices.

For constructing such judgement matrices, the DMs have to give a preference degree of one alternative over another when comparing each pair of alternatives and $n(n-1)$ times of judgements are required if a complete PR with n alternatives is constructed. However, due to the complexity and uncertainty of real world, time pressure or not possessing a sufficient level of knowledge of part of the problem [1,2,46–48], DMs sometimes may have difficulty providing complete judgments. As a result, providing incomplete PRs with some missing or unknown preference values become a realistic choice to express DMs' preferences. Up to now, a series of models and methods with incomplete PRs have been developed, specially on how to estimate the unknown preference values and obtain the priority vectors. For example, Alonso et al. [3] proposed a procedure for finding out the missing information in an expert's incomplete FPR based on additive consistency. Herrera-Viedma et al. [20] proposed an iterative procedure to estimate the missing information in an expert's incomplete FPR based on additive consistency property. Alonso et al. [1] put forward a general procedure for estimating the missing information of incomplete PRs with several formats. Herrera-Viedma et al. [19] presented a characterization of the consistency property defined by the additive or multiplicative transitivity property of the FPRs. Lee [23] proposed a method for estimating unknown preference values based on the additive consistency. Chen et al. [6] discussed the drawbacks of Lee's method and presented an improved method for group decision making (GDM) using incomplete FPRs. Xu [49] defined the concepts of incomplete FPRs, additive consistent incomplete FPRs and multiplicative consistent incomplete FPRs, and then proposed two goal programming models for obtaining the priority vectors for incomplete FPRs. Gong [18] developed a least-square model for obtaining the collective priority vectors for incomplete PRs. Xu and Chen [50] developed a simple method for deriving the ranking of the alternatives from an incomplete reciprocal relation based on additive transitivity. Xu et al. [43] gave a definition of multiplicative consistent for incomplete FPRs and extended the logarithmic least squares method for deriving priorities from group incomplete FPRs. Liu et al. [24] proposed a method for determining the priority weights of FPRs, and presented a least square completion and inconsistency repair methods for dealing with incomplete and inconsistent FPRs.

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Most of the existing literature about incomplete PRs has two prominent characteristics: (1) Many studies are based on the basic assumption that the PRs satisfy reciprocity properties, that is, FPRs and MPRs are additive and multiplicative reciprocal, respectively. (2) The estimating of unknown values of incomplete PRs is mainly based on an iterative procedure proposed by Alonso et al. [1], where three cases of the multiplicative or additive consistency property among three elements of incomplete PRs under are considered.

However, in real decision making situations, it is not uncommon that preferences provided by DMs do not fully comply with any transitivity or even reciprocity properties [4,5]. And the existing models and methods on reciprocal FPRs and MPRs are not suitable for solving decision making problems with FPRs and MPRs not satisfying reciprocity property. For example, given a MPR not satisfying reciprocity property, the weight vector derived by geometric mean method denoted by (12) can not preserve the original information of the given MPR as much as possible. In addition, more than three cases of the consistency property among elements of incomplete PRs can be applied to estimate its unknown values. Based on these considerations, this paper develops new decision making models with incomplete MPRs and FPRs without the requirement of satisfying reciprocity property. The idea of the method reflects as follows: a procedure to determine unknown preference values in incomplete PRs not satisfying reciprocity property is developed based on more comprehensive multiplicative consistency given by Proposition 3.1 and the corresponding method to obtain the weight vector is presented, which preserves the information of the complete PR as much as possible. As a result, fewer times of iteration is needed to estimate the unknown values if the incomplete PRs are acceptable [51], i.e., all of the unknown preference values can be estimated, and the corresponding complete PRs derived may be of higher consistency level.

The rest of this paper is structured as follows. In Section 2, a brief introduction to the basic notions is provided. Section 3 describes some Propositions on complete MPRs, based on which a method for estimating unknown preference values in an incomplete PR is proposed. In Section 4, two models for GDM with incomplete MPRs and FPRs are presented, respectively. In Section 5, the comparison with other similar methods is provided. Section 6 gives the conclusions.

2. Preliminaries

For simplicity, let $X = (x_1, x_2, \ldots, x_n)$ be a finite set of alternatives and $I = \{1, 2, \ldots, n\}$ be the set of index [7,17,30].

Definition 1. A FPR $P = (p_{ij})_{n\times n}$ on the set X is a fuzzy set on the product set $X \times X$, which is characterized by a membership function $\mu_P : X \times X \to [0, 1]$, where p_{ij} denotes the preference degree of the alternative x_i over x_j, $p_{ii} = 0.5$, $1 \le i \le n$. Specially, $p_{ij} = 0$ indicates that x_j is absolutely preferred to x_i; $p_{ij} = 0.5$ indicates indifference between x_i and x_j; $p_{ij} > 0.5$ indicates that x_i is preferred to x_j; $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j.

P is called an additive reciprocal FPR if the following condition is satisfied [31]:

$$p_{ij} + p_{ji} = 1,\ \forall \mathrm{i}, \mathrm{j} \in \mathrm{I}. \tag{1}$$

Definition 2. An additive reciprocal FPR $P = (p_{ij})_{n\times n}$ is additively consistent, if the following additive transitivity is satisfied

$$p_{ij} = p_{ik} - p_{jk} + 0.5,\ \forall \mathrm{i}, \mathrm{j}, \mathrm{k} \in \mathrm{I}. \tag{2}$$

And $P = (p_{ij})_{n\times n}$ is multiplicatively consistent, if the following multiplicative transitivity is satisfied

$$\frac{p_{ji}p_{kj}}{p_{ij}p_{jk}} = \frac{p_{ki}}{p_{ik}},\ \forall \mathrm{i}, \mathrm{j}, \mathrm{k} \in \mathrm{I}. \tag{3}$$

Note that Eq. (3) is given based on the assumptions: $p_{ij} \ne 0$ and $p_{ij} \ne 1, \forall i, j \in I$ [27,28].

Definition 3. A MPR A on a set of alternatives X is represented by a matrix $A \subset X \times X$, $A = (a_{ij})_{n\times n}$, where a_{ij} is the preference ratio of alternative x_i over x_j, $a_{ij} > 0$, $a_{ii} = 1$, $\forall i, j \in I$. Specially, $a_{ij} < 1$ indicates that x_j is preferred to x_i; $a_{ij} = 1$ indicates indifference between x_i and x_j; $a_{ij} > 1$ indicates that x_i is preferred to x_j.

A is called a reciprocal MPR if the following condition is satisfied [29]:

$$a_{ij}a_{ji} = 1,\ \forall \mathrm{i}, \mathrm{j} \in \mathrm{I}. \tag{4}$$

Definition 4. A reciprocal MPR $A = (a_{ij})_{n\times n}$ is called a consistent MPR, if the following multiplicative transitivity is satisfied

$$a_{ij} = a_{ik}a_{kj},\ \forall \mathrm{i}, \mathrm{j}, \mathrm{k} \in \mathrm{I}. \tag{5}$$

Herrera-Viedma et al. [19] gave the following characterization of multiplicative consistency.

Proposition 2.1. For a reciprocal MPR $A = (a_{ij})_{n\times n}$, the following statements are equivalent:

(i) $a_{ij}a_{jk} = a_{ik}\ \forall i, j, k$.
(ii) $a_{ij}a_{jk} = a_{ik}\ \forall i < j < k$.
(iii) $a_{ij} = a_{ii+1}a_{i+1i+2}\ldots a_{j-1j}\ \forall i < j$.

3. Determine unknown preference values in incomplete PRs

In this section, some Propositions on complete MPR are firstly introduced.

Note that a consistent MPR $A = (a_{ij})_{n\times n}$ can be precisely characterized by a weight vector $w = (w_1, w_2, \ldots, w_n)$ $(w_i > 0, \sum_{i=1}^{n} w_i = 1)$ such that $a_{ij} = \frac{w_i}{w_j}$, i.e., $\log a_{ij} = \log\frac{w_i}{w_j}$. Accordingly, a reasonable weight vector of an inconsistent MPR $A = (a_{ij})_{n\times n}$ is supposed to have this characterization as far as possible, i.e., the weight vector obtained should preserve the original information in $A = (a_{ij})_{n\times n}$ as much as possible. Then, given a complete MPR $A = (a_{ij})_{n\times n}$, its weight vector $w = (w_1, w_2, \ldots, w_n)$ can be obtained by the following optimization model:

$$\min D(A) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\log a_{ij} - \log\frac{w_i}{w_j}\right)^2 \tag{6}$$

$$\text{s.t. } w_i > 0, i \in I$$

$$\sum_{i=1}^{n} w_i = 1 .$$

Proposition 3.1. If MPR $A = (a_{ij})_{n\times n}$ is complete, then the optimal solution of (6) is

$$w_i = \frac{\prod_{j=1}^{n}\left(a_{ij}/a_{ji}\right)^{1/2n}}{\sum_{i=1}^{n}\prod_{j=1}^{n}\left(a_{ij}/a_{ji}\right)^{1/2n}}. \tag{7}$$

Proof. To determine the weight vector of $A = (a_{ij})_{n\times n}$, we construct the following function

$$f(w) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\log a_{ij} - \log\frac{w_i}{w_j}\right)^2$$

By differentiating $f(w)$ with respect to w_i, and setting the partial derivative equal to zero, we derive the following equation:

$$\frac{\partial f(w)}{\partial w_i} = \sum_{j=1}^{n}\left(loga_{ij} - logw_i + logw_j\right)\frac{-1}{w_i}$$

$$+\sum_{j=1}^{n}\left(loga_{ji} - logw_j + logw_i\right)\frac{1}{w_i} = 0, \text{ i.e.,}$$

$$logw_i = \frac{1}{n}\sum_{j=1}^{n} logw_j - \frac{1}{2n}\sum_{j=1}^{n} log\frac{a_{ji}}{a_{ij}}. \quad (8)$$

Similarly, we can also derive

$$logw_k = \frac{1}{n}\sum_{j=1}^{n} logw_j - \frac{1}{2n}\sum_{j=1}^{n} log\frac{a_{jk}}{a_{kj}}. \quad (9)$$

According to (8) and (9), we have

$$logw_i - logw_k = \frac{1}{2n}\sum_{j=1}^{n} log\frac{a_{jk}}{a_{kj}} - \frac{1}{2n}\sum_{j=1}^{n} log\frac{a_{ji}}{a_{ij}} = \frac{1}{2n}\sum_{j=1}^{n} log\frac{a_{ij}a_{jk}}{a_{kj}a_{ji}},$$

which can be rewritten as

$$w_i = w_k\frac{\prod_{j=1}^{n}\left(a_{ij}/a_{ji}\right)^{1/2n}}{\prod_{j=1}^{n}\left(a_{kj}/a_{jk}\right)^{1/2n}}, \text{ i} \in \text{I}. \quad (10)$$

Summing both sides of Eq. (10) with respect to i, we derive

$$\sum_{i=1}^{n} w_i = 1 = w_k\frac{\sum_{i=1}^{n}\prod_{j=1}^{n}\left(a_{ij}/a_{ji}\right)^{1/2n}}{\prod_{j=1}^{n}\left(a_{kj}/a_{jk}\right)^{1/2n}}, \text{ i.e.,}$$

$$w_k = \frac{\prod_{j=1}^{n}\left(a_{kj}/a_{jk}\right)^{1/2n}}{\sum_{i=1}^{n}\prod_{j=1}^{n}\left(a_{ij}/a_{ji}\right)^{1/2n}}. \quad (11)$$

This completes the proof of Proposition 3.1.□

It is easy to find that the weigh vector derived by this paper has great difference from those derived by Fedrizzi and Brunelli [15], Wang and Fan [35], which are obtained from fuzzy preference relation and supposed to satisfy reciprocity property. (10) is derived from multiplicative preference relation and there is no requirement of reciprocity.

Specially, if $A = (a_{ij})_{n\times n}$ is a reciprocal MPR, then (11) equals to the geometric mean method [12]:

$$w_i = \frac{\prod_{j=1}^{n}\left(a_{ij}\right)^{1/n}}{\sum_{i=1}^{n}\prod_{j=1}^{n}\left(a_{ij}\right)^{1/n}}. \quad (12)$$

Based on Proposition 2.1 [19], we gave the following characterization of multiplicative consistency.

Proposition 3.2. For a reciprocal MPR $A = (a_{ij})_{n\times n}$, the following statements are equivalent:

(i) $a_{ij}a_{jk} = a_{ik}\ \forall i, j, k.$
(ii) $a_{ij} = a_{ik_1}a_{k_1k_2}\ldots a_{k_tj}\ \forall i, j, k_l \in I, l = 1, 2, \ldots, t.$

Proof. (i) ⇒ (ii). The mathematical induction is used to prove this part of the proposition.

Let $t = 1$, then by (i), we have $a_{ik_1}a_{k_1j} = a_{ij}$.

In addition, if the hypothesis is true for $t = m$, i.e., $a_{ij} = a_{ik_1}a_{k_1k_2}\ldots a_{k_mj}$, then, it is true for $t = m + 1$:

$$a_{ik_1}a_{k_1k_2}\ldots a_{k_mk_{m+1}}a_{k_{m+1}j} = \left(a_{ik_1}a_{k_1k_2}\ldots a_{k_mk_{m+1}}\right)a_{k_{m+1}j}$$
$$= a_{ik_{m+1}}a_{k_{m+1}j} = a_{ij}.$$

(ii) ⇒ (i). In the case $t = 1$, (ii) reduces to (i), and therefore (i) is true.

This completes the proof of Proposition 3.2.□

Specially, if $k_1 = i + 1, k_2 = i + 2, \ldots, k_t = i + t = j - 1$, then the condition (ii) in Proposition 3.2 is equivalent to the condition (iii) in Proposition 2.1.

3.1. Determine unknown preference values in incomplete MPRs

Unless otherwise specified, in what follows, we suppose that all of the incomplete PRs are acceptable [51], i.e., all of the unknown preference values can be estimated. Based on multiplicative consistency property, we develop a procedure to determine unknown preference values in an incomplete MPR. Suppose some sets are defined as follows:

$$C = \left\{(i, j)|i, j \in I \wedge i \neq j\right\},$$

$$MV = \left\{(i, j) \in C|a_{ij} \text{is unknown}\right\},$$

$$EV = C\backslash MV,$$

where MV is the set of pairs of alternatives for unknown preference values, EV is the set of pairs of alternatives for preference values provided by DMs and the symbol "\" denotes the exclusion. Based on this, the proposed Algorithm I is presented as follows.

Step 1. If $(i, j) \in MV$ and $(j, i) \in EV$, then the unknown preference value a_{ij} is derived by

$$a_{ij} = 1/a_{ji}. \quad (13)$$

Accordingly, the corresponding sets are updated as follows:

$$MV = MV\backslash(i, j),$$

$$EV = EV \cup (i, j).$$

Step 2. If $(i, j), (j, i) \in MV$, then the unknown preference value a_{ij} is derived by some other known preference values as follows:

$$H_{ij} = \left\{(k_1, k_2, \ldots, k_t)|(i, k_1) \in EV, (k_1, k_2) \in EV, \ldots, (k_t, j) \in EV \text{ and } (i, j) \in MV\right\},$$

$$a_{ij} = \prod_{(k_1,k_2,\ldots,k_t)\in H_{ij}}\left(a_{ik_1}a_{k_1k_2}\ldots a_{k_tj}\right)^{1/\#H_{ij}}, \quad (14)$$

$$a_{ji} = 1/a_{ij}, \quad (15)$$

where $\#H_{ji}$ denotes the cardinality of the set H_{ij}.

Step 3. Construct a MPR $\overline{A} = (\overline{a}_{ij})_{n\times n}$, where

$$\overline{a}_{ij} = \prod_{l=1}^{n}\left(\frac{a_{il}a_{lj}}{a_{li}a_{jl}}\right)^{1/2n}. \quad (16)$$

Proposition 3.3. The MPR $\overline{A}=(\overline{a}_{ij})_{n\times n}$ constructed in Step 3 of Algorithm I is a consistent MPR.

Proof. Based on (16), we have

$$\overline{a}_{ij}\overline{a}_{jk}=\prod_{l=1}^{n}\left(\frac{a_{il}a_{lj}}{a_{li}a_{jl}}\right)^{1/2n}\prod_{l=1}^{n}\left(\frac{a_{jl}a_{lk}}{a_{lj}a_{kl}}\right)^{1/2n}=\prod_{l=1}^{n}\left(\frac{a_{il}a_{lj}}{a_{li}a_{jl}}\frac{a_{jl}a_{lk}}{a_{lj}a_{kl}}\right)^{1/2n}$$
$$=\prod_{l=1}^{n}\left(\frac{a_{il}a_{lk}}{a_{li}a_{kl}}\right)^{1/2n}=\overline{a}_{ik},$$

$$\overline{a}_{ij}\overline{a}_{ji}=\prod_{l=1}^{n}\left(\frac{a_{il}a_{lj}}{a_{li}a_{jl}}\right)^{1/2n}\prod_{l=1}^{n}\left(\frac{a_{jl}a_{li}}{a_{lj}a_{il}}\right)^{1/2n}=\prod_{l=1}^{n}\left(\frac{a_{il}a_{lj}}{a_{li}a_{jl}}\frac{a_{jl}a_{li}}{a_{lj}a_{il}}\right)^{1/2n}=$$

1 and $\overline{a}_{ii}=\prod_{l=1}^{n}\left(\frac{a_{il}a_{li}}{a_{li}a_{il}}\right)^{1/2n}=1$, which means $\overline{A}=(\overline{a}_{ij})_{n\times n}$ is a consistent MPR.

This completes the proof of Proposition 3.3□

Taking Proposition 3.1 and Proposition 3.3 together, we conclude that $\overline{A}=(\overline{a}_{ij})_{n\times n}$is not only a consistent MPR but also preserves the information of $A=(a_{ij})_{n\times n}$as much as possible.

Proposition 3.4. Let $\overline{w}=(\overline{w}_1,\overline{w}_2,\ldots,\overline{w}_n)$ be the priority vector of $\overline{A}=(\overline{a}_{ij})_{n\times n}$ constructed in Step 3 of Algorithm I, and $w=(w_1,w_2,\ldots,w_n)$ be the priority vector of the complete MPR $A=(a_{ij})_{n\times n}$ derived in Step 2 of Algorithm I, then $\overline{w}=w$.

Proof. Since $\overline{A}=(\overline{a}_{ij})_{n\times n}$ is a consistent MPR, then by (12), we have

$$\overline{w}_i=\frac{\prod_{j=1}^{n}(\overline{a}_{ij})^{1/n}}{\sum_{i=1}^{n}\prod_{j=1}^{n}(\overline{a}_{ij})^{1/n}}.$$

Thus, we have

$$\frac{\overline{w}_i}{\overline{w}_k}=\frac{\prod_{j=1}^{n}(\overline{a}_{ij})^{1/n}}{\prod_{j=1}^{n}(\overline{a}_{kj})^{1/n}}=\frac{\prod_{j=1}^{n}\left(\prod_{l=1}^{n}\left(\frac{a_{il}a_{lj}}{a_{li}a_{jl}}\right)^{1/2n}\right)^{1/n}}{\prod_{j=1}^{n}\left(\prod_{l=1}^{n}\left(\frac{a_{kl}a_{lj}}{a_{lk}a_{jl}}\right)^{1/2n}\right)^{1/n}}$$
$$=\prod_{j=1}^{n}\left(\prod_{l=1}^{n}\left(\frac{a_{il}a_{lk}}{a_{li}a_{kl}}\right)^{1/2n}\right)^{1/n}$$
$$=\left(\left(\prod_{l=1}^{n}\left(\frac{a_{il}a_{lk}}{a_{li}a_{kl}}\right)^{1/2n}\right)^{1/n}\right)^{n}=\prod_{l=1}^{n}\left(\frac{a_{il}a_{lk}}{a_{li}a_{kl}}\right)^{1/2n}.$$

That is,

$$\overline{w}_i=\overline{w}_k\frac{\prod_{l=1}^{n}(a_{il}/a_{li})^{1/2n}}{\prod_{l=1}^{n}(a_{kl}/a_{lk})^{1/2n}},\ \mathrm{i}\in\mathrm{I}. \tag{17}$$

Summing both sides of Eq. (17) with respect to i, we derive

$$\sum_{i=1}^{n}\overline{w}_i=1=\overline{w}_k\frac{\sum_{i=1}^{n}\prod_{l=1}^{n}(a_{il}/a_{li})^{1/2n}}{\prod_{l=1}^{n}(a_{kl}/a_{lk})^{1/2n}},\ \text{i.e.,}$$

$$\overline{w}_k=\frac{\prod_{l=1}^{n}(a_{kl}/a_{lk})^{1/2n}}{\sum_{i=1}^{n}\prod_{l=1}^{n}(a_{il}/a_{li})^{1/2n}}.$$

According to (11), we have

$\overline{w}_k=w_k$, $\mathrm{k}\in\mathrm{I}$.

This completes the proof of Proposition 3.4.□

Proposition 3.4 indicates that the transformation process in Step 3 of Algorithm I doesn't change the weight vector of the PR $A=(a_{ij})_{n\times n}$, which also indicates $\overline{A}=(\overline{a}_{ij})_{n\times n}$ can preserves the information of $A=(a_{ij})_{n\times n}$ as much as possible in the transformation process.

Example 1. Let $A=(a_{ij})_{4\times 4}$ be an incomplete MPR as follows:

$$A=\begin{bmatrix}1 & 3 & 5 & 4\\ 1/2 & 1 & x & x\\ x & x & 1 & x\\ 1/4 & x & x & 1\end{bmatrix}.$$

By using Algorithm I, we derive the following complete MPR and its corresponding consistent MPR:

$$A=\begin{bmatrix}1 & 3 & 5 & 4\\ 1/2 & 1 & 5/2 & 2\\ 1/5 & 3/5 & 1 & 4/5\\ 1/4 & 3/4 & 5/4 & 1\end{bmatrix},$$
$$\overline{A}=\begin{bmatrix}1 & \sqrt{6} & 5 & 4\\ \sqrt{6}/6 & 1 & 5\sqrt{6}/6 & 2\sqrt{6}/3\\ 1/5 & \sqrt{6}/5 & 1 & 4/5\\ 1/4 & \sqrt{6}/4 & 5/4 & 1\end{bmatrix}.$$

Wu and Xu [39] proposed the following equation to construct a consistent MPR $G=(g_{ij})_{n\times n}$:

$$g_{ij}=\prod_{l=1}^{n}(a_{il}a_{lj})^{1/n}. \tag{18}$$

If we take (18), we derive the following PR:

$$G=\begin{bmatrix}1 & 3 & 5.5334 & 4.4267\\ 0.5 & 1 & 2.6864 & 2\\ 0.2213 & 0.6 & 1 & 0.8853\\ 0.2767 & 0.3164 & 1.3834 & 1\end{bmatrix}.$$

It is easy to find G is not a consistent MPR. In fact, if A is a reciprocal MPR, that is, $a_{ij}a_{ji}=1,\forall i,j\in I$, then (16) is equivalent to (18). In such a case, we can derive the same consistent MPR by either (16) or (18).

3.2. Determine unknown preference values in incomplete FPRs

In what follows, based on transformation function between MPRs and FPRs [7], we develop another procedure to determine unknown preference values in an incomplete FPR. Suppose some sets are defined as follows:

$$C = \left\{(i,j)|i,j \in I \wedge i \neq j\right\},$$

$$MV = \left\{(i,j) \in C|p_{ij} \text{is unknown}\right\},$$

$$EV = C\backslash MV,$$

where MV is the set of pairs of alternatives for unknown preference values, EV is the set of pairs of alternatives for preference values provided by DMs and the symbol "\" denotes the exclusion. Based on this, the proposed Algorithm II to determine unknown preference values is presented as follows.

Step 1. Based on the transformation function proposed by Chiclana et al. [7], transform the incomplete FPRs $P = (p_{ij})_{n\times n}$ into the corresponding incomplete MPRs $A = (a_{ij})_{n\times n}$ as follows:

$$a_{ij} = p_{ij}/(1-p_{ij}),\ a_{ji} = p_{ji}/(1-p_{ji}),\ \text{(i, j), (j, i)} \in \text{EV},$$

$$a_{kl} = p_{kl}/(1-p_{kl}),\ \text{(k, l)} \in \text{EV, (l, k)} \in \text{MV},$$

$$a_{lk} = (1-p_{kl})/p_{kl},\ \text{(k, l)} \in \text{EV, (l, k)} \in \text{MV}.$$

Accordingly, the corresponding sets are updated as follows:

$$MV = MV\backslash(l,k),$$

$$EV = EV \cup (l,k)$$

Step 2. Refer to Step 2 of Algorithm I.

Step 3. Refer to Step 3 of Algorithm I.

Step 4. Transform the MPR $\overline{A} = (\overline{a}_{ij})_{n\times n}$ into the corresponding FPR $\overline{P} = (\overline{p}_{ij})_{n\times n}$ as follows:

$$\overline{p}_{ij} = \overline{a}_{ij}/\left(\overline{a}_{ij}+1\right). \tag{19}$$

Property 3.5. The FPR $\overline{P} = (\overline{p}_{ij})_{n\times n}$ obtained in Step 4 of Algorithm II is multiplicatively consistent.

Proof. Based on (16), we have $\overline{p}_{ii} = \overline{a}_{ii}/(\overline{a}_{ii}+1) = 0.5$,

$$\overline{p}_{ij} + \overline{p}_{ji} = \overline{a}_{ij}/\left(\overline{a}_{ij}+1\right) + \overline{a}_{ji}/\left(\overline{a}_{ji}+1\right) = \overline{a}_{ij}/\left(\overline{a}_{ij}+1\right) + \left(1/\overline{a}_{ij}\right)/\left(1/\overline{a}_{ij}+1\right) = 1.$$

and

$$\begin{aligned}\frac{\overline{p}_{ji}\overline{p}_{kj}}{\overline{p}_{ij}\overline{p}_{jk}} &= \frac{\overline{a}_{ji}/\left(\overline{a}_{ji}+1\right)}{\overline{a}_{ij}/\left(\overline{a}_{ij}+1\right)}\frac{\overline{a}_{kj}/\left(\overline{a}_{kj}+1\right)}{\overline{a}_{jk}/\left(\overline{a}_{jk}+1\right)} = \frac{\overline{a}_{kj}\overline{a}_{ji}}{\overline{a}_{ij}\overline{a}_{jk}}\frac{\left(\overline{a}_{ij}+1\right)\left(\overline{a}_{jk}+1\right)}{\left(\overline{a}_{ji}+1\right)\left(\overline{a}_{kj}+1\right)}\\ &= \frac{\overline{a}_{ki}}{\overline{a}_{ik}}\frac{\overline{a}_{ij}\left(1+\overline{a}_{ji}\right)\overline{a}_{jk}\left(1+\overline{a}_{kj}\right)}{\left(\overline{a}_{ji}+1\right)\left(\overline{a}_{kj}+1\right)}\\ &= \frac{\overline{a}_{ki}}{\overline{a}_{ik}}\overline{a}_{ij}\overline{a}_{jk} = \frac{\overline{a}_{ki}}{\overline{a}_{ik}}\overline{a}_{ik} = \frac{\overline{a}_{ki}}{\overline{a}_{ik}}\frac{\overline{a}_{ik}(\overline{a}_{ki}+1)}{(\overline{a}_{ki}+1)} = \frac{\overline{a}_{ki}}{\overline{a}_{ik}}\frac{(\overline{a}_{ik}+1)}{(\overline{a}_{ki}+1)}\\ &= \frac{\overline{a}_{ki}/(\overline{a}_{ki}+1)}{\overline{a}_{ik}/(\overline{a}_{ik}+1)} = \frac{\overline{p}_{ki}}{\overline{p}_{ik}},\end{aligned}$$

which means $\overline{P} = (\overline{p}_{ij})_{n\times n}$ is multiplicatively consistent.

This completes the proof of Proposition 3.5.□

4. The models for GDM with incomplete PRs

In this section, two methods for GDM with incomplete MPRs and FPRs are presented, respectively. Let $d = (d_1, d_2, \ldots, d_t)$ be the set of DMs, whose weight vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_t)$, with $\omega_r \in (0, 1)$ and $\sum_{r=1}^{t}\omega_r = 1$.

Suppose each DM d_r provides his/her preference values by incomplete MPR $A^r = (a^r_{ij})_{n\times n}$. Then, the Algorithm III for GDM method with incomplete MPRs is proposed as follows:

Step 1. For each incomplete MPR $A^r = (a^r_{ij})_{n\times n}$, define the following sets:

$$C = \left\{(i,j)|i,j \in I \wedge i \neq j\right\},$$

$$MV^r = \left\{(i,j) \in C|a^r_{ij} \text{is unknown}\right\},$$

$$EV^r = C\backslash MV^r,$$

where MV^r is the set of pairs of alternatives for unknown preference values, EV^r is the set of pairs of alternatives for known preference values provided by DM d_r and the symbol "\" denotes the exclusion.

Step 2. If $(i,j) \in MV^r$ and $(j,i) \in EV^r$, then the unknown preference value a^r_{ij} is derived by

$$a^r_{ij} = 1/a^r_{ji}. \tag{20}$$

Accordingly, the corresponding sets are updated as follows:
$MV^r = MV^r\backslash(i,j)$,
$EV^r = EV^r \cup (i,j)$..

Step 3. For $(i,j), (j,i) \in MV^r$, determine the unknown preference value a^r_{ij} by other known preference values as follows:

$$H^r_{ij} = \left\{(k_1,k_2,\ldots,k_t)|(i,k_1) \in EV^r,(k_1,k_2) \in EV^r,\ldots,(k_t,j) \in EV^r \text{and} (i,j) \in MV^r\right\},$$

$$a^r_{ij} = \prod_{(k_1,k_2,\ldots,k_t)\in H^r_{ij}}\left(a^r_{ik_1}a^r_{k_1k_2}\ldots a^r_{k_tj}\right)^{1/\#H^r_{ij}}, \tag{21}$$

$$a^r_{ji} = 1/a^r_{ij}, \tag{22}$$

where $\#H^r_{ji}$ denotes the cardinality of the set H^r_{ij}.

Step 4. Construct a MPR $\overline{A}^r = (\overline{a}^r_{ij})_{n\times n}$, where

$$\overline{a}^r_{ij} = \prod_{l=1}^{n}\left(\frac{a^r_{il}a^r_{lj}}{a^r_{li}a^r_{jl}}\right)^{1/2n}. \tag{23}$$

Step 5. Derive the collective PR $\overline{A}^c = (\overline{a}^c_{ij})_{n\times n}$ for all $\overline{A}^r$, r = 1, 2,. . .,t, where

$$\overline{a}^c_{ij} = \prod_{r=1}^{t}\left(\overline{a}^r_{ij}\right)^{\omega_r}. \tag{24}$$

Step 6. Obtain the priority vector for $\overline{A}^c = (\overline{a}^c_{ij})_{n\times n}$ as follows:

$$w_i = \prod_{j=1}^{n}\left(\overline{a}^c_{ij}\right)^{1/n}/\sum_{i=1}^{n}\prod_{j=1}^{n}\left(\overline{a}^c_{ij}\right)^{1/n}. \tag{25}$$

Property 4.1. The MPR $\overline{A}^c = (\overline{a}^c_{ij})_{n\times n}$ derived in Step 5 of Algorithm III is a consistent MPR.

Proof. According to Proposition 3.3, it follows that $\overline{A}^r = (\overline{a}^r_{ij})_{n\times n}$ $(r = 1, 2, \ldots, t)$ is consistent MPR, that is, $\overline{a}^r_{ij}\overline{a}^r_{jk} = \overline{a}^r_{ik}, \forall i, j \in I$.

Then, we have $\overline{a}^c_{ij}\overline{a}^c_{jk} = \prod_{r=1}^{t}\left(\overline{a}^r_{ij}\right)^{\omega_r}\prod_{r=1}^{t}\left(\overline{a}^r_{jk}\right)^{\omega_r} = \prod_{r=1}^{t}\left(\overline{a}^r_{ij}\overline{a}^r_{jk}\right)^{\omega_r} = \prod_{r=1}^{t}\left(\overline{a}^r_{ik}\right)^{\omega_r} = \overline{a}^c_{ik}$, $\overline{a}^c_{ij}\overline{a}^c_{ji} = \prod_{r=1}^{t}\left(\overline{a}^r_{ij}\right)^{\omega_r}\prod_{r=1}^{t}\left(\overline{a}^r_{ji}\right)^{\omega_r} = \prod_{r=1}^{t}\left(\overline{a}^r_{ij}\overline{a}^r_{ji}\right)^{\omega_r} = 1$, and $\overline{a}^c_{ii} = \prod_{r=1}^{t}\left(\overline{a}^r_{ii}\right)^{\omega_r} = 1$, which means $\overline{A}^c = (\overline{a}^c_{ij})_{n\times n}$ is a consistent MPR.

This completes the proof of Proposition 4.1.□

In Algorithm III, we firstly derive the consistent MPRs $\overline{A}^r (r = 1, 2, \ldots, t)$ from their corresponding complete MPRs A^r, and then obtain the collective PR $\overline{A}^c = (\overline{a}^c_{ij})_{n\times n}$ for all $\overline{A}^r$. In addition, we can take opposite steps, i.e., we firstly derive the collective PR $A^c = (a^c_{ij})_{n\times n}$ for all A^r, and then obtain the consistent MPR $\overline{A^c} = (\overline{a^c_{ij}})_{n\times n}$ of A^c. The following Property 4.2 proves that $\overline{A}^c = \overline{A^c}$.

Property 4.2. Let $A^c = (a^c_{ij})_{n\times n}$ be the collective PR of A^r $(r = 1, 2, \ldots, t)$ with positive weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_t)$, that is,

$$a^c_{ij} = \prod_{r=1}^{t}\left(a^r_{ij}\right)^{\omega_r}, \tag{26}$$

and $\overline{A^c} = (\overline{a^c_{ij}})_{n\times n}$ be the consistent MPR derived by

$$\overline{a^c_{ij}} = \prod_{l=1}^{n}\left(\frac{a^c_{il}a^c_{lj}}{a^c_{li}a^c_{jl}}\right)^{1/2n}, \tag{27}$$

then $\overline{A}^c = \overline{A^c}$.

Proof. According to (26) and (27), we have

$$\overline{a^c_{ij}} = \prod_{l=1}^{n}\left(\frac{\prod_{r=1}^{t}\left(a^r_{il}\right)^{\omega_r}\prod_{r=1}^{t}\left(a^r_{lj}\right)^{\omega_r}}{\prod_{r=1}^{t}\left(a^r_{li}\right)^{\omega_r}\prod_{r=1}^{t}\left(a^r_{jl}\right)^{\omega_r}}\right)^{1/2n} = \prod_{l=1}^{n}\left(\prod_{r=1}^{t}\left(\frac{a^r_{il}a^r_{lj}}{a^r_{li}a^r_{jl}}\right)^{\omega_r}\right)^{1/2n} = \prod_{r=1}^{t}\left(\prod_{l=1}^{n}\left(\frac{a^r_{il}a^r_{lj}}{a^r_{li}a^r_{jl}}\right)^{1/2n}\right)^{\omega_r}$$

$$= \prod_{r=1}^{t}\left(\prod_{l=1}^{n}\left(\frac{a^r_{il}a^r_{lj}}{a^r_{li}a^r_{jl}}\right)^{1/2n}\right)^{\omega_r} = \prod_{r=1}^{t}\left(\overline{a}^r_{ij}\right)^{\omega_r} = \overline{a}^c_{ij}.$$

This completes the proof of Proposition 4.2.□

Alternatively, if each DM d_r provides his/her preference values by incomplete FPR $P^r = (p^r_{ij})_{n\times n}$, the Algorithm IV for GDM method with incomplete FPRs is proposed as follows:

Step 1. For each incomplete FPR $P^r = (p^r_{ij})_{n\times n}$, define the following sets:

$$C = \left\{(i, j) | i, j \in I \wedge i \neq j\right\},$$

$$MV^r = \left\{(i, j) \in C | p^r_{ij} \text{ is unknown}\right\},$$

$$EV^r = C \backslash MV^r,$$

where MV^r is the set of pairs of alternatives for unknown preference values, EV^r is the set of pairs of alternatives for known preference values provided by DM d_r and the symbol "\" denotes the exclusion.

Step 2. Transform the incomplete FPRs $P^r = (p^r_{ij})_{n\times n}$ into the corresponding incomplete MPRs $A^r = (a^r_{ij})_{n\times n}$ as follows:

$$a^r_{ij} = p^r_{ij}/(1 - p^r_{ij}),\ a^r_{ji} = p^r_{ji}/(1 - p^r_{ji}),\ (\text{i, j}), (\text{j, i}) \in \text{EV}^r, \tag{28}$$

$$a^r_{kl} = p^r_{kl}/(1 - p^r_{kl}),\ (\text{k, l}) \in \text{EV}^r, (\text{l, k}) \in \text{MV}^r, \tag{29}$$

$$a^r_{lk} = (1 - p^r_{kl})/p^r_{kl},\ (\text{k, l}) \in \text{EV}^r, (\text{l, k}) \in \text{MV}^r. \tag{30}$$

Accordingly, the corresponding sets are updated as follows:

$$MV^r = MV^r \backslash (l, k),$$

$$EV^r = EV^r \cup (l, k)$$

Step 3. Refer to Step 3 of Algorithm III.

Step 4. Refer to Step 4 of Algorithm III.

Step 5. Refer to Step 5 of Algorithm III.

Step 6. Transform the collective MPR $\overline{A^c} = (\overline{a^c_{ij}})_{n\times n}$ into the corresponding FPR $\overline{P}^c = (\overline{p}^c_{ij})_{n\times n}$ as follows:

$$\overline{p}^c_{ij} = \overline{a^c_{ij}} \Big/ \left(\overline{a^c_{ij}} + 1\right). \tag{31}$$

Step 7. Obtain the positive priority vector for $\overline{P}^c = (\overline{p}^c_{ij})_{n\times n}$ as follows:

$$w_i = \left(\prod_{j=1}^{n}\frac{\overline{p}^c_{ij}}{\overline{p}^c_{ji}}\right)^{1/n} \Big/ \sum_{i=1}^{n}\left(\prod_{j=1}^{n}\frac{\overline{p}^c_{ij}}{\overline{p}^c_{ji}}\right)^{1/n}. \tag{32}$$

Property 4.3. The FPR $\overline{P}^c = (\overline{p}^c_{ij})_{n\times n}$ obtained in Step 6 of Algorithm IV is multiplicatively consistent.

Proof. According to Proposition 4.2, it follows that the collective PR $\overline{A}^c = (\overline{a}^c_{ij})_{n\times n}$ is a consistent MPR.

Based on (31), we have $\overline{p}^c_{ii} = \overline{a^c_{ii}} \Big/ \left(\overline{a^c_{ii}} + 1\right) = 0.5$, $\overline{p}^c_{ij} + \overline{p}^c_{ji} = \overline{a^c_{ij}} \Big/ \left(\overline{a^c_{ij}} + 1\right) + \overline{a^c_{ji}} \Big/ \left(\overline{a^c_{ji}} + 1\right) = 1$, and

$$\frac{\overline{p}^c_{ji}\overline{p}^c_{kj}}{\overline{p}^c_{ij}\overline{p}^c_{jk}} = \frac{\overline{a^c_{ji}} \Big/ \left(\overline{a^c_{ji}} + 1\right) \overline{a^c_{kj}} \Big/ \left(\overline{a^c_{kj}} + 1\right)}{\overline{a^c_{ij}} \Big/ \left(\overline{a^c_{ij}} + 1\right) \overline{a^c_{jk}} \Big/ \left(\overline{a^c_{jk}} + 1\right)} = \frac{\overline{a^c_{kj}}\,\overline{a^c_{ji}}}{\overline{a^c_{ij}}\,\overline{a^c_{ji}}}\frac{\left(\overline{a^c_{ij}} + 1\right)\left(\overline{a^c_{jk}} + 1\right)}{\left(\overline{a^c_{ji}} + 1\right)\left(\overline{a^c_{kj}} + 1\right)}$$

$$= \frac{\overline{a^c_{ki}}}{\overline{a^c_{ik}}}\frac{\overline{a^c_{ij}}\left(1 + \overline{a^c_{ji}}\right)\overline{a^c_{jk}}\left(1 + \overline{a^c_{kj}}\right)}{\left(\overline{a^c_{ji}} + 1\right)\left(\overline{a^c_{kj}} + 1\right)}$$

$$= \frac{\overline{a^c_{ki}}}{\overline{a^c_{ik}}}\overline{a^c_{ij}}\,\overline{a^c_{jk}} = \frac{\overline{a^c_{ki}}}{\overline{a^c_{ik}}}\overline{a^c_{ik}} = \frac{\overline{a^c_{ki}}}{\overline{a^c_{ik}}}\frac{\overline{a^c_{ik}}\left(\overline{a^c_{ki}} + 1\right)}{\left(\overline{a^c_{ki}} + 1\right)} = \frac{\overline{a^c_{ki}}}{\overline{a^c_{ik}}}\frac{\left(\overline{a^c_{ik}} + 1\right)}{\left(\overline{a^c_{ki}} + 1\right)}$$

$$= \frac{\overline{a^c_{ki}} \Big/ \left(\overline{a^c_{ki}} + 1\right)}{\overline{a^c_{ik}} \Big/ \left(\overline{a^c_{ik}} + 1\right)} = \frac{\overline{p}^c_{ki}}{\overline{p}^c_{ik}},$$

which mean $\overline{P}^c = (\overline{p}^c_{ij})_{n\times n}$ is multiplicatively consistent.

This completes the proof of Proposition 4.3.□

In what follows, we apply the GDM model proposed to solve one practical problem, which is adapted from literature [25].

Example 2. Suppose a distributed health expert team consists of three members:d_1, d_2 and d_3, who are responsible for collaboratively observing the SARS outbreak and epidemic for a region. At present, one of their main works is to construct the evaluation index system for epidemic situation. Each team member proposes some criteria and communicates with each other. Finally, four of them are chosen as criteria: x_1 (The number of infected), x_2 (The atypical presentations), x_3 (The epidemic time), x_4 (The number of death). Based on the criteria proposed, each member provides a multiplicative preference matrix of the relative importance of these criteria, shown as follows:

$$A^1=\begin{bmatrix}1&3&2&4\\x&1&3&x\\x&1/2&1&x\\x&x&x&1\end{bmatrix},\ A^2=\begin{bmatrix}1&2&x&x\\x&1&4&x\\x&x&1&1/2\\x&x&1&1\end{bmatrix},$$

$$A^3=\begin{bmatrix}1&1/2&x&x\\x&1&4&x\\x&x&1&1/2\\x&x&x&1\end{bmatrix}$$

Step 1. Define the corresponding sets for each MPR. Taking A^1 as an example, we have

$$MV^1=\left\{(2,1),(2,4),(3,1),(3,4),(4,1),(4,2),(4,3)\right\},$$

$$EV^1=\left\{(1,2),(1,3),(1,4),(2,3),(3,2)\right\}.$$

Step 2. For $(i,j)\in MV^r$ and $(j,i)\in EV^r$, by (20), we determine the unknown preference valuea_{ij}^r, and update the corresponding sets. For A^1, we have $a_{21}^1=1/3$,$a_{31}^1=1/2$, $a_{41}^1=1/4$, and

$$MV^1=\left\{(2,4),(3,4),(4,2),(4,3)\right\},$$

$$EV^1=\left\{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\right\}.$$

Step 3. For $(i,j),(j,i)\in MV^r$, by (21), we determine the unknown preference value a_{ij}^r by some other known preference values. Taking a_{24}^1 as an example, we have

$$H_{24}^1=\left\{(1),(3,1)\right\},$$

$$a_{24}^1=\left[\left(a_{21}^1a_{14}^1\right)\left(a_{23}^1a_{31}^1a_{14}^1\right)\right]^{1/2}=\left[\left(\frac{1}{3}\times 4\right)\left(3\times\frac{1}{2}\times 4\right)\right]^{1/2}$$
$$=2\sqrt{2}.$$

As a result, we can construct the following three complete MPRs:

$$A^1=\begin{bmatrix}1&3&2&4\\1/3&1&3&2\sqrt{2}\\1/2&1/2&1&2\sqrt{3}/3\\1/4&\sqrt{3}/4&3\sqrt{2}/4&1\end{bmatrix},\ A^2=\begin{bmatrix}1&2&8&4\\1/2&1&4&2\\1/8&1/4&1&1/2\\1/8&1/4&1&1\end{bmatrix},$$

$$A^3=\begin{bmatrix}1&1/2&2&1\\2&1&4&2\\1/2&1/4&1&1/2\\1&1/2&2&1\end{bmatrix}.$$

Step 4. Based on (23), we can derive the following consistent MPRs:

$$\overline{A}^1=\begin{bmatrix}1&1.8416&3.2581&4\\0.5430&1&1.7691&2.1720\\0.3069&0.5652&1&1.2276\\0.25&0.4604&0.8145&1\end{bmatrix},$$

$$\overline{A}^2=\begin{bmatrix}1&2&8&5.6569\\0.5&1&4&2.8285\\0.125&0.25&1&0.7071\\0.1768&0.3536&1.4144&1\end{bmatrix},$$

$$\overline{A}^3=\begin{bmatrix}1&1/2&2&1\\2&1&4&2\\1/2&1/4&1&1/2\\1&1/2&2&1\end{bmatrix}.$$

Step 5. Suppose the members have the same importance degree, that is, ω = (1/3, 1/3, 1/3), then we have the following collective PR by (24):

$$\overline{A}^c=\begin{bmatrix}1&1.2257&3.7356&2.8284\\0.8159&1&3.0479&2.3077\\0.2677&0.3281&1&0.7572\\0.3536&0.4334&1.3209&1\end{bmatrix}.$$

Step 6. According to (25), we obtain the priority vector of $\overline{A}^c$ as follows:

w = (0.4103, 0.3348, 0.1098, 0.1451), which means x_1 (The number of infected) is the most important criterion.

5. Discussion and comparison with other methods

Based on Proposition 2.1, Herrera-Viedma et al. [19] proposed a method to construct consistent MPRs from a set of n-1 preference values$\left\{a_{12},a_{23},\ldots,a_{n-1n}\right\}$, which facilitates experts the expression of consistent MPRs. Taking the incomplete MPR A^3 in Example 2 as an example, by Proposition 2.1, we can construct the following consistent MPR

$$A^3=\begin{bmatrix}1&1/2&2&1\\2&1&4&2\\1/2&1/4&1&1/2\\1&1/2&2&1\end{bmatrix},$$

which equals to the one obtained in Example 2.

As Proposition 3.2 illustrates the condition (ii) in Proposition 3.2 is equivalent to the condition (iii) in Proposition 2.1 if $k_1=i+1$, $k_2=i+2$,. . .,$k_t=i+t=j-1$. That is, Proposition 2.1 can be seen a special case of Proposition 3.2. For example, if DM provides the following MPR:

$$A^{3\prime}=\begin{bmatrix}1 & 1/2 & x & 1\\ x & 1 & 4 & x\\ x & x & 1 & x\\ x & x & x & 1\end{bmatrix},$$

then we can derive its corresponding complete consistent MPR not by Proposition 2.1 but by Proposition 3.2.

Example 3. Consider the following MPR A in Example 1 and MPR A^4 (also investigated by Alonso et al. [1]):

$$A=\begin{bmatrix}1 & 3 & 5 & 4\\ 1/2 & 1 & x & x\\ x & x & 1 & x\\ 1/4 & x & x & 1\end{bmatrix},\ A^4=\begin{bmatrix}1 & 0.80 & 1.55 & 1\\ 1.25 & 1 & x & 3.74\\ 0.65 & x & 1 & 1.93\\ 1 & 0.33 & 0.52 & 1\end{bmatrix}.$$

Alonso et al. [3] defined a consistency measure for FPRs based on the additive transitivity property for FPRs. Alonso et al. [1] extended this consistency measure to MPRs and proposed an iterative procedure for estimating missing values in incomplete MPRs based on the following equations (Please refer to [1] for more detail).

$$ca_{ik}^{j1}=a_{ij}a_{jk};\ Ca_{ik}^{j2}=a_{jk}/a_{ji};\ Ca_{ik}^{j3}=a_{ij}a_{jk} \tag{33}$$

$$ca_{ik}=\prod_l\left(\prod_j ca_{ik}^{jl}\right)^{1/\left(\#ca_{ik}^{j1}+\#ca_{ik}^{j2}+\#ca_{ik}^{j3}\right)} \tag{34}$$

where $\#ca_{ik}^{jl}$ is number of j satisfies (33), l = 1, 2, 3.

Taking the incomplete MPR A in Example 1 as an example, by the iterative procedure of Alonso et al. [1], we derive the following MPR after the first iteration:

$$A'=\begin{bmatrix}1 & 3 & 5 & 4\\ 1/2 & 1 & 2.0412 & 1.7472\\ x & 0.6 & 1 & 0.8\\ 1/4 & 0.6552 & 1.25 & 1\end{bmatrix}$$

where $a_{23}=\left[(a_{21}a_{13})\left(a_{13}/a_{12}\right)\right]^{1/2}=2.0412$,$a_{24}=\left[(a_{21}a_{14})\left(a_{14}/a_{12}\right)\left(a_{21}/a_{41}\right)\right]^{1/3}=1.7472$,
$a_{32}=\left(a_{12}/a_{13}\right)=0.6$,$a_{34}=\left(a_{14}/a_{13}\right)=0.8$,$a_{42}=\left[(a_{41}a_{12})\left(a_{12}/a_{14}\right)\left(a_{41}/a_{21}\right)\right]^{1/3}=0.6552$,
$a_{43}=\left[(a_{41}a_{13})\left(a_{13}/a_{14}\right)\right]^{1/2}=1.25$.

Then, we obtain the final MPR after the second iteration:

$$A'=\begin{bmatrix}1 & 3 & 5 & 4\\ 1/2 & 1 & 2.0412 & 1.7472\\ 0.3580 & 0.6 & 1 & 0.8\\ 1/4 & 0.6552 & 1.25 & 1\end{bmatrix}$$

where
$a_{31}=\left[\left(a_{32}/a_{12}\right)\left(a_{34}/a_{14}\right)\left(a_{21}/a_{23}\right)\left(a_{41}/a_{43}\right)(a_{32}a_{21})\right]^{1/5}=0.3580$.

$A'$is not equivalent to the complete MPR A derived in Example 1, although both of these two methods are based on multiplicative consistency property. Consistency measures the level of agreement among the preference values provided by the individual DMs. Generally speaking, the corresponding complete MPRs derived are supposed to be of high consistency. The lack of consistency in decision making with preference relations generally results in inconsistent conclusions. Considering that complete MPRs A and A' are not reciprocal, we apply (6) to measure the consistent levels of complete MPRs A and A', where the corresponding weight vectors are determined by (7). Obviously, the lower value of (6) the higher consistency level of MPR A. Based on this idea, the corresponding results are presented as follows:

min $D(A)$ = 0.2466, where the weight vector w = (0.5381, 0.2197, 0.1076, 0.1345),

min $D(A')$ = 0.3469, where the weight vector w = (0.5170, 0.2213, 0.1227, 0.1390),

which indicate MPR A is more consistent than A'.

Similarly, by the iterative procedure of Alonso et al. [1], the following corresponding complete MPR from A^4 is derived:

$$A^{4\prime}=\begin{bmatrix}1 & 0.80 & 1.55 & 1\\ 1.25 & 1 & 1.87 & 3.74\\ 0.65 & 0.56 & 1 & 1.93\\ 1 & 0.33 & 0.52 & 1\end{bmatrix}$$

By using Algorithm I, we derive the following corresponding complete MPR:

$$A^4=\begin{bmatrix}1 & 0.80 & 1.55 & 1\\ 1.25 & 1 & 1.8366 & 3.74\\ 0.65 & 0.5445 & 1 & 1.93\\ 1 & 0.33 & 0.52 & 1\end{bmatrix}$$

And the results are presented as follows:

min $D(A^{4\prime})$ = 0. 5687, where the weight vector w = (0.2478, 0.3913, 0.2136, 0.1473),

min $D(A^4)$ = 0. 5680, where the weight vector w = (0.2477, 0.3917, 0.2133, 0.1472),

which indicate MPR A^4 is more consistent than $A^{4\prime}$.

The iterative procedure of Alonso et al. [1] mainly considers the multiplicative consistency property among three elements under three cases, which are displayed by (33). Another case, $ca_{ik}^{j4}=1/a_{ji}a_{kj}$, is not considered. For example, in the second iteration of the above example, $a_{31}=1/a_{12}a_{23}$ is not considered in the process of estimating a_{31}. Apart from the multiplicative consistency property among three known preference values, the method proposed by this paper also takes the multiplicative consistency property among more than three values into account. Proposition 3.2 illustrates this point. For example, in Example 2, $a_{24}^1=a_{23}^1a_{31}^1a_{14}^1$ is also applied to estimate a_{24}^1 for A^1, where a_{23}^1, a_{31}^1 and a_{14}^1 are all known. As a result, the corresponding complete MPR derived by this paper is more consistent than that derived by other methods and fewer times of iteration is needed to estimate the unknown values if the incomplete PRs are acceptable.

6. Conclusions

Considering the fact that DMs sometimes provide their preferences which do not fully comply with reciprocity properties, this paper investigates the decision models on incomplete MPRs and FPRs without the requirement of satisfying reciprocity property. For a complete MPR $A=(a_{ij})_{n\times n}$, a method to construct its corresponding consistent MPR is proposed, which preserves the original information in $A=(a_{ij})_{n\times n}$ as much as possible. In addition, this paper presents a new characterization of the multiplicative consistency condition, based on which a method to estimate unknown preference values in an incomplete MPR is proposed. On this basis, the final $\overline{A}=(\overline{a}_{ij})_{n\times n}$ derived by (16) is not only a consistent MPR but also has the same weight vector with$A=(a_{ij})_{n\times n}$. It is worth mentioning that the methods proposed are also suitable for solving decision making problems with reciprocal MPRs and FPRs. In this situation, the method proposed derives the same results with existing methods. Furthermore, this paper mainly investigates the consistency for incomplete FPRs and MPRs not satisfying reciprocity property, thus, a corresponding study on consensus issues in GDM process needs to be carried out in future works.

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