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Note on vertex and total proper connection numbers

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Abstract

This note introduces the vertex proper connection number of a graph and provides a relationship to the chromatic number of minimally connected subgraphs. Also a notion of total proper connection is introduced and a question is asked about a possible relationship between the total proper connection number and the vertex and edge proper connection numbers.

Keywords: Colored connectivity; Proper connection; Vertex proper connection

1. Introduction

All graphs considered in this work are simple, finite and undirected. Unless otherwise noted, by a coloring of a graph, we mean a vertex-coloring, not necessarily proper.

Now well studied, the *(edge) rainbow k-connection number* of a graph is the minimum number of colors c such that the edges of the graph can be colored so that between every pair of vertices, there exist k internally disjoint rainbow edge-colored paths. See [1,2] for surveys of results about the rainbow connection number. Note that the rainbow 1-connection number is related, at least conceptually, to the diameter of the graph.

The *total rainbow k-connection number*, defined in [3], is defined to be the minimum number of colors c such that the edges and vertices of the graph can be colored with c colors so that between every pair of vertices, there exist k internally disjoint rainbow paths where here rainbow means all interior vertices and edges have distinct colors. Note that we cannot require the end-vertices of the paths to also have distinct colors as that would reduce the problem to edge rainbow k-connectivity since every vertex would then be required to have a distinct color.

The *edge proper connection number* $\mathrm{pc}_k(G)$, defined in [4] and further studied in [5], is defined to be the minimum number of colors c such that the edges of the graph G can be colored with c colors such that between each pair of vertices, there exist k internally disjoint, properly edge-colored paths. One feature of edge-proper connection that makes the results extremely complicated is that proper edge-colored paths are not transitive in the sense that if there is a proper path from u to v and a proper path from v to w, there may not be a proper path from u to w. For example, let G be a path on three vertices, uvw and color both edges red.

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In this work, we consider a vertex version of the edge proper connection number. For a positive integer k, a colored graph G is called *(vertex) properly k-connected* if, between every pair of vertices, there exist at least k internally disjoint properly colored paths. Note that each path, including end-vertices, must be properly colored. Given a graph G, the *vertex proper k-connection number* of the graph G, denoted $\text{vpc}_k(G)$, is the minimum number of colors needed to produce a properly k-connected coloring of G. For ease of notation, let $\text{vpc}(G) = \text{vpc}_1(G)$.

The function $\text{vpc}_k(G)$ is clearly well defined if and only if $\kappa(G) \geq k$. Also note that $\text{vpc}_k(G) \leq \chi(G)$ for every k-connected graph G. Furthermore, the following fact is immediate.

Fact 1. *For all $k \geq 2$ and every k-connected graph G,* $\text{vpc}_k(G) \geq \text{vpc}_{k-1}(G)$.

A graph G is called *minimally k-connected* if G is k-connected but the removal of any edge from G leaves a graph that is not k-connected. A classical result of Mader [6] (also found in [7]) will immediately give us one of our upper bounds.

Theorem 1 ([6,7])**.** *A minimally k-connected graph is $k + 1$ colorable and this bound is sharp.*

2. General classification

Our first observation demonstrates the transitivity of the vertex proper connection, a fact that is not true in the case of edge proper connection.

Fact 2. *In a colored graph G, if there is a proper path from u to v and a proper path from v to w, then there is a proper path from u to w.*

Proof. The proof is trivial if the u–v path and the v–w path intersect only at v so suppose the paths intersect elsewhere and let x be the first vertex on the path from u to v that is also on the v–w path. Note that we may have $x = u$. Then the subpath of the u–v path that goes from u to x and the subpath of the v–w path that goes from x to w is a properly colored path and completes the proof. ■

Clearly the addition of edges cannot increase the vertex proper connection number of a graph so the following fact is trivial.

Fact 3. *Given a positive integer k and a k-connected graph G, if H is a spanning k-connected subgraph of G, then* $\text{vpc}_k(G) \leq \text{vpc}_k(H)$.

Our main result solidifies the link between the vpc_k function and the chromatic number of the graph. It turns out that $\text{vpc}_k(G)$ always equals the chromatic number of a particular subgraph of G. Let

$$s\chi_k(G) = \min\{\chi(H) : H \text{ is a } k\text{-connected spanning subgraph of } G\}.$$

Theorem 2 (*Classification*)**.** *Given a k-connected graph G,* $\text{vpc}_k(G) = s\chi_k(G)$.

Proof. Given a k-connected spanning subgraph H of G with chromatic number ℓ, color this subgraph properly with ℓ colors. Then between every pair of vertices in H, there are at least k internally disjoint properly colored paths. Thus, using Fact 3, $\text{vpc}_k(G) \leq \text{vpc}_k(H) = \ell$ so $\text{vpc}_k(G) \leq s\chi_k(G)$.

Now let $\ell = \text{vpc}_k(G)$ and consider an ℓ-coloring of G which is properly k-connected. Let $\mathscr{P}$ be the set of all proper paths between pairs of vertices (k paths for each pair of vertices). Then the subgraph H of G induced on all the edges of $\mathscr{P}$ spans G, is k-connected and has chromatic number at most ℓ. This means $\text{vpc}_k(G) \geq s\chi_k(G)$, completing the proof. ■

3. Consequences of Theorem 2

Theorem 2 shows that every statement about vpc_k is a statement about the chromatic number of a minimally k-connected subgraph. Particularly, if G is minimally k-connected, then $\text{vpc}_k(G) = \chi(G)$. When the graph is bipartite, we get the following easy observation.

Corollary 3. *If G is k-connected and bipartite, then for all $t \leq k$, we have* $\text{vpc}_t(G) = 2$.

In light of the classification theorem, we immediately get equivalent colored "fan lemma" and "disjoint paths between k-sets" versions of the definition of vertex proper connectivity.

Corollary 4. *A colored graph G is properly k-connected if and only if for every vertex v and k-set of vertices $\{u_1, u_2, \ldots, u_k\}$, there exists a set of properly colored paths $\{P_1, P_2, \ldots, P_k\}$ where P_i goes from v to u_i and $P_i \cap P_j = \{v\}$ for all i, j.*

Corollary 5. *A colored graph G is properly k-connected if and only if for every $2k$-set of vertices $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$, there exists a set of properly colored paths $\{P_1, P_2, \ldots, P_k\}$ where P_i goes from u_i to v_j for some j and $P_i \cap P_\ell = \emptyset$ for all i, ℓ.*

Theorem 2, along with Theorem 1, also gives us the following general upper bound. The sharpness of Theorem 1 and Corollary 3 yield the sharpness of both bounds here.

Corollary 6. *If G is k-connected, then for $t \leq k$, we have $2 \leq \text{vpc}_t(G) \leq t + 1$ and both bounds are sharp.*

When $k = 1$, Corollary 6 reduces to the following.

Corollary 7. *For every connected graph G on at least 2 vertices,* $\text{vpc}(G) = 2$.

4. Total proper connection

A natural definition of a total proper connection number is the following. Let $\text{tpc}(G)$ be the minimum number of colors needed to color the vertices and edges of G so that between every pair of vertices u, v, there is a path $P = P_{u,v}$ such that the vertices of P induce a properly (vertex-)colored path and the edges of P also induce a properly (edge-) colored path. Furthermore, we define $\text{tpc}_k(G)$ to be the minimum number of colors needed to produce k internally disjoint such paths between every pair of vertices.

One might think that $\text{tpc}_k(G)$ might simply be the maximum of $\text{pc}_k(G)$ and $\text{vpc}_k(G)$ but this is not obvious even when $k = 1$ since the edge path (for pc) and the vertex path (for vpc) must be the same path. Indeed, in Question 1, we ask whether this equality holds in general. Our results concerning the function tpc support a positive answer to this question.

Question 1. *Is it true that* $\text{tpc}_k(G) = \max\{\text{pc}_k(G), \text{vpc}_k(G)\}$*?*

First we recall a result of Borozan et al. [4] which was originally stated in a stronger form.

Theorem 8 *([4]).* *If G is bipartite and 2-connected, then* $\text{pc}(G) = 2$.

Proposition 1. *If $\kappa(G) \geq 3$, then* $\text{tpc}(G) = 2$.

Proof. With $\kappa(G) \geq 3$, there is a spanning 2-connected bipartite subgraph B. Color the vertices with two colors according to this subgraph. By Theorem 8, the proper connection number of B is 2. Color the edges of B with 2 colors to be properly connected. For any pair of vertices in B, there is a properly edge-colored path between them which induces a properly vertex-colored path as well since the vertices are properly colored. This means $\text{tpc}(B) = 2$. Since $B \subseteq G$, we must also have $\text{tpc}(G) = 2$ as well. ■

Using a similar argument and Corollary 3, we easily get the following result.

Corollary 9. *If G is k-connected and bipartite, then for $t \leq k$,* $\text{tpc}_t(G) = \max\{\text{pc}_t(G), \text{vpc}_t(G)\} = \text{pc}_t(G)$.

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