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Distributions of amplitude and phase for bivariate distributions

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ABSTRACT

The distribution of amplitude, its moments and the distribution of phase are derived for thirty four flexible bivariate distributions, including the correlated bivariate normal distribution. The results in part extend those given in Coluccia (2013).

1. Introduction

Let (X, Y) be a random vector defined on the real space $(-\infty, +\infty) \times (-\infty, +\infty)$. Let $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan(Y/X)$. The distributions of R and Θ arise in many areas of the IEEE literature: radar communications, positioning related applications, fading channels, etc. R is usually referred to as the amplitude and Θ the phase.Some published examples in the IEEE literature where the distributions of R and Θ arise directly are: analog data-transmission [7]; radiation-pattern of an offset fed paraboloidal reflector antenna [19]; microwave pulse generation [8]; photonic beamforming based on programmable phase shifters [37]; radiation characteristics of multimode concentric circular microstrip patch antennas [32]; transmission schemes for Rayleigh block fading channels [9]; broadband microwave photonic splitters [25]; performance of transmit beamforming codebooks [13]; AC-AC converters [39]; modulation of ultrawideband monocycle pulses on a silicon photonic chip [35].

Several papers have derived the distributions of R and Θ: Aalo et al. [1] and Dharmawansa et al. [11] derived the distributions of R and Θ when X and Y are correlated and non-identical normal random variables with non-zero means; Yacoub [36] derived the distributions of R and Θ when X and Y are independent and identical Nakagami type random variables; Coluccia [10] derived the moments of R when X and Y are uncorrelated normal random variables with zero means and unequal variances; and so on.

Almost all of the known papers have supposed X and Y are normally distributed. However, non-normal distributions are becoming increasingly popular in the IEEE literature: hyperbolic distribution in network modeling [24]; Laplace distribution in signal processing [14]; Cauchy distribution in segmentation of noisy colour images [34]; Kotz type distribution for multilook polarimetric radar data [20]; Student's t distribution in medical image segmentation [28]; logistic distribution for networked video quality assessment [38]; Gumbel distribution for peak sidelobe level for arrays of randomly placed antennas [23]; skew normal distribution for statistical static timing analysis [33]; and so on. Also, we are not aware of any paper giving expressions for moments of R when X and Y are correlated random variables.

The aim of this note is to derive the distribution of R, its moments and the distribution of Θ for a wide range of bivariate distributions, including the correlated bivariate normal distribution. We consider thirty four flexible bivariate distributions in total. These include eleven bivariate normal distributions, eight bivariate t distributions, five bivariate Laplace distributions, two bivariate hyperbolic distributions, two bivariate Gumbel distributions and one bivariate logistic distribution.

The contents of this note are organized as follows. Section 2 derives the distribution of R, its moments and the distribution of Θ when X and Y are correlated normal random variables with zero means and unequal variances. These results extend those given in Coluccia [10]. Section 3 derives the same when X and Y are correlated random variables following thirty three other bivariate distributions. Details of the derivations are not given. They can be obtained from the corresponding author. To the best of our knowledge, the derived expressions for the distribution of R, its moments and the distribution of Θ are all new and original. Section 4 discusses simulation of R and Θ for the bivariate distributions considered.

It is hoped that the details given in Section 3 could be a useful reference for the IEEE community. They could also encourage more

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non-normal distributions being applied to real problems in the EEE. A future work is to extend the results in Section 3 for multivariate distributions, that is to derive the distributions of $\sqrt{X_1^2+X_2^2+\cdots+X_p^2}$ and $(X_1/\sqrt{X_1^2+X_2^2+\cdots+X_p^2}, X_2/\sqrt{X_1^2+X_2^2+\cdots+X_p^2}, \ldots, X_{p-1}/\sqrt{X_1^2+X_2^2+\cdots+X_p^2})$ given a distribution for $(X_1, X_2, \ldots, X_p), p > 2$.

The expressions given in Sections 2 and 3 involve various special functions, including the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt$$

for $a > 0$; the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x t^{a-1} \exp(-t) dt$$

for $a > 0$ and $x > 0$; the beta function defined by

$$B(a,b) = \int_0^\infty t^{a-1}(1-t)^{b-1} dt$$

for $a > 0$ and $b > 0$; the error function defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt$$

for $x > 0$; the complementary error function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt$$

for $-\infty < x < +\infty$; the parabolic cylinder function defined by

$$D_\nu(x) = \frac{\exp\left(-x^2/4\right)}{\Gamma(-\nu/2)} \int_0^\infty t^{-\frac{\nu}{2}-1}(1+2t)^{\frac{\nu-1}{2}} \exp\left(-x^2 t\right) dt$$

for $\nu < 0$ and $x^2 > 0$; the Euler number of order n defined by

$$E_{2n} = (-1)^n 2^{2n+1} \int_0^\infty t^{2n} \,\mathrm{sech}\,(\pi t) dt;$$

the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \sum_{k=0}^\infty \frac{1}{\Gamma(k+\nu+1) k!} \left(\frac{x}{2}\right)^{2k+\nu};$$

the modified Bessel function of the second kind of order ν defined by

$$K_\nu(x) = \begin{cases} \frac{\pi \csc(\pi\nu)}{2}\left[I_{-\nu}(x) - I_\nu(x)\right], & \text{if } \nu \neg \in \mathbb{Z}, \\ \lim\limits_{\mu \to \nu} K_\mu(x), & \text{if } \nu \in \mathbb{Z}; \end{cases}$$

the confluent hypergeometric function defined by

$${}_1F_1(\alpha;\beta;z) = \sum_{k=0}^\infty \frac{(\alpha)_k}{(\beta)_k} \frac{z^k}{k!},$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ denotes the ascending factorial; the Gauss hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^\infty \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{z^k}{k!};$$

the ${}_2F_2$ hypergeometric function defined by

$${}_2F_2(\alpha,\beta;\gamma,\delta;z) = \sum_{k=0}^\infty \frac{(\alpha)_k(\beta)_k}{(\gamma)_k(\delta)_k} \frac{z^k}{k!};$$

the ${}_4F_2$ hypergeometric function defined by

$${}_4F_2(a,b,c,d;e,f;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k(c)_k(d)_k}{(e)_k(f)_k} \frac{z^k}{k!};$$

the ${}_4F_3$ hypergeometric function defined by

$${}_4F_3(a,b,c,d;e,f,g;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k(c)_k(d)_k}{(e)_k(f)_k(g)_k} \frac{z^k}{k!};$$

and, the Appell function of the first kind defined by

$$F_1(a,b,c,d;z,\xi) = \sum_{k=0}^\infty \sum_{\ell=0}^\infty \frac{(a)_{k+\ell}(b)_k(c)_\ell z^k \xi^\ell}{(d)_{k+\ell} k! \ell!}.$$

These special functions are well known and well established in the mathematics literature. Some details of their properties can be found in Prudnikov et al. [29] and Gradshteyn and Ryzhik [17]. In-built routines for computing them are widely available in packages like Maple, Matlab and Mathematica. For example, the in-built routines in Mathematica for the stated special functions are: GAMMA[a] for the gamma function, GAMMA[a]-GAMMA[a,x] for the incomplete gamma function, Beta[a,b] for the beta function, Erc[x] for the error function, Erfc[x] for the complementary error function, ParabolicCylinderD[nu,x] for the parabolic cylinder function, EulerE[n] for the Euler number of order n, BesselI[nu,x] for the modified Bessel function of the first kind of order ν, BesselK[nu,x] for the modified Bessel function of the second kind of order ν, Hypergeometric1F1[alpha,beta,z] for the confluent hypergeometric function, Hypergeometric2F1[alpha,beta,gamma,z] for the Gauss hypergeometric function, HypergeometricPFQ [{alpha,beta}, {gamma,delta}, z] for the ${}_2F_2$ hypergeometric function, HypergeometricPFQ [{a,b,c,d}, {e,f}, z] for the ${}_4F_2$ hypergeometric function, HypergeometricPFQ [{a,b,c,d}, {e,f,g}, z] for the ${}_4F_3$ hypergeometric function, AppellF1[a,b,c,d,z,xi] for the Appell function of the first kind. Mathematica like other algebraic manipulation packages allows for arbitrary precision, so the accuracy of computations is not an issue.

In-built routines for most of the stated special functions are also available in the freely available R software [30]: gamma(a) in the base package for the gamma function, gamma(a)*pgamma(x,shape=a) in the base package for the incomplete gamma function, beta(a,b) in the base package for the beta function, erf(x) in the contributed package NORMT3 for the error function, erfc(x) in the contributed package NORMT3 for the complementary error function, besselI(x,nu) in the base package for the modified Bessel function of the first kind of order ν, besselK(x,nu) in the base package for the modified Bessel function of the second kind of order ν, kummerM(z,alpha,beta) in the contributed package AsianOptions for the confluent hypergeometric function, hypergeo(alpha,beta,gamma,z) in the contributed package hypergeo for the Gauss hypergeometric function, genhypergeo(U = c(alpha,beta), L = c(gamma,beta), z) in the contributed package hypergeo for the ${}_2F_2$ hypergeometric function, genhypergeo(U = c(a,b,c,d), L = c(e,f), z) in the contributed package hypergeo for the ${}_4F_2$ hypergeometric function, genhypergeo(U = c(a,b,c,d), L = c(e,f,g), z) in the contributed package hypergeo for the ${}_4F_3$ hypergeometric function, F1(a,b,c,d,z,xi) in the contributed package tolerance for the Appell function of the first kind.

2. Bivariate normal case

Here, we derive the pdf of R, the pdf of Θ and $E(R^p)$ when (X, Y) has the bivariate normal pdf

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}\sigma_x\sigma_y} \exp\left[-\left(\frac{x}{\sigma_x}\right)^2 - \left(\frac{y}{\sigma_y}\right)^2 - 2\rho\frac{x}{\sigma_x}\frac{y}{\sigma_y}\right] \quad (1)$$

for $-\infty < x < +\infty, -\infty < y < +\infty$, $\sigma_x > 0, \sigma_y > 0$ and $-1 < \rho < 1$. We can write (1) in the form

$$f(x,y) = C \exp\left(-ax^2 - by^2 - 2cxy\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0, b > 0$ and $-\infty < c < +\infty$, where C denotes the normalizing constant. The corresponding joint pdf of R and Θ can be expressed as

$$\begin{aligned} f(r,\theta) &= Cr\exp\left[-r^2\left(a\sin^2\theta + b\cos^2\theta + 2c\sin\theta\cos\theta\right)\right] \\ &= Cr\exp\left\{-r^2\left[a + (b-a)\cos^2\theta + c\sin(2\theta)\right]\right\} \\ &= Cr\exp\left\{-r^2\left(a + \frac{b-a}{2}[1+\cos(2\theta)] + c\sin(2\theta)\right)\right\} \\ &= Cr\exp\left\{-r^2\frac{b+a}{2} - r^2\frac{b-a}{2}\cos(2\theta) - cr^2\sin(2\theta)\right\} \end{aligned} \tag{2}$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$. Using the fact

$$\int_0^{2\pi}\exp(x\cos\theta + y\sin\theta)d\theta = 2\pi I_0\left(\sqrt{x^2+y^2}\right),$$

we obtain the pdf of R as

$$f_R(r) = 2\pi Cr\exp\left(-\alpha r^2\right)I_0\left(r^2\beta\right)$$

for $r > 0$, where $\alpha = \frac{a+b}{2}$, $\beta = \sqrt{\gamma^2+c^2}$ and $\gamma = \frac{b-a}{2}$. A straightforward integration of (2) shows that the pdf of Θ is

$$f_\Theta(\theta) = C[a+b+(b-a)\cos(2\theta)+2c\sin(2\theta)]^{-1}$$

for $0 < \theta < 2\pi$. An application of Eq. (2.15.3.2) in Prudnikov et al. [29] volume 2, shows that the pth moment of R can be expressed as

$$E(R^p) = C\pi\alpha^{-\frac{p}{2}-1}\Gamma\left(\frac{p}{2}+1\right){}_2F_1\left(\frac{p}{4}+1, \frac{p}{4}+\frac{1}{2}; 1; \frac{\beta^2}{\alpha^2}\right) \tag{3}$$

for $p > 0$. By the transformation formulas of the Gauss hypergeometric function, three equivalent representation of (3) are

$$E(R^p) = C\pi\alpha^{\frac{p}{2}}\left(\alpha^2-\beta^2\right)^{-\frac{1}{2}-\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right){}_2F_1\left(-\frac{p}{4}, -\frac{p}{4}+\frac{1}{2}; 1; \frac{\beta^2}{\alpha^2}\right), \tag{4}$$

$$E(R^p) = C\pi\alpha\left(\alpha^2-\beta^2\right)^{-1-\frac{p}{4}}\Gamma\left(\frac{p}{2}+1\right){}_2F_1\left(1+\frac{p}{4}, -\frac{p}{4}+\frac{1}{2}; 1; \frac{\beta^2}{\beta^2-\alpha^2}\right) \tag{5}$$

and

$$E(R^p) = C\pi\left(\alpha^2-\beta^2\right)^{-\frac{1}{2}-\frac{p}{4}}\Gamma\left(\frac{p}{2}+1\right){}_2F_1\left(-\frac{p}{4}, \frac{p}{4}+\frac{1}{2}; 1; \frac{\beta^2}{\beta^2-\alpha^2}\right) \tag{6}$$

for $p > 0$. If p is a positive integer and a multiple of 4 then (4) and (6) reduce to the elementary forms

$$E(R^p) = C\pi\alpha^{\frac{p}{2}}\left(\alpha^2-\beta^2\right)^{-\frac{1}{2}-\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right)\sum_{k=0}^{p/4}\frac{\left(-\frac{p}{4}\right)_k\left(-\frac{p}{4}+\frac{1}{2}\right)_k}{k!^2}\left(\frac{\beta^2}{\alpha^2}\right)^k$$

and

$$E(R^p) = C\pi\left(\alpha^2-\beta^2\right)^{-\frac{1}{2}-\frac{p}{4}}\Gamma\left(\frac{p}{2}+1\right)\sum_{k=0}^{p/4}\frac{\left(-\frac{p}{4}\right)_k\left(\frac{p}{4}+\frac{1}{2}\right)_k}{k!^2}\left(\frac{\beta^2}{\beta^2-\alpha^2}\right)^k,$$

respectively.

3. The collection

Here, we tabulate expressions for the pdf of R, the pdf of Θ and $E(R^p)$ when (X, Y) follows thirty four flexible bivariate distributions. For some distributions, the derivation of $E(R^p)$ was not possible. Details of the derivations for all of the distributions can be obtained from the corresponding author.

Bivariate normal distribution ([6], Chapter 11) has the joint pdf specified by

$$f(x,y) = C\exp\left(-ax^2 - by^2 - 2cxy\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0, b > 0$ and $-\infty < c < +\infty$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp\left[-\alpha r^2 - \frac{b-a}{2}\cos(2\theta)r^2 - c\sin(2\theta)r^2\right],$$

$$f_R(r) = 2\pi Cr\exp\left(-\alpha r^2\right)I_0\left(r^2\beta\right),$$

$$f_\Theta(\theta) = C[a+b+(b-a)\cos(2\theta)+2c\sin(2\theta)]^{-1}$$

and

$$E(R^p) = C\pi\alpha^{-\frac{p}{2}-1}\Gamma\left(\frac{p}{2}+1\right){}_2F_1\left(\frac{p}{4}+1, \frac{p}{4}+\frac{1}{2}; 1; \frac{\beta^2}{\alpha^2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$, where $\alpha = \frac{a+b}{2}$, $\beta = \sqrt{\gamma^2+c^2}$ and $\gamma = \frac{b-a}{2}$.

Bivariate normal distribution with non-zero means ([6], Chapter 11) has the joint pdf specified by

$$f(x,y) = C\exp\left(-ax^2 - by^2 - 2cxy - dx - ey\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0, b > 0, -\infty < c < +\infty, d > 0$ and $e > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp\left[-\alpha r^2 - \gamma\cos(2\theta)r^2 - c\sin(2\theta)r^2 - dr\sin\theta - er\cos\theta\right],$$

$$f_R(r) = Cr\exp\left(-\alpha r^2\right)\int_0^{2\pi}\exp\left[-\gamma\cos(2\theta)r^2 - c\sin(2\theta)r^2 - dr\sin\theta - er\cos\theta\right]d\theta$$

and

$$\begin{aligned} f_\Theta(\theta) &= C[a+b+(b-a)\cos(2\theta)+2c\sin(2\theta)]^{-1} \\ &\cdot\exp\left\{\frac{(d\sin\theta+e\cos\theta)^2}{4[a+b+(b-a)\cos(2\theta)+2c\sin(2\theta)]}\right\} \\ &\cdot D_{-2}\left(\frac{d\sin\theta+e\cos\theta}{\sqrt{a+b+(b-a)\cos(2\theta)+2c\sin(2\theta)}}\right) \end{aligned}$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$, where $\alpha = \frac{a+b}{2}$ and $\gamma = \frac{b-a}{2}$.

Conditionally specified bivariate normal distribution ([6], Eq. (6.8)) has the joint pdf specified by

$$f(x,y) = C\exp\left(-\frac{x^2+y^2+cx^2y^2}{2}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty$ and $-\infty < c < +\infty$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp\left[-\frac{r^2+\left(cr^4/4\right)\sin^2(2\theta)}{2}\right],$$

$$f_R(r) = 2Cr\exp\left(-\frac{r^2}{2}\right)\sum_{i=0}^{\infty}\frac{\left(-cr^4\right)^i}{8^i i!}B\left(\frac{1}{2}, i+\frac{1}{2}\right),$$

$$f_\Theta(\theta) = \frac{C}{\sqrt{c}|\sin(2\theta)|}\exp\left[\frac{1}{4c\sin^2(2\theta)}\right]D_{-1}\left(\frac{1}{\sqrt{c}|\sin(2\theta)|}\right)$$

and

$$E(R^p) = 2^{1+\frac{p}{2}}C\sum_{i=0}^{\infty}\frac{(-c)^i}{2^i i!}B\left(\frac{1}{2}, i+\frac{1}{2}\right)\Gamma\left(2i+\frac{p}{2}+1\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Bivariate skew normal distribution ([6], Eq. (7.17)) has the joint pdf specified by

$$f(x,y) = 2\phi(x)\phi(y)\Phi(\alpha x + \beta y)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < \alpha < +\infty$ and $-\infty < \beta < +\infty$, where $\phi(\cdot)$ and $\Phi(\cdot)$ denote, respectively, the pdf and cdf of a standard normal random variable. For this distribution,

$$f(r,\theta) = \frac{r}{\pi}\exp\left(-\frac{r^2}{2}\right)\Phi(\alpha r\sin\theta + \beta r\cos\theta),$$

$$f_R(r) = r\exp\left(-\frac{r^2}{2}\right),$$

$$f_\Theta(\theta) = \frac{1}{2\pi} + \frac{\alpha\sin\theta + \beta\cos\theta}{2\pi\sqrt{1 + (\alpha\sin\theta + \beta\cos\theta)^2}}$$

and

$$E(R^p) = 2^{\frac{p}{2}}\Gamma\left(\frac{p}{2} + 1\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$. The given expression for $f_\Theta(\theta)$ follows from $\Phi(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$ and by writing

$$\begin{aligned}
f_\Theta(\theta) &= \int_0^\infty \frac{r}{\pi}\exp\left(-\frac{r^2}{2}\right)\Phi(\alpha r\sin\theta + \beta r\cos\theta)dr \\
&= \int_0^\infty \frac{r}{\pi}\exp\left(-\frac{r^2}{2}\right)\left[\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\alpha r\sin\theta + \beta r\cos\theta}{\sqrt{2}}\right)\right]dr \\
&= \frac{1}{2\pi}\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)dr + \frac{1}{2\pi}\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)\mathrm{erf}\left(\frac{\alpha r\sin\theta + \beta r\cos\theta}{\sqrt{2}}\right)dr \\
&= \frac{1}{2\pi} + \frac{1}{2\pi}\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)\mathrm{erf}\left(\frac{\alpha r\sin\theta + \beta r\cos\theta}{\sqrt{2}}\right)dr
\end{aligned}$$

and applying Eq. (2.8.5.9) in Prudnikov et al. [29] volume 2, to calculate the integral. The given expression for $f_R(r)$ follows from $\Phi(x) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$ and by writing

$$\begin{aligned}
f_R(r) &= \frac{r}{\pi}\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\Phi(\alpha r\sin\theta + \beta r\cos\theta)d\theta \\
&= \frac{r}{\pi}\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\left[\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\alpha r\sin\theta + \beta r\cos\theta}{\sqrt{2}}\right)\right]d\theta \\
&= r\exp\left(-\frac{r^2}{2}\right) + \frac{r}{2\pi}\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\mathrm{erf}\left(\frac{\alpha r\sin\theta + \beta r\cos\theta}{\sqrt{2}}\right)d\theta \\
&= r\exp\left(-\frac{r^2}{2}\right) + \frac{r}{\pi^{3/2}}\exp\left(-\frac{r^2}{2}\right) \\
&\quad \cdot\int_0^{2\pi}\sum_{k=0}^\infty \frac{(-1)^k}{k!(2k+1)2^{k+\frac{1}{2}}}\left(\frac{\alpha r\sin\theta + \beta r\cos\theta}{\sqrt{2}}\right)^{2k+1}d\theta \\
&= r\exp\left(-\frac{r^2}{2}\right) + \frac{r}{\pi^{3/2}}\exp\left(-\frac{r^2}{2}\right) \\
&\quad \cdot\sum_{k=0}^\infty \frac{(-1)^k}{k!(2k+1)4^{k+\frac{1}{2}}}\int_0^{2\pi}(\alpha r\sin\theta + \beta r\cos\theta)^{2k+1}d\theta \\
&= r\exp\left(-\frac{r^2}{2}\right) + \frac{r}{\pi^{3/2}}\exp\left(-\frac{r^2}{2}\right) \\
&\quad \cdot\sum_{k=0}^\infty \frac{(-1)^k r^{2k+1}}{k!(2k+1)4^{k+\frac{1}{2}}}\sum_{\ell=0}^{2k+1}\binom{2k+1}{\ell}\alpha^{2k+1-\ell}\beta^\ell\int_0^{2\pi}(\sin\theta)^{2k+1-\ell}(\cos\theta)^\ell d\theta \\
&= r\exp\left(-\frac{r^2}{2}\right),
\end{aligned}$$

where the last step follows by noting that the integrand is an odd function.

Bivariate skew normal distribution ([3], Eq. (6.5)) has the joint pdf specified by

$$f(x,y) = C\exp\left(-\frac{x^2+y^2}{2}\right)\Phi(ax + by + cxy)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < a < +\infty, -\infty < b < +\infty$ and $-\infty < c < +\infty$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp\left(-\frac{r^2}{2}\right)\Phi\left(ar\sin\theta + br\cos\theta + \frac{cr^2}{2}\sin(2\theta)\right),$$

$$f_R(r) = Cr\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\Phi\left(ar\sin\theta + br\cos\theta + \frac{cr^2}{2}\sin(2\theta)\right)d\theta$$

and

$$f_\Theta(\theta) = C\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)\Phi\left(ar\sin\theta + br\cos\theta + \frac{cr^2}{2}\sin(2\theta)\right)dr$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate skew normal distribution ([3], Eq. (4.11)) has the joint pdf specified by

$$f(x,y) = C\exp\left(-\frac{x^2+y^2}{2}\right)\Phi(ax + by + c)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < a < +\infty, -\infty < b < +\infty$ and $-\infty < c < +\infty$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp\left(-\frac{r^2}{2}\right)\Phi(ar\sin\theta + br\cos\theta + c),$$

$$f_R(r) = Cr\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\Phi(ar\sin\theta + br\cos\theta + c)d\theta$$

and

$$\begin{aligned}
f_\Theta(\theta) &= \frac{C}{2} + \frac{C}{2}\mathrm{erf}\left(\frac{c}{\sqrt{2}}\right) + \frac{C}{2}\frac{a\sin\theta + b\cos\theta}{\sqrt{(a\sin\theta + b\cos\theta)^2 + 1}} \\
&\cdot\exp\left[-\frac{c^2}{2(a\sin\theta + b\cos\theta)^2 + 2}\right]\mathrm{erfc}\left(\frac{c(a\sin\theta + b\cos\theta)}{\sqrt{2(a\sin\theta + b\cos\theta)^2 + 2}}\right)
\end{aligned}$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate skew normal distribution ([4], Eq. (5.1)) has the joint pdf specified by

$$f(x,y) = C\exp\left(-\frac{x^2+y^2}{2}\right)\Phi\left(a + bx + cy + dx^2 + ey^2 + fxy\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < a < +\infty, -\infty < b < +\infty$, $-\infty < c < +\infty, -\infty < d < +\infty, -\infty < e < +\infty$ and $-\infty < f < +\infty$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp\left(-\frac{r^2}{2}\right) \times\Phi\left(a + br\sin\theta + cr\cos\theta + dr^2\sin^2\theta + er^2\cos^2\theta + \frac{fr^2}{2}\sin(2\theta)\right),$$

$$f_R(r) = Cr\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi} \times\Phi\left(a + br\sin\theta + cr\cos\theta + dr^2\sin^2\theta + er^2\cos^2\theta + \frac{fr^2}{2}\sin(2\theta)\right)d\theta$$

and

$$f_\Theta(\theta)=C\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)\times\Phi\left(a+br\sin\theta+cr\cos\theta+dr^2\sin^2\theta+er^2\cos^2\theta+\frac{fr^2}{2}\sin(2\theta)\right)dr$$

for $r>0$ and $0\leqslant\theta\leqslant 2\pi$.

Bivariate skew normal distribution ([6], page 525) has the joint pdf specified by

$$f(x,y)=C\exp\left(-\frac{x^2+y^2}{2a^2}\right)\Phi(\alpha x)\Phi(\beta y)$$

for $-\infty<x<+\infty, -\infty<y<+\infty, a>0, -\infty<\alpha<+\infty$ and $-\infty<\beta<+\infty$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta)=Cr\exp\left(-\frac{r^2}{2a^2}\right)\Phi(\alpha r\sin\theta)\Phi(\beta r\cos\theta),$$

$$f_R(r)=\frac{C\pi r}{2}\exp\left(-\frac{r^2}{2a^2}\right),$$

$$f_\Theta(\theta)=\frac{Ca^2}{4}+\frac{Ca^3\alpha\sin\theta}{4\sqrt{\alpha^2a^2\sin^2\theta+1}}+\frac{Ca^3\beta\cos\theta}{4\sqrt{\beta^2a^2\cos^2\theta+1}}+\frac{Ca^3\alpha\sin\theta}{2\pi\sqrt{\alpha^2a^2\sin^2\theta+1}}\arctan\frac{\beta a\cos\theta}{\sqrt{\alpha^2a^2\sin^2\theta+1}}+\frac{Ca^3\beta\cos\theta}{2\pi\sqrt{\beta^2a^2\cos^2\theta+1}}\arctan\frac{\alpha a\sin\theta}{\sqrt{\beta^2a^2\cos^2\theta+1}}$$

and

$$E(R^p)=C\pi a^{p+2}2^{\frac{p}{2}-1}\Gamma\left(\frac{p}{2}+1\right)$$

for $r>0, 0\leqslant\theta\leqslant 2\pi$ and $p>0$.

Conditionally specified bivariate skew normal distribution ([6], Eq. (6.78)) has the joint pdf specified by

$$f(x,y)=2\phi(x)\phi(y)\Phi(\lambda xy)$$

for $-\infty<x<+\infty, -\infty<y<+\infty$ and $-\infty<\lambda<+\infty$. For this distribution,

$$f(r,\theta)=\frac{r}{\pi}\exp\left(-\frac{r^2}{2}\right)\Phi\left(\frac{\lambda r^2}{2}\sin 2\theta\right),$$

$$f_R(r)=r\exp\left(-\frac{r^2}{2}\right),$$

$$f_\Theta(\theta)=\frac{1}{2\pi}\left\{1+\exp\left[\frac{1}{2\lambda^2\sin^2(2\theta)}\right]\mathrm{erfc}\left(\frac{1}{\sqrt{2}\lambda\sin(2\theta)}\right)\right\}$$

and

$$E(R^p)=2^{\frac{p}{2}}\Gamma\left(\frac{p}{2}+1\right)$$

for $r>0, 0\leqslant\theta\leqslant 2\pi$ and $p>0$.

Bivariate alpha-skew-normal distribution [26] has the joint pdf specified by

$$f(x,y)=C\left[1-(1-ax-by)^2\right]\exp\left(-x^2-y^2-2\rho xy\right)$$

for $-\infty<x<+\infty, -\infty<y<+\infty, -\infty<a<+\infty, -\infty<b<+\infty$ and $-1\leqslant b\leqslant 1$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta)=Cr\left[2-2r(a\sin\theta+b\cos\theta)+r^2(a\sin\theta+b\cos\theta)^2\right]\times\exp\left[-r^2-\rho r^2\sin(2\theta)\right],$$

$$f_\Theta(\theta)=\frac{C}{1+\rho\sin(2\theta)}-\frac{\sqrt{\pi}C(a\sin\theta+b\cos\theta)}{2[1+\rho\sin(2\theta)]^{3/2}}+\frac{C(a\sin\theta+b\cos\theta)^2}{2[1+\rho\sin(2\theta)]^2},$$

$$f_R(r)=2Cr\exp\left(-r^2\right)\sum_{k=0}^\infty\frac{\left(-2\rho r^2\right)^k}{k!^2}\left[1+(-1)^k\right]\Gamma^2\left(\frac{k+1}{2}\right)+2abCr^3\exp\left(-r^2\right)\sum_{k=0}^\infty\frac{\left(-2\rho r^2\right)^k}{k!(k+1)!}\left[1+(-1)^{k+1}\right]\Gamma^2\left(\frac{k+2}{2}\right)+\left(a^2+b^2\right)Cr^3\exp\left(-r^2\right)\sum_{k=0}^\infty\frac{\left(-2\rho r^2\right)^k}{k!(k+1)!}\left[1+(-1)^k\right]\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+3}{2}\right)$$

and

$$E(R^p)=C\sum_{k=0}^\infty\frac{(-2\rho)^k}{k!^2}\left[1+(-1)^k\right]\Gamma\left(\frac{p}{2}+k+1\right)\Gamma^2\left(\frac{k+1}{2}\right)+2abC\sum_{k=0}^\infty\frac{(-2\rho)^k}{k!(k+1)!}\Gamma\left(\frac{p}{2}+k+2\right)\left[1+(-1)^{k+1}\right]\Gamma^2\left(\frac{k+2}{2}\right)+\left(a^2+b^2\right)C\sum_{k=0}^\infty\frac{(-2\rho)^k}{k!(k+1)!}\Gamma\left(\frac{p}{2}+k+2\right)\left[1+(-1)^k\right]\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k+3}{2}\right)$$

for $r>0$ and $0\leqslant\theta\leqslant 2\pi$.

Bivariate generalized skew-symmetric normal distribution ([16, Eq. (1)) has the joint pdf specified by

$$f(x,y)=C\exp\left(-\frac{x^2+y^2}{2}\right)\Phi\left(\frac{axy}{1+bx^2y^2}\right)$$

for $-\infty<x<+\infty, -\infty<y<+\infty, -\infty<a<+\infty$ and $b\geqslant 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta)=Cr\exp\left(-\frac{r^2}{2}\right)\Phi\left(\frac{2ar^2\sin(2\theta)}{4+br^4\sin^2(2\theta)}\right),$$

$$f_R(r)=Cr\exp\left(-\frac{r^2}{2}\right)\int_0^{2\pi}\Phi\left(\frac{2ar^2\sin(2\theta)}{4+br^4\sin^2(2\theta)}\right)d\theta$$

and

$$f_\Theta(\theta)=C\int_0^\infty r\exp\left(-\frac{r^2}{2}\right)\Phi\left(\frac{2ar^2\sin(2\theta)}{4+br^4\sin^2(2\theta)}\right)dr$$

for $r>0$ and $0\leqslant\theta\leqslant 2\pi$.

Bivariate Kotz type distribution ([6], Section 13.6.1) has the joint pdf specified by

$$f(x,y)=C\left(x^2+y^2\right)^{N-1}\exp\left[-\phi\left(x^2+y^2\right)^s\right]$$

for $-\infty<x<+\infty, -\infty<y<+\infty, N>0, s>0$ and $\phi>0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta)=Cr^{2N-1}\exp\left(-\phi r^{2s}\right),$$

$$f_R(r)=2\pi Cr^{2N-1}\exp\left(-\phi r^{2s}\right),$$

$$f_\Theta(\theta)=\frac{1}{2\pi}$$

and

$$E(R^p)=\pi Cs^{-1}\phi^{-\frac{2N+p}{2s}}\Gamma\left(\frac{2N+p}{2s}\right)$$

for $r>0, 0\leqslant\theta\leqslant 2\pi$ and $p>0$.

Bivariate *t* distribution ([22], Chapter 1) has the joint pdf specified by

$$f(x,y)=C\left(1+ax^2+by^2+2cxy\right)^{-\frac{\nu+2}{2}}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0, b > 0, -\infty < c < +\infty$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\left\{1 + \frac{a+b}{2}r^2 + \left[\frac{b-a}{2}\cos(2\theta) + c\sin(2\theta)\right]r^2\right\}^{-\frac{\nu+2}{2}},$$

$$f_R(r) = \frac{C}{2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{-\frac{\nu+2}{2}}{k}\binom{k}{\ell}c^{\ell}\left(\frac{b-a}{2}\right)^{k-\ell}\left[1+(-1)^k+(-1)^{\ell}+(-1)^{k-\ell}\right] \cdot B\left(\frac{\ell+1}{2},\frac{k-\ell+1}{2}\right)r^{2k+1}\left[1+\frac{a+b}{2}r^2\right]^{-\frac{\nu+2}{2}-k},$$

$$f_\Theta(\theta) = \frac{2C}{\nu[a+b+(b-a)\cos(2\theta)+2c\sin(2\theta)]}$$

and

$$E(R^p) = \frac{2^{\frac{p}{2}-1}C}{(b+a)^{\frac{p}{2}+1}}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{-\frac{\nu+2}{2}}{k}\binom{k}{\ell}(2c)^{\ell}\frac{(b-a)^{k-\ell}}{(b+a)^k}\left[1+(-1)^k+(-1)^{\ell}+(-1)^{k-\ell}\right] \cdot B\left(\frac{\ell+1}{2},\frac{k-\ell+1}{2}\right)B\left(k+\frac{p}{2}+1,\frac{\nu-p}{2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < \nu$.

Bivariate Cauchy distribution ([22], Chapter 1) has the joint pdf specified by

$$f(x,y) = C\left(1 + ax^2 + by^2\right)^{-\frac{3}{2}}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0$ and $b > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\left\{1 + ar^2 + \frac{b-a}{2}[1+\cos(2\theta)]r^2\right\}^{-\frac{3}{2}},$$

$$f_R(r) = C\sum_{k=0}^{\infty}\binom{-\frac{3}{2}}{k}\left(\frac{b-a}{2}\right)^k\left[1+(-1)^k\right]B\left(\frac{1}{2},\frac{k+1}{2}\right)r^{2k+1}\left[1+\frac{a+b}{2}r^2\right]^{-\frac{3}{2}-k},$$

$$f_\Theta(\theta) = \frac{2C}{a+b+(b-a)\cos(2\theta)}$$

and

$$E(R^p) = \frac{2^{\frac{p}{2}}C}{(b+a)^{\frac{p}{2}+1}}\sum_{k=0}^{\infty}\binom{-\frac{3}{2}}{k}\left(\frac{b-a}{b+a}\right)^k\left[1+(-1)^k\right]B\left(\frac{1}{2},\frac{k+1}{2}\right)B\left(k+\frac{p}{2}+1,\frac{1-p}{2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < 1$.

Bivariate skew t distribution [5] has the joint pdf specified by

$$f(x,y) = \frac{1}{\pi}\left(1+\frac{x^2+y^2}{\nu}\right)^{-\frac{\nu+2}{2}}T_{\nu+2}\left((ax+by)\sqrt{\frac{\nu+2}{x^2+y^2+\nu}}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < a < +\infty, -\infty < b < +\infty$ and $\nu > 0$, where $T_a(\cdot)$ denotes the cdf of a standard Student's t random variable with degree of freedom a. For this distribution,

$$f(r,\theta) = \frac{r}{\pi}\left(1+\frac{r^2}{\nu}\right)^{-\frac{\nu+2}{2}}T_{\nu+2}\left(r(a\sin\theta+b\cos\theta)\sqrt{\frac{\nu+2}{r^2+\nu}}\right),$$

$$f_R(r) = r\left(1+\frac{r^2}{\nu}\right)^{-\frac{\nu+2}{2}},$$

$$f_\Theta(\theta) = \frac{1}{2\pi} + \frac{\nu\Gamma\left(\frac{\nu+3}{2}\right)}{2\pi^{3/2}\Gamma\left(\frac{\nu+2}{2}\right)}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\nu+3}{2}\right)_k}{k!(2k+1)}(a\sin\theta+b\cos\theta)^{2k+1}B\left(k+\frac{3}{2},\frac{\nu}{2}\right)$$

and

$$E(R^p) = 2^{-1}\nu^{\frac{p}{2}+1}B\left(\frac{p}{2}+1,\frac{\nu-p}{2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < \nu$.

Bivariate skew Cauchy distribution [5] has the joint pdf specified by

$$f(x,y) = \frac{1}{\pi}(1+x^2+y^2)^{-\frac{3}{2}}T_3\left((ax+by)\sqrt{\frac{3}{x^2+y^2+1}}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < a < +\infty$ and $-\infty < b < +\infty$. For this distribution,

$$f(r,\theta) = \frac{r}{\pi}(1+r^2)^{-\frac{3}{2}}T_3\left(r(a\sin\theta+b\cos\theta)\sqrt{\frac{3}{r^2+1}}\right),$$

$$f_R(r) = r(1+r^2)^{-\frac{3}{2}},$$

$$f_\Theta(\theta) = \frac{1}{2\pi} + \frac{1}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k(k+1)}{2k+1}(a\sin\theta+b\cos\theta)^{2k+1}B\left(k+\frac{3}{2},\frac{1}{2}\right)$$

and

$$E(R^p) = 2^{-1}B\left(\frac{p}{2}+1,\frac{1-p}{2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < 1$.

Standard bivariate t distribution ([22], Chapter 1) has the joint pdf specified by

$$f(x,y) = C(a^2+x^2+y^2)^{-\frac{\nu+2}{2}}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr(a^2+r^2)^{-\frac{\nu+2}{2}},$$

$$f_R(r) = 2\pi Cr(a^2+r^2)^{-\frac{\nu+2}{2}},$$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

and

$$E(R^p) = \pi Ca^{p-\nu}B\left(\frac{p}{2}+1,\frac{\nu-p}{2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < \nu$.

Conditionally specified bivariate t distribution ([22], Eq. (4.26)) has the joint pdf specified by

$$f(x,y) = C\left(a+bx^2+by^2+cx^2y^2\right)^{-\frac{\nu+1}{2}}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0, b > 0, -\infty < c < +\infty$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\left[a+br^2+\frac{c}{4}r^4\sin^2(2\theta)\right]^{-\frac{\nu+1}{2}},$$

$$f_R(r) = 2Cr\left(a+br^2\right)^{-\frac{\nu+1}{2}}\sum_{k=0}^{\infty}\binom{-\frac{\nu+1}{2}}{k}\left[\frac{cr^4}{4\left(a+br^2\right)}\right]^kB\left(\frac{1}{2},k+\frac{1}{2}\right)$$

and

$$f_\Theta(\theta) = \frac{C}{\sqrt{c}\nu a^{\nu/2}|\sin(2\theta)|}\,{}_2F_1\left(\frac{1}{2},\frac{\nu}{2};\frac{\nu}{2}+1;1-\frac{b^2}{ac\sin^2(2\theta)}\right)$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate poly t distribution [12] has the joint pdf specified by

$$f(x,y) = C\left(1+\alpha x^2+\alpha y^2\right)^{-\frac{\mu+1}{2}}\left(1+\beta x^2+\beta y^2\right)^{-\frac{\nu+1}{2}}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha > 0, \beta > 0, \mu > 0$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\left(1+\alpha r^2\right)^{-\frac{\mu+1}{2}}\left(1+\beta r^2\right)^{-\frac{\nu+1}{2}},$$

$$f_R(r) = 2\pi Cr\left(1+\alpha r^2\right)^{-\frac{\mu+1}{2}}\left(1+\beta r^2\right)^{-\frac{\nu+1}{2}},$$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

and

$$E(R^p) = \pi\beta^{-\frac{p}{2}-1}CB\left(\frac{p}{2}+1,\frac{\mu+\nu-p}{2}\right){}_2F_1\left(\frac{\mu+1}{2},\frac{p}{2}+1;\frac{\mu+\nu}{2}+1;1-\frac{\alpha}{\beta}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < \mu+\nu$.

Bivariate poly Cauchy distribution [12] has the joint pdf specified by

$$f(x,y) = C\left(1+\alpha x^2+\alpha y^2\right)^{-1}\left(1+\beta x^2+\beta y^2\right)^{-1}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha > 0$ and $\beta > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\left(1+\alpha r^2\right)^{-1}\left(1+\beta r^2\right)^{-1},$$

$$f_R(r) = 2\pi Cr\left(1+\alpha r^2\right)^{-1}\left(1+\beta r^2\right)^{-1},$$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

and

$$E(R^p) = \pi\beta^{-\frac{p}{2}-1}CB\left(\frac{p}{2}+1,\frac{2-p}{2}\right){}_2F_1\left(1,\frac{p}{2}+1;2;1-\frac{\alpha}{\beta}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p < 2$.

Bivariate heavy tailed distribution ([6], Eq. (9.22)) has the joint pdf specified by

$$f(x,y) = C\left(1+x^2\right)^{-\frac{\alpha}{2}}\left(1+y^2\right)^{-\frac{\beta}{2}}\left(1+x^2+y^2\right)^{-\frac{\gamma}{2}}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha > 0, \beta > 0$ and $\gamma > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\left(1+r^2\sin^2\theta\right)^{-\frac{\alpha}{2}}\left(1+r^2\cos^2\theta\right)^{-\frac{\beta}{2}}\left(1+r^2\right)^{-\frac{\gamma}{2}},$$

$$f_R(r) = 2\pi Cr\left(1+r^2\right)^{-\frac{\gamma+\beta}{2}}F_1\left(\frac{1}{2},\frac{\alpha}{2},\frac{\beta}{2},1;-r^2,\frac{r^2}{1+r^2}\right)$$

and

$$f_\Theta(\theta) = C\int_0^\infty\left(1+2y\sin^2\theta\right)^{-\frac{\alpha}{2}}\left(1+2y\cos^2\theta\right)^{-\frac{\beta}{2}}(1+2y)^{-\frac{\gamma}{2}}dy$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Standard symmetric bivariate Laplace distribution ([21], Eq. (5.1.2)) has the joint pdf specified by

$$f(x,y) = \frac{1}{\pi}K_0\left(\sqrt{2(x^2+y^2)}\right)$$

for $-\infty < x < +\infty$ and $-\infty < y < +\infty$. For this distribution,

$$f(r,\theta) = \frac{1}{\pi}rK_0\left(\sqrt{2}r\right),$$

$$f_R(r) = 2rK_0\left(\sqrt{2}r\right),$$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

and

$$E(R^p) = 2^{\frac{p}{2}}\Gamma^2\left(\frac{p}{2}+1\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

General symmetric bivariate Laplace distribution ([21], Eq. (5.2.2)) has the joint pdf specified by

$$f(x,y) = C\left(x^2+y^2\right)^{\frac{\nu}{2}}K_\nu\left(\sqrt{2(x^2+y^2)}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr^{1+\nu}K_\nu\left(\sqrt{2}r\right),$$

$$f_R(r) = 2\pi Cr^{1+\nu}K_\nu\left(\sqrt{2}r\right),$$

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

and

$$E(R^p) = \pi C2^{\frac{p+\nu}{2}}\Gamma\left(1+\nu+\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Asymmetric bivariate Laplace distribution ([21], Eq. (6.5.3)) has the joint pdf specified by

$$f(x,y) = C\exp(\alpha x+\beta y)\left(x^2+y^2\right)^{\frac{\nu}{2}}K_\nu\left(\gamma\sqrt{x^2+y^2}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha < 0, \beta < 0, \gamma > 0$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr^{1+\nu}\exp(\alpha r\sin\theta+\beta r\cos\theta)K_\nu(\gamma r),$$

$$f_R(r) = 2\pi Cr^{1+\nu}I_0\left(r\sqrt{\alpha^2+\beta^2}\right)K_\nu(\gamma r),$$

$$f_\Theta(\theta) = \frac{\sqrt{\pi}C(2\gamma)^\nu\Gamma(2+2\nu)}{\Gamma\left(\frac{5}{2}+\nu\right)(\gamma-\alpha\sin\theta-\beta\cos\theta)^{2+2\nu}}{}_2F_1\left(2+2\nu,\frac{1}{2}+\nu;\frac{5}{2}+\nu;-\frac{\gamma+\alpha\sin\theta+\beta\cos\theta}{\gamma-\alpha\sin\theta-\beta\cos\theta}\right)$$

and

$$E(R^p) = 2^{\nu+p+1}\pi C\gamma^{-2-\nu-p}\Gamma\left(1+\nu+\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right){}_2F_1\left(1+\nu+\frac{p}{2},1+\frac{p}{2};1;\frac{\alpha^2+\beta^2}{\gamma^2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Standard asymmetric bivariate Laplace distribution ([21], page 302) has the joint pdf specified by

$$f(x,y) = C\exp(\alpha x+\beta y)K_0\left(\gamma\sqrt{x^2+y^2}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha < 0, \beta < 0$ and $\gamma > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r,\theta) = Cr\exp(\alpha r\sin\theta+\beta r\cos\theta)K_0(\gamma r),$$

$$f_R(r) = 2\pi CrI_0\left(r\sqrt{\alpha^2+\beta^2}\right)K_0(\gamma r),$$

$$f_\Theta(\theta) = \frac{4C}{3(\gamma - \alpha\sin\theta - \beta\cos\theta)^2}\, {}_2F_1\left(2, \frac{1}{2}; \frac{5}{2}; -\frac{\gamma + \alpha\sin\theta + \beta\cos\theta}{\gamma - \alpha\sin\theta - \beta\cos\theta}\right)$$

and

$$E(R^p) = 2^{p+1}\pi C\gamma^{-2-p}\Gamma^2\left(1 + \frac{p}{2}\right){}_2F_1\left(1 + \frac{p}{2}, 1 + \frac{p}{2}; 1; \frac{\alpha^2 + \beta^2}{\gamma^2}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Bivariate poly Laplace distribution [2] has the joint pdf specified by

$$f(x, y) = C\exp(\alpha x + \beta y)(x^2 + y^2)^\lambda K_\mu\left(\gamma\sqrt{x^2 + y^2}\right)K_\nu\left(\gamma\sqrt{x^2 + y^2}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha > 0, \beta > 0, \gamma > 0, \lambda > 0, \mu > 0$ and $\nu > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r, \theta) = Cr^{1+2\lambda}\exp(\alpha r\sin\theta + \beta r\cos\theta)K_\mu(\gamma r)K_\nu(\gamma r),$$

$$f_R(r) = 2\pi Cr^{1+2\lambda}I_0\left(r\sqrt{\alpha^2 + \beta^2}\right)K_\mu(\gamma r)K_\nu(\gamma r),$$

$$f_\Theta(\theta) = C\int_0^\infty r^{1+2\lambda}\exp(\alpha r\sin\theta + \beta r\cos\theta)K_\mu(\gamma r)K_\nu(\gamma r)dr$$

and

$$\begin{aligned} E(R^p) = & \frac{2^{2\lambda+p}\pi C}{\gamma^{2+2\lambda+p}\Gamma(2+2\lambda+p)}\Gamma\left(1+\lambda+\frac{p+\mu+\nu}{2}\right)\Gamma\left(1+\lambda+\frac{p+\mu-\nu}{2}\right) \\ & \cdot\Gamma\left(1+\lambda+\frac{p-\mu+\nu}{2}\right)\Gamma\left(1+\lambda+\frac{p-\mu-\nu}{2}\right) \\ & \cdot {}_4F_3\left(1+\lambda+\frac{p+\mu+\nu}{2}, 1+\lambda+\frac{p+\mu-\nu}{2}, 1+\lambda \right. \\ & +\frac{p-\mu+\nu}{2}, 1+\lambda+\frac{p-\mu-\nu}{2}; 1+\lambda \\ & \left. +\frac{p}{2}, 1+\lambda+\frac{p+1}{2}, 1; \frac{\alpha^2+\beta^2}{4\gamma^2}\right) \end{aligned}$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Bivariate hyperbolic distribution ([6], Section 13.14) has the joint pdf specified by

$$f(x, y) = C\exp\left[-\alpha(x^2 + y^2) - \beta x - \gamma y\right]$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha > 0, \beta > 0$ and $\gamma > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r, \theta) = Cr\exp(-\alpha r^2 - \beta r\sin\theta - \gamma r\cos\theta),$$

$$f_R(r) = 2\pi Cr\exp(-\alpha r^2)I_0\left(r\sqrt{\beta^2 + \gamma^2}\right),$$

$$f_\Theta(\theta) = \frac{C}{2\alpha}\exp\left[\frac{(\beta\sin\theta + \gamma\cos\theta)^2}{8\alpha}\right]D_{-2}\left(\frac{\beta\sin\theta + \gamma\cos\theta}{\sqrt{2\alpha}}\right)$$

and

$$E(R^p) = \pi C\alpha^{-\frac{p}{2}-1}\Gamma\left(\frac{p}{2} + 1\right){}_1F_1\left(\frac{p}{2} + 1; 1; \frac{\beta^2 + \gamma^2}{4\alpha}\right)$$

for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Bivariate hyperbolic secant distribution ([6], Chapter 13) has the joint pdf specified by

$$f(x, y) = \frac{1}{4}(1 + cxy)\operatorname{sech}\left(\frac{\pi x}{2}\right)\operatorname{sech}\left(\frac{\pi y}{2}\right)$$

for $-\infty < x < +\infty, -\infty < y < +\infty$ and $-1 < c < 1$. For this distribution,

$$f(r, \theta) = \frac{r}{4}\left(1 + \frac{c}{2}r^2\sin 2\theta\right)\operatorname{sech}\left(\frac{\pi r\sin\theta}{2}\right)\operatorname{sech}\left(\frac{\pi r\cos\theta}{2}\right),$$

$$f_R(r) = \frac{r}{2}\sum_{i=0}^\infty\sum_{j=0}^\infty\frac{E_{2i}E_{2j}\pi^{2i+2j}}{(2i)!(2j)!4^{i+j}}B\left(i + \frac{1}{2}, j + \frac{1}{2}\right)r^{2i+2j},$$

$$\begin{aligned} f_\Theta(\theta) = & \pi^{-2}\sum_{i=0}^\infty\sum_{j=0}^\infty(-1)^{i+j}\left[\left(i + \frac{1}{2}\right)\sin\theta + \left(j + \frac{1}{2}\right)\cos\theta\right]^{-2} \\ & \cdot\left\{1 + 3c\pi^{-2}\sin(2\theta)\left[\left(i + \frac{1}{2}\right)\sin\theta + \left(j + \frac{1}{2}\right)\cos\theta\right]^{-2}\right\} \end{aligned}$$

if $\sin\theta\cos\theta \geqslant 0$, and

$$\begin{aligned} f_\Theta(\theta) = & \pi^{-2}\sum_{i=0}^\infty\sum_{j=0}^\infty(-1)^{i+j}\left[\left(i + \frac{1}{2}\right)\sin\theta - \left(j + \frac{1}{2}\right)\cos\theta\right]^{-2} \\ & \cdot\left\{1 + 3c\pi^{-2}\sin(2\theta)\left[\left(i + \frac{1}{2}\right)\sin\theta - \left(j + \frac{1}{2}\right)\cos\theta\right]^{-2}\right\} \end{aligned}$$

if $\sin\theta\cos\theta < 0$ for $r > 0, 0 \leqslant \theta \leqslant 2\pi$ and $p > 0$.

Conditionally specified bivariate Gumbel distribution ([6], Section 12.13.1) has the joint pdf specified by

$$f(x, y) = C\exp\left[-x - y - \exp(-x) - \exp(-y) - \theta\exp(-x - y)\right]$$

for $-\infty < x < +\infty, -\infty < y < +\infty$ and $0 < \theta < 1$, where C denotes the normalizing constant. For this distribution,

$$\begin{aligned} f(r, \theta) = & Cr\exp\left[-r\sin\theta - r\cos\theta - \exp(-r\sin\theta) - \exp(-r\cos\theta)\right. \\ & \left.-\theta\exp(-r\sin\theta - r\cos\theta)\right], \end{aligned}$$

$$f_R(r) = 2\pi Cr\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty\frac{(-1)^{i+j+k}\theta^k}{i!j!k!}I_0\left(r\sqrt{(1 + i + k)^2 + (1 + j + k)^2}\right)$$

and

$$f_\Theta(\theta) = C\sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty\frac{(-1)^{i+j+k}\theta^k}{i!j!k!}\left[(1 + i + k)\sin\theta + (1 + j + k)\cos\theta\right]^{-2}$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate logistic distribution ([6], Section 2.3.1) has the joint pdf specified by

$$f(x, y) = 2\exp(-\alpha x - \beta y)[1 + \exp(-\alpha x) + \exp(-\beta y)]^{-3}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, \alpha > 0$ and $\beta > 0$. For this distribution,

$$f(r, \theta) = 2r\exp(-\alpha r\sin\theta - \beta r\cos\theta)[1 + \exp(-\alpha r\sin\theta) + \exp(-\beta r\cos\theta)]^{-3},$$

$$f_R(r) = 4\pi r\sum_{k=0}^\infty\sum_{\ell=0}^k\binom{-3}{k}\binom{k}{\ell}I_0\left(r\sqrt{\alpha^2(\ell + 1)^2 + \beta^2(k - \ell + 1)^2}\right)$$

and

$$f_\Theta(\theta) = 2\sum_{k=0}^\infty\sum_{\ell=0}^k\binom{-3}{k}\binom{k}{\ell}[(1 + \ell)\alpha\sin\theta + (1 + k - \ell)\beta\cos\theta]^{-2}$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate Gumbel type distribution ([6], Eq. (13.27)) has the joint pdf specified by

$$f(x, y) = C\exp\left\{-a(x^2 + y^2) - b\exp\left[-a(x^2 + y^2)\right]\right\}$$

for $-\infty < x < +\infty, -\infty < y < +\infty, a > 0$ and $b > 0$, where C denotes the normalizing constant. For this distribution,

$$f(r, \theta) = Cr\exp\left\{-ar^2 - b\exp(-ar^2)\right\},$$

$$f_R(r) = 2\pi Cr \exp\left\{-ar^2 - b\exp\left(-ar^2\right)\right\},$$

$$f_\Theta(\theta) = \frac{1}{2\pi},$$

$$E(R^p) = \frac{\pi C}{a(-a)^{p/2}}\left(\frac{\partial}{\partial\alpha}\right)^{p/2}\left[b^{-\alpha-1}\gamma(\alpha+1,b)\right]\Bigg|_{\alpha=0}$$

if p is even, and

$$E(R^p) = \frac{\pi C}{a(-a)^{p/2}}\left(\frac{\partial}{\partial\alpha}\right)^{(p-1)/2}\left[\int_0^1 \sqrt{\log t}\, t^\alpha \exp(-bt)dt\right]\Bigg|_{\alpha=0}$$

if p is odd, where $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate skew elliptical distribution ([3], Section 10) has the joint pdf specified by

$$f(x,y) = Cg(x,y)H(a + bx + cy)$$

for $-\infty < x < +\infty, -\infty < y < +\infty, -\infty < a < +\infty, -\infty < b < +\infty$ and $-\infty < c < +\infty$, where C denotes the normalizing constant, g denotes a valid joint pdf and H denotes a valid univariate cdf. For this distribution,

$$f(r,\theta) = Crg(r\sin\theta, r\cos\theta)H(a + br\sin\theta + cr\cos\theta),$$

$$f_R(r) = Cr\int_0^{2\pi} g(r\sin\theta, r\cos\theta)H(a + br\sin\theta + cr\cos\theta)d\theta$$

and

$$f_\Theta(\theta) = C\int_0^\infty rg(r\sin\theta, r\cos\theta)H(a + br\sin\theta + cr\cos\theta)dr$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate Sarmanov distribution [31] has the joint pdf specified by

$$f(x,y) = g_1(x)g_2(y)\{1 + \alpha\theta_1(x)\theta_2(y)\}$$

for $-\infty < x < +\infty, -\infty < y < +\infty$ and $-1 \leqslant \alpha \leqslant 1$, where g_1, g_2 are valid probability density functions and θ_1, θ_2 are bounded nonconstant functions such that

$$\int_{-\infty}^{+\infty}\theta_1(x)g_1(x)dx = 0, \qquad \int_{-\infty}^{+\infty}\theta_2(y)g_2(y)dy = 0.$$

For this distribution,

$$f(r,\theta) = rg_1(r\sin\theta)g_2(r\cos\theta)\{1 + \alpha\theta_1(r\sin\theta)\theta_2(r\cos\theta)\},$$

$$f_R(r) = r\int_0^{2\pi} g_1(r\sin\theta)g_2(r\cos\theta)\{1 + \alpha\theta_1(r\sin\theta)\theta_2(r\cos\theta)\}d\theta$$

and

$$f_\Theta(\theta) = \int_0^\infty rg_1(r\sin\theta)g_2(r\cos\theta)\{1 + \alpha\theta_1(r\sin\theta)\theta_2(r\cos\theta)\}dr$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

Bivariate Farlie-Gumbel-Morgenstern distribution [15,18,27] has the joint pdf specified by

$$f(x,y) = g_1(x)g_2(y)\{1 + \alpha[1 - 2G_1(x)][1 - 2G_2(y)]\}$$

for $-\infty < x < +\infty, -\infty < y < +\infty$ and $-1 \leqslant \alpha \leqslant 1$, where g_1, g_2 are valid probability density functions and G_1, G_2 are the corresponding cumulative distribution functions. For this distribution,

$$f(r,\theta) = rg_1(r\sin\theta)g_2(r\cos\theta)\{1 + \alpha[1 - 2G_1(r\sin\theta)][1 - 2G_2(r\cos\theta)]\},$$

$$f_R(r) = r\int_0^{2\pi} g_1(r\sin\theta)g_2(r\cos\theta)\{1 + \alpha[1 - 2G_1(r\sin\theta)][1 - 2G_2(r\cos\theta)]\}d\theta$$

and

$$f_\Theta(\theta) = \int_0^\infty rg_1(r\sin\theta)g_2(r\cos\theta)\{1 + \alpha[1 - 2G_1(r\sin\theta)][1 - 2G_2(r\cos\theta)]\}dr$$

for $r > 0$ and $0 \leqslant \theta \leqslant 2\pi$.

4. Simulation

Simulation of R and Θ from the stated bivariate distributions is simple given there are algorithms for simulating (X, Y):

- simulate (X, Y) from the stated pdf $f(x, y)$;
- set $R = \sqrt{X^2 + Y^2}$;
- set $\Theta = \arctan(Y/X)$.

Algorithms for simulating from each of the stated bivariate distributions are available in the literature. For example, a random vector (X, Y) having the bivariate normal distribution with means $\left(\mu_x, \mu_y\right)$, variances $\left(\sigma_x^2, \sigma_y^2\right)$ and correlation coefficient ρ can be simulated as

- simulate Z_1 and Z_2 independently from a standard normal distribution;
- set $X = (1 - \rho^2)\sigma_x Z_1 + \rho\sigma_x Z_2 + \mu_x$;
- set $Y = \sigma_y Z_2 + \mu_y$.

Similarly, a random vector (X, Y) having the bivariate t distribution with means $\left(\mu_x, \mu_y\right)$, scale parameters $\left(\sigma_x^2, \sigma_y^2\right)$, correlation coefficient ρ and degree of freedom ν can be simulated as

- simulate Z_1 and Z_2 independently from a standard normal distribution;
- set $P = (1 - \rho^2)\sigma_x Z_1 + \rho\sigma_x Z_2$;
- set $Q = \sigma_y Z_2$;
- simulate U independently from a chisquare distribution with degree of freedom ν;
- set $X = \mu_x + P\sqrt{\frac{\nu}{U}}$;
- set $Y = \mu_y + Q\sqrt{\frac{\nu}{U}}$.

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