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# The converse of a theorem by Bayer and Stillman

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Bayer–Stillman showed that $reg(I) = reg(gin_\tau(I))$ when $\tau$ is the graded reverse lexicographic order. We show that the reverse lexicographic order is the unique monomial order $\tau$ satisfying $reg(I) = reg(gin_\tau(I))$ for all ideals $I$. We also show that if $gin_{\tau_1}(I) = gin_{\tau_2}(I)$ for all $I$, then $\tau_1 = \tau_2$.


## 1. Introduction

If we have a homogeneous ideal $I$ and a monomial term order $\tau$, then there is a Zariski open dense subset $U$ of coordinate transformations where the initial ideal is fixed [1,11]. This initial ideal is called the generic initial ideal denoted $gin_\tau(I)$ or simply $gin(I)$ if the monomial order is specified before. It can be shown that the generic initial ideal is Borel-fixed. Then, we can analyze the structure of $gin(I)$ by the good combinatorial properties of Borel-fixed ideals. For example, the minimal free resolution is given by the Eliahou–Kervaire theorem and the regularity is given by the maximum degree of a minimal generator [1,10]. Also, the Betti numbers of an ideal $I$ are bounded by the Betti numbers of generic initial ideals [3,5].

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A well known result of Conca on generic initial ideals is that if $I$ is Borel-fixed, then $gin_\tau(I) = I$ for any $\tau$ [5]. There are more results on the algebraic properties and the structure of specific monomial ideals [4,5,13]. In the case where $I$ is not a monomial ideal however, these methods are not directly applicable. In this paper, we introduce the notion of $\tau$-segment ideals, which is the generalization of lex-segment ideals. We show that if $in_\tau(I)$ is a $\tau$-segment ideal, then $gin_\tau(I) = in_\tau(I)$. Here, we do not require $I$ to be a monomial ideal. Consequently, we will construct an ideal which has different generic initial ideals for two given monomial orders. This implies that the generic initial ideals fully characterize monomial term orders.

When regarding the degree complexity of an ideal, the regularity of an ideal is a good invariant. An ideal $I$ is $m$-regular if the $j$th syzygy module of $I$ is generated in degrees $\leq m + j$, for all $j \geq 0$. The regularity of $I$, $reg(I)$, is defined as the least $m$ for which $I$ is $m$-regular [9]. Since graded Betti numbers are upper-semicontinuous in flat families, we have $reg(in_\tau(I)) \geq reg(I)$ for any $\tau$ [14]. However, Bayer and Stillman showed that $reg(in_\tau(I)) = reg(I)$ in general coordinates and when $\tau$ is the graded reverse lexicographic order(rlex) [1]. This means that rlex is an optimal order for the computation of the Gröbner Basis. Bayer and Stillman also suggested a method of refining monomial orders by the reverse lexicographic order, which will give faster computation [2]. We show that for any other monomial order $\tau$ besides rlex, there exists an ideal $I$ such that $reg(gin_\tau(I)) > reg(I)$. This implies that the graded reverse lexicographic order is the unique optimal monomial order that gives minimum regularity.

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## 2. Notation and terminology

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field $K$ with $charK = 0$. Let $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ be the vector notation. For a homogeneous ideal $I$, let $\mathcal{G}(I)$ be a Gröbner basis of $I$.

In this paper, we assume all monomial orders to be graded multiplicative orders with $x_1 > x_2 > \cdots > x_n$. A monomial order $\tau$ is graded if $deg(f) > deg(g)$ implies $f >_\tau g$. A monomial order $\tau$ is multiplicative if $f >_\tau g$ implies $fh >_\tau gh$. Then $fh >_\tau gh$ also implies $f >_\tau g$. Let rlex denote the graded reverse lexicographic order and lex denote the graded lexicographic order. Define the Borel order as a partial order on monomials by $fx_i >_{\text{Borel}} fx_j$ if $i < j$ and $f$ is a monomial.

Let $B = \{f_1, \dots, f_k\} \subset S_d$ be a set and $V = K\langle f_1, \dots, f_k\rangle \subset S_d$ be the vector space spanned by $B$. Then, define $in_\tau(B) = \{in_\tau(f_1), \dots, in_\tau(f_k)\}$ and $in_\tau(V) = K\langle in_\tau(f) \mid f \in V\rangle$.

**Definition 2.1.** Let $M$ be a finitely generated graded $S$-module and

$$0 \to \oplus_j S(-a_{lj}) \to \cdots \to \oplus_j S(-a_{1j}) \to \oplus_j S(-a_{0j}) \to M \to 0$$

be a minimal graded free resolution of $M$. We say that $M$ is $d$-regular if $a_{ij} \le d + i$ for all $i, j$. Let the regularity of $M$, denoted $reg(M)$, be the least $d$ such that $M$ is $d$-regular.

**Remark 2.2.** The regularity of an ideal $I$ is defined by the minimal free resolution of the following form.

$$0 \to \oplus_j S(-a_{lj}) \to \cdots \to \oplus_j S(-a_{1j}) \to \oplus_j S(-a_{0j}) \to I \to 0$$

Then the minimal free resolution of $M = S/I$ follows from that of $I$.

$$0 \to \oplus_j S(-a_{lj}) \to \cdots \to \oplus_j S(-a_{1j}) \to \oplus_j S(-a_{0j}) \to S \to S/I \to 0$$

Hence have $reg(S/I) = reg(I) - 1$. Note that if $I$ has a minimal generator of degree $d$, then $reg(I) \ge d$.

## 3. Generic initial ideals and $\tau$-segment ideals

The notion of generic initial ideals was introduced by Galligo [11]. He showed that generic initial ideals have a good combinatorial property called the Borel-fixedness. Since then, generic initial ideals have been studied extensively in commutative algebra and geometry. We introduce the theorem of Galligo. For a more detailed introduction, see [8].

**Definition 3.1.** A monomial ideal $I$ is Borel-fixed if $m \in I$ and $m\frac{x_i}{x_j} \in S$ for $i < j$ implies $m\frac{x_i}{x_j} \in I$.

**Theorem 3.2** *(Galligo, Bayer–Stillman). For a given ideal $I$ and monomial term order $\tau$, there exists a Zariski open subset $U$ of $GL_n$ such that $gin_\tau(I) := in_\tau(gI)$ is constant over all $g \in U$ and $gin_\tau(I)$ is Borel-fixed.*

We will say that $I$ is in general coordinates if $id \in U$ where $in_\tau(gI)$ is fixed for $g \in U$. For example, Conca showed for any $\tau$ if $I$ is Borel-fixed, then $gin_\tau(I) = I$ and thus $I$ is in general coordinates. However, if $I$ is not a monomial ideal, we cannot use similar methods because there is no concept of Borel-fixedness. Taking the initial ideal also does not work well because syzygy computations are not preserved under coordinate transformations. We slightly extend Conca's results to some non-monomial ideals using the notion of $\tau$-segment ideals. This is a generalization of $Seg_\tau(I)$ introduced in [6] that we do not require the ideal to be a $\tau$-segment in every degree. In the rlex case, it is also known as the weakly rlex property. We show that if $in_\tau(I)$ is a $\tau$-segment ideal, we have $gin_\tau(I) = in_\tau(I)$.

**Definition 3.3.** Let $B = \{f_1, \dots, f_k\}$ be a set of monomials with $deg(f_i) = d_i$. If $g \in B$ for all monomials $g \in S$ such that $deg(g) = d_i$ for some $i$ and $g >_\tau f$ for some $f \in B$, call $B$ a $\tau$-segment. If an ideal $I = (f_1, \dots, f_k)$ is generated by a $\tau$-segment $B = \{f_1, \dots, f_k\}$, then call $I$ a $\tau$-segment ideal.

**Example 3.4.** Let $S = K[x, y, z]$ and $\mathbf{w} = (10, 5, 3)$ be a graded weight order with tie breaking by lex. The ideal $I = (x^2, xy, y^5) \subset S$ is a $\mathbf{w}$-segment ideal generated in degrees 2 and 5. The bases of $I_2 = K\langle x^2, xy\rangle$ and $I_5 = K\langle f \mid deg(f) = 5, f \geq_{\mathbf{w}} xyz^3\rangle$ are both $\mathbf{w}$-segments. $I_3$, $I_4$ are not $\mathbf{w}$-segments since $y^3 >_{\mathbf{w}} xyz \in I_3$ and $y^4 >_{\mathbf{w}}, xyz^2 \in I_4$ but $y^3, y^4 \notin I$.

When $\tau$ is the graded lexicographic order, the lex-segment ideals have good combinatorial properties [15]. If $I$ is a lex-segment ideal, then the generating set of $I_d$ is a lex-segment for every $d$. There follows a one-to-one correspondence with lex-segment ideals and Hilbert functions satisfying a particular growth criterion by Gotzmann. For $\tau \neq$ lex, there always exists some $d$ where $I_d$ is not a $\tau$-segment. For general $\tau$, the $\tau$-segments and $\tau$-segment ideals have the following property.

**Lemma 3.5.** *Let $\tau$ be any graded monomial order.*
*(a) A $\tau$-segment is Borel fixed.*
*(b) A $\tau$-segment ideal is Borel fixed.*

**Proof.** (a) Let $B$ be a $\tau$-segment. Let $f \in B$ and $f\frac{x_i}{x_j} \in S$ for $i < j$. Then we have $f\frac{x_i}{x_j} >_\tau f$ since $x_j f\frac{x_i}{x_j} = x_i f >_\tau x_j f$. By the definition of $\tau$-segments, $f\frac{x_i}{x_j} \in B$. So $B$ is Borel-fixed.
(b) Let $I = (f_1, f_k)$ be a $\tau$-segment ideal. Suppose $F = hf_t$ is a monomial in $I$ for some $t$ and $F\frac{x_i}{x_j} = hf_t\frac{x_i}{x_j} \in S$ for $i < j$. If $f_t\frac{x_i}{x_j} \in S$, we have $f_t\frac{x_i}{x_j} \in I$ by the definition of $\tau$-segment ideals. Otherwise if $f_t\frac{x_i}{x_j} \notin S$, we have $h\frac{x_i}{x_j} \in S$. Therefore, $F = h\frac{x_i}{x_j}f_t \in I$. $\square$

Let $in_\tau(I)$ be a $\tau$-segment ideal where $I$ is a homogeneous ideal. Since $\tau$-segment ideals are Borel-fixed, $in_\tau(I)$ is already in general coordinates. Moreover, we show that if $in_\tau(I)$ is a $\tau$-segment, then $gin_\tau(I) = in_\tau(I)$. This means that $I$ is also in general coordinates.

**Lemma 3.6.** *If $in_\tau(I)$ is a $\tau$-segment ideal, then $gin_\tau(I) = in_\tau(I)$.*

**Proof.** We shall prove that $gin(I)_d = in(I_d)$ for all $d$. Let $in_\tau(I)$ be a $\tau$-segment ideal with minimal generators in degree $d_1, \dots, d_t$.

First suppose that $d = d_i$ for some $i$. Let $M_1 > M_2 > \dots$ be the total ordering of degree $d$ monomials with respect to $\tau$. Since $in(I)$ is a $\tau$-segment ideal, we have $in(I)_d = \langle M_1, \dots, M_r\rangle$ for some $r$. Then, $\wedge^r(in(I_d)) = \langle M_1 \wedge \dots \wedge M_r\rangle$. Let $g = [g_{ij}] \in GL(S_1)$ be a coordinate transformation. Since the dimensions of $I_d$ and $in(gI)_d$ are the same, the degree $d$ part of $gI$ is given by $\wedge^r(gI)_d$.

We have $\wedge^r(gI)_d = \langle g(M_1) \wedge \dots \wedge g(M_r)\rangle = \langle P_d(g_{11}, \dots, g_{nn})\ M_1 \wedge \dots \wedge M_r +$ lower terms$\rangle$ for some $P_d(g_{11}, \dots, g_{nn})$. However, $\wedge^r(in(I_d)) = \langle M_1 \wedge \dots \wedge M_r\rangle$. This is the largest standard exterior monomial in $\wedge^r(S_d)$, which means that $P_d(g_{11}, \dots, g_{nn})$ of $M_1 \wedge \dots \wedge M_r$ is nonvanishing for $g = id$. Hence $U_d = \{g \mid P_d(g_{11}, \dots, g_{nn}) \neq 0\}$ is a nonempty Zariski open subset where $in(gI)$ is fixed. Therefore $gin(I)_d = in(I_d)$.

Now let $d \neq d_1, \dots, d_t$. Since there are no degree $d$ elements of the Gröbner basis, we have $in(I_d) = in(I_{d-1})S_1$. Then, $gin(I)_d \supset gin(I)_{d-1}S_1 = in(I_{d-1})S_1 = in(I_d)$. Since $in(I)$ and $gin(I)$ have the same dimension in every degree, we have $gin(I)_d = in(I_d)$. Since $gin(I)_d = in(I_d)$ for every $d$, we conclude that $gin(I) = in(I)$. $\square$

**Remark 3.7.** Even if $in(I)$ is Borel-fixed, $gin(I)$ may differ from $in(I)$. Let $S = K[x, y, z]$ and $I = (x^3, x^2y + xy^2, x^2z)$. Then $in_{\text{rlex}}(I) = (x^3, x^2y, x^2z, xy^3, xy^2z)$ but $gin_{\text{rlex}}(I) = (x^3, x^2y, xy^2, x^2z^2)$.

Now we have a class of ideals which are already in general coordinates. We use this lemma for the proof of our main results. The following theorem shows that generic initial ideals fully characterize monomial orders.

**Theorem 3.8.** *$gin_{\tau_1}(I) = gin_{\tau_2}(I)$ for all ideals $I \subset S$, if and only if $\tau_1 = \tau_2$.*

**Proof.** One way is trivial. For the other way, we show that if $\tau_1 \neq \tau_2$ then there exists some $I$ such that $gin_{\tau_1}(I) \neq gin_{\tau_2}(I)$. Let $x_1^d = M_1 >_{\tau_1} M_2 >_{\tau_1} \dots$ be the total ordering of degree $d$ monomials with respect to $\tau_1$ and $x_1^d = M'_1 >_{\tau_2} M'_2 >_{\tau_2} \dots$ be the total ordering of degree $d$ monomials with respect to $\tau_2$. Let $k$ be the least integer such that $M_k \neq M'_k$. Define the ideal $I = (M_1, \dots, M_{k-1}, M_k + M'_k)$.

By symmetry, it suffices to show that $gin_{\tau_1}(I_d) = (M_1, \dots, M_{k-1}, M_k)$. We use Buchberger's algorithm on $I$. Since $I$ is generated by degree $d$ homogeneous elements, all syzygies have degree larger than $d$. Then, $in_{\tau_1}(I)_d$ is generated by the initial parts of the degree $d$ elements of the Gröbner basis. These are just the initial terms of the generators of $I$. Then $in_{\tau_1}(I)_d = \langle M_1, \dots, M_k\rangle$. Since $M_1, \dots, M_k$ are the largest $k$ monomials in degree $d$ with respect to $\tau_1$, $in_{\tau_1}(I_d)$ is a $\tau_1$-segment. By Lemma 3.6, we have $gin_{\tau_1}(I)_d = in_{\tau_1}(I_d) = \langle M_1, \dots, M_k\rangle$. $\square$

## 4. The reverse lexicographic order

We have $reg(I) = reg(gI)$ for any ideal $I$ and a coordinate transformation $g \in GL_n$ because the Betti tables of $I$ and $gI$ coincide. However, taking the initial ideal does not commute with coordinate transformation because syzygy calculations are not preserved under coordinate transformations.

Where $reg(I) \leq reg(in_\tau(I))$ for any order $\tau$, the following theorem of Bayer and Stillman shows that the graded reverse lexicographic order gives the lowest possible regularity for generic initial ideals.

**Theorem 4.1** *(Bayer–Stillman). [1] If $I$ is a homogeneous ideal, then $reg(I) = reg(gin_{\mathrm{rlex}}(I))$.*

Thus the graded reverse lexicographic order is an optimal order in Gröbner basis computation. Conversely, we show that if $reg(I) = reg(gin_\tau(I))$ for all ideals $I \subset S$, then $\tau = \mathrm{revlex}$. This proves the unique optimality of the graded reverse lexicographic order in Gröbner basis computation. However, this does not show that general coordinates give the lowest regularity. If $I = (x^2+y^2, xyz) \subset S = K[x, y, z]$, we have $reg(in_{\mathrm{lex}}(I)) = 4$ but $reg(gin_{\mathrm{lex}}(I)) = 5$. Before the main theorem, we give a characterization of the graded reverse lexicographic order.

**Lemma 4.2.** $\tau = \mathrm{rlex}$ *if and only if $x_{k-1}^{d+1} > x_1^d x_k$ for all $k, d$.*

**Proof.** One way is trivial. We show that if $x_{k-1}^{d+1} >_\tau x_1^d x_k$ for all $k$, then $\tau$ is the reverse lexicographic order. Let $f = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$, $g = x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$ be degree $d+1$ polynomials. If $K$ is the largest $i$ such that $a_i \neq b_i$, let $a_K < b_K$. We show that $f >_\tau g$.

Since $\tau$ is multiplicative, the term order is preserved under factoring out common terms. We factor out $c = x_K^{a_K}$. Any monomial order $\tau$ with $x_1 >_\tau \cdots >_\tau x_n$ includes the Borel order in the way that if $M >_{Borel} N$ then $M >_\tau N$. We have $f/c = x_1^{a_1} \ldots x_{K-1}^{a_{K-1}} >_\tau x_{K-1}^{d+1-a_K} >_\tau x_1^{d-a_K} x_K >_\tau x_1^{b_1} x_2^{b_2} \ldots x_K^{b_K - a_K} = g/c$. Therefore, $f >_\tau g$. This is the defining property of the reverse lexicographic order. Hence $\tau$ is the reverse lexicographic order. □

**Lemma 4.3** *(Conca). [5] Let $I$ be a Borel-fixed ideal and let $m_1, \ldots, m_k$ be its monomial generators. Let $g \in GL_n$ be a generic matrix. Then $gI$ is generated by polynomials $f_1, \ldots, f_k$ of the form $f_i = m_i + h_i$ such that the monomials in $h_i$ are smaller than $m_i$ in the Borel-order. The polynomials $f_1, \ldots, f_k$ form a Gröbner basis of $gI$ with respect to any term order.*

Now we prove our main theorem.

**Theorem 4.4.** *If $reg(gin_\tau(I)) = reg(I)$ for all homogeneous ideals $I \subset S$, then* $\tau = \mathrm{rlex}$.

**Proof.** Suppose $\tau \neq$ rlex. By Lemma 4.2, there exist some $k$, $d$ such that $x_1^d x_k > x_{k-1}^{d+1}$. We show that $reg(gin_{\mathrm{rlex}}(I)) \neq reg(gin_\tau(I))$ for the ideal $I = (x_1^{d+1}, \ldots, x_{k-2}x_{k-1}^d, x_{k-1}^{d+1} + x_1^d x_k)$. This ideal $I$ is generated by $x_{k-1}^{d+1} + x_1^d x_k$ and all degree $d+1$ monomials in $K[x_1, \ldots, x_{k-1}]$ except $x_{k-1}^{d+1}$.

First, consider the graded reverse lexicographic case. Let $x_1^{d+1} = M_1 >_{\mathrm{rlex}} M_2 >_{\mathrm{rlex}} \cdots >_{\mathrm{rlex}} M_{L+1} = x_{k-1}^{d+1}$ be the total ordering of degree $d+1$ monomials in $K[x_1, \ldots, x_{k-1}]$. Then we can write $I = (M_1, \ldots, M_L, x_{k-1}^{d+1} + x_1^d x_k)$. We use Buchberger's algorithm and show that no syzygy is added to the Gröbner basis. The syzygies for the first $L$ generators are 0. Also for any possible syzygy $S = f_1 M_i - f_2(x_{k-1}^{d+1} + x_1^d x_k) = f_2 x_1^d x_k$, we have $f_2 x_1^d x_k \in (x_1, \ldots, x_{k-1})^{d+1}$ since $f_2 \mid M_i$ and $M_i \in (x_1, \ldots, x_{k-1})$. Therefore, $\{M_1, \ldots, M_L, x_{k-1}^{d+1} + x_1^d x_k\}$ is a Gröbner basis of $I$. Consequently, $in_{\mathrm{rlex}}(I) = (x_1, \ldots, x_{k-1})^{d+1}$. Since this is a rlex-segment ideal, we have $gin_{\mathrm{rlex}}(I) = (x_1, \ldots, x_{k-1})^{d+1}$ by Lemma 3.6. Then $reg(gin_{\mathrm{rlex}}(I)) = d+1$, which is the maximum degree of a minimal generator of $gin_{\mathrm{rlex}}(I)$.

Now, let $\tau \neq$ rlex with $x_1^d x_k >_\tau x_{k-1}^{d+1}$. Let $I' = (M_1, \ldots, M_L)$ and $M_0 = x_1^d x_k + x_{k-1}^{d+1}$. Then, $in_\tau(g(\wedge^{L+1} I_{d+1})) = in_\tau(g(M_1) \wedge g(M_2) \wedge \cdots \wedge g(M_L) \wedge g(M_0))$. Take $g$ a general coordinate for $I_{d+1}$ and $I'_{d+1}$. Since $I'$ is Borel-fixed, $in_\tau(g(\wedge^L I'_{d+1})) = M_1 \wedge \cdots \wedge M_L$. This means that $g(M_1) \wedge \cdots \wedge g(M_L) = P(g)(M_1 \wedge \cdots \wedge M_L) + (\text{lower terms})$ for $P(g) \not\equiv 0$. We take a generic $g$ such that $g(M_0)$ has nonzero coefficients for all degree $d+1$ monomials. This can be done by expanding $g(M_0)$ and taking the coordinate transformation avoiding the zero locus of each coefficient of the monomial terms. Since $x_1^d x_k$ is the largest degree $d+1$ monomial besides $M_1, \ldots, M_L$, we obtain $in_\tau(g(\wedge^{L+1} I_{d+1})) = M_1 \wedge \cdots \wedge M_L \wedge x_1^d x_k$. This exterior monomial may not be in standard form because we don't know the order in $\tau$.

We observe that $S = x_{k-1}^{d+2} = x_{k-1}(x_1^d x_k + x_{k-1}^{d+1}) - x_k(x_1^d x_{k-1}) \in I$. Then we add this redundant basis so that $I = (M_1, \ldots, M_L, M_0, x_{k-1}^{d+2})$. Let $J = (M_1, \ldots, M_L, x_{k-1}^{d+2})$ then $J$ is Borel-fixed. By Lemma 4.3, $\mathcal{G}(g(J)) = \{M_1 + N_1, \ldots, M_L + N_L, x_{k-1}^{d+2} + N_{L+1}\}$ where the $N_i$ are linear sums of terms smaller than $M_i$ in Borel-order. Then $gI = (M_1 + N_1, \ldots, M_L + N_L, g(M_0), x_{k-1}^{d+2} + N_{L+1})$.

Since we have shown that $in_\tau(g(\wedge^{L+1} I_{d+1})) = M_1 \wedge \cdots \wedge M_L \wedge x_1^d x_k$, we rewrite this as $gI = (M_1 + N_1, \ldots, M_L + N_L, x_1^d x_k + N_0, x_{k-1}^{d+2} + N_{L+1})$. The syzygy $S = x_{k-1}(x_1^d x_k + x_{k-1}^{d+1}) - x_k(x_1^d x_{k-1}) = x_{k-1}^{d+2}$ in $I$ is not reducible by $M_1, \ldots, M_L, M_0$ using $\tau$. Since the initial terms of the generators of $gI$ and $I$ coincide, we also cannot reduce $x_{k-1}^{d+2} + N_{L+1}$ by lower degree generators of $gI$. Hence, this is a proper Gröbner basis element of $gI$. Consequently, $gin_\tau(I) = in_\tau(gI)$ has a generator of degree $d+2$ and therefore has regularity $\geq d+2$. $\square$

**Example 4.5.** Let $K$ be a field with any characteristic. Let $S = K[x_1, \ldots, x_6]$ and $I = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3 + x_1^2 x_3)$. Then, $gin_{\mathrm{lex}}(I) = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_1^3 x_3) + (x_2^4)$ and $gin_{\mathrm{rlex}}(I) = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)$. We have $reg(gin_{\mathrm{lex}}(I)) = 4$ and $reg(gin_{\mathrm{rlex}}(I)) = 3$.

Using the theorem, we directly obtain the converse statement of Bayer and Stillman.

**Corollary 4.6.** *If $reg(gin_\tau(I)) = reg(I)$ for all ideals $I \subset S$, then $\tau = \text{rlex}$.*

**Proof.** This follows from the result of Bayer–Stillman: $reg(gin_{\text{rlex}}(I)) = reg(I)$ [1]. $\square$

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