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Fluctuations of the fission energy generated in a multiplying system

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ABSTRACT

The purpose of this work is to elaborate a master equation formalism for the evolution of the probability distribution of the cumulative energy generated by fissions in a multiplying system with delayed neutrons. The formalism accounts for the fact that the fission energy χ is also a random variable, thus the fluctuations of the total energy generated are due to both the fluctuations of the number of fissions, as well as to the fluctuations of the energy per fission. By comparing to the case where the fission energy is taken as constant, the significance of the fluctuations of the fission energy can be assessed. The first two moments of the cumulative fission energy are determined explicitly, and the time dependence of the expectation and the variance is calculated for different reactivities. As expected, the variance of the energy per fission does not play a significant role in the variance of the cumulative fission energy.

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1. Introduction

Regarding the random character of neutron multiplication in subcritical systems driven by an extraneous source, so far only the fluctuations of the discrete random variable representing the detector counts have been investigated (Feynman et al., 1956; Williams, 1974; Pázsit and Pál, 2008). For instance, the subcritical reactivity can be determined from the dependence of the variance to mean of the number of detections on the measurement time period T. Theoretically, one can also calculate the moments of the total number of absorptions or fissions; however, these cannot be measured.

A continuous random variable of interest is the cumulative energy developed in a chain started by one neutron, or in a subcritical system driven by a source. In particular, the fluctuations of the cumulative energy by fissions in a slightly supercritical system driven by a weak source, or the fluctuations of the time it takes for the system to generate a certain amount of energy, can be of interest, since these can show large variations (Williams, 1974).

The question can also be interesting from the practical point of view. Safety regulations determine how much power a reactor may be operated with. Reactors are operated with constant power; however, the generated energy is a random variable with a certain variance. To ensure with a certain confidence that one does not exceed the allowed power limit, one has to operate the reactor with a mean power which is below the nominal value with two or three standard deviations of the fluctuations of the generated power. Intuitively these fluctuations can be expected to be small, but it is advisable to investigate the question theoretically.

If the energy generated in fission was constant, then the cumulative generated energy would be also a discrete variable. This variable would be proportional to the number of fissions in the system, the (constant) fission energy being the scaling factor. This is then a classic case, and correspondingly, results on the moments of the number of fissions in such systems are found in the literature (Pázsit and Pál, 2008).

The situation becomes though different if the fission energy becomes a continuous random variable. In this case the fluctuations in the energy generated in the fission chain will fluctuate not only because of the fluctuations in the number of fissions during the time period concerned, but also due to the fluctuations in the energy generated in the individual fissions. It is an interesting question how much the existence of the fluctuations of the fission energy influences the fluctuations of the energy generated.

To incorporate the fluctuations of the fission energy, the traditional treatment has to be modified from handling only discrete random variables to be able to accommodate also the continuous energy variable. Such a formalism has become recently available. In connection with the description of the stochastic properties of fission chamber signals, a master equation technique was elaborated by the present authors for the distribution of the sum of continuous random functions, associated with the discrete detection events, for both independent and correlated events (Pál et al., 2014; Pál and Pázsit, 2015). The same methodology, used for the description of the statistics of neutron detector signals, with due modifications, can also be used here to calculate the cumulative fission energy generated in a fission chain initiated either by a single

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neutron or by an extraneous source, as it will be developed in the forthcoming sections. In the numerical computations the well known 200 MeV mean energy per fission has been used, while the second moment of energy per fission has been estimated from the data published in the literature (Crouch, 1977). A comparison with the case of constant fission energy makes it possible to assess the significance of the fluctuations of the fission energy in the fluctuations of the total cumulative energy developed in a fission chain.

2. Basic theory

The novel aspect of the present work is to account for the fluctuations of the fission energy when calculating the distribution of the total energy generated in a fission chain. Denote the energy released in one fission reaction by the random variable $\boldsymbol{\chi}$, and the probability distribution function of this random variable by

$$\mathscr{P}\{\boldsymbol{\chi} \leqslant E\} = H(E) = \int_0^E h(E')\,dE'. \tag{1}$$

It is assumed here that all energy production per fission is prompt, whereas in reality about 6% is delayed. The concrete analytical form of the probability density function $h(E)$ is not known, but this does not bring about difficulties, because in the quantitative calculations only the first moment

$$Q_1 = \int_0^\infty E\,h(E)\,dE$$

and the second moment

$$Q_2 = \int_0^\infty E^2\,h(E)\,dE$$

are needed.

Define the random function $\boldsymbol{\eta}(t)$ as the cumulative fission energy at the time instant $t \geqslant 0$. Introduce the probabilities

$$\mathscr{P}\{\boldsymbol{\eta}(t) \leqslant E,\, t \mid \boldsymbol{n}(0) = 1\} = P(E, t \mid \boldsymbol{n}(0) = 1) = \int_0^E p(E', t \mid \boldsymbol{n}(0) = 1)\,dE' \tag{2}$$

and

$$\mathscr{P}\{\boldsymbol{\eta}(t) \leqslant E,\, t \mid \boldsymbol{c}(0) = 1\} = P(E, t \mid \boldsymbol{c}(0) = 1) = \int_0^E p(E', t \mid \boldsymbol{c}(0) = 1)\,dE' \tag{3}$$

that the random sum of the fission energy released in the time interval $(0, t]$ is not larger than E, provided that at time $t = 0$ one neutron and no precursors, as well as one precursor and no neutrons were present in the multiplying system, respectively.

The derivation of the backward equation for $p(E, t \mid \boldsymbol{n}(0) = 1)$ goes as follows. If at $t = 0$ one single neutron exists in the system, then in the time interval $(0, t]$ the following three mutually exclusive events can take place:

- the neutron will not have any reaction;
- on its first reaction, the neutron is captured in the subcritical medium with intensity λ_c;
- the first reaction of the neutron is a fission in the subcritical medium with intensity λ_f.

The total intensity of a reaction in the system is $\lambda_r = \lambda_c + \lambda_f$. Further, denote by $f(k, \ell)$ the probability that in a fission reaction $k \geqslant 0$ neutrons and $\ell \geqslant 0$ precursors of the same type are produced, and assume that the number of neutrons and that of precursors are independent, i.e.

$$f(k, \ell) = f_k^{(p)} f_\ell^{(d)}. \tag{4}$$

For later use, introduce the generating functions

$$q^{(p)}(z) = \sum_{k=0}^\infty f_k^{(p)} z^k \quad \text{and} \quad q^{(d)}(z) = \sum_{\ell=0}^\infty f_\ell^{(d)} z^\ell. \tag{5}$$

With these preliminaries, the following integral backward equation can be written down for the distribution $p(E, t \mid \boldsymbol{n}(0) = 1)$:

$$\begin{aligned} p(E,t \mid \boldsymbol{n}(0)=1) &= e^{-\lambda_r t}\,\delta(E) + \lambda_c \int_0^t e^{-\lambda_r(t-t')}\,\delta(E)\,dt' \\ &+ \lambda_f \int_0^t e^{-\lambda_r(t-t')}\, f^{(p)}(k) f^{(d)}(\ell) \times h(E_0)\, U_k(E_1, t' \\ &\mid \boldsymbol{n}(0)=1)\, V_\ell(E_2, t' \mid \boldsymbol{c}(0) \\ &= 1)\, dE_0\, dE_1\, dE_2\, dt', \sum_k \sum_\ell \iiint_{E_0+E_1+E_2=E} \end{aligned} \tag{6}$$

where

$$U_k(E_1, t' \mid \boldsymbol{n}(0) = 1) = [1 - \Delta(k)]\,\delta(E_1) + \Delta(k) \times \int \cdots \int_{E_{11}+\cdots+E_{1k}=E_1} \prod_{j=1}^k p(E_{1j}, t' \mid \boldsymbol{n}(0) = 1)\,dE_{1j} \tag{7}$$

and

$$V_\ell(E_2, t' \mid \boldsymbol{c}(0) = 1) = [1 - \Delta(\ell)]\,\delta(E_2) + \Delta(\ell) \times \int \cdots \int_{E_{21}+\cdots+E_{2\ell}=E_2} \prod_{j=1}^\ell p(E_{2j}, t' \mid \boldsymbol{c}(0) = 1)\,dy_{2j}. \tag{8}$$

Here, $U_k(E_1, t' \mid \boldsymbol{n}(0) = 1)$ stands for the probability that the k prompt neutrons, generated by the single starting neutron in the fission at time $t - t'$, will jointly generate E_1 cumulative energy during the time interval t', whereas $V_\ell(E_2, t' \mid \boldsymbol{c}(0) = 1)$ is the same for the l delayed neutrons generated in the same fission to generate a cumulative energy E_2. Note that in (6) the summation for k and ℓ starts from zero, because even if the number of the neutrons, or of the precursors in the first fission reaction is zero, hence the corresponding chain dies out, this single fission will already generate a random energy larger than zero.

In a similar manner, taking into account the two mutually exclusive events that the delayed neutron precursor will not decay or will decay with intensity λ, the following equation can be derived for the case when the branching process is started by one precursor:

$$p(E, t \mid \boldsymbol{c}(0) = 1) = e^{-\lambda t}\,\delta(E) + \lambda \int_0^t e^{-\lambda(t-t')}\, p(E, t' \mid \boldsymbol{n}(0) = 1)\,dt', \tag{9}$$

which connects the density function $p(E, t \mid \boldsymbol{c}(0) = 1)$ with $p(E, t \mid \boldsymbol{n}(0) = 1)$.

Define the characteristic functions by Laplace transforms:

$$\tilde{h}(\omega) = \int_0^\infty e^{-\omega E}\, h(E)\,dE, \tag{10}$$

$$g(\omega, t \mid \boldsymbol{n}(0) = 1) = \int_0^\infty e^{-\omega E}\, p(E, t \mid \boldsymbol{n}(0) = 1)\,dE \tag{11}$$

and

$$g(\omega, t \mid \boldsymbol{c}(0) = 1) = \int_0^\infty e^{-\omega E}\, p(E, t \mid \boldsymbol{c}(0) = 1)\,dE. \tag{12}$$

From (6) and (9) one obtains the equations of characteristic functions $g(\omega, t \mid \boldsymbol{n}(0) = 1)$ and $g(\omega, t \mid \boldsymbol{c}(0) = 1)$ in the following form:

$$g(\omega,t \mid \boldsymbol{n}(0)=1) = e^{-\lambda_r t} + \lambda_c \int_0^t e^{-\lambda_r(t-t')}\, dt' + \lambda_f \int_0^t e^{-\lambda_r(t-t')}\, \widetilde{h}(\omega)\, q^{(p)} \times [g(\omega,t' \mid \boldsymbol{n}(0)=1)]\, q^{(d)}[g(\omega,t' \mid \boldsymbol{c}(0)=1)], \quad (13)$$

and

$$g(\omega,t \mid \boldsymbol{c}(0)=1) = e^{-\lambda t} + \lambda \int_0^t e^{-\lambda(t-t')}\, g(\omega,t' \mid \boldsymbol{n}(0)=1)\, dt'. \quad (14)$$

It is seen that the dynamics of the branching is compressed into the non-linear functions $q^{(p)}[\ldots]$ and $q^{(d)}[\ldots]$.

2.1. Expectation of the cumulative fission energy generated by one neutron

The expectation of the cumulative energy generated in a chain induced by one neutron during the time interval $(0,t]$ can be determined from the characteristic function (6) as

$$\mathbf{E}\{\boldsymbol{\eta}(t)|\boldsymbol{n}(0)=1\} = m_1(t,|\boldsymbol{n}(0)=1) = -\left[\frac{\partial g(\omega,t\,|\boldsymbol{n}(0)=1)}{\partial\omega}\right]_{\omega=0}. \quad (15)$$

After elementary calculations one obtains the following integral equation:

$$m_1(t \mid \boldsymbol{n}(0)=1) = \lambda_f \int_0^t e^{-\lambda_r(t-t')} \Big[Q_1 + q_1^{(p)}\, m_1(t' \mid \boldsymbol{n}(0)=1) + q_1^{(d)}\, m_1(t' \mid \boldsymbol{c}(0)=1)\Big]\, dt', \quad (16)$$

where the expectation $m_1(t' \mid \boldsymbol{c}(0)=1)$ is obtained from Eq. (14) as

$$m_1(t' \mid \boldsymbol{c}(0)=1) = \lambda \int_0^t e^{-\lambda(t-t')}\, m_1(t' \mid \boldsymbol{n}(0)=1)\, dt'. \quad (17)$$

In (16) the following notations were introduced:

$$Q_1 = -\left[\frac{\partial \widetilde{h}(\omega)}{\partial\omega}\right]_{\omega=0}, \quad q_1^{(p)} = \left[\frac{dq^{(p)}(z)}{dz}\right]_{z=1} = v_p \quad \text{and} \quad q_1^{(d)} = \left[\frac{dq^{(d)}(z)}{dz}\right]_{z=1} = v_d. \quad (18)$$

The integral Eq. (16) can be readily solved by applying the method of Laplace transformation. Using the expressions

$$\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1) = \int_0^\infty e^{-st}\, m_1(t \mid \boldsymbol{n}(0)=1)\, dt \quad (19)$$

and

$$\widetilde{m}_1(s \mid \boldsymbol{c}(0)=1) = \int_0^\infty e^{-st}\, m_1(t \mid \boldsymbol{c}(0)=1)\, dt, \quad (20)$$

respectively, one obtains from (16) the following algebraic equation:

$$\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1) = \frac{\lambda_f}{s+\lambda_r}\left[\frac{Q_1}{s} + q_1^{(p)}\,\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1) + q_1^{(d)}\,\widetilde{m}_1(s \mid \boldsymbol{c}(0)=1)\right], \quad (21)$$

where

$$\widetilde{m}_1(s \mid \boldsymbol{c}(0)=1) = \frac{\lambda}{s+\lambda}\,\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1). \quad (22)$$

Applying the conventional notations:

$$q_1^{(p)} = v_p = (1-\beta)\,v, \qquad \text{and} \qquad q_1^{(d)} = v_d = \beta\,v, \quad (23)$$

where

$$v = v_p + v_d \qquad \text{and} \qquad \beta = \frac{v_d}{v} \approx 0.0064, \quad (24)$$

and remembering that

$$\lambda_r = \lambda_c + \lambda_f, \quad (25)$$

by also accounting for (22), Eq. (21) can be rewritten in the form:

$$\left\{s + \lambda_c + \lambda_f\left[1 - (1-\beta)\,v - \beta\,v\,\frac{\lambda}{s+\lambda}\right]\right\}\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1) = \frac{Q_1}{v\Lambda}\frac{1}{s}, \quad (26)$$

where

$$\Lambda = \frac{1}{v\,\lambda_f} \quad (27)$$

is the prompt neutron generation time. Introducing the reactivity

$$\rho = \frac{(v-1)\,\lambda_f - \lambda_c}{v\,\lambda_f}, \quad (28)$$

after some manipulations Eq. (26) is obtained as

$$\frac{(s+\lambda)\,(s+\frac{\beta-\rho}{\Lambda}) - \lambda\,\frac{\beta}{\Lambda}}{s+\lambda}\,\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1) = \frac{Q_1}{v\Lambda}\frac{1}{s}. \quad (29)$$

By using the traditional notation

$$\alpha = \frac{\beta-\rho}{\Lambda}, \quad (30)$$

for the prompt neutron decay constant, the negative roots of the equation

$$(s+\lambda)\,(s+\alpha) - \lambda\,\frac{\beta}{\Lambda} = s^2 + (\lambda+\alpha)\,s - \lambda\,\frac{\rho}{\Lambda} = 0 \quad (31)$$

are given by

$$s_1 = \frac{1}{2}\left[\lambda + \alpha + \sqrt{(\lambda+\alpha)^2 + 4\,\frac{\lambda\rho}{\Lambda}}\right] \quad (32)$$

and

$$s_2 = \frac{1}{2}\left[\lambda + \alpha - \sqrt{(\lambda+\alpha)^2 + 4\,\frac{\lambda\rho}{\Lambda}}\right]. \quad (33)$$

Hence, from Eq. (29) one obtains

$$\widetilde{m}_1(s \mid \boldsymbol{n}(0)=1) = \frac{Q_1}{v\Lambda}\frac{1}{s}\frac{s+\lambda}{(s+s_1)\,(s+s_2)}. \quad (34)$$

The inverse Laplace transform of (34) is obtained as

$$m_1(t\,|\,\boldsymbol{n}(0)=1) = \frac{Q_1\lambda}{\Lambda v s_1 s_2} + \frac{Q_1}{\Lambda v(s_1-s_2)}\left[\left(1-\frac{\lambda}{s_2}\right)e^{-s_2 t} - \left(1-\frac{\lambda}{s_1}\right)e^{-s_1 t}\right], \quad (35)$$

$\rho \neq 0$,

which gives the expectation of the cumulative fission energy in the time interval $(0,t]$ in a non-critical multiplying system, generated by one neutron injected at $t=0$.

In order to show some quantitative results, we need to choose the first two moments Q_1 and Q_2, respectively. As mentioned already, in the numerical calculations we will use the value $Q_1 = 0.2$ GeV. For the second moment Q_2, by applying the data of fission-product yields of thermal neutron-induced fissions of ^{235}U (Crouch, 1977), we estimated an approximate value $Q_2 \approx 0.0400162$ GeV^2. This indicates that the variance of the energy generated per fission, i.e. $Q_2 - Q_1^2$, is rather small: the relative standard deviation is approximately 2%. This can be expected on physical grounds: the total number of nucleons in the primary fission products varies only about 1%, because of the variation in the number of prompt free neutrons, and the binding energy per nucleon in the relevant range of mass numbers is an almost linear function of the mass number.

Fig. 1 shows the time dependence of the expectation of the cumulative fission energy generated by one injected neutron for three different reactivities of the multiplying medium. The time dependence is characterised by two different domains. For short times, the effect of the prompt chain is seen which, after the chain died out, leads to a plateau of the cumulative energy generated. The second region starts at times corresponding to the time constant of the delayed neutron precursors. When the delayed chain also dies out, the cumulative energy reaches a second, higher plateau, which is equal to the asymptotic value of the cumulative energy generated by one injected neutron. It is interesting to note that in systems close to critical, the delayed chain generates larger energy (through generating more fissions) than the prompt chain. For $\rho = -0.003$, the energy generated in the delayed chain is twice as large as in the prompt chain.

For further insight, the initial time dependence of the process is shown in Fig. 2. It is worth to note that in the present calculations, the asymptotic mean of the total fission energy generated by one neutron at reactivities $\rho = -0.007, -0.005$ and -0.003 equals to 12, 16, and 27 GeV, respectively.

It is also worth to note the influence of the delayed neutrons, which manifests itself in Fig. 1 by the two plateaus at two different time scales. For systems close to critical, the cumulative fission energy remains small for an initial time period, up to the decay time of the precursors, after which a second, larger increase of the cumulative energy follows. This is rather different from the case when the process is entirely based on the prompt neutrons only. For an illustration, the time dependence of the mean cumulative fission energy in a multiplying system without precursors is shown for three negative reactivities in Fig. 3.

For critical systems, that is if

$$\rho = 0,$$

one obtains

$$m_1^{(cr)}(t \mid \boldsymbol{n}(0) = 1) = Q_1 \frac{\beta + (\beta + \lambda\Lambda)\lambda t - \beta \exp\{-(\lambda + \beta/\Lambda)t\}}{\nu(\beta + \lambda\Lambda)^2}, \tag{36}$$

which shows that after a sufficiently large time interval, the expectation of the cumulative fission energy increases unbounded in a linear manner.

To show the temporal evolution of the expectation of the cumulative fission energy in a critical system, a quantitative realisation of Eq. (36) is shown in Fig. 4, and its initial part is shown in Fig. 5.

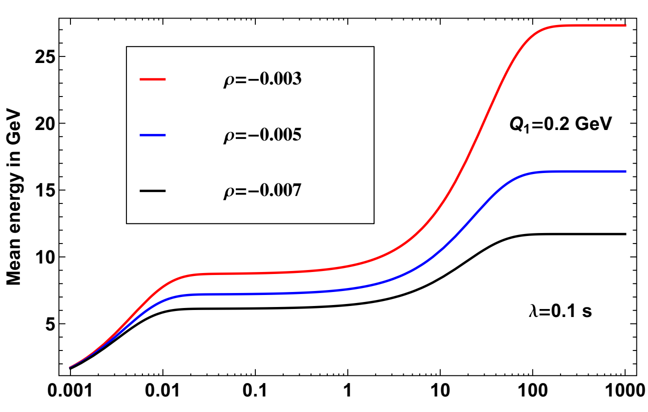

Fig. 1. Time dependence of the expectation of the cumulative fission energy generated by one neutron injected into a subcritical multiplying system with different reactivities.

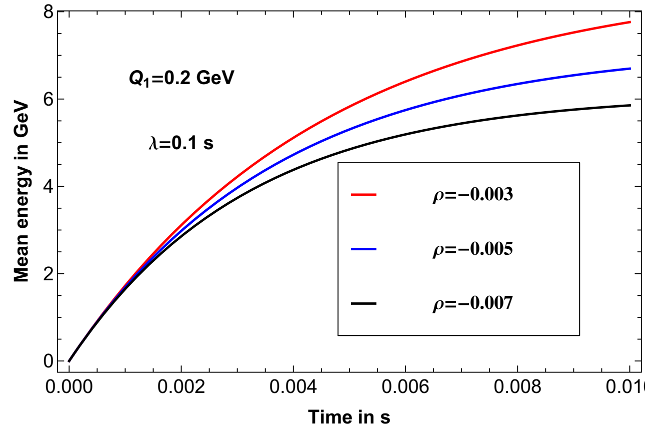

Fig. 2. The initial time dependence of the expectation of the cumulative fission energy.

For subcritical systems, accounting for the relation

$$s_1 s_2 = -\lambda \frac{\rho}{\Lambda},$$

one obtains

$$\lim_{t\to\infty} m_1(t \mid \boldsymbol{n}(0) = 1) = \frac{Q_1 \lambda}{\Lambda \nu s_1 s_2} = -\frac{Q_1}{\nu \rho}, \qquad \rho < 0. \tag{37}$$

In contrast, one notes that $\lim_{t\to\infty} m_1(t \mid \boldsymbol{n}(0) = 1) = \infty$, when $\rho \geqslant 0$.

2.2. Variance of the cumulative fission energy generated by one neutron

A suitable quantity to characterise the stochastic behaviour of the cumulative fission energy generated by one neutron in a multiplying system is the variance

$$\begin{aligned}\mathbf{D}^2\{\boldsymbol{\eta}(t) \mid \boldsymbol{n}(0) = 1\} &= \mathbf{E}\{\boldsymbol{\eta}^2(t) \mid \boldsymbol{n}(0) = 1\} \\ &\quad - [\mathbf{E}\{\boldsymbol{\eta}(t) \mid \boldsymbol{n}(0) = 1\}]^2 \\ &= m_2(t \mid \boldsymbol{n}(0) = 1) - [m_1(t \mid \boldsymbol{n}(0) = 1)]^2. \end{aligned} \tag{38}$$

Since the expectation $m_1(t \mid \boldsymbol{n}(0) = 1)$ has already been determined in (35), to obtain the variance it remains to calculate the second moment $m_2(t \mid \boldsymbol{n}(0) = 1)$. This can be calculated from Eq. (13) as

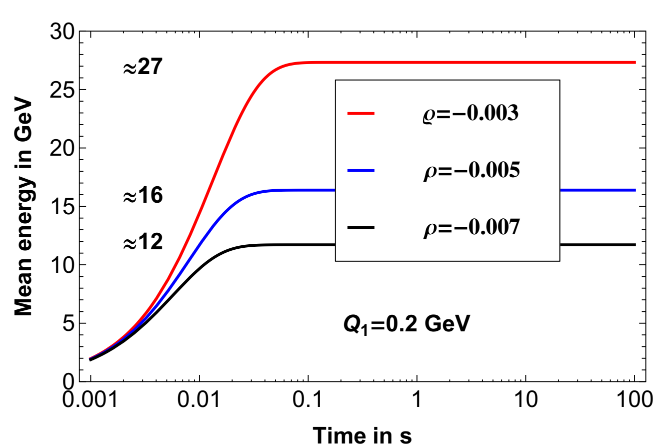

Fig. 3. Time dependence of the mean cumulative fission energy in multiplying system without precursors at three reactivities.

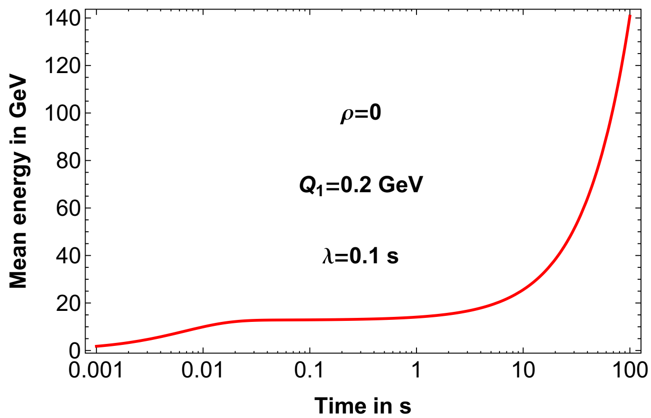

Fig. 4. The time dependence of the expectation of the cumulative fission energy in a critical multiplying system.

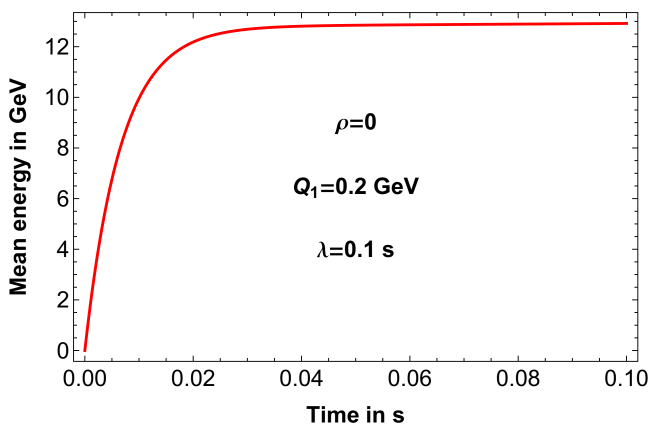

Fig. 5. The initial part of the time dependence of the expectation of the cumulative fission energy in a critical multiplying system.

$$m_2(t\,|\,\mathbf{n}(0)=1)=\left[\frac{\partial^2 g(\omega,t\,|\,\mathbf{n}(0)=1)}{\partial\omega^2}\right]_{\omega=0}$$
$$=\lambda_f\int_0^t e^{-\lambda_r(t-t')}\Big\{Q_2+q_1^{(p)}m_2(t'\,|\,\mathbf{n}(0)=1)+q_1^{(d)}m_2(t\,|\,\mathbf{c}(0)=1)+2q_1^{(p)}Q_1m_1(t'\,|\,\mathbf{n}(0)=1)+2q_1^{(d)}Q_1m_1(t'\,|\,\mathbf{c}(0)=1)+q_2^{(p)}[m_1(t'\,|\,\mathbf{n}(0)=1)]^2+2q_1^{(p)}q_1^{(d)}m_1(t'\,|\,\mathbf{n}(0)=1)m_1(t'\,|\,\mathbf{c}(0)=1)+q_2^{(d)}[m_1(t'\,|\,\mathbf{c}(0)=1)]^2\Big\}dt' \quad (39)$$

where the parameters $q_1^{(p)}=\nu_p$ and $q_1^{(d)}=\nu_d$ are the first moments of the neutron and precursor multiplicities, respectively, and further

$$Q_2=\int_0^\infty E^2h(E)\,dE, \quad (40)$$

and

$$m_j(t\,|\,\mathbf{c}(0)=1)=\lambda\int_0^t e^{-\lambda(t-t')}m_j(t\,|\,\mathbf{n}(0)=1)\,dt', \qquad j=1,2. \quad (41)$$

From (41) one obtains

$$m_1(t\,|\,\mathbf{c}(0)=1)=\frac{Q_1\lambda}{\Lambda\nu s_1s_2}+\frac{Q_1}{\Lambda\nu(s_1-s_2)}\left[\frac{\lambda}{s_1}e^{-s_1t}-\frac{\lambda}{s_2}e^{-s_2t}\right], \quad (42)$$

which will be needed in the forthcoming. The second factorial moments $q_2^{(p)}$ and $q_2^{(d)}$ of the prompt and delayed neutron multiplicities, respectively, are determined from the generating functions $q^{(p)}(z)$ and $q^{(d)}(z)$, defined by (5) as

$$\left[\frac{d^2q^{(p)}(z)}{dz^2}\right]_{z=1}=q_2^{(p)}=\langle\nu_p(\nu_p-1)\rangle, \quad (43)$$

$$\left[\frac{d^2q^{(d)}(z)}{dz^2}\right]_{z=1}=q_2^{(d)}=\langle\nu_d(\nu_d-1)\rangle. \quad (44)$$

Next, let us introduce the function

$$R(t)=q_2^{(p)}[m_1(t'\,|\,\mathbf{n}(0)=1)]^2+2q_1^{(p)}q_1^{(d)}m_1(t'\,|\,\mathbf{n}(0)=1)m_1(t'\,|\,\mathbf{c}(0)=1)+q_2^{(d)}[m_1(t'\,|\,\mathbf{c}(0)=1)]^2 \quad (45)$$

and write down the Laplace transform of the integral Eq. (39). By using the notations

$$\tilde{m}_1(s\,|\,\mathbf{n}(0)=1)=\int_0^\infty e^{-st}m_1(t\,|\,\mathbf{n}(0)=1)\,dt, \quad (46)$$

$$\tilde{m}_2(s\,|\,\mathbf{n}(0)=1)=\int_0^\infty e^{-st}m_2(t\,|\,\mathbf{n}(0)=1)\,dt, \quad (47)$$

$$\tilde{R}(s)=\int_0^\infty e^{-st}R(t)\,dt, \quad (48)$$

and performing some rearrangements, we obtain the following:

$$\left\{s+\lambda_c+\lambda_f\left[1-(1-\beta)\nu-\beta\nu\frac{\lambda}{s+\lambda}\right]\right\}\tilde{m}_2(s\,|\,\mathbf{n}(0)=1)=\frac{1}{\nu\Lambda}\left[\frac{Q_2}{s}+2q_1^{(p)}Q_1\tilde{m}_1(s\,|\,\mathbf{n}(0)=1)+2q_1^{(d)}Q_1\tilde{m}_1(s\,|\,\mathbf{c}(0)=1)+\tilde{R}(s)\right], \quad (49)$$

from which, by applying the same method as in the previous Section, the formula

$$\tilde{m}_2(s\,|\,\mathbf{n}(0)=1)=\frac{s+\lambda}{(s+s_1)(s+s_2)}\times\frac{1}{\nu\Lambda}\left[\frac{Q_2}{s}+2q_1^{(p)}Q_1\tilde{m}_1(s\,|\,\mathbf{n}(0)=1)+2q_1^{(d)}Q_1\tilde{m}_1(s\,|\,\mathbf{c}(0)=1)+\tilde{R}(s)\right] \quad (50)$$

can be derived, where $\tilde{m}_1(s\,|\,\mathbf{n}(0)=1$ is given by (34) and

$$\tilde{m}_1(s\,|\,\mathbf{c}(0)=1)=\frac{Q_1}{\nu\Lambda}\frac{1}{s}\frac{\lambda}{(s+s_1)(s+s_2)}, \quad (51)$$

respectively. After elementary but lengthy calculations for $\tilde{R}(s)$ one obtains

$$\tilde{R}(s)=\frac{2Q_1^2}{\nu^2\Lambda^2}\times\Bigg\{\langle\nu_p(\nu_p-1)\rangle\frac{s^3+(s_1+s_2+3\lambda)s^2+[s_1(s_2+2\lambda)+(2s_2+3\lambda)\lambda]s+2(s_1+s_2)\lambda^2}{s(s+s_1)(s+2s_1)(s+s_2)(s+s_1+s_2)(s+2s_2)}+\nu_p\nu_d\frac{[3s+2(s_1+s_2)](s+2\lambda)\lambda}{s(s+s_1)(s+2s_1)(s+s_2)(s+s_1+s_2)(s+2s_2)}+\langle\nu_d(\nu_d-1)\rangle\frac{[3s+2(s_1+s_2)]\lambda^2}{s(s+s_1)(s+2s_1)(s+s_2)(s+s_1+s_2)(s+2s_2)}\Bigg\}. \quad (52)$$

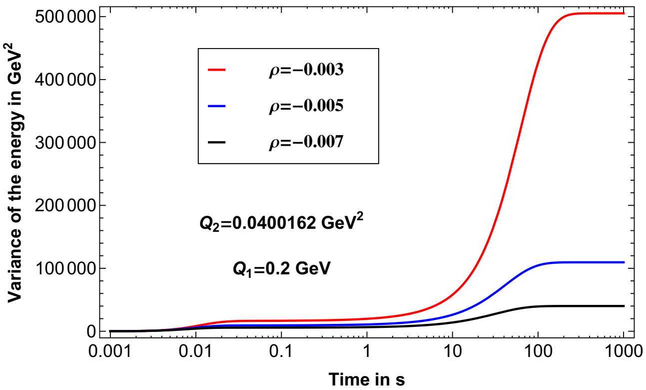

Fig. 6. The time dependence of the variance of the cumulative fission energy generated by one neutron in multiplying systems with different subcritical reactivities.

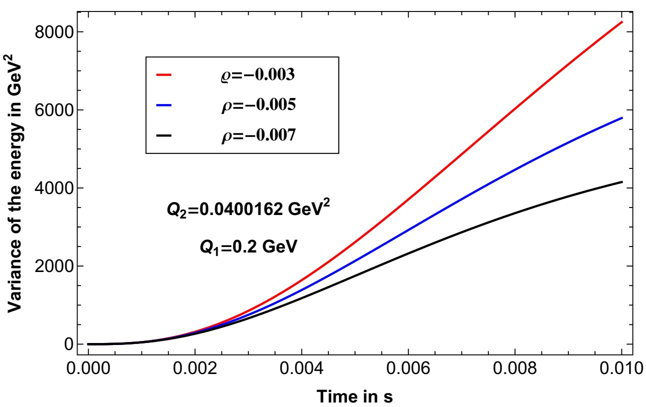

Fig. 7. The initial part of the time dependence of the variance of the cumulative fission energy generated by one neutron in different subcritical multiplying systems.

In order to obtain the second moment $m_2(t \mid \boldsymbol{n}(0) = 1)$, we have to determine the inverse Laplace transform $\widetilde{m}_2(s \mid \boldsymbol{n}(0) = 1)$. This can be performed analytically with the symbolic manipulation code Mathematica (Wolfram Research, 2014), but it results in an extremely long formula, which will not be listed here. To get some insight it is much more expedient to plot the variance of $\boldsymbol{\eta}(t)$. The time dependence of the variance is given by the expression

$$\mathbf{D}^2\{\boldsymbol{\eta}(\mathrm{t}) \mid \boldsymbol{n}(0) = 1\} = m_2(t \mid \boldsymbol{n}(0) = 1) - [m_1(t \mid \boldsymbol{n}(0) = 1)]^2 = \mathbb{V}(t \mid \boldsymbol{n}(0) = 1), \quad (53)$$

$\rho \neq 0$,

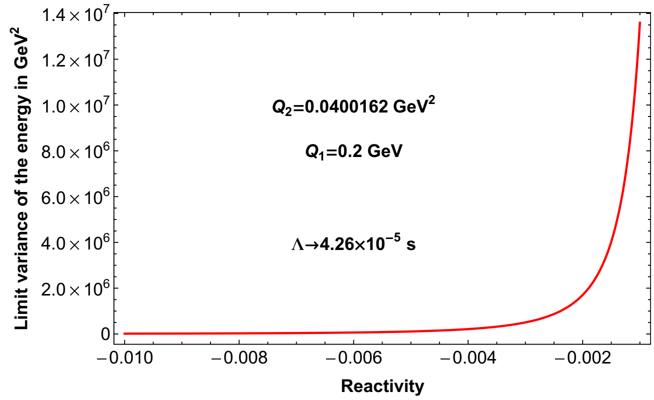

Fig. 8. Dependence of the limit variance of the energy $\mathbb{V}(\infty)$ on the negative reactivity.

Fig. 6 illustrates how, for large times, the precursors modify the time dependence of the variance of the cumulative fission energy generated by one neutron in three subcritical multiplying systems. At the same time Fig. 7 clearly shows that at the beginning of the process, the prompt neutrons play the decisive role.

It is also interesting to calculate the dependence of the limit variance

$$\lim_{t\to\infty}\mathbf{D}^2\{\boldsymbol{\eta}(\mathrm{t}) \mid \boldsymbol{n}(0) = 1\} = \mathbb{V}(\infty)$$

on the reactivity $\rho < 0$ in subcritical systems. This is obtained as

$$\mathbb{V}(\infty) = -\frac{1}{\nu^3\rho^3}\left\{Q_2\,\nu^2\rho^2 + Q_1^2\left[\langle\nu_p(\nu_p-1)\rangle + \langle\nu_d(\nu_d-1)\rangle + \nu\left(2\beta\nu - 2\beta^2\nu + \rho - 2\nu\rho\right)\right]\right\}. \quad (54)$$

Fig. 8 shows the dependence of the limit variance of the energy $\mathbb{V}(\infty)$ on the negative reactivity. Naturally, in critical and supercritical systems, i.e. for $\rho \geqslant 0$, the limit variance is infinite. Calculation of the time dependence of the variance in a critical multiplying system shows the nature of this divergence. After lengthy calculations one obtains that

$$\lim_{\rho\to 0}\mathbb{V}(t \mid \boldsymbol{n}(0) = 1) = \mathbb{V}^{(cr)}(t \mid \boldsymbol{n}(0) = 1) = \mathbb{V}_0^{(cr)}(t) + \mathbb{V}_1^{(cr)}(t)\exp\left\{-\left(\lambda + \frac{\beta}{\Lambda}\right)t\right\} + \mathbb{V}_2^{(cr)}(t)\exp\left\{-2\left(\lambda + \frac{\beta}{\Lambda}\right)t\right\}. \quad (55)$$

In order to simplify the formulae, let us introduce the notation

$$\chi = \lambda + \frac{\beta}{\Lambda}. \quad (56)$$

With this the result can be written as

$$\begin{aligned}\mathbb{V}_0^{(cr)}(t) = A_0\Big\{&6Q_2\chi^4(\chi - \lambda + \chi\lambda t)\Lambda^2\nu^2 + Q_1^2\big[\langle\nu_p(\nu_p-1)\rangle\lambda^2(-27\lambda + 30\chi(1+\lambda t) - 12\chi^2 t(2+\lambda t) + 2\chi^3 t^2(3+\lambda t)\\ &+\langle\nu_d(\nu_d-1)\rangle\left(-27\lambda^3 + 30\chi\lambda^2(2+\lambda t) - 3\chi^2\lambda(13+16\lambda t + 4\lambda^2 t^2) + 2\chi^3(3 + 9\lambda t + 6\lambda^2 t^2 + \lambda^3 t^3)\right)\\ &+2\nu(27(-1+\beta)\beta\lambda^3\nu - 15\chi(-1+\beta)\beta\lambda^2(3+2\lambda t)\nu + 3\chi^4\Lambda(-1+\lambda^2 t^2(\nu-1)\\ &-2(-1+\beta)\nu - 2\lambda t(1+(-2+\beta)\nu)) + 3\chi^2\lambda(4(-1+\beta)\beta\nu + 4(-1+\beta)\beta\nu + 4\beta(-1+\beta)\nu\lambda^2 t^2\\ &+\lambda(12t\beta(-1+\beta)\nu + \Lambda(6\nu-1))) + \chi^3\lambda(-t(-1+\beta)\beta\times(6+9\lambda t + 2\lambda^2 t^2)\nu + 6\Lambda(1+2(-2+\beta)\nu + \lambda t(1-2\nu))))\big]\Big\},\end{aligned}$$

$$\mathbb{V}_1^{(cr)}(t) = A_1 \Big\{ Q_2\,\chi^4\,(\lambda-\chi)\,\Lambda^2\,\nu^2 - Q_1^2\,[\langle\nu_p(\nu_p-1)\rangle\,\lambda^2\,\big(4\,\chi-\chi^3\,t^2-4\,\lambda+\chi^2\,t(2+\lambda t)\big)+\langle\nu_d(\nu_d-1)\rangle(\chi-\lambda)\big(-4\,\chi\,\lambda+4\,\lambda^2\,\chi^3\,t\,(2+\lambda t)$$
$$-\chi^2\,\lambda t\,(4+\lambda t)\big)+\nu\big(-12\,\chi\,(\beta-1)\,\beta\,\lambda^2\,\nu+8\,(\beta-1)\,\beta\,\lambda^3\,\nu-2\,\chi^5\,t\,(\beta-1)\,\Lambda\,\nu-2\,\chi^2\,\lambda\,\big(-(\beta-1)\,\beta\,\nu+t^2\,(\beta-1)\,\beta\,\lambda^2\,\nu$$
$$+\lambda\,(\Lambda+3\,t\,(\beta-1)\,\beta\,\nu+))]\}\chi^3\,\lambda\,(t\,(\beta-1)\,\beta(4+3\,\lambda t)\,\nu+2\,\Lambda\,(2+2\,(\beta-2)\,\nu+\lambda t(1+\nu)))+\chi^4\,\big(-t^2\,(\beta-1)\,\lambda\,\nu$$
$$+2\,\Lambda\,(-1+\nu-\beta\,\nu+\lambda t\,(-1+(\beta-2)\,\nu)))]\}$$

and

$$\mathbb{V}_2^{(cr)}(t) = A_2\,Q_1^2\,\{\langle\nu_p(\nu_p-1)\rangle\,\lambda^2\,(\lambda-2\,\chi)$$
$$-\langle\nu_d(\nu_d-1)\rangle\,(\chi-\lambda)^2\,(2\,\chi-\lambda)-2\,(\chi-\lambda)\,\nu\,[\chi^3\,\Lambda-\chi^2\,\lambda\,\Lambda$$
$$+2\,\chi\,(\beta-1)\,\beta\,\lambda\,\nu]\},$$

where

$$A_0 = \frac{1}{6\,\chi^6\,\Lambda^3\,\nu^3}, \qquad A_1 = \frac{1}{\chi^6\,\Lambda^3\,\nu^3} \qquad \text{and} \qquad A_2 = \frac{1}{2\,\chi^6\,\Lambda^3\,\nu^3}.$$

Fig. 9 shows the time dependence of the variance of the fission energy in a critical multiplying system. The rapid divergence of the variance starts around the time corresponding to the precursor decay time.

In order to show the initial time dependence of the variance, where the prompt neutrons play the deceiving role, the beginning of the time dependence of the variance is shown in Fig. 10.

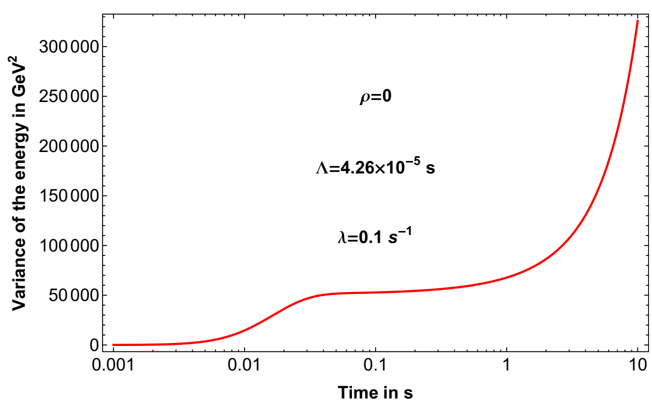

Fig. 9. Time dependence of the variance of the fission energy in a critical system.

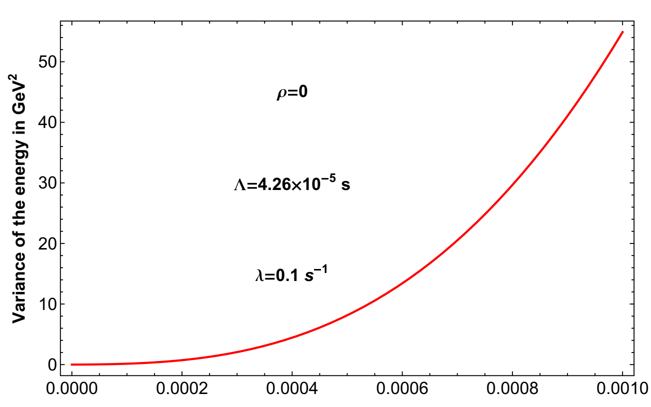

Fig. 10. Initial part of the time dependence of the variance of the fission energy in critical multiplying system.

3. Discussion

It is interesting to quantify the influence of the variation of the fission energy per fission to the total cumulative fission energy released in the process, compared to the case when the energy released in each fission is constant, and the only source of the fluctuations of the cumulative fission energy is the fluctuations of the number of fissions.

To this order, we consider the case when the fission energy is constant, the constant being equal to the first moment Q_1. Hence the probability density function $h_c(E)$ can be written as

$$h_c(E) = \delta(E - Q_1). \tag{57}$$

The expectation is Q_1, whereas the second moment $Q_2 = Q_1^2$. The fluctuation of the cumulative fission energy in this case, when the fission energy is constant, can simply be obtained by replacing Q_2 with Q_1^2.

A quantitative comparison is shown in Fig. 11. It is seen that the difference tends to saturate with increasing time. Moreover, a comparison with the value of the variance with random fission energy, Fig. 11 shows that the relative difference between the variances is extremely small. In other words, as the fission chain develops, the significance of the variations in the cumulative fission energy arising from the fluctuations of the fission energy in individual fissions diminishes in comparison to the fluctuations due to the fluctuations of the number of fission events. Since this latter is related to the branching process, one can say as the chain develops, the statistics of the cumulative energy is more and more dominated by the statistics of the branching properties of the process. Similar results were obtained also when changing the numerical value of the second moment Q_2, up to much larger (i.e. non-physical, several tens of percents) relative standard deviations of the individual fission

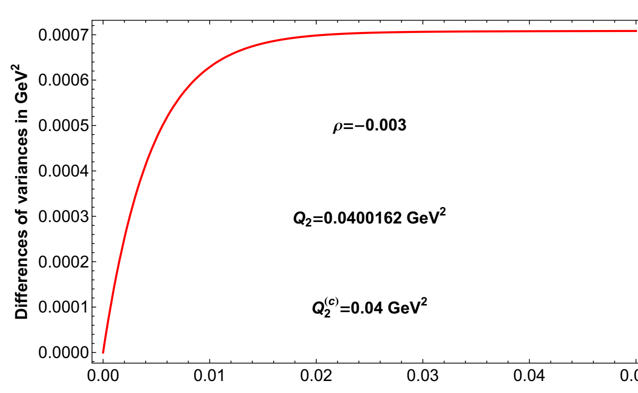

Fig. 11. Time dependence of the difference in the variance of the cumulative fission energy with constant and random fission energies.

energy; the results remained quantitatively very similar, leading to the same conclusions.

4. Conclusions

In order to get an insight into the stochastic behaviour of the cumulative fission energy production generated by one starting neutron in a neutron multiplying system, a backward generating function equation was derived, which made it possible to calculate the time dependence of the moments of the cumulative fission energy for the case the fission energy in the individual fissions is a random variable. The expectation and the variance of the cumulative fission energy was determined in systems of various reactivities. In order to assess the significance of the random fission energy, a comparison was made with the case when the fission energy is constant. The difference proved to be very minute, from which one can conclude that the variance of the cumulative fission energy is mainly due to the variance of the number of fissions. This also means that in calculations of the higher order moments of the cumulative fission energy, the fluctuations in the energy generated in individual fissions can be safely neglected.

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