Physica A 406 (2014) 67–72

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

The returns and risks of investment portfolio in a financial market☆

Jiang-Cheng Li, Dong-Cheng Mei *

Department of Physics, Yunnan University, Kunming, 650091, China

HIGHLIGHTS

- The model of the investment portfolio in financial market was established.
- The agreements between the results of our model and Dow Jones Industrial Average were found.
- The effects of dispersion of investment portfolio are analyzed.
- The roles of investment period on risks and returns are discussed.

ARTICLE INFO

Article history:
Received 21 December 2013
Received in revised form 23 February 2014
Available online 20 March 2014

Keywords:
Financial markets
Heston model
Equity portfolio

ABSTRACT

The returns and risks of investment portfolio in a financial system was investigated by constructing a theoretical model based *on the Heston model*. After the theoretical model and analysis of portfolio were calculated and analyzed, we find the following: (i) The statistical properties (i.e., the probability distribution, the variance and loss rate of equity portfolio return) between simulation results of the theoretical model and the real financial data obtained from Dow Jones Industrial *Average are in good* agreement; (ii) The maximum *dispersion of the investment* portfolio is associated with the maximum stability of the equity portfolio return and minimal investment risks; (iii) An *increase of the investment* period and a worst investment period are associated with a decrease of stability of the equity portfolio return and a maximum investment risk, respectively.

1. Introduction

The researches of financial markets as complex systems in statistical physics have obtained more and more attentions and given rise to a new field called "econophysics" in recent years [1,2]. The geometric Brownian motion *model was early* proposed to describe the stochastic dynamics of stock prices [3,4]. However, this geometric Brownian motion model cannot agree with some statistical characteristics from actual financial data, such as, the fat tails [5,6], long range memory and clustering of volatility [7]. Afterwards, many valuable *models have been developed* to make up these deficiencies, such as the ARCH model [8], GARCH model [9], and Heston model [10]. In particular, the Heston model *is well consistent with* statistical characteristics of stock prices obtained from actual financial data, for example, the probability distribution of returns for Dow-Jones data [11] and for the three major stock-market indexes (Nasdaq, S&P500, and Dow-Jones) [12], the exponential distribution of financial returns obtained from actual financial data [13], the probability density distribution of the logarithmic returns of the empirical high-frequency data of DAX and its stocks [14] or the typical price fluctuations of the Brazilian

☆ This work was supported by the National Natural Science Foundation of China (Grant No. 11165016) and the program of IRTSTYN in Yunnan Province.
* Corresponding author. Tel.: +86 08715032577; fax: +86 08715035570.
E-mail addresses: meidch@ynu.edu.cn, meidch@126.com (D.-C. Mei).

http://dx.doi.org/10.1016/j.physa.2014.03.005
0378-4371/

São Paulo Stock Exchange Index [15]. Meanwhile, the Heston model has been widely used to analyze dynamics of stock price in actual financial markets, for example, the mean escape time in a modified Heston model with a cubic nonlinearity [16,17], the statistical properties of the hitting times in different models for stock market evolution [18,19], exact expressions for the survival probability and the mean exit time [20,21], the effects of the delay time in Heston model [22,23], and the stochastic resonance of the stock prices [24].

In the actual trade in financial market, rational investors try to choose an appropriate investment portfolio, due to lower investment risks and higher stability of returns. Since Merton [25] used a stochastic optimal control approach of geometric Brownian motion model to analyze the investment optimization, investment portfolio has been discussed in an incomplete semimartingale market by Kramkov and Schachermayer [26,27] and Bouchard et al. [28], in accounting by Kothari [29], in risk-based indifference pricing [30] and so on. Hence, investment portfolio need to be further investigated in actual market. Even though portfolio with Heston model is discussed in some studies, such as bond portfolio selection problem [31] and financial planning [32], the studies are deficient in dispersion of investment portfolio, investment period and analysis of real financial data, etc. In order to make up the deficiencies, we use the Heston model to construct a model for the investment portfolio due to the model more close to reality.

In this paper, we establish a model for investment portfolio with the Heston model to discuss the returns and risks of investment portfolio. In addition, the remainder of this paper is organized as follows. In Section 2, a model for investment portfolio is introduced with the Heston model. In Section 3, the returns and risks are discussed in a investment portfolio of two uncorrelated stocks. In Section 4, a brief conclusion ends the paper.

2. The investment portfolio with the Heston model

In the actual financial investment, equity portfolio consists of the price and quantity of n stocks held by investors, and then we define the equity portfolio $S(t)$ as

$$S(t) = \sum_{i=1}^{n} n_i(t) S_i(t), \tag{1}$$

where at time t, $S_i(t)$ is the price of the ith stock and $n_i(t)$ is the quantity of the ith stock ($i = 1, 2, 3 \ldots n$), and then the rate of equity portfolio $C(t)$ can be written as:

$$\begin{aligned} C(t) &= \frac{S(t)}{S(0)} \\ &= \frac{\sum\limits_{i=1}^{n} n_i(t) S_i(t)}{S(0)}. \end{aligned} \tag{2}$$

Let us define the initial rate of the i-th stock on the total equity portfolio

$$r_i = \frac{n_i(0) S_i(0)}{S(0)}. \tag{3}$$

From Eqs. (2) and (3), we can find $C(0) = \sum_{i=1}^{n} r_i = 1$. For $\frac{dn_i(t)}{\mathrm{d}t} = 0$, fixing the logarithmic stock price $x_i(t) = \ln S_i(t)$, and employing the Heston model [10,11] to describe the stock price *dynamics,* Eq. (2) *becomes*

$$C(t) = \sum_{i=1}^{n} r_i \exp(x_i(t) - x_i(0)), \tag{4}$$

where

$$\mathrm{d}x_i(t) = \left(\mu_i - \frac{\nu_i(t)}{2}\right) \mathrm{d}t + \sqrt{\nu_i(t)}\, \mathrm{d}W_i(t), \tag{5}$$

and

$$\mathrm{d}\nu_i(t) = a_i(b_i - \nu_i(t))\, \mathrm{d}t + c_i \sqrt{\nu_i(t)}\, \mathrm{d}Z_i(t). \tag{6}$$

Here the subscript i indicates the ith stock, μ_i is the growth rate, $\nu(t)_i$ is the volatility of the stock price, a_i is the mean reversion of the $\nu(t)_i$, b_i is the long-run variance, c_i is *the volatility of volatility, that is* the amplitude of volatility fluctuations [33], $\mathrm{d}W_i(t)$ and $\mathrm{d}Z_i(t)$ are correlated *Wiener processes with the following statistical properties*

$$\begin{aligned} &\langle \mathrm{d}W_i(t) \rangle = \langle \mathrm{d}Z_i(t) \rangle = 0, \\ &\langle \mathrm{d}W_i(t)\, \mathrm{d}W_j(t') \rangle = \langle \mathrm{d}Z_i(t)\, \mathrm{d}Z_j(t') \rangle = \delta_{i,j} \delta(t - t')\, \mathrm{d}t, \\ &\langle \mathrm{d}W_i(t)\, \mathrm{d}Z_j(t') \rangle = \rho_{i,j} \delta(t - t')\, \mathrm{d}t, \\ &\langle \mathrm{d}Z_i(t') d_j W(t) \rangle = \lambda_{i,j} \delta(t - t')\, \mathrm{d}t, \end{aligned} \tag{7}$$

$\rho_{i,j}$ denotes the cross correlation coefficient between $\mathrm{d}W_i(t)$ and $\mathrm{d}Z_j(t)$, and $\lambda_{i,j}$ denotes the cross correlation coefficient between $\mathrm{d}Z_i(t)$ and $\mathrm{d}W_j(t)$.

3. The investment portfolio of two stocks

In order to facilitate the analysis and discussion *with the model* (Eqs. (4)–(6)), *we analyze the returns* and risks of investment portfolio of two uncorrelated stocks. *By using the method of* Refs. [24,11], the μ_1 and μ_2 can be simplified, *and then* Eqs. (4)–(6) *become*

$$C(t) = r_1 \exp(x_1(t) - x_1(0)) + (1 - r_1)\exp(x_2(t) - x_2(0)), \tag{8}$$

$$\mathrm{d}x_1(t) = -\frac{\nu_1(t)}{2}\,\mathrm{d}t + \sqrt{\nu_1(t)}\,\mathrm{d}W_1(t),$$
$$\mathrm{d}\nu_1(t) = a_1(b_1 - \nu_1(t))\,\mathrm{d}t + c_1\sqrt{\nu_1(t)}\,\mathrm{d}Z_1(t), \tag{9}$$

and

$$\mathrm{d}x_2(t) = -\frac{\nu_2(t)}{2}\,\mathrm{d}t + \sqrt{\nu_2(t)}\,\mathrm{d}W_2(t),$$
$$d_2\nu(t) = a_2(b_2 - \nu_2(t))\,\mathrm{d}t + c_2\sqrt{\nu_2(t)}\,\mathrm{d}Z_2(t). \tag{10}$$

We consider $\mathrm{d}W_i(t)$ *and* $\mathrm{d}Z_i(t)$ $(i = 1, 2)$ *as uncorrelated Wiener processes with the following statistical properties*

$$\langle \mathrm{d}W_i(t)\rangle = \langle \mathrm{d}Z_i(t)\rangle = 0,$$
$$\langle \mathrm{d}W_i(t)\,\mathrm{d}W_j(t')\rangle = \langle \mathrm{d}Z_i(t)\,\mathrm{d}Z_j(t')\rangle = \delta_{i,j}\delta(t - t')\,\mathrm{d}t,$$
$$\langle \mathrm{d}W_i(t)\,\mathrm{d}Z_i(t')\rangle = \langle \mathrm{d}Z_i(t')\,\mathrm{d}W_i(t')\rangle = 0, \tag{11}$$

where for $i \neq j$, $\delta_{i,j} = 0$, for $i = j$, $\delta_{i,j} = 1$.

Afterwards we define the equity portfolio return as,

$$\Delta C = \frac{C(t + \triangle t) - C(t)}{C(t)} = \frac{C_{i+1} - C_i}{C_i}, \tag{12}$$

where C_i *is the rate of the equity* portfolio of ith time point ($i = 0, 1, 2, 3 \ldots$).

When investors only buy a stock (i.e., $r_1 = 1.0$ or $r_2 = 1.0$), equity portfolio return ΔC (i.e., Eq. (12)) regresses toward stock price returns, and Eq. (4) also regresses toward the Heston model. In this condition, the agreements between statistical properties of the Heston model and actual financial markets have been demonstrated in Refs. [12–23].

For the investment portfolio of two stocks, the probability density function (PDF) of equity portfolio return calculated by theoretical simulating results with Eqs. (8)–(10) and that obtained from the real market data are shown in Fig. 1.

As one can see the agreement between real and theoretical data is very good, and the leptokurtic nature *is observed. We use the Box–Muller method* to generate random processes with a Gaussian distribution for simulating *the noise sources in* Eqs. (8)–(12), *with time integration step* $\triangle t = 0.01$ (as a trading day) *and then the PDF of equity portfolio* is calculated by using the *forward Euler* method [34]. The parameters are $a_i = 100.0$, $b_i = 0.01$ and $c_i = 2.5$ ($i = 1, 2.$) with $r_1 = 0.5$. For the real *market data, we use the daily adjusted* closing values of 30 stocks in Dow Jones Industrial Average (ˆDJI) from 2 January 1970 (or offering date) to 3 June 2013 [35]. *We choose two stocks* to constitute our investment portfolio with $r_1 = 0.5$ based on Eqs. (2) and (3), and then calculate the PDF of equity portfolio return via Eq. (12).

In order to analyze the effects of dispersion of investment portfolio on the returns, we calculate the variance $\delta_{\Delta C}$ of equity portfolio return as a function of the initial rate r_1 *of the first stock. The results are shown in* Fig. 2. *As* $r_1 \to 0.5$, *we find that both real data and theoretical results show a monotonical decreasing behavior of the variance* $\delta_{\Delta C}$. *This means that there is* an optimal r_1 ($r_1 = 0.5$) concerning the maximum stability of the equity portfolio return. Obviously, the same statistical features and good agreements are observed between real data and theoretical results. From a financial point of view, an increase in dispersion of investment portfolio (i.e., $r_1 \to 0.5$) enhances *stability of the equity portfolio* return (i.e., $\delta_{\Delta C}$ decreases). This is reminiscent of the noise enhanced stability effect [36]. In other words, the maximum dispersion of investment portfolio for the case of two stocks corresponding to the value of the initial rate $r = 0.5$ is associated with the maximum stability of the equity portfolio return.

To understand the roles of dispersion of investment portfolio on the risks, we use the loss rate of equity portfolio return $P_{\Delta C<\Delta C_0}$. Here $P_{\Delta C<\Delta C_0}$ is the probability that equity portfolio return ΔC is less than the risk boundary ΔC_0 (i.e., loss is greater than the margin of investors). We fix the $\Delta C_0 = -0.02$ (i.e., loss rate 2%) and calculate the $P_{\Delta C<-0.02}$ *as shown in* Fig. 3. As one can see that as $r_1 \to 0.5$, the loss rate $P_{\Delta C<-0.02}$ monotonically decreases, i.e., there is an optimal r_1 ($r_1 = 0.5$) concerning the minimal loss rate of equity portfolio return. From a financial point of view, an increase in dispersion of investment portfolio (i.e., $r_1 \to 0.5$) reduces risks of equity portfolio return (i.e., $\delta_{\Delta C}$ decreases). In other words, the maximum dispersion of investment portfolio is associated with the minimal risks of equity portfolio return. In addition, we can observe the same statistical features and good agreements between real data and theoretical *results in* Fig. 3. *In the analyses of* Figs. 1–3, we only consider the case of a trading day, therefore we next discuss the influences of investment period of portfolio on returns

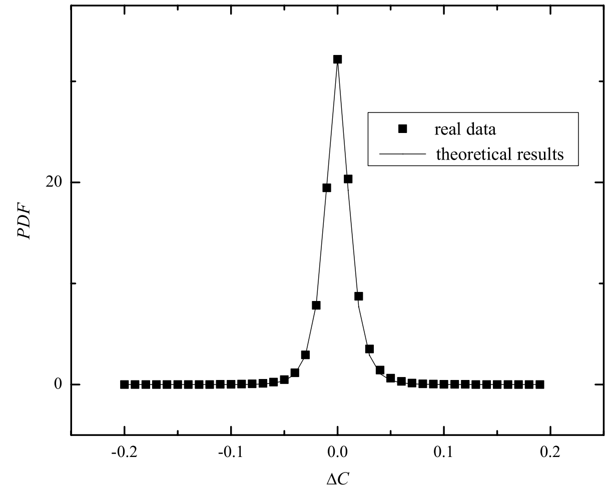

Fig. 1. The PDF as a function of ΔC for theoretical results (Eqs. (8)–(10)) and for real market data [35].

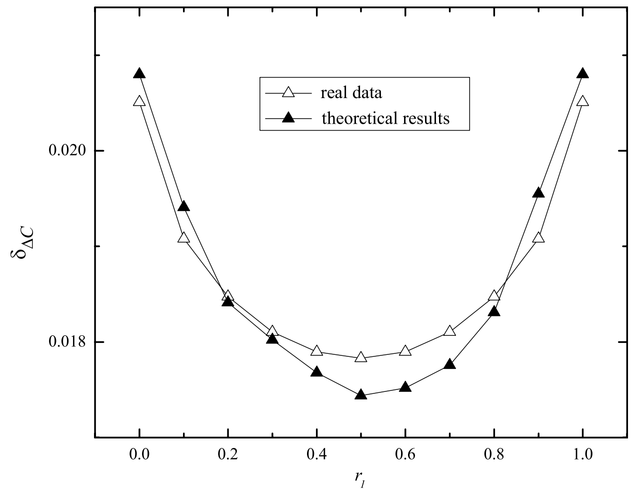

Fig. 2. The variance $\delta_{\Delta C}$ as a function of the initial rate r_1 of first stock for theoretical results and for real market data. The other parameters are the same as Fig. 1.

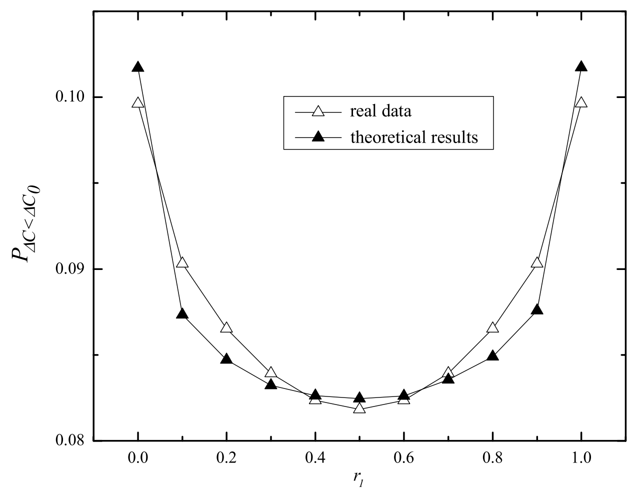

Fig. 3. The loss rate $P_{\Delta C<\Delta C_0}$ as a function of r_1 for theoretical results and for real market data. The other parameters are the same as Fig. 1.

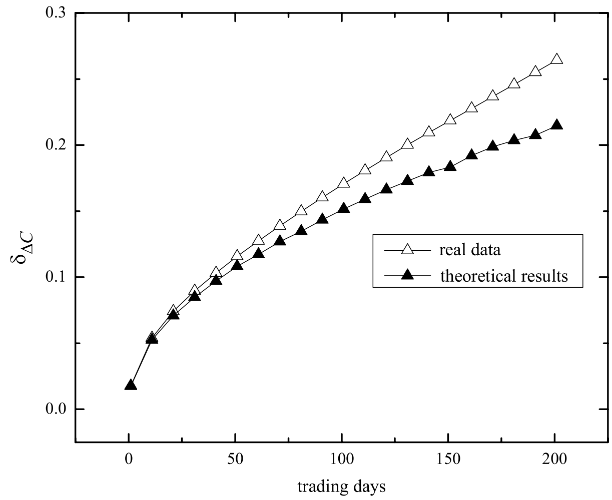

Fig. 4. The variance $\delta_{\Delta C}$ versus the trading days for theoretical results and for real market data. The other parameters are the same as Fig. 1.

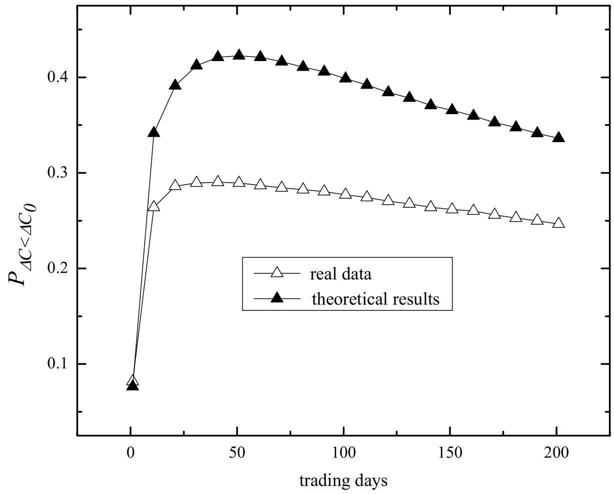

Fig. 5. The loss rate $P_{\Delta C<\Delta C_0}$ versus the trading days for theoretical results and for real market data. The other parameters are the same as Fig. 1.

and risks. To understand the roles of investment period of portfolio on returns, the variance $\delta_{\Delta C}$ as a function of the trading days is presented in Fig. 4. Both real data and theoretical results clearly *show that the variance* $\delta_{\Delta C}$ *monotonically* increases. From a financial point of view, an increase in investment period of portfolio monotonically reduces the stability of the equity portfolio return.

In the end of this section, to discuss the effects of investment period of portfolio on risks, the loss rate $P_{\Delta C<\Delta C_0}$ as a function of the trading days *is calculated. The results are shown in* Fig. 5. *Obviously,* both real data and theoretical results clearly *indicate that the loss rate* $P_{\Delta C<\Delta C_0}$ first increases and then decreases, i.e., there is a worst investment period (about 45 trading days) matching the maximum loss rate. From a financial point of view, there is a worst investment period concerning the maximum risks. From Figs. 4–5, we observe a qualitative agreement between real data and theoretical results. The discrepancy between empirical and theoretical results increases with the increasing of trading day. The same discrepancy between empirical and theoretical cumulative distributions can be found in Refs. [11,12,24]. Every stock has a characteristic and hierarchical order, and the stocks in the real market have a diversity and hierarchical distribution [37,38]. The increasing discrepancy in Figs. 4–5 is due to the increase of diversity and hierarchical order with an increase of trading day. In addition, the good agreement in Fig. 1 is due to the shorter trading days (i.e., a trading day), and the good agreement in Figs. 4–5 is also observed for shorter trading days (about trading days < 10).

4. Conclusions

In this paper, we propose a model for investment portfolio based on the Heston model, and then we investigate the roles of *investment portfolio in a financial market based on the Heston model. The returns and risks* of investment portfolio of two stocks are discussed by calculating the equity portfolio return from the theoretical and real data. The maximum dispersion of investment portfolio maximally enhances the stability of the equity portfolio return and reduces investment risks. An increase of investment period of portfolio is associated with a decrease of stability of the equity portfolio return, and a maximum investment risk is induced by a worst investment period. In addition, we compare the probability density function, the variance and loss rate of equity portfolio return from theoretical simulating results with that from ˆDJI data and *find good agreement*.

References

[1] P.W. Anderson, K.J. Arrow, D. Pines, The Economy as an Evolving Complex System, Addison Wesley Longman, 1988;
P.W. Anderson, K.J. Arrow, D. Pines, The Economy as an Evolving Complex System II, Addison Wesley Longman, 1997.
[2] R.N. Mantegna, H.E. Stanley, An Introduction to Econophysics: Correlations and Complexity in Finance, Cambridge University Press, Cambridge, 2000.
[3] J.C. Hull, Options, Futures, Other Derivatives, Prentice-Hall, New Jersey, 1997.
[4] P. Wilmott, Derivatives, John Willey and Sons, New York, 1998.
[5] J.P. Bouchaud, M. Potters, Theory of Financial Risks, Cambridge University Press, Cambridge, 2000.
[6] J. Voit, The Statistical Mechanics of Financial Markets, Springer, Berlin, 2001.
[7] M.M. Dacorogna, R. Gencay, U.A. Müller, R.B. Olsen, O.V. Pictet, An Introduction to High-Frequency Finance, Academic Press, New York, 2001.
[8] R.F. Engle, Econometrica 50 (1982) 987.
[9] T. Bollerslev, J. Econometrics 31 (1986) 307.
[10] S.L. Heston, Rev. Financ. Stud. 6 (1993) 327.
[11] A.A. Drăgulescu, V.M. Yakovenko, Quant. Finance 2 (2002) 443.
[12] A.C. Silva, V.M. Yakovenko, Physica A 324 (2003) 303.
[13] A.C. Silva, R.E. Prange, V.M. Yakovenko, Physica A 344 (2004) 227.
[14] R. Remer, R. Mahnke, Physica A 344 (2004) 236.
[15] R. Vicente, et al., Physica A 361 (2006) 272.
[16] G. Bonanno, D. Valenti, B. Spagnolo, Eur. Phys. J. B 53 (2006) 405.
[17] G. Bonanno, D. Valenti, B. Spagnolo, Phys. Rev. E 75 (2007) 016106.
[18] D. Valenti, B. Spagnolo, G. Bonanno, Physica A 382 (2007) 311.
[19] B. Spagnolo, D. Valenti, Int. J. Bifurcation Chaos 18 (2008) 2775.
[20] J. Masoliver, J. Perelló, Phys. Rev. E 78 (2008) 056104.
[21] J. Masoliver, J. Perelló, Phys. Rev. E 80 (2009) 016108.
[22] J.C. Li, D.C. Mei, Physica A 392 (2013) 763.
[23] J.C. Li, D.C. Mei, Phys. Lett. A 377 (2013) 663.
[24] J.C. Li, D.C. Mei, Phys. Rev. E 88 (2013) 012811.
[25] R.C. Merton, J. Econom. Theory 3 (1971) 373.
[26] D. Kramkov, W. Schachermayer, Ann. Appl. Probab. 9 (1999) 904.
[27] D. Kramkov, W. Schachermayer, Ann. Appl. Probab. 13 (2003) 1504.
[28] B. Bouchard, N. Touzi, A. Zeghal, Ann. Appl. Probab. 14 (2004) 678.
[29] S.P. Kothari, J. Account. Econ. 31 (2001) 105.
[30] R.J. Elliott, T.K. Siu, Commun. Stoch. Anal. 4 (2010) 51.
[31] J. Liu, Rev. Financ. Stud. 20 (2007) 1.
[32] S. Graf, A. Kling, J. Ruß, Eur. Actuar. J. 2 (2012) 77.
[33] J. Cox, J. Ingersoll, S. Ross, Econometrica 53 (1985) 385.
[34] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, Numerical Recipes, Cambridge University Press, Cambridge, 1992.
[35] Yahoo Finance, http://finance.yahoo.com/q/cp?s=%5EDJI.
[36] N.V. Agudov, B. Spagnolo, Phys. Rev. E 64 (2001) 035102;
A. Fiasconaro, B. Spagnolo, S. Boccaletti, Phys. Rev. E 72 (2005) 061110;
B. Spagnolo, A.A. Dubkov, N.V. Agudov, Acta Phys. Pol. B 5 (4) (2004) 1419–1436.
[37] G. Bonanno, G. Caldarelli, F. Lillo, R.N. Mantegna, Phys. Rev. E 68 (2003) 046130.
[38] G. Caldarelli, S. Battiston, D. Garlaschelli, M. Catanzaro, Lecture Notes in Phys. 650 (2004) 399.