DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **16**, Number **4**, November **2011**

doi:10.3934/dcdsb.2011.16.1071

pp. **1071–1082**

SYNCHRONIZATION OF CHAOTIC SYSTEMS WITH TIME-VARYING COUPLING DELAYS

Tingwen Huang

Texas A&M University at Qatar
Doha, P.O.Box 23874, Qatar

Guanrong Chen

Department of Electronic Engineering
City University of Hong Kong
Hong Kong, China

Juergen Kurths

Institute for Physics, University of Potsdam
Am Neuen Palais, Gebude 19, D-14415 Potsdam, Germany

Dedicated to the 65th Birthday of Professor Kok Lay Teo

Abstract. In this paper, we study the complete synchronization of a class of *time-varying* delayed coupled chaotic systems using feedback control. In terms of Linear Matrix Inequalities, a sufficient condition is obtained through using a Lyapunov-Krasovskii functional and differential equation inequalities. The conditions can be easily verified and implemented. We present two simulation examples to illustrate the effectiveness of the proposed method.

1. **Introduction.** In this paper, we address the complete synchronization of a class of *time-varying* delayed chaotic systems. Chaos synchronization is a basic focus in nonlinear science due to its extensive applications in secure communications, biological science, neural networks, automatic control, etc. Since the 1990s, chaos control and synchronization have sparked increasing interest of many researchers, and many schemes have been developed. The readers are referred to the review monographs [1],[6], where the authors presented the main ideas involved in the field of chaos synchronization as well as many potential applications. So far, a lot of research on this subject has been done, and many fundamental results have been reported on synchronization and control of chaotic systems by scholars from physics, engineering, biology, and mathematics, etc. Various control schemes have been developed to synchronize chaotic systems such as drive-response, coupling control, adaptive control, feedback control, observer-based control, impulsive control, intermittent control, to name some typical ones.

Many delayed systems in various research fields including biology, chemistry, nonlinear optics, economics, and epidemiology, have been found to be chaotic. For

2000 *Mathematics Subject Classification.* Primary: 34A20, 34G20.

Key words and phrases. Synchronization, Chaotic System, Time-varying Delay.

The work described in this paper was partially supported by the National Natural Science Foundation of China (Grant No. 60974020 and No. 10971240), and by the Hong Kong RGC under the GRF Grant CityU1114/11E. .

instance, in hardware implementation, time delays occur due to finite switching speeds of communications. Time delay may lead to some complex dynamic behaviors such as chaos, oscillation, divergence, and instability. There are rather few reports on the synchronization for chaotic systems with time-varying delays. In [12], Lu et al. proposed an adaptive scheme for stabilization and synchronization of chaotic Lure systems with time-varying delays using the invariant principle of functional differential equations. While in [22], Zhen et al. proposed to use sliding model control to achieve synchronization for time-varying chaotic systems. The sliding model synchronization controller is designed to satisfy the external disturbance vector with an unknown upper bound. In this paper, we obtain some criteria for synchronization of systems with coupling time-varying delays using feedback control. The derived conditions for the synchronization can be easily verified.

2. **Problem formulation and preliminaries.** Consider the following chaotic system with time-varying delays:

$$\begin{aligned}&\frac{dx_i(t)}{dt}=\sum_{j=1}^{n}a_{ij}x_i(t)+\sum_{j=1}^{n}b_{ij}f_j(x_j(t))+\sum_{j=1}^{n}c_{ij}f_j(x_j(t-\tau_j(t))),\\&x_i(t)=\varphi_i(t),-\tau\le t\le 0,\quad i=1,\cdots,n,\end{aligned}\tag{1}$$

where $f_i, i=1,\cdots,n$, are nonlinear functions; $\tau_i(t)\ge 0, i=1,\cdots,n$, are the the time-varying delays with $\tau_i(t)\le\tau$ in which τ is a positive constant.

For model (1), we can write it in the following matrix-vector form:

$$\begin{aligned}&\frac{dx}{dt}(t)=Ax(t)+Bf(x(t))+Cf(x(t-\tau(t)))\\&x(t)=\varphi(t),-\tau\le t\le 0,\end{aligned}\tag{2}$$

where $x(t)=(x_1(t),\cdots,x_n(t))^T$, $\varphi(t)=(\varphi_1(t),\cdots,\varphi_n(t))^T$, $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$, $C=(c_{ij})_{n\times n}$.

Assume that the nonlinear functions $f_i, i=1,\cdots,n$, are bounded and satisfy Lipschitz's condition:

H: $f_i, i=1,\cdots,n$, are bounded functions defined on R and satisfy

$$|f_i(x)-f_i(y)|\le l_i|x-y|\tag{3}$$

for all $x,y\in R$, where l_i are positive constants, $i=1,\cdots,n$.

For convenience, let $A^T, A^{-1}, \lambda_m(A), \lambda_M(A)$ represent the transpose, inverse, minimum eigenvalue, and maximum eigenvalue of a square matrix A, respectively. The vector norm is Euclidian, denoted by $||.||$. And $A>0$ $(<0,\le 0,\ge 0)$ represents a symmetrical positive (negative, semi-negative, semi-positive) definite matrix A.

Let the chaotic system (2) be the master system and its unidirectionally coupled copy

$$\begin{aligned}&\frac{dy}{dt}(t)=Ay(t)+Bf(y(t))+Cf(y(t-\tau(t)))+u(t)\\&u(t)=K(x(t)-y(t))\\&x(t)=\phi(t),-\tau\le t\le 0,\end{aligned}\tag{4}$$

be the slave system, where $K = diag(k_1, \cdots, k_n)$ and $u(t)$ is a feedback control to be designed.

Let $e(t) = y(t) - x(t)$ be the error between the two systems. Complete synchronization between the master and slave systems means that $e(t) \to 0$, as $t \to \infty$. By systems (2) and (4), we have the following error system:

$$\begin{aligned}\frac{de}{dt}(t) &= -(K - A)e(t) + B(f(y(t)) - f(x(t))) \\ &\quad + C(f(y(t - \tau(t))) - f(x(t - \tau(t)))).\end{aligned} \tag{5}$$

We are looking for simple conditions to guarantee that $e(t)$ approaches to 0, i.e., complete synchronization.

Definition 1. Systems (2) and (6) are said to be exponentially synchronized if there exist constants $\lambda > 0$ and $M > 0$ such that

$$||e(t)|| \leq M \parallel \phi(0) - \varphi(0) \parallel_\tau e^{-\lambda t} \tag{6}$$

for all $t \geq 0$, where $\parallel \phi(t) - \varphi(t) \parallel_\tau = \sup_{s \in (t-\tau, t]} \parallel \phi(s) - \varphi(s) \parallel$.

To prove the main result for exponential synchronization, we present several lemmas first.

Lemma 1. Given any real matrices A, B, C with appropriate dimensions, and a positive symmetric matrix C. Then, for any scalar $\varepsilon > 0$, the following inequality holds:

$A^T B + B^T A \leq \varepsilon A^T C A + \frac{1}{\varepsilon} B^T C^{-1} B$.

Lemma 2. (Schur complement, Boyd et al. [1]). The following LMI:

$$\begin{bmatrix} A(x) & B(x) \\ B^T(x) & C(x) \end{bmatrix} > 0,$$

where $A(x) = A^T(x)$, $C(x) = C^T(x)$, and $B(x)$ depends affinely on x, is equivalent to one of the following conditions:

(i) $A(x) > 0$, $C(x) - B^T(x)A(x)^{-1}B(x) > 0$;

(i) $C(x) > 0$, $A(x) - B(x)C(x)^{-1}B^T(x) > 0$.

3. **Synchronization criterion.** In this section, using the Lyapunov method and LMI techniques, we establish a sufficient condition for synchronization of chaotic systems with time-varying delays. The following is the main theorem.

Theorem 1. Suppose that there exist positive constants α, β, γ and a positive diagonal matrix P such that

(i) the following LMI holds:

$$\begin{bmatrix} 2P(K - A) - \gamma P - \frac{1}{\alpha} B^T B - \frac{1}{\beta} C^T C & \sqrt{\alpha} PL \\ \sqrt{\alpha} LP & I_n \end{bmatrix} > 0,$$

(ii) $c \equiv \frac{\beta}{\gamma} \lambda_M(P) \max_{1 \leq i \leq n} \{l_i^2\} < 1$;

where I_n is an n-order identity matrix, $L = diag(l_i)_{n \times n}$, l_i being the Lipschitz constants in (3). Then, systems (2) and (6) are exponentially synchronized under assumption H.

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(e(t)) = e^T(t) Q e(t), \tag{7}$$

where $Q = P^{-1}$. It is clear that

$$\lambda_m(Q)||e(t)||^2 \leq V(e(t)) \leq \lambda_M(Q)||e(t)||^2. \tag{8}$$

For convenience, define

$$\sigma = \sup_{-\tau \le t \le \tau}\{||e(t)||\}\ ,\ \varsigma = \sup_{-\tau \le t \le 2\tau}\{||e(t)||\}.$$

Because $e(t)$ in the error system (5) is continuous, the existence of σ, ς is guaranteed. In the following, we take the time derivative of $V(e(t))$ along the trajectory of the error system (5) and estimate it. We have

$$\begin{aligned}
\dot{V}(e(t)) &= 2e^T(t)Q[-(K-A)e(t) + B(f(y(t)) - f(x(t))) \\
&\quad + C(f(y(t-\tau(t))) - f(x(t-\tau(t))))] \\
&\le -2e^T(t)Q(K-A)e(t) + 2e^T(t)QB(f(y(t)) - f(x(t))) \\
&\quad + 2e^T(t)QC(f(y(t-\tau(t))) - f(x(t-\tau(t)))) \\
&\le -2e^T(t)Q(K-A)e(t) + \frac{1}{\alpha}e^T(t)QBB^TQe(t) \\
&\quad + \alpha(f(y(t)) - f(x(t)))^T(f(y(t)) - f(x(t))) + \frac{1}{\beta}e^T(t)QCC^TQe(t) \\
&\quad + \beta(f(y(t-\tau(t))) - f(x(t-\tau(t))))^T(f(y(t-\tau(t))) - f(x(t-\tau(t)))) \\
&\le -2e^T(t)Q(K-A)e(t) + \frac{1}{\alpha}e^T(t)QBB^TQe(t) \\
&\quad + \alpha e^T(t)L^2e(t) + \frac{1}{\beta}e^T(t)QCC^TQe(t) \\
&\quad + \beta e^T(t-\tau(t))L^2e(t-\tau(t)) \\
&= e^T(t)[-2Q(K-A) + \frac{1}{\alpha}QBB^TQ + \alpha L^2 + \frac{1}{\beta}QCC^TQ]e(t) \\
&\quad + \beta e^T(t-\tau(t))L^2e(t-\tau(t)) \\
&= e^T(t)Q[-2(K-A)P + \frac{1}{\alpha}BB^T + \alpha L^2P^2 + \frac{1}{\beta}CC^T]Qe(t) \\
&\quad + \beta e^T(t-\tau(t))L^2e(t-\tau(t)) \\
&\le -\gamma V(e(t)) + e^T(t)Q[\gamma P - 2(K-A)P + \frac{1}{\alpha}BB^T \\
&\quad + \alpha L^2P^2 + \frac{1}{\beta}CC^T]Qe(t) + \beta e^T(t-\tau(t))L^2e(t-\tau(t)).
\end{aligned} \tag{9}$$

By condition (i), we have

$$\begin{aligned}
\dot{V}(e(t)) &\le -\gamma V(e(t)) + \beta e^T(t-\tau(t))L^2e(t-\tau(t)) \\
&\le -\gamma V(e(t)) + \beta \max_{1\le i\le n}\{l_i^2\}||e(t-\tau(t))||^2.
\end{aligned} \tag{10}$$

which implied that

$$\begin{aligned} V(e(t)) &\leq V(e(\tau))e^{-\gamma(t-\tau)} \\ &\quad +\beta \max_{1\leq i\leq n}\{l_i^2\}\int_\tau^t e^{-\gamma(t-s)}||e(s-\tau(s))||^2 ds \\ &\leq \lambda_M(Q)\sigma^2 e^{-\gamma(t-\tau)} \\ &\quad +\beta \max_{1\leq i\leq n}\{l_i^2\}\int_\tau^t e^{-\gamma(t-s)}||e(s-\tau(s))||^2 ds. \end{aligned} \tag{11}$$

By inequalities (8) and (11), we obtain the following:

$$\begin{aligned} ||e(t)||^2 \leq \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\gamma(t-\tau)} \\ +\beta \max_{1\leq i\leq n}\{l_i^2\}\int_\tau^t e^{-\gamma(t-s)}||e(s-\tau(s))||^2 ds]. \end{aligned} \tag{12}$$

Next, we prove that $e(t)$ satisfies

$$||e(t)|| \leq \left(\frac{1}{1-\eta}\right)^{\frac{1}{2}} Me^{-\frac{\epsilon}{2}(t-\tau)}, \quad t \geq \tau, \tag{13}$$

where $M = \left(\frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 + \frac{\beta}{\gamma}\max_{1\leq i\leq n}\{l_i^2\}\varsigma^2 e^{\gamma\tau}]\right)^{\frac{1}{2}}$, and ϵ is a constant satisfying $0 < \epsilon < \gamma$ and $\eta = \frac{1}{\lambda_m(Q)}\frac{1}{\gamma-\epsilon}\beta\max_{1\leq i\leq n}\{l_i^2\}e^{\epsilon\tau} < 1$, noticing that $\lambda_M(P) = \frac{1}{\lambda_m(Q)}$. The existence of ϵ is guaranteed by condition (ii) of this theorem.

We also notice that inequality (13) is equivalent to the following inequality:

$$||e(t)|| \leq \rho\left(\frac{1}{1-\eta}\right)^{\frac{1}{2}} Me^{-\frac{\epsilon}{2}(t-\tau)}, \quad \rho > 1, \quad t \geq \tau, \tag{14}$$

It is obvious that when $t = \tau$, inequality (14) follows from (12).

Now, suppose that there exist $t_0 > \tau$ and $\rho_0 > 1$ such that

$$||e(t_0)|| = \rho_0\left(\frac{1}{1-\eta}\right)^{\frac{1}{2}} Me^{-\frac{\epsilon}{2}(t_0-\tau)}, \tag{15}$$

and, for any $t \in [\tau, t_0)$,

$$||e(t)|| \leq \rho_0\left(\frac{1}{1-\eta}\right)^{\frac{1}{2}} Me^{-\frac{\epsilon}{2}(t-\tau)}. \tag{16}$$

Here, it is noted that t_0 is either in interval $(\tau, 2\tau]$ or $(2\tau, \infty)$.

First, consider the case of $t_0 \in (\tau, 2\tau]$. By inequality (12), we get

$$
\begin{aligned}
&||e(t_0)||^2 \leq \frac{1}{\lambda_m(Q)}\Bigg[\lambda_M(Q)\sigma^2 e^{-\gamma(t_0-\tau)} \\
&+\beta \max_{1\leq i\leq n}\{l_i^2\}\int_{\tau}^{t_0} e^{-\gamma(t_0-s)}||e(s-\tau(s))||^2 ds\Bigg] \\
\leq\ & \frac{1}{\lambda_m(Q)}\Bigg[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&+\beta \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2\int_{\tau}^{t_0} e^{-\gamma(t_0-s)} ds\Bigg] \\
=\ & \frac{1}{\lambda_m(Q)}\Bigg[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&+\frac{\beta}{\gamma} \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2(1-e^{-\gamma(t_0-\tau)})\Bigg] \\
<\ & \frac{1}{\lambda_m(Q)}\Bigg[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} + \frac{\beta}{\gamma}\max_{1\leq i\leq n}\{l_i^2\}\varsigma^2\Bigg] \\
<\ & \frac{1}{\lambda_m(Q)}\Bigg[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&+\frac{\beta}{\gamma} \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2 e^{\gamma\tau} e^{-\gamma(t_0-\tau)}\Bigg] \\
<\ & \frac{1}{\lambda_m(Q)}\Bigg[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&+\frac{\beta}{\gamma} \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2 e^{\gamma\tau} e^{-\epsilon(t_0-\tau)}\Bigg] \\
=\ & M^2 e^{-\epsilon(t_0-\tau)} \\
<\ & \frac{\rho_0^2}{1-\eta} M^2 e^{-\epsilon(t_0-\tau)}.
\end{aligned}
\tag{17}
$$

It is contradicted to equation (15).
Second, consider the case of $t_0 \in (2\tau, \infty)$. From inequality (12), we have

$$
\begin{aligned}
&||e(t_0)||^2 \leq \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\gamma(t_0-\tau)} \\
&+\beta \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2\int_{\tau}^{2\tau} e^{-\gamma(t_0-s)} ds \\
&+\beta \max_{1\leq i\leq n}\{l_i^2\}\int_{2\tau}^{t_0} e^{-\gamma(t_0-s)}||e(s-\tau(s))||^2 ds] \\
\leq\ & \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&+\beta \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2\int_{\tau}^{2\tau} e^{-\gamma(t_0-s)} ds \\
&+\beta\rho_0^2\left(\frac{1}{1-\eta}\right)M^2 \max_{1\leq i\leq n}\{l_i^2\}\int_{2\tau}^{t_0} e^{-\gamma(t_0-s)}e^{-\epsilon(s-\tau(s)-\tau)} ds]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&\quad +\beta \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2 \int_{\tau}^{2\tau} e^{-\gamma(t_0-s)}ds \\
&\quad +\beta\rho_0^2\left(\frac{1}{1-\eta}\right)M^2 \max_{1\leq i\leq n}\{l_i^2\}\int_{2\tau}^{t_0} e^{-\gamma(t_0-s)}e^{-\epsilon(s-2\tau)}ds] \\
&= \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&\quad +\frac{\beta}{\gamma} \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2(e^{-\gamma(t_0-\tau)}-e^{-\gamma(t_0-2\tau)}) \\
&\quad +\beta\rho_0^2\left(\frac{1}{1-\eta}\right)M^2 \max_{1\leq i\leq n}\{l_i^2\}\frac{1}{\gamma-\epsilon}(e^{-\epsilon(t_0-2\tau)}-e^{-\gamma(t_0-2\tau)})] \\
&\leq \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&\quad +\frac{\beta}{\gamma} \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2(e^{-\gamma(t_0-\tau)}-e^{-\gamma(t_0-2\tau)}) \\
&\quad +\beta\rho_0^2\left(\frac{1}{1-\eta}\right)M^2 \max_{1\leq i\leq n}\{l_i^2\}\frac{1}{\gamma-\epsilon}(e^{-\epsilon(t_0-2\tau)}-e^{-\gamma(t_0-2\tau)})] \\
&\leq \frac{1}{\lambda_m(Q)}[\lambda_M(Q)\sigma^2 e^{-\epsilon(t_0-\tau)} \\
&\quad +\frac{\beta}{\gamma}e^{\gamma\tau} \max_{1\leq i\leq n}\{l_i^2\}\varsigma^2 e^{-\epsilon(t_0-\tau)} \\
&\quad +\beta\rho_0^2\left(\frac{1}{1-\eta}\right)M^2 e^{\epsilon\tau} \max_{1\leq i\leq n}\{l_i^2\}\frac{1}{\gamma-\epsilon}e^{-\epsilon(t_0-\tau)}] \\
&\leq [M^2+\frac{\rho_0^2}{1-\eta}M^2\eta]e^{-\epsilon(t_0-\tau)} \\
&< \frac{\rho_0^2}{1-\eta}M^2 e^{-\epsilon(t_0-\tau)}.
\end{aligned}
\tag{18}
$$

The above result is contradicted to the assumption (15). Thus, inequality (14) is correct,so (13) is correct, i.e., the systems (2) and (6) are exponentially synchronized. The proof is complete. □

Let $\alpha=\beta=1$. Then, we have the following corollary.

Corollary 1. Suppose that there exist a diagonal matrix $P=diag(p_1,\cdots,p_n)>0$ and a positive γ, such that
(i) the following LMI holds:

$$
\begin{bmatrix} 2P(K-A)-\gamma P-B^TB-C^TC & PL \\ LP & I_n \end{bmatrix} > 0;
$$

(ii) $c \equiv \frac{1}{\gamma}\max_{1\leq i\leq n}\{p_i\}\max_{1\leq i\leq n}\{l_i^2\}<1$;
where I_n is an n-order identity matrix, $L=diag(l_i)_{n\times n}$ and l_i being the Lipschitz constants in (3). Then, the systems (2) and (6) are exponentially synchronized under the assumption H.

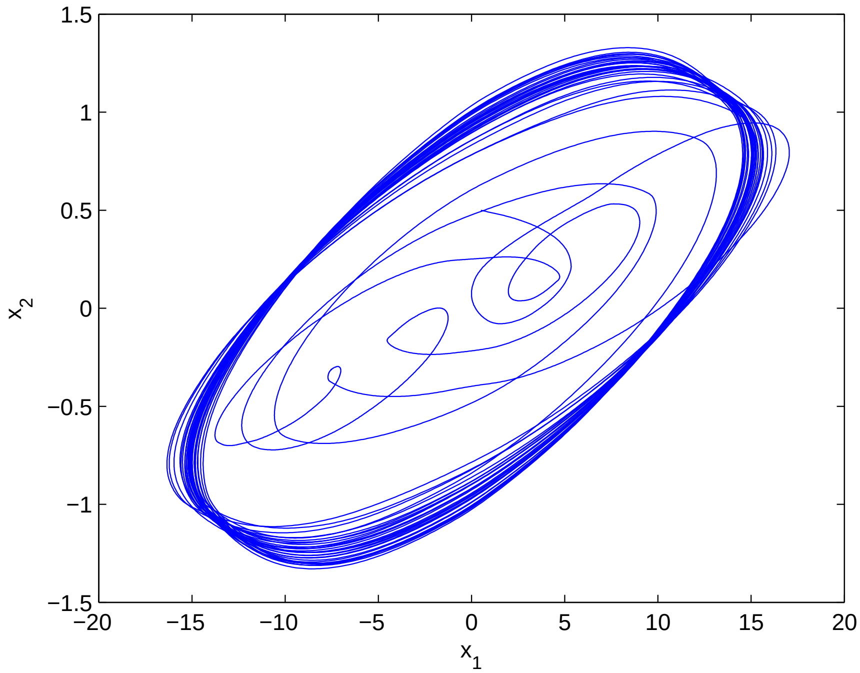

FIGURE 1. Transient behavior of x_2 in Example 1.

4. **Simulations.** In this section, we provide two examples to demonstrate that the new results on synchronization of chaotic systems are effective.

Example 1. Consider the following chaotic master system (shown in Fig. 1):

$$\frac{dx}{dt}(t) = Ax(t) + Bf(x(t)) + Cf(x(t-\tau(t))), \tag{19}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1+\frac{\pi}{4} & 20 \\ 0.1 & 1+\frac{\pi}{4} \end{bmatrix}, \ C = \begin{bmatrix} \frac{-1.3\sqrt{2}\pi}{4} & 0.1 \\ 0.1 & \frac{-1.3\sqrt{2}\pi}{4} \end{bmatrix},$$

$\tau(t) = 1 + 2sin^2 t$, and $f(x) = 0.5(|x+1| - |x-1|)$. Let its response system be

$$\frac{dy}{dt}(t) = Ay(t) + Bf(y(t)) + Cf(y(t-\tau(t))) + K(x(t) - y(t)), \tag{20}$$

where $K = diag(k_1, k_2)$. For a small enough scalar $\epsilon > 0$, let $P = I$, $\gamma = \beta + \epsilon$, $\alpha = \sqrt{439.0741}$, $\beta = \sqrt{2.3837}$ in Theorem 1. When $k_1 > 21.498$, $k_2 > 21.498$, the conditions in Theorem 1 are satisfied. Thus, systems (19) and (20) are synchronized. Fig. 2 and Fig. 3 show the error curves of e_1 and e_2 with the initial conditions $\varphi_{x1}(s) = \varphi_{x2}(s) = 0.5$, $\varphi_{y1}(s) = \varphi_{y2}(s) = 0.48$. From those graphs, one observes that the synchronization between two systems are achieved effectively.

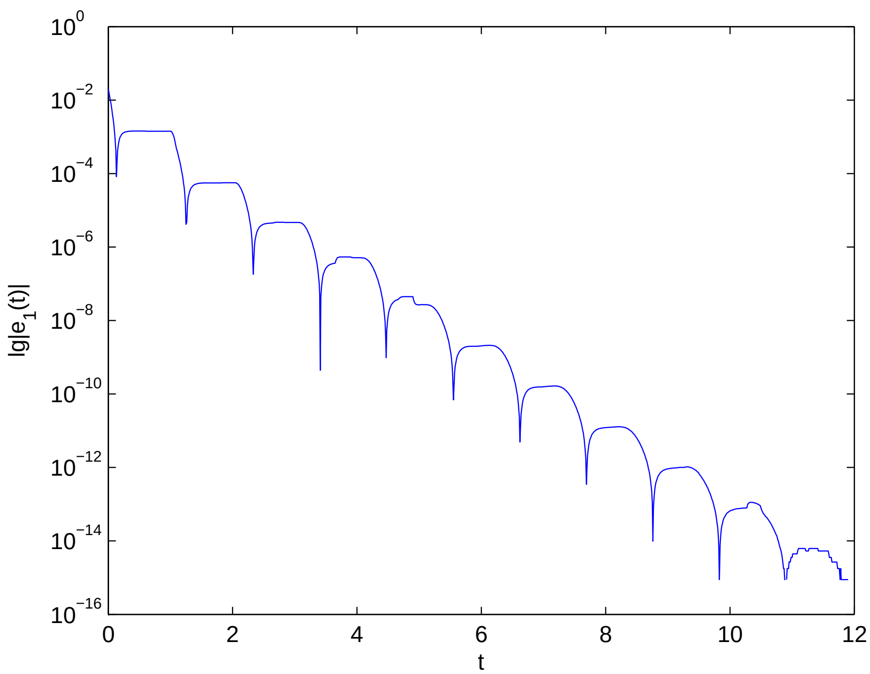

FIGURE 2. Error e_1 in Example 1.

Example 2. Consider the following master chaotic system (shown in Fig. 4):

$$\frac{dx}{dt}(t) = ax(t) + cf(x(t - \tau(t))), \tag{21}$$

where $a = -0.16$, $c = 0.2$,, $\tau(t) = 25 + sin^2 t$, and

$$f(x) = \begin{cases} 0, & |x| \geq \frac{4}{3} \\ -1.5x - 2, & -\frac{4}{3} < x - 0.8 \\ x, & |x| \leq -0.8 \\ -1.5x + 2, & 0.8 < x < \frac{4}{3}. \end{cases} \tag{22}$$

Let the response system be

$$\frac{dx}{dt}(t) = ax(t) + cf(x(t - \tau(t))) + k(x(t) - y(t)). \tag{23}$$

Choosing $\alpha = \gamma = 1$, $\beta = 0.25$, $P = 1$, we can see that all of the conditions of Theorem 1 are satisfied when $k > 1.868$. The error curve with the initial conditions $\varphi_x(s) = 0.9$, $\varphi_y(s) = 0.8$, and the feedback strength $k = 1.9$, is shown in Fig. 5.

REFERENCES

[1] S. Boccaletti, J. Kurths, G. Osipov, D. L. Vallares and C. S. Zhou, *The synchronization of chaotic systems*, Phys. Rep., **366** (2002), 1–101.

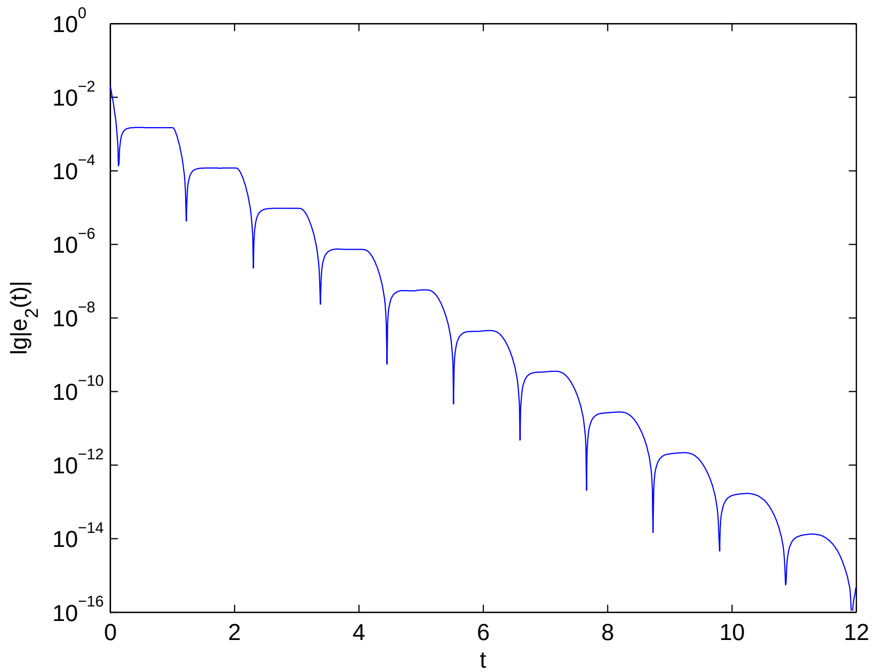

FIGURE 3. Error e_2 in Example 1.

[2] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, "Linear Matrix Inequalities In Systems And Control Theory," SIAM Studies in Applied Mathematics, **15**, SIAM, Philadephia, PA, 1994.

[3] J. Cao and J. Lu, *Adaptive synchronization of neural networks with or without time-varying delay*, Chaos, **16** (2006), 013133, 6 pp.

[4] P. Colet and R. Roy, *Digital communication with synchronized chaotic lasers*, Opt. Lett., **19** (1994), 2056.

[5] K. Gopalsamy, "Stability and Oscillations in Delay Differential Equations of Population Dynamics," Mathematics and its Applications, **74**, Kluwer Academic Publishers Group, Dordrecht, 1992.

[6] H. G. Schuster, ed., "Handbook of Chaos Control: Foundations and Applications," Wiley-VCH, Weinheim, 1999.

[7] H. Huang, G. Feng and Y. Sun, *Robust synchronization of chaotic systems subject to parameter uncertainties*, Chaos, **19** (2009), 033128.

[8] T. Huang, C. Li and X. Liu, *Synchronization of chaotic systems with delay using intermittent linear state feedback*, Chaos, **18** (2008), 033122, 8 pp.

[9] C. Li, G. Feng and X. Liao, *Stabilization of nonlinear systems via periodically intermittent control*, IEEE Trans. Circuits and Systems II, **54** (2006), 1019–1023.

[10] C. Li, X. Liao and K. Wong, *Chaotic lag synchronization of coupled time-delayed systems and its application in secure communication*, Physica D, **194** (2004), 187–202.

[11] X. Liu, T. Chen and W. Lu, *Cluster synchronization for linearly coupled complex networks*, Journal of Industrial and Management Optimization (JIMO), **7** (2011), 87–101.

[12] J. Lu, J. Cao and D. Ho, *Adaptive stabilization and synchronization for chaotic lure systems with time-varying delay*, IEEE Transactions on Circuits and Systems I: Regular Papers, **55** (2008), 1347–1356.

[13] L. M. Pecora and T. L. Carroll, *Synchronization in chaotic systems*, Phys. Rev. Lett., **64** (1990), 821–824.

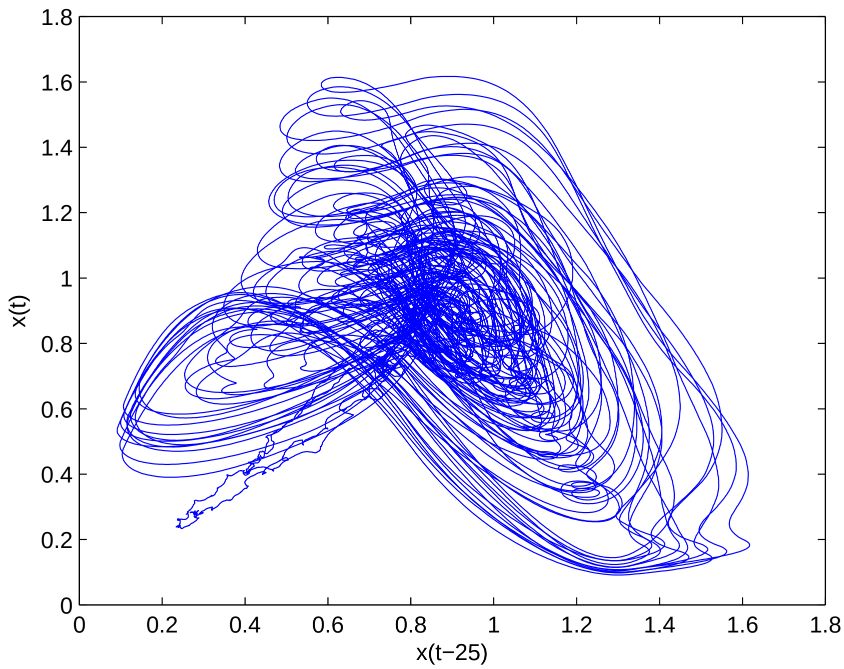

FIGURE 4. Transient behavior of x in Example 2.

[14] J. Qing, *Projective synchronization of a new hyperchaotic Lorenz system*, Physics Letters A, **370** (2007), 40–45.

[15] F. Rogister, D. Pieroux, M. Sciamanna, P. Megret and M. Blondel, *Anticipating synchronization of two chaotic laser diodes by incoherent optical coupling and its application to secure communications*, Optics Communications, **207** (2002), 295–306.

[16] M. Rosenblum and A. Pikovsky, *Phase synchronization of chaotic oscillators*, Phys. Rev. Lett., **76** (1996), 1804–1807.

[17] N. Rulkov and M. Sushchik, *Generalized synchronization of chaos in directionally coupled chaotic systems*, Phys. Rev. E, **51** (1995), 980–994.

[18] S. Sivaprakasam and P. Spencer, *Regimes of chaotic synchronization in external-cavity laser diodes*, IEEE Journal of Quantum Electronics, **38** (2002), 1155–1161.

[19] Q. Song and J. Cao, *Global dissipativity analysis on uncertain neural networks with mixed time-varying delays*, Chaos, **18** (2008), 043126, 10 pp.

[20] K. Thornburg, M. Moller, R. Roy and T. Carr, *Chaos and coherence in coupled lasers*, Phys. Rev. E, **55** (1997), 3865.

[21] J. Wang, Z. Yang, T. Huang and M. Xiao, L*ocal and global exponential synchronization of complex delayed dynamical networks with general topology*, Discrete and Continuous Dynamical Systems-Series B, **16** (2011), 393–408.

[22] R. Zhen, X. Wu and J. Zhang, "Sliding Model Synchronization Controller Design for Chaotic Neural Network with Time-Varying Delay," Proceedings of the 8th World Congress on Intelligent Control and Automation, China, (2010), 3914–3919.

Received October 2010; revised May 2011.

E-mail address: tingwen.huang@qatar.tamu.edu
E-mail address: gchen@ee.cityu.edu.hk
E-mail address: jkurths@agnld.uni-potsdam.de

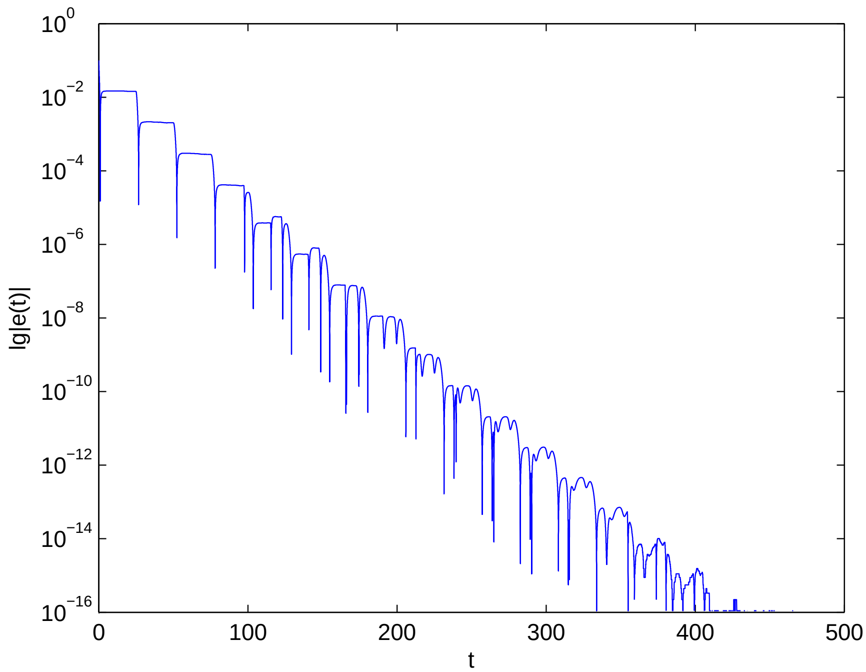

Figure 5. Error e in Example 2.