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# Linear time-varying flatness-based control of Anti-lock Brake System (ABS)

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***Abstract*—In this paper, a flatness-based control strategy for linear time-varying systems is proposed in order to track a desired trajectory. The flatness-based control is designed by using two observers: a reduced order observer with a constant estimator error gain and an exact observer for designing a polynomial two-degrees-of-freedom controller without resolving Bezout equation in time varying framework. The proposed approach is illustrated with the control of an Anti-lock Brake System (ABS) and led to track a given trajectory for the wheel slip.**

***Index Terms*—Linear time-varying systems, trajectory linearization, flatness, path tracking, exact observer, polynomial controller, reduced order observer.**

## I. Introduction

In the control theory, the study of linear time-varying (LTV) systems has been important since this situation is encountered not only when some parameters of the system vary with time, but also when the system to be controlled is nonlinear and the problem is approached by linearizing this system around a desired trajectory which leads to an LTV model.

For finite-dimensional and time-invariant linear systems, a well-known control design technique is obtained by polynomial two-degrees-of-freedom controllers [5], [11], [20] which were introduced fifty years ago by Horowitz [8]. More details are given in the reference therein and in the following these controllers will be denoted as RST controllers [7]. Whatever the chosen design method, this powerful method is based on pole placement and presents one drawback: it needs to know where to place all the poles of the closed-loop system at the outset.

Following [4], by the use of flatness design control principles, the problem of pole placement which consists in imposing closed loop system dynamics can be related to the tracking problem, to design an RST (two-degree-of-freedom) controller with very natural choices of closed loop poles. In this design, a solution of the Bezout equation is obtained depending on the planned trajectories.

The RST design controller problem is not easy to transcribe in the case of LTV systems due to the fact that the coefficients do not commute with the time derivative operator. Besides, the structure of the set of the poles of the closed-loop system is more complex.

In this case, the pole placement problem was solved recently by Marinescu [1], who proposes some technical methods for factorization of linear time-varying transfert matrices. These key points lead to solve Bezout equation written in the time-varying framework.

In order to overcome these two points in LTV framework, namely the choice of desired poles at the outset and the determination of solution for the Bezout equation, we propose in this paper to extend the flatness-based control strategy developed in [4] to the case of time-varying systems. It will be seen that applying the guideline induced by a flatness based control to a LTV system leads to express it in a natural RST form.

This control strategy is be compared to the flatness-based control based on the use of a reduced order observer. The paper is organized as follows: in section II, some background notions about SISO LTV systems and flatness-based control strategy are presented. In section III, a reduced order observer for the state vector are presented. In section IV, the polynomial controller design based on exact observer is proposed. The state vector constituted by the flat output and its derivatives and the designed observer is without dynamics. In section V, the proposed strategy is illustrated on the control of an Anti-lock Brake System (ABS).

## II. Background notions

### A. SISO linear time-varying systems

For finite-dimensional, several input-output descriptions have been introduced for LTV systems. Here, a time-varying linear system is described by the following state space model of dimension $n$:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t)\end{aligned} \tag{1}$$

The matrices $A(t)$, $B(t)$ and $C(t)$ whose coefficients depend on the time are of dimensions $(n \times n)$, $(n \times 1)$ and $(1 \times n)$, respectively. If the system (1) is completely controllable and by applying the algorithm presented in [12], [13], we obtain the controllable form of (1) given by:

$$\begin{aligned}\dot{Z}(t) &= \bar{A}(t)Z(t) + \bar{B}(t)u(t)\\ y(t) &= \bar{C}(t)Z(t)\end{aligned} \tag{2}$$

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with:

$$\bar{A}(t)=\begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ -\psi_0(t) & -\psi_1(t) & \cdots & -\psi_{n-1}(t) \end{pmatrix} \tag{3}$$

$$\bar{B}=\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T, \bar{C}=\begin{pmatrix} \gamma_0(t) & \cdots & \gamma_0(t) \end{pmatrix}$$

### B. Short survey on flatness

The flatness property, which was introduced by Fliess *et al.* in (1992) [14], for continuous-time nonlinear systems, leads to interesting results for control design. This system property was widely introduced and used in literature [2], [6], [9], [10]. The existence of a variable called a flat output permits to define all other system variables. Let us consider the nonlinear system described by the following differential equation:

$$\dot{x}(t)=f(x(t),u(t)) \tag{4}$$

where $x(t)\in\Re^n$ is the state vector and $u(t)\in\Re^m$ is the input vector. Roughly speaking, this system is called differentially flat if there exists a variable $z(t)\in\Re^m$ of the form:

$$z(t)=h(x(t),u(t),\dot{u}(t),...,u^{(r)}(t)) \tag{5}$$

such that the state and the input of the system are given by:

$$x(t)=\mathbf{A}(z(t),\dot{z}(t),...,z^{(\alpha)}(t)) \tag{6}$$

$$u(t)=\mathbf{B}(z(t),\dot{z}(t),...,z^{(\alpha+1)}(t)) \tag{7}$$

where $\alpha$ is an integer. The variable $z(t)$ is called the flat output of the system.

*1) Implication for the LTV systems:* Let us consider the controllable state space equation (2) and let us denote by $z_i(t)$ the $i$-th component of $Z(t)$. The variable $z_1(t)$, denoted as $z(t)$, can be considered for this system as a flat output. Then, the state vector of the controllable form $Z(t)$ is composed by the flat output and its derivatives.

*2) Tracking control and pole placement:* For a given planned trajectory of the flat output, $z_d(t)$, the control law based on flatness is as follows:

$$u(t)=z_d^{(n)}(t)+\sum_{i=0}^{n-1}k_i(z_d^{(i)}(t)-z^{(i)}(t))+\psi_i(t)z^{(i)}(t) \tag{8}$$

and by introducing the polynomial:

$$K(p)=p^n+\sum_{i=0}^{n-1}k_ip^i \tag{9}$$

where the $k_i$ are chosen such that $K(p)$ is a Hurwitz polynomial, the control $u(t)$ can be written as:

$$u(t)=K(p)z_d(t)+\sum_{i=0}^{n-1}(\psi_i(t)-k_i)z^{(i)}(t) \tag{10}$$

By applying this control, the tracking error verifies:

$$\lim_{t\to\infty}(z_d(t)-z(t))=0 \tag{11}$$

and the closed-loop dynamics are given by the roots of $K(p)$. This strategy differs from the usual pole placement for linear time-varying systems obtained by a time-varying state feedback.

By denoting: $\psi-k=\begin{pmatrix}\psi_0(t)-k_0 \\ \vdots \\ \psi_{n-1}(t)-k_{n-1}\end{pmatrix}$

the previous control can be written as:

$$u(t)=K(p)z_d(t)+(\psi-k)^TZ(t) \tag{12}$$

where:

$$Z(t)=\begin{pmatrix} z(t) & \dot{z}(t) & \cdots & z^{(n-1)}(t)\end{pmatrix}^T \tag{13}$$

is the state vector of the controllable form.

To implement the control (12), the vector $Z(t)$ must be estimated with an observer. In the next sections, two type of observers are considered.

## III. Reduced order observer

Let us consider the observable form of the state equation (1) given by the following relation:

$$\begin{aligned}\dot{x}_o(t)&=\begin{pmatrix} 0 & \cdots & 0 & -\tau_0(t) \\ 1 & \ddots & \vdots & -\tau_1(t) \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -\tau_{n-1}(t)\end{pmatrix}x_o(t)+\begin{pmatrix}\sigma_0(t)\\ \sigma_1(t) \\ \vdots \\ \sigma_{n-1}(t)\end{pmatrix}u(t)\\ y(t)&=\begin{pmatrix} 0 & \cdots & 0 & 1\end{pmatrix}x_o(t)\end{aligned} \tag{14}$$

As in (14), the system output is the last component, a reduced order observer is then used to estimate the state vector $x_o(t)$. Let us group the $n-1$ first components of $x_o(t)$ in $\chi(t)$ then[1]:

$$\begin{aligned}\dot{\chi}&=\begin{pmatrix} & & & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0\end{pmatrix}\chi-\begin{pmatrix}\tau_0\\ \tau_1 \\ \vdots \\ \tau_{n-2}\end{pmatrix}y+\begin{pmatrix}\sigma_0\\ \sigma_1 \\ \vdots \\ \sigma_{n-2}\end{pmatrix}u,\\ \dot{y}&+\tau_{n-1}(t)y-\sigma_{n-1}(t)u=\begin{pmatrix} 0 & \cdots & 0 & 1\end{pmatrix}\chi\end{aligned}$$

The observer for this system is then given by:

$$\begin{aligned}\dot{\hat{\chi}}=&\begin{pmatrix} & & & 0 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0\end{pmatrix}\hat{\chi}-\begin{pmatrix}\tau_0\\ \tau_1 \\ \vdots \\ \tau_{n-2}\end{pmatrix}y+\begin{pmatrix}\sigma_0\\ \sigma_1 \\ \vdots \\ \sigma_{n-2}\end{pmatrix}u\\ &+\Gamma\left(\dot{y}+\tau_{n-1}y-\sigma_{n-1}u-\begin{pmatrix} 0 & \cdots & 0 & 1\end{pmatrix}\hat{\chi}\right)\end{aligned}$$

To overcome the output derivation, we are led to propose the following reduced order observer with the introduction of a

[1]For space reasons, we dropped the time argument.

new variable $\zeta(t)=\hat{\chi}(t)-\begin{pmatrix}\lambda_0(t) & \cdots & \lambda_{n-2}(t)\end{pmatrix}^T y(t)$:

$$\dot{\zeta} = \begin{pmatrix} & & -\lambda_0 \\ 1 & & -\lambda_1 \\ & \ddots & \vdots \\ & 1 & -\lambda_{n-2}\end{pmatrix}\zeta + \begin{pmatrix}\sigma_0-\sigma_{n-1}\lambda_0 \\ \sigma_1-\sigma_{n-1}\lambda_1 \\ \vdots \\ \sigma_{n-2}-\sigma_{n-1}\lambda_{n-2}\end{pmatrix}u + \begin{pmatrix}-\tau_0-\dot{\lambda}_0+(\tau_{n-1}-\lambda_{n-2})\lambda_0 \\ \lambda_0-\tau_1-\dot{\lambda}_1+(\tau_{n-1}-\lambda_{n-2})\lambda_1 \\ \vdots \\ \lambda_{n-3}-\tau_{n-2}-\dot{\lambda}_{n-2}+(\tau_{n-1}-\lambda_{n-2})\lambda_{n-2}\end{pmatrix}y$$

As the error dynamics are given by the matrix:

$$\begin{pmatrix} & & -\lambda_0(t) \\ 1 & & -\lambda_1(t) \\ & \ddots & \vdots \\ & 1 & -\lambda_{n-2}(t)\end{pmatrix}$$

we choose for all $i$, $\lambda_i(t)$ as constant parameters to give an asymptotic observer. The observation of $x_o(t)$ is then deduced:

$$\hat{x}_o(t)=\begin{pmatrix}\zeta(t)+\begin{bmatrix}\lambda_0(t) & \cdots & \lambda_{n-2}(t)\end{bmatrix}^T y(t) \\ y(t)\end{pmatrix} \quad (15)$$

In this solution, the difficulty appears in the choice of the observers' poles in the LTV framework. To overcome this point, an enlightening ideas suggested in [15] and applied in [4] can be used. The realization of this controller, using the exact observer, will be the subject of the next part.

## IV. Exact state space observer

Let us consider the model (2) where the first component of the state vector $Z(t)$ is the system flat output. By successive derivations of the output plant $y(t)$ until the order $(n-1)$, we get:

$$Y(t)=O(t)Z(t)+M(t)U(t) \quad (16)$$

where:

- $Y(t)=\begin{pmatrix}y(t) & \cdots & y^{(n-1)}(t)\end{pmatrix}^T$,
- $U(t)=\begin{pmatrix}u(t) & \cdots & u^{(n-2)}(t)\end{pmatrix}^T$,
- $O(t)$ is the observability matrix of the pair $\left(\bar{A}(t),\bar{C}(t)\right)$ and it is given by:

$$O(t)=\begin{pmatrix}\bar{C}_1(t) & \cdots & \bar{C}_n(t)\end{pmatrix}^T \quad (17)$$

  such that:

$$\bar{C}_1(t)=\bar{C}(t)$$
$$\bar{C}_i(t)=\dot{\bar{C}}_{i-1}(t)+\bar{C}_{i-1}(t)\bar{A}(t) \text{ for } i=2 \text{ to } n$$

- The matrix $M(t)$ has the following expression:

$$M(t)=\begin{pmatrix}0 & 0 & \cdots & 0 \\ M_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ M_{n-2} & \cdots & M_1 & 0 \\ M_{n-1} & M_{n-1,2} & \cdots & M_1\end{pmatrix} \quad (18)$$

  with:
  - $M_1(t)=\bar{C}_1(t)\bar{B}$,
  - $M_i(t)=\dot{M}_{i-1}(t)+\bar{C}_i(t)\bar{B}$, for $i=2$ to $n-1$,
  - $M_{n-1,2}(t)=M_{n-2}(t)+\sum_{i=1}^{n-3}M_i^{(n-2-i)}(t)$,
  - $M_{n-1,3}(t) = M_{n-3}(t)+\sum_{i=1}^{n-4}(n-i-2)M_i^{(n-3-i)}(t)$,
  - etc.

As the pair $(\bar{A}(t),\bar{C}(t))$ is observable, the matrix $O(t)$ is of rank $n$ and the state vector can be written as:

$$Z(t)=O^{-1}(t)Y(t)-O^{-1}(t)M(t)U(t) \quad (19)$$

Taking into account the state space equation (2) and avoiding variable derivations, we get:

$$Z(t)=p^{-1}\left(\bar{A}(t)Z(t)\right)+p^{-1}\left(\bar{B}u(t)\right) \quad (20)$$

where $p^{-1}$ stands for the integration operator:

$$p^{-1}h(t)=\int_{-\infty}^{t}h(\tau)d\tau \quad (21)$$

with $h(-\infty)=0$. This last hypothesis ensures commutativity between $p$ and $p^{-1}$.
By rewriting this equation to the order $(n-1)$, the equation(20) becomes:

$$\begin{aligned}Z(t)=&\,p^{-1}\left(\bar{A}(t)p^{-1}(\bar{A}(t)\dots p^{-1}(\bar{A}(t)Z(t))\right)+\\ &p^{-1}\left(\bar{A}(t)\dots p^{-1}(\bar{A}(t)\bar{B}p^{-1}u(t))\right)+\\ &p^{-1}\left(\bar{A}(t)\bar{B}p^{-1}u(t)\right)+\bar{B}p^{-1}u(t)\end{aligned} \quad (22)$$

If the term $Z(t)$ is replaced in this equation by the one given in (19), we get:

$$\begin{aligned}Z(t)=&\,p^{-1}\left(\bar{A}(t)\dots p^{-1}(\bar{A}(t)O^{-1}(t)Y(t)\right.\\ &\left.-\bar{A}(t)O^{-1}(t)M(t)U(t))\right)+p^{-1}\left(\bar{A}(t)\dots\right.\\ &\left.p^{-1}(\bar{A}(t)\bar{B}p^{-1}u(t))\right)+p^{-1}\left(\bar{A}(t)\bar{B}p^{-1}u(t)\right)+\bar{B}p^{-1}u(t)\end{aligned} \quad (23)$$

To eliminate the terms containing the derivatives of the plant output $y(t)$ in $Y(t)$, we proceed by using successive integrations by parts leading to the following expression of the state vector:

$$\begin{aligned}Z(t)=&\,p^{-n+1}\left(\Theta_1(t)y(t)\right)+\cdots+p^{-1}\left(\Theta_{n-1}(t)y(t)\right)+\\ &\left(\Theta_n(t)y(t)\right)+p^{-n+1}\left(\Delta_1(t)u(t)\right)+\cdots+\\ &p^{-1}\left(\Delta_{n-1}(t)u(t)\right)+p^{-1}\left(\bar{A}(t)\dots p^{-1}(\bar{A}(t)\bar{B}p^{-1}u(t))\right)\\ &+\cdots+p^{-1}\left(\bar{A}(t)\bar{B}p^{-1}u(t)\right)+\bar{B}p^{-1}u(t)\end{aligned} \quad (24)$$

where $\Theta_j(t) = \begin{pmatrix} \theta_{1j}(t) & \cdots & \theta_{nj}(t) \end{pmatrix}^T$ and $\Delta_j(t) = \begin{pmatrix} \delta_{1j}(t) & \cdots & \delta_{nj}(t) \end{pmatrix}^T$. The components $\theta_{ij}(t)$ and $\delta_{ij}(t)$ are function of the parameters $\psi_i(t)$ and their derivatives. The control law (12) can be written in the RST form:

$$R\left(p^{-1}, u\left(t\right)\right) = K(p)z_d(t) - S(p^{-1}, y(t)) \tag{25}$$

where:

$$S(p^{-1}, y(t)) = (k-\psi)\left(p^{-n+1}\left(\Theta_1(t)y(t)\right) + \cdots + \Theta_n(t)y(t)\right)$$

$$R(p^{-1}, u(t)) = u\left(t\right) + (k-\psi)\left(p^{-n+1}\left(\Delta_1(t)u(t)\right) + \cdots \left(+p^{-1}\left(\Delta_{n-1}(t)u(t)\right)\right)\right) + (k-\psi)\left(\left(p^{-1}\bar{A}(t)\ldots +p^{-1}(\bar{A}(t)\bar{B}p^{-1}u(t))\cdots + \bar{B}p^{-1}u(t)\right)\right.$$

The proposed control design can be seen as an RST controller without resolution of a Bezout identity. Now the design is focused in the choice of the trajectory $z_d(t)$ to follow and the traking dynamis with $K(p)$.
This regulator-observer permits to the output system to track a desired trajectory without using an observer dynamics then the problem of pole placement, which consists in imposing closed-loop system dynamics, can be related to tracking.

## V. Application to Anti-lock Braking System (ABS) in vehicle

As an illustrative example of the proposed strategy, the control of the wheel slip in an Anti-lock Brake System is studied. The considered process is an Anti-lock Brake System (ABS), used to control the slip of each wheel of a vehicle to prevent it from locking such that a high friction is achieved and steerability is maintained. The main objective of this control system is the prevention of wheel-lock while braking and the maintaining of the wheel slip the nearest possible to 0. The problem of wheel slip control is better explained by looking at a quarter car model. A mathematical model of the wheel slip dynamics is given by [16], [17]:

$$\begin{array}{l} \dot{\lambda}\left(t\right) = -\frac{1}{v(t)}\left[\frac{1}{m}\left(1-\lambda\left(t\right)\right) + \frac{r^2}{J}\right]F(\lambda) + \frac{1}{v(t)}\frac{r}{J}T\left(t\right) \\ \dot{v}\left(t\right) = -\frac{1}{m}F(\lambda) \end{array} \tag{26}$$

where:

| | | |
|---|---|---|
| $\omega\left(t\right)$ | angular speed of the wheel | (rad/s) |
| $v\left(t\right)$ | horizontal speed | (m/s) |
| $T\left(t\right)$ | brake-acceleration torque | (N.m) |
| $m$ | mass of the quarter car | (450 kg) |
| $r$ | wheel radius | (0.32 m) |
| $J$ | wheel inertia | (1kg.m$^2$) |
| $g$ | acceleration of gravity | (9.81 m/$s^2$) |

and $\lambda\left(t\right)$ is the wheel slip given by:

$$\lambda\left(t\right) = \frac{v\left(t\right) - r\omega\left(t\right)}{v\left(t\right)} \tag{27}$$

The input signal $T\left(t\right)$ is a brake-acceleration torque applied to the wheel, it is expressed in (N.m), and the output is the vehicle speed $v\left(t\right)$. The longitudinal slip $\lambda\left(t\right)$ is defined by the normalized difference between $v\left(t\right)$ and the speed of the wheel perimeter $\omega\left(t\right)r$. $F(\lambda)$ is the friction force, which depends on the normal force, steering angle, road surface, tyre

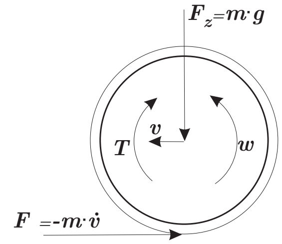


Fig. 1. Quarter car forces and torques.

characteristics and velocity of the car. The friction or adhesion coefficient $\mu\left(\lambda\right)$ is defined as the ratio of the frictional force acting in the wheel plane $F(\lambda)$ and the wheel ground contact force $F_Z$:

$$\mu\left(\lambda\right) = \frac{F\left(\lambda\right)}{F_Z} \tag{28}$$

The calculation of friction force can be carried out using the Burckhardt method [19]:

$$\mu\left(\lambda\right) = c_1.\left(1 - e^{-c_2.\lambda(t)}\right) - c_3\lambda\left(t\right) \tag{29}$$

The parameters $c_1$, $c_2$ and $c_3$ are given for various road surfaces.In the case of asphalt and dry road, the friction force is given by:

$$F(\lambda) = mg\left[1.28 \times \left(1 - \exp\left(-24\lambda\left(t\right)\right)\right) - 0.52\lambda\left(t\right)\right] \tag{30}$$

To design a control law which maintains the wheel slip the nearest possible to 0, we perform, in the next development, an approximation to a friction force $F(\lambda)$ by applying the Taylor series for this function with a first-order approximation at $\lambda = 0$ to obtain:

$$F(\lambda) = a\lambda\left(t\right) \tag{31}$$

where $a = 30.2 \times mg$. The equation of the system (26) becomes:

$$\begin{array}{l} \dot{\lambda}\left(t\right) = -\dfrac{1}{v\left(t\right)}\left[\dfrac{1}{m}\left(1-\lambda\left(t\right)\right) + \dfrac{r^2}{J}\right]a\lambda\left(t\right) + \dfrac{1}{v\left(t\right)}\dfrac{r}{J}T\left(t\right) \\ \dot{v}\left(t\right) = -\dfrac{a}{m}\lambda\left(t\right) \end{array} \tag{32}$$

By analyzing the equation (32), we remark that the input and the output system are function of a finite number of derivatives of the horizontal speed $v(t)$. By denoting $z(t) = v(t)$, we obtain:

$$\begin{array}{l} T\left(t\right) = \dfrac{J}{r}\left(\dfrac{-m}{a}z\left(t\right)\ddot{z}\left(t\right) - \dot{z}\left(t\right)\left(1 + \dfrac{mr^2}{J}\right) - \dfrac{-m}{a}\dot{z}^2\left(t\right)\right) \\ \lambda\left(t\right) = -\dfrac{m}{a}\dot{z}\left(t\right) \end{array} \tag{33}$$

Then the vehicle speed is a flat output of the considered non-linear model. For the considered system, a desired trajectory $\left(T_d\left(t\right), \lambda_d\left(t\right), v_d\left(t\right)\right)$ is defined and the following variables are given: $\delta T\left(t\right) = T_d\left(t\right) - T\left(t\right)$, $\delta\lambda\left(t\right) = \lambda_d\left(t\right) - \lambda\left(t\right)$ and $\delta v\left(t\right) = v_d\left(t\right) - v\left(t\right)$. The drawback of the flatness control is that all system variables are carried out from the flat output trajectory. This situation is critical for the flatness control because the dynamics of the output system can not be well controlled. To deny this critical point, a tyre slip is imposed and all the system variables are designed from this trajectory which is given by:

$$\lambda_d(t) = 0.04 \times (-\cos(\pi t) + 1) \ \text{ if } t \in [0, 2] \tag{34}$$

From equation (32), the desired flat output are given by:

$$\begin{aligned} z_d(t) &= \int_0^t -\tfrac{\alpha}{m}\lambda_d(h)\,dh \\ &= -\left(\tfrac{0.04\times\alpha}{m}\right)\cdot\left(\tfrac{-\sin(\pi t)}{\pi} + t\right) + z_{d0} \end{aligned} \tag{35}$$

where $z_{d0} = 36.11$ m/s is the initial condition for the horizontal speed. The brake-acceleration torque $T_d(t)$ is deduced from equation (33). The figures 2 and 3 show the desired trajectories for the input, tyre slip and the flat output of the nonlinear system. The linearized model of (26) around this desired trajectory is given by:

$$\begin{aligned} \delta\dot{\lambda} &= \frac{-\dot{\lambda}_d}{v_d}\delta v - \left(a\left(\frac{1-\lambda_d}{m} + \frac{r^2}{J}\right) - \frac{a\lambda_d}{m}\right)\frac{\delta\lambda}{v_d} + \frac{r}{Jv_d}\delta T \\ \delta\dot{v} &= -a\frac{\delta\lambda}{m} \end{aligned} \tag{36}$$

To design the closed-loop control which allows to track variable reference trajectories, the following state space representation of the system is considered:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\delta T(t) \\ \delta\lambda &= C(t)x(t) \end{aligned} \tag{37}$$

with $x(t) = \begin{pmatrix} \delta\lambda & \delta v \end{pmatrix}^T$ is the state vector such that:

$$\begin{aligned} A(t) &= \begin{pmatrix} -\frac{a}{v_d}\left(\frac{1}{m} + \frac{r^2}{J} - \frac{2\lambda_d}{m}\right) & \frac{-\dot{\lambda}_d}{v_d} \\ -\frac{a}{m} & 0 \end{pmatrix} \\ B(t) &= \begin{pmatrix} \frac{r}{Jv_d} & 0 \end{pmatrix}^T, C(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \end{aligned} \tag{38}$$

For the model equation (36), it can be seen that $\delta v$ is a flat output of the linearized system.
The linearization around a reference trajectory leads to a LTV system and its controllability matrix is given by:

$$\mathscr{K}(t) = \begin{pmatrix} \frac{r}{Jv_d} & \frac{-r\dot{v}_d}{Jv_d^2} + \frac{ar}{Jv_d^2}\left(\frac{1}{m} + \frac{r^2}{J} - \frac{2\lambda_d}{m}\right) \\ 0 & -\frac{ar}{mJv_d} \end{pmatrix}$$

where $\mathscr{K}(t)$ has rank 2 because $\frac{ar^2}{mJ^2v_d^2} \neq 0 \ \forall\, t \geq 0$. Then, the system (37) is completely controllable and following [2], the time-varying linearized system (37) is flat. The observability matrix of the pair $(A(t), C(t))$ is given by:

$$O_{(A(t),C(t))} = \begin{pmatrix} 0 & 1 \\ -\frac{a}{m} & 0 \end{pmatrix}$$

which has rank 2 $\forall\ t \geq 0$. The system is then observable and its controllable canonical form is obtained by applying the algorithm presented in section II:

$$\begin{aligned} \delta\dot{Z}(t) &= \bar{A}(t)\,\delta Z(t) + \bar{B}\delta T(t) \\ \delta\lambda(t) &= \bar{C}\delta Z(t) \end{aligned} \tag{39}$$

with $\bar{A}(t) = \begin{pmatrix} 0 & 1 \\ -\psi_0(t) & -\psi_1(t) \end{pmatrix}, \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\delta Z(t) = P(t)x(t)$ and $\delta Z(t) = \begin{pmatrix} \delta z(t) & \delta\dot{z}(t) \end{pmatrix}^T$. The previous control law (12) can be written as:

$$T(t) = T_d(t) + (k_1 - \Psi_1(t))\,\delta\dot{z}(t) + (k_0 - \Psi_0(t))\,\delta z(t)$$

which leads to:

$$\delta T(t) = \Lambda(t)\delta Z(t) \tag{40}$$

with $\Lambda(t) = \begin{bmatrix} (\Psi_0(t) - k_0) & (\Psi_1(t) - k_1) \end{bmatrix}$.
From equation (16), we deduce:

$$\delta Y(t) = O(t)\delta Z(t) + M(t)\delta T(t) \tag{41}$$

with: $\delta Y(t) = \begin{pmatrix} \delta v(t) & \delta\dot{v}(t) \end{pmatrix}^T$

$$O(t) = \begin{pmatrix} \bar{C}(t) \\ \dot{\bar{C}}(t) + \bar{C}(t)\bar{A}(t) \end{pmatrix}, M(t) = \begin{pmatrix} 0 \\ \bar{C}(t)\bar{B} \end{pmatrix}$$

The equation (39) can be written as:

$$\delta Z(t) = p^{-1}\left[\bar{A}(t)\delta Z(t) + \bar{B}\delta T(t)\right] \tag{42}$$

By replacing the expression of $\delta Z(t)$, deduced from equation (41), in equation (41) in the right side of the equation (42), we get:

$$\begin{aligned} &\delta Z(t) = p^{-1}\left[\bar{A}(t)O^{-1}(t)\delta Y(t)\right] - \\ &p^{-1}\left[\bar{A}(t)O^{-1}(t)M(t)\delta T(t)\right] + \bar{B}p^{-1}\delta T(t) \end{aligned} \tag{43}$$

with:

$$\bar{A}(t)O^{-1}(t) = \begin{pmatrix} \alpha_1(t) & \alpha_2(t) \\ \alpha_3(t) & \alpha_4(t) \end{pmatrix}, \bar{A}(t)O^{-1}(t)M(t) = \begin{pmatrix} \beta_1(t) \\ \beta_2(t) \end{pmatrix}$$

By using integration by parts, it leads to the following expression of the state vector:

$$\begin{aligned} \delta Z(t) = &\begin{pmatrix} \alpha_2(t) \\ \alpha_4(t) \end{pmatrix}\delta v(t) + p^{-1}\left[\begin{pmatrix} \alpha_1(t) - \dot{\alpha}_2(t) \\ \alpha_3(t) - \dot{\alpha}_4(t) \end{pmatrix}\delta v(t)\right] \\ &+ p^{-1}\left[\begin{pmatrix} -\beta_1 \\ 1 - \beta_2 \end{pmatrix}\delta T(t)\right] \end{aligned} \tag{44}$$

By rewriting the expression (40), the following form is obtained:

$$\begin{aligned} \delta T(t) = \Lambda(t) \times &\left[\begin{pmatrix} \alpha_2(t) \\ \alpha_4(t) \end{pmatrix}\delta v(t) + \right. \\ &\left. p^{-1}\left[\begin{pmatrix} \alpha_1(t) - \dot{\alpha}_2(t) \\ \alpha_3(t) - \dot{\alpha}_4(t) \end{pmatrix}\delta v(t)\right] + p^{-1}\left[\begin{pmatrix} -\beta_1 \\ 1 - \beta_2 \end{pmatrix}\delta T(t)\right]\right] \end{aligned} \tag{45}$$

and then:

$$\delta T(t) = S\left(p^{-1}, \delta v(t)\right) + R\left(p^{-1}, \delta T(t)\right) \tag{46}$$

with:

$$\begin{aligned} S\left(p^{-1}, \delta v(t)\right) = \Lambda(t) \times &\left[\begin{pmatrix} \alpha_2(t) \\ \alpha_4(t) \end{pmatrix}\delta v(t) + \right. \\ &\left. p^{-1}\left[\begin{pmatrix} \alpha_1(t) - \dot{\alpha}_2(t) \\ \alpha_3(t) - \dot{\alpha}_4(t) \end{pmatrix}\delta v(t)\right]\right] \end{aligned} \tag{47}$$

$$R\left(p^{-1}, \delta T(t)\right) = \Lambda(t) \times p^{-1}\left[\begin{pmatrix} -\beta_1 \\ 1 - \beta_2 \end{pmatrix}\delta T(t)\right] \tag{48}$$

By applying the previous control strategy to the new state space representation and by considering the tracking model set to be a second order model with a time response of 0.005 s, the simulation results are obtained in figure 2, where the trajectories of the nonlinear

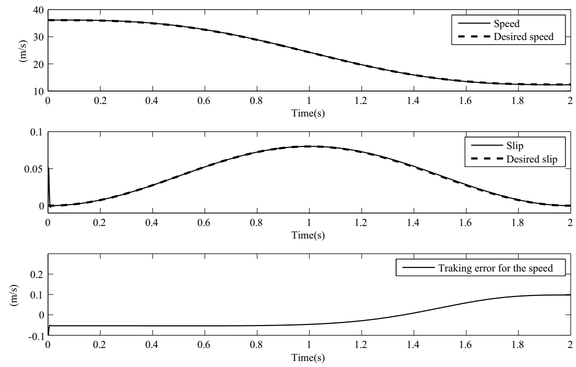


Fig. 2. The output $v(t)$ and the slip $\lambda(t)$ trajectories of the nonlinear system and the traking error for the speed by the use of an exact observer

system follow the desired trajectories with a good performance. The observable forme of the state equation (38) is given by:

$$\dot{x}_o(t) = \begin{pmatrix} 0 & -\tau_0(t) \\ 1 & -\tau_1(t) \end{pmatrix} x_o(t) + \begin{pmatrix} -\frac{ra}{Jmv_d} \\ 0 \end{pmatrix} \delta T(t)$$
$$\delta v(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x_o(t) \tag{49}$$

where:

$$\tau_0(t) = \frac{a}{mv_d^2}\left(\dot{\lambda}_d v_d + \left(1 + \frac{r^2}{Jm} - 2\lambda_d\right)\dot{v}_d\right)$$
$$\tau_1(t) = \frac{a}{mv_d}\left(1 - 2\lambda_d + \frac{mr^2}{J}\right) \tag{50}$$

Following section III, the estimated state vector of the observable form is given by:

$$\hat{x}_o(t) = \begin{pmatrix} \zeta(t) + \lambda_0 \delta v(t) \\ \delta v(t) \end{pmatrix} \tag{51}$$

$$\dot{\zeta}(t) = \lambda_0 \zeta(t) + \left(-\frac{ra}{Jmv_d}\right)\delta T(t) + (\tau_1(t) - \lambda_0)\lambda_0 \tag{52}$$

By replacing $\delta\hat{x}_o(t)$ into the control law (40) leads to:

$$\delta T(t) = \Lambda(t) P(t) x_o(t) \tag{53}$$

where $P(t)$ is the change of variable from the observable form to the controllable form. With a constant dynamics observer $\lambda_0 = 10$ and by considering the tracking model set to be a second order model with a time response of 0.005 s, the results are obtained the in figure 3. These results point out the effectiveness of the use of the flatness-based approach for the LTV systems in a path tracking context.

## VI. Conclusion

In this paper, we have underlined the advantage of the use of a reduced order observer in order to design a flatness-based control for tracking a desired trajectory in the case of LTV systems. This advantage consists in the calculation of the error estimator gain which is found constant. A method with a direct calculation of the state vector which contains the flat output and its derivatives is proposed and leads to a control law which can be seen as an RST controller but without resolution of the Bezout equation. This regulator-observer permits to the output system to track a desired trajectory without using observer dynamics. The control law applied on an Anti-lock Brake System (ABS) gives a high level performances in terms of the tracking of the wheel slip.

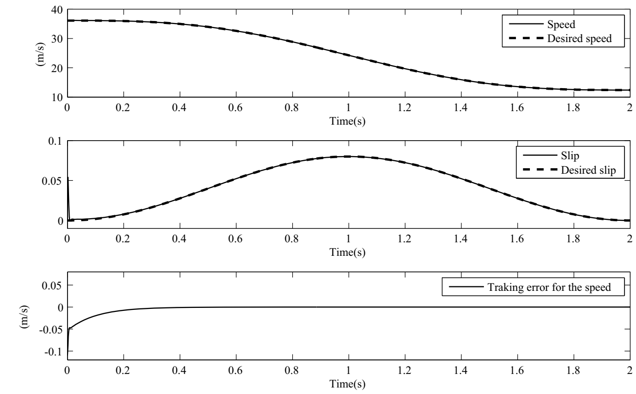


Fig. 3. The output $v(t)$ and the slip $\lambda(t)$ trajectories of the nonlinear system and the traking error for the speed by the use of a reduced order observer

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