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Application of Improved $\mathcal{L}_2$-Gain Synthesis on LPV Missile Autopilot Design

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Abstract

This paper presents an improved finite-dimensional linear matrix inequality (LMI) formulation for the $\mathcal{L}_2$-gain synthesis of a piecewise-affine linear parameter-varying (PALPV) system. The new formulation is then used to design a missile autopilot to investigate issues such as the effectiveness, reliability, and conservatism. Our approach is based on a nonsmooth dissipative systems framework using a continuous, quasi-piecewise-affine parameter-dependent Lyapunov function (QPAL) which has been shown to yield a less conservative, guaranteed result than previously published techniques based on quasi-affine parameter-dependent Lyapunov functions (QALs) or Luré-Postnikov Lyapunov functions. The results in this paper improve the computational efficiency by using a different "convexifying" technique. We also generalize our previous results by eliminating the restriction on the number of parameters.

1 Introduction

A missile autopilot design problem has attracted the attention of many researchers because it involves a large variation in the system dynamics and stringent performance requirements [13,15]. While the gain-scheduling approach has been successfully applied to many interesting problems [17], there are several potential difficulties when this approach is applied to the advanced missile autopilot design. For example, some parameters, such as the angle-of-attack, are arbitrarily fast varying, which violates the heuristic guideline that the scheduling parameters should be slowly varying [17]. A complicated scheduling scheme is also required to account for nonlinearities in the system that have been ignored in the design process [13]. These potential difficulties have recently been overcome using a linear parameter-varying (LPV) control approach [2,9,16,18].

An LPV system is characterized as a linear system that depends on time-varying parameters that are assumed to be exogenous signals. It is further assumed that the trajectory of the parameter is not known in advance but is constrained *a priori* to lie in some known, bounded set, and its value can be measured in real time. Associated with the LPV system have been various synthesis tools such as the scaled small-gain framework [3,9,14,16] and the dissipative systems framework using smooth parameter-dependent Lyapunov functions (PDLFs) [1,4,10,18]. The authors have also proposed a dissipative systems framework using a nonsmooth *quasi-piecewise affine parameter-dependent Lyapunov function (QPAL)* for a PALPV system. The results in [12] showed that while the QPAL approach can be computationally intensive, it leads to less conservative, guaranteed synthesis results than previously published techniques based on QALs or Luré-Postnikov Lyapunov functions.

This paper extends our previous results [12] in an attempt to reduce the computational complexity by using the multi-convexity approach [7] in the convexifying step of the algorithm development rather than the $\mathcal{S}$-procedure [6]. Furthermore, this paper generalizes the previous results by eliminating the restriction on the number of parameters. This new approach is then applied to the missile autopilot design problem. For a comparison, we also consider other controller design techniques, such as a "naive" gain-scheduling [13], complex-μ controller [15], and the LPV control design based on the gridding technique [18]. For this application, we show that the PALPV system can yield a more accurate model than the typical LFT because it provides a better approximation of the nonlinear missile dynamics. The QPAL approach reduces conservatism of the synthesis result because of the richness of the PDLFs used in the synthesis. Furthermore, we address the potential difficulty in selecting a PDLF for the gridding technique in the LPV control [18] and show that our new technique can reduce this specific difficulty.

This paper is organized as follows: Section 2 sets up the problem definitions for the synthesis. Section 3 summarizes the $\mathcal{L}_2$-gain synthesis formulation from [11]. Section 4 shows the missile autopilot design process and the simulation result. Notations are fairly standard. For notational convenience, we make some definitions for index: $l \triangleq i_1 \cdots i_s$ and $\mathbf{1}_s \triangleq 1 \cdots 1$, *i.e.*, s number of 1's, with $\mathbf{1}_0 \triangleq \emptyset$. For example, $X_{i\mathbf{1}_2} \triangleq X_{i11}$ since $\mathbf{1}_2 = 11$ and $X_{i\mathbf{1}_0} \triangleq X_i$. $[A(\theta)]_l \triangleq A_l(\theta_l)$. Let $\mathcal{W} \subset \mathbf{R}^m$, $\mathcal{U} \subset \mathbf{R}^u$ and $\mathcal{Y} \subset \mathbf{R}^y$. The parameter set are defined $\mathcal{F}^s \triangleq \{\theta \in C^1(\mathcal{I}, \mathbf{R}^s) : \theta(t) \in \mathcal{P},\ \dot\theta(t) \in \Omega,\ \forall t \in \mathcal{I}\}$ where $\mathcal{P} = [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_s, \overline{\theta}_s]$ and $\Omega = [-\nu_1, \nu_1] \times \cdots \times [-\nu_s, \nu_s]$.

2 Problem Definitions

We define a PALPV system for the general case of $s \geq 3$ (see [11]). The parameter space, $\mathcal{P} \triangleq [\underline{\theta}_1, \overline{\theta}_1] \times \cdots \times [\underline{\theta}_s, \overline{\theta}_s]$, is partitioned into $m_1 \times \cdots \times m_s$ closed hyper-rectangles with width $\Delta\theta_1 \times \cdots \times \Delta\theta_s$, where $\Delta\theta_k = (\overline{\theta}_k - \underline{\theta}_k)/m_k$ (see Fig 1). In this case, each hyper-rectangle is represented by $\mathcal{P}_l \triangleq \mathcal{P}_{i_1 \cdots i_s}$ where the index l defines the subregion of interest. Given each subspace $\mathcal{P}_l$, we introduce a new local coordinate $\hat\theta_l \triangleq \left[\hat\theta_{l1} \cdots \hat\theta_{ls}\right]^T$ ($\in \hat{\mathcal{F}}^s$) measured from the center of $\mathcal{P}_l$. Here, $\hat{\mathcal{F}}^s$ is equivalent to $\mathcal{F}^s$ when $\mathcal{P}$ is replaced by $\hat{\mathcal{P}} \triangleq [-\frac{\Delta\theta_1}{2}, \frac{\Delta\theta_1}{2}] \times \cdots \times [-\frac{\Delta\theta_s}{2}, \frac{\Delta\theta_s}{2}]$. For each $\mathcal{P}_l$, an ALPV system is described by a set of nominal dynamics at

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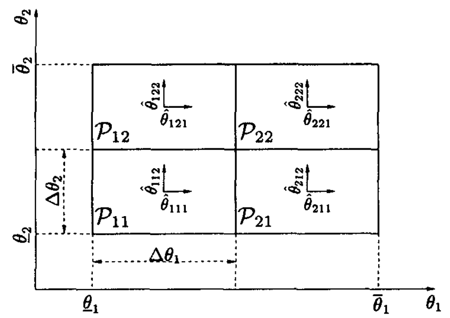

Fig. 1: Parameter subspaces for $m_1 = m_2 = 2$

the center of the parameter subspace and affine parameter-dependent terms. A PALPV system is a system that switches between ALPV systems: for $(x, \theta, w, u) \in \mathcal{D} \times \mathcal{F}^s \times \mathcal{W} \times \mathcal{U}$,

$$\left[\begin{array}{c} \dot{x} \\ \hline z_1 \\ z_2 \\ \hline y \end{array}\right] = \sum_{i_1=1}^{m_1} \cdot \sum_{i_s=1}^{m_s} \alpha_l(\theta) \left[\mathcal{G}(\hat{\theta})\right]_l \left[\begin{array}{c} x \\ \hline w_1 \\ w_2 \\ \hline u \end{array}\right], \qquad (1)$$

$$\left[\mathcal{G}(\hat{\theta})\right]_l = \left[\begin{array}{c|cc|c} A(\hat{\theta}) & B_{w1}(\hat{\theta}) & B_{w2} & B_u(\hat{\theta}) \\ \hline C_{z1}(\hat{\theta}) & 0 & 0 & 0 \\ C_{z2} & 0 & 0 & I_{n_u} \\ \hline C_y(\hat{\theta}) & 0 & I_{n_y} & 0 \end{array}\right]_l,$$

where all $\hat{\theta}_l$-dependent system matrices are affine dependent in $\hat{\theta}_l$, *e.g.*, $A_l(\hat{\theta}_l) = A_{l0} + \sum_{k=1}^{s} \hat{\theta}_{lk} A_{lk}$. $\alpha_l(\theta)$ is the switching function such that each $\mathcal{G}_l$ is well defined over the corresponding $\mathcal{P}_l$ (see [11]). With $w^T = [w_1^T \ w_2^T]$ and $z^T = [z_1^T \ z_2^T]$, define $[B_w(\hat{\theta})]_l = [B_{w1}(\hat{\theta}) \ B_{w2}]_l$ and $[C_z(\hat{\theta})]_l^T = [C_{z1}(\hat{\theta})^T \ C_{z2}^T]_l$. In this paper, we consider a special form of an LPV controller that for $(x_c, \theta, y) \in \mathcal{D} \times \mathcal{F}^s \times \mathcal{Y}$,

$$\left[\begin{array}{c} \dot{x}_c \\ u \end{array}\right] = \sum_{i_1=1}^{m_1} \cdot \sum_{i_s=1}^{m_s} \alpha_l(\theta) \left[\begin{array}{cc} A_c(\hat{\theta}, \dot{\hat{\theta}}) & B_c(\hat{\theta}) \\ C_c(\hat{\theta}) & 0 \end{array}\right]_l \left[\begin{array}{c} x_c \\ y \end{array}\right].$$

A key tenet of the LPV control design is that $\hat{\theta}(\cdot)$ and $\dot{\hat{\theta}}(\cdot)$ are assumed to be measurable in real time. $[A_c(\hat{\theta}, \dot{\hat{\theta}})]_l$, $[B_c(\hat{\theta})]_l$ and $[C_c(\hat{\theta})]_l$ are continuous matrix functions on $\hat{\theta}_l$ and $\dot{\hat{\theta}}_l$. Note that these unknown controller dynamics are allowed to be discontinuous on the boundary of $\mathcal{P}_l$.

3 $\mathcal{L}_2$-Gain Synthesis

The problem is to design an LPV controller that minimizes an upper bound of the $\mathcal{L}_2$-gain (γ_2) of the closed-loop system. The synthesis process follows the standard three steps (see [11,12]): [Step I] Formulate the analysis problem of the closed-loop system; [Step II] Eliminate the unknown controller dynamics and "convexify" to transform into finite-dimensional LMIs using the multiconvexity approach [7]; [Step III] Construct the central controller dynamics analytically. For the dissipative systems framework, we consider the nonsmooth QPAL. A key contribution of this paper is to extend the QPAL to the general case of $s \geq 3$ [11]. The QPAL, $V(x_{cl}, \theta) = x_{cl}^T P_{cl}(\theta) x_{cl}$, is defined such that

$$P_{cl}(\theta) = \sum_{i_1=1}^{m_1} \cdot \sum_{i_s=1}^{m_s} \alpha_l(\theta) \left[\hat{P}_{cl}(\hat{\theta})\right]_l,$$

$$\left[\hat{P}_{cl}(\hat{\theta})\right]_l = \left[\begin{array}{cc} \hat{X}(\hat{\theta}) & \hat{R}(\hat{\theta}) \\ \hat{R}(\hat{\theta})^T & \hat{R}(\hat{\theta})^T \left(\hat{X}(\hat{\theta}) - \hat{Y}^{-1}(\hat{\theta})\right)^{-1} \hat{R}(\hat{\theta}) \end{array}\right]_l$$

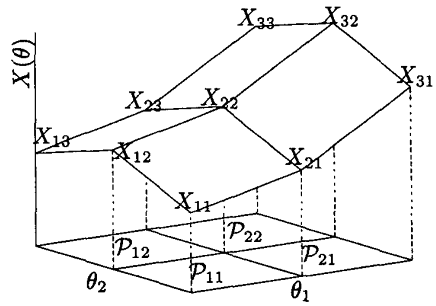

Fig. 2: $X(\theta)$ for $m_1 = m_2 = 2$

- Continuous, piecewise-affine $X(\theta)$ (see Fig 2):

$$X(\theta) = \sum_{i_1=1}^{m_1} \cdot \sum_{i_s=1}^{m_s} \alpha_l(\theta) [\hat{X}(\hat{\theta})]_l$$

and over $\mathcal{P}_l$, $\left[\hat{X}(\hat{\theta})\right]_l = X_{l0} + \sum_{k=1}^{s} \hat{\theta}_{lk} X_{lk}$ where

$$\begin{aligned} X_{l0} &= \frac{1}{2}\left(X_{(i_1+1)\mathbf{1}_{(s-1)}} + X_{i_1\mathbf{1}_{(s-1)}} + \cdots \right. \\ &\quad \left. + X_{\mathbf{1}_{(s-1)}(i_s+1)} + X_{\mathbf{1}_{(s-1)}i_s} - 2(s-1)X_{\mathbf{1}_s}\right), \\ X_{lk} &= \frac{X_{\mathbf{1}_{(k-1)}(i_k+1)\mathbf{1}_{(s-k)}} - X_{\mathbf{1}_{(k-1)}i_k\mathbf{1}_{(s-k)}}}{\Delta\theta_k}. \end{aligned}$$

- Continuous, piecewise-affine $Y(\theta)$. (note $Y(\theta)$ is also similarly defined with $Y_{(i_1+1)\mathbf{1}_{(s-1)}}, \cdots, Y_{\mathbf{1}_{(s-1)}i_s}$)
- Continuous, piecewise-smooth $R(\theta)$:

$$R(\theta) = \sum_{i_1=1}^{m_1} \cdot \sum_{i_s=1}^{m_s} \alpha_l(\theta) [\hat{R}(\hat{\theta})]_l$$

and over $\mathcal{P}_l$, $\left[\hat{R}(\hat{\theta})\hat{Z}(\hat{\theta})^T\right]_l = \left[I - \hat{X}(\hat{\theta})\hat{Y}(\hat{\theta})\right]_l$.

Note that the $[\hat{X}(\hat{\theta})]_l$ and $[\hat{Y}(\hat{\theta})]_l$ can be constructed from only $X_{a\mathbf{1}_{(s-1)}}$, $Y_{a\mathbf{1}_{(s-1)}}$, $\cdots$, $X_{\mathbf{1}_{(s-1)}b}$, $Y_{\mathbf{1}_{(s-1)}b}$ ($a = 1, \cdots, m_1 + 1$, $\cdots$, $b = 1, \cdots, m_s + 1$). We present only the results of the $\mathcal{L}_2$-gain synthesis (see [11]). For simplicity, we define $[\hat{A}(\hat{\theta})]_l = [A(\hat{\theta}) - B_u(\hat{\theta})C_{z2}]_l$ and $[\tilde{A}(\hat{\theta})]_l = [A(\hat{\theta}) - B_{w2}C_y(\hat{\theta})]_l$, which are affine in $\hat{\theta}_l$. We also define $[L_X(A, \eta, \tau)]_l = [\hat{X}(\eta)A(\eta) + A(\eta)^T\hat{X}(\eta) + \dot{\hat{X}}(\tau)]_l$ and $[L_Y(A, \eta, \tau)]_l = [A(\eta)\hat{Y}(\eta) + \hat{Y}(\eta)A(\eta)^T - \dot{\hat{Y}}(\tau)]_l$. Here, let Θ and Ψ be the set of 2^s vertices of $\hat{\mathcal{P}}$ and the set of 2^s vertices of Ω, respectively.

Proposition 3.1 [11] *Suppose there exist M_{lk}, N_{lk} (> 0), $(X_{a\mathbf{1}_{(s-1)}}, Y_{a\mathbf{1}_{(s-1)}})$ ($a = 1, \cdots, m_1 + 1$), $\cdots$, and $(X_{\mathbf{1}_{(s-1)}b}, Y_{\mathbf{1}_{(s-1)}b})$ ($b = 1, \cdots, m_s + 1$) such that for all l,*

$$\left[\begin{array}{cc} \hat{X}(\eta) & I \\ I & \hat{Y}(\eta) \end{array}\right]_l > 0,$$

$$\left[\begin{array}{cc} \left(\begin{array}{c} L_Y(\hat{A}, \eta, \tau) \\ -\gamma B_u(\eta)B_u(\eta)^T \\ +1/\gamma B_w(\eta)B_w(\eta)^T \end{array}\right) & \hat{Y}(\eta)C_{z1}(\eta)^T \\ C_{z1}(\eta)\hat{Y}(\eta) & -\gamma I \end{array}\right]_l + \sum_{k=1}^{s} \eta_k^2 M_{lk} < 0$$

$$\begin{bmatrix} \begin{pmatrix} L_X(\tilde{A},\eta,\tau) \\ -\gamma C_y(\eta)^T C_y(\eta) \\ +1/\gamma C_z(\eta)^T C_z(\eta) \end{pmatrix} & \hat{X}(\eta) B_{w1}(\eta) \\ B_{w1}(\eta)^T \hat{X}(\eta) & -\gamma I \end{bmatrix}_l + \sum_{k=1}^{s} \eta_k^2 N_{lk} < 0$$

for all $(\eta,\tau) \in \Theta \times \Psi$ *and*

$$\begin{bmatrix} Y_{lk}\hat{A}_{lk}^T + \hat{A}_{lk} Y_{lk} & Y_{lk} C_{z1lk}^T \\ C_{z1lk} Y_{lk} & 0 \end{bmatrix} + M_{lk} \geq 0,$$

$$\begin{bmatrix} \tilde{A}_{lk}^T X_{lk} + X_{lk}\tilde{A}_{lk} & X_{lk} B_{w1lk} \\ B_{w1lk}^T X_{lk} & 0 \end{bmatrix} + N_{lk} \geq 0$$

for all k. *Then the closed-loop system is uniformly asymptotically stable about* $x = 0$ *for all* $\theta \in \mathcal{F}^s$ *and the* $\mathcal{L}_2$*-gain* (γ_2) *is less than* γ, *i.e.,* $\gamma_2 < \gamma$.

Proposition 3.2 [11] *Suppose* γ, $[\hat{X}(\hat{\theta})]_l$ *and* $[\hat{Y}(\hat{\theta})]_l$ *are solutions of Proposition 3.1. Then a central controller, which makes the* $\mathcal{L}_2$*-gain of the QPALPV system less than* γ, *is given as follows (for simplicity, omit dependency on the* $\hat{\theta}$ *and* $\dot{\hat{\theta}}$*): for* $\theta(t) \in \mathcal{P}_l$,

$$\begin{aligned} [A_c]_l &= \left[\hat{R}^{-1}\left(\hat{A}_c + \hat{X}\dot{\hat{Y}} + \hat{R}\dot{\hat{Z}}^T - \hat{X}A\hat{Y}\right.\right. \\ &\quad \left.\left. -\hat{X}B_u\hat{C}_c - \hat{\mathcal{B}}_c C_y \hat{Y}\right)\hat{Z}^{-T}\right]_l, \\ [B_c]_l &= \left[\hat{R}^{-1}\hat{\mathcal{B}}_c\right]_l, \quad [C_c]_l = \left[\hat{\mathcal{C}}_c \hat{Z}^{-T}\right]_l, \end{aligned}$$

where

$$\begin{aligned} \left[\hat{\mathcal{A}}_c\right]_l &= \left[-A^T - \gamma^{-1}\hat{X}B_{w1}B_{w1}^T + C_y^T B_{w2}^T\right. \\ &\quad \left. -\gamma^{-1}C_{z1}^T C_{z1}\hat{Y} + C_{z2}^T B_u^T\right]_l, \\ \left[\hat{\mathcal{B}}_c\right]_l &= -\left[\gamma C_y^T + \hat{X}B_{w2}\right]_l, \quad \left[\hat{\mathcal{C}}_c\right]_l = -\left[\gamma B_u^T + C_{z2}\hat{Y}\right]_l. \end{aligned}$$

Remark 3.1 By minimizing γ, we solve a suboptimal solution for γ_2. The minimization is a standard convex optimization subject to LMI constraints. Generally speaking, these LMIs yields a larger number of LMIs than the typical dissipative systems approach using the quadratic Lyapunov function. Furthermore, to eliminate $\dot{\theta}_l(t)$ from the controller formulation, we use a method that constrains $X(\theta)$ or $Y(\theta)$ to be constant [1].

4 Autopilot Design

By applying the $\mathcal{L}_2$-gain synthesis technique to a missile autopilot design problem, this section investigates the effectiveness, reliability, and conservatism of our new technique.

4.1 Missile Model and Performance Objective

We consider the benchmark pitch-axis missile dynamics [13, 18]. The missile model can be easily formulated as

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} f_1 & 1 \\ f_2 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} f_3 \\ f_4 \end{bmatrix}\delta_c, \quad (2)$$

$$\begin{bmatrix} \eta \\ q \end{bmatrix} = \begin{bmatrix} f_5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} f_6 \\ 0 \end{bmatrix}\delta_c, \quad (3)$$

where f_1 to f_6 are nonlinear functions of M and α. We also consider 2^{nd}-order actuator dynamics describing the tail deflection [13, 18]. η_c and η are, respectively, command and actual normal acceleration in g's, while other variables are standard. M and α are assumed to be scheduling parameters in the process of modeling the missile dynamics as an LPV system. The prescribed bounds on these parameters are $2 \leq M(t) \leq 4$ with $-1.5 \leq \dot{M}(t) \leq 1.5$, and $-20 \leq \alpha(t) \leq 20$ with $-\infty \leq \dot{\alpha}(t) \leq \infty$, *i.e.*, $\mathcal{P} = [2 \;\; 4] \times [-20 \;\; 20]$ and $\Omega = [-1.5 \;\; 1.5] \times [-\infty \;\; \infty]$. The assumption on the rate of M is made to reflect that the real Mach number is relatively slowly varying [9, 5]. The angle-of-attack $\alpha(t)$ is actually one of the states in the missile model so the actual variation of $\alpha(t)$ could be larger than the prescribed value. An iterative design process might be necessary to design an LPV controller for which the closed-loop system satisfies the prescribed bounds.

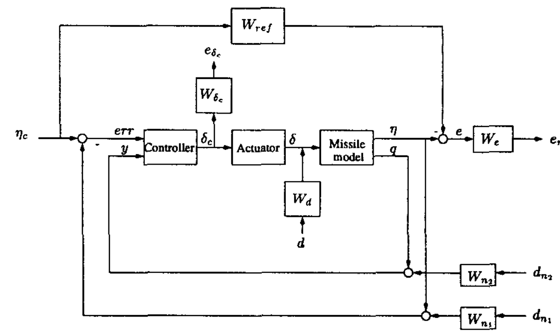

Fig. 3: Weighted open-loop interconnection

The goals are **(1) Robust Stability:** maintain stability over the prescribed bounds of parameters, **(2) Robust Performance:** track step commands in $\eta_c(t)$ with time constant no greater than 0.35 sec, maximum overshoot no greater than 10%, and steady-state error no greater than 1%, and **(3) Bandwidth:** maximum tail deflection rate for 1g step command in $\eta_c(t)$ does not exceed 25 deg/sec. We use rational weighting functions to characterize the overall closed-loop performance objective (similar to those in [18]).

$$W_{ref}(s) = \frac{144(-0.05s+1)}{s^2 + 2 \times 0.8 \times 12s + 144}, \; W_e(s) = \frac{17.321}{s + 0.0577},$$

$$W_{\delta_c}(s) = \frac{s + 0.25}{25(0.005s+1)}, W_{n1}(s) = W_{n2}(s) = 0.001,$$

$W_d(s) = 0.01$, $\text{Act}(s) = 1$. Note that a strictly proper W_e is used for the augmented open-loop system dynamics to satisfy the assumption that B_{w2} and C_{z2} are parameter-independent. W_d and W_{δ_c} are included to yield a stable LPV controller. The open-loop interconnection for the synthesis is shown in Fig. 3. In this figure, the missile model varies from one synthesis approach to another because of the different assumptions about the open-loop dynamics. For example, the LFT-μ approach uses an LFT model of the missile dynamics, while our proposed approach uses a PALPV model. Note that the direct feedforward term from d to η is small enough that it can be neglected and then the augmented open-loop system dynamics satisfy the assumption that C_{z2} is constant. A low-pass filter $W_d(s)$ would eliminate the direct feedforward term but increases the size of the augmented system dynamics.

4.2 PALPV Modeling

This section investigates a simple process to derive PALPV models for the synthesis from the original missile dynamics with a different number of partitions (N). For this analysis, the LFT model corresponds to a PALPV model with $N = 1$.

Table 1: Relative RMS of $E_k(M,\alpha)$ in (%)

	RMS(E_1)	RMS(E_2)	RMS(E_5)
$N=1$	4.59	30.8	6.87
$N=3$	0.55	3.83	0.86
$N=5$	0.22	1.50	0.34

The gridding technique for the LPV control in [18] uses a set of the original nonlinear missile dynamics at prescribed (dense) grid points. The PALPV modeling process is outlined below. Functions f_1 – f_6 are symmetric in α, so we consider the half parameter space $\overline{\mathcal{P}} = [2\ \ 4] \times [0\ \ 20]$. First, partition $\overline{\mathcal{P}}$ into $N \times N$ parameter subspaces $\mathcal{P}_l$'s. Over each $\mathcal{P}_l$, an approximate of each function f_k is assumed of the form,

$$\left[\hat{f}_k(\hat{M},\hat{\alpha})\right]_l = \hat{f}_{kl0} - \hat{M}_l\hat{f}_{kl1} - \hat{\alpha}_l\hat{f}_{kl2},$$

where $\hat{M}_l$ and $\hat{\alpha}_l$ ($|\hat{M}_l| \le 1/N$ and $|\hat{\alpha}_l| \le 10/N$) are values of M and α measured from the local coordinates with the origin at the center of $\mathcal{P}_l$. The approximate functions are then fit to the actual functions over each $\mathcal{P}_l$ by finding $\{\hat{f}_{kl0}, \hat{f}_{kl1}, \hat{f}_{kl2}\}$ such that

$$\min \|f_k(M,\alpha) - \left[\hat{f}_k(\hat{M},\hat{\alpha})\right]_l\|_2 \quad \forall (M,\alpha) \in \mathcal{P}_l.$$

This optimization problem is infinite-dimensional, but in practice it is sufficient to solve it over a dense finite grid (*e.g.*, 50 × 50). A post-analysis was performed to calculate the approximation error ($E_k(M,\alpha)$) from the fit. To compare the approximation errors, we normalize them as relative RMS errors defined as the ratio of RMS gain of $E_k(M,\alpha)$ to that of $f_k(M,\alpha)$ (see Table 1). As expected, the LFT model ($N=1$) yields large modeling errors, especially on $f_2(M,\alpha)$. To design a reliable controller, the LFT model for the synthesis should take into account these large approximation errors as extra uncertainties. In contrast, large N leads to a PALPV model that is a good approximation of the original missile dynamics. In this case, all $E_k(M,\alpha)$'s are small enough that they can be ignored. We can derive a PALPV system with uncertainty to represent the missile model:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \sum_{i_1=1}^{N}\sum_{i_2=1}^{N} \alpha_l(M,\alpha) \begin{bmatrix} A+\Delta A & B \\ C & D \end{bmatrix}_l \begin{bmatrix} x \\ u \end{bmatrix} \quad (4)$$

and $[A]_l$, $[B]_l$, $[C]_l$ and $[D]_l$ are affine in $\hat{M}_l$ and $\hat{\alpha}_l$. Furthermore, $[\Delta A]_l = \delta_l \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $-30.6 \le \delta_l(M,\alpha) \le 123.2$ for $N=1$ and $\delta_l(M,\alpha) \approx 0$ for $N \ge 3$ because the relative RMS error is small. We select the model with $N=3$ as the PALPV model for the synthesis. Note that the LFT model is obtained by converting the above model with $N=1$ to an LFT form with the minimum size of uncertainties using singular value decomposition [9].

4.3 Autopilot Design and Simulation

We design pitch-axis missile autopilots using several techniques: naive gain-scheduling (NGS) [13], complex-μ (C-μ), LPV control techniques, such as our approach based on Proposition 3.1 (QPAL) and the typical gridding technique (GRID) [18]. The NGS technique linearly interpolates the gain, zeros, and poles of the nine nominal $\mathcal{H}_\infty$ controllers to yield a gain-scheduling controller. The C-μ technique designs a robust controller with the LFT model developed above. In this case, a constant scaling matrix is used to treat the structured time-varying uncertainties. The QPAL approach is based on the PALPV model with $N=3$ and a special class of QPAL such that X is constant and $Y(M)$ is piecewise-affine in M (see remark 3.1). The GRID approach uses the quadratic Lyapunov function with constant X and Y. All LMI-related computations are performed with the MATLAB *LMI Control Toolbox* [8]. Except for NGS, all other techniques can provide guaranteed $\mathcal{L}_2$-gains: $\mathcal{L}_2$-gain from QPAL is 1.44, GRID is 3.10, and C-μ is 37.7. Our QPAL approach yields the best guaranteed performance because of the accurate modeling and the use of a general class of PDLFs. This result is also comparable to other results [5]. As expected, the C-μ approach yields the worst guaranteed performance. The very poor performance of the C-μ controller is primarily due to the large uncertainty for $E_2(M,\alpha)$ associated with the LFT model ($\mathcal{L}_2$-gain from the LFT model without $E_2(M,\alpha)$ is improved to 6.11, but it is not guaranteed).

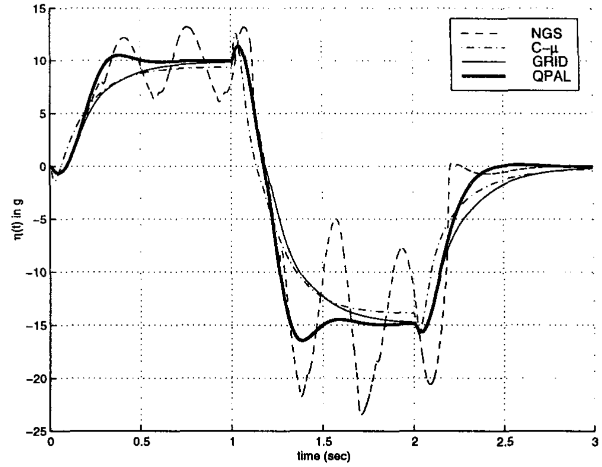

Fig. 4: Normal acceleration $\eta(t)$

We perform nonlinear numerical simulations to verify the performance of these controllers. The Mach number profile is assumed $M(t) = -0.05\sin\left(\frac{\pi}{6}t\right) + 2.1$. $\alpha(t)$ is estimated by a simple nonlinear static estimator [18] which is a polynomial approximation of an inverse of the output equation (Eq. 3). The simulation result is shown in Fig. 4. The controller from NGS yields the worst performance. This result is due to the fact that the $\mathcal{H}_\infty$ controllers have been optimized at local nominal models and clearly indicates that the naive gain-scheduling can be sensitive to coupling and other nonlinear effects that are not included in the control design models [13]. The simulations of the other three cases are consistent with the synthesis results. The conservatism of the C-μ controller ($\gamma_{\text{opt}} = 37.7$) is captured in terms of a relatively large steady-state error during 1 – 2 (sec). In contrast, our QPAL controller yields the best performance and satisfies the design objectives for various input commands.

4.4 Comparison of LPV control techniques

We further investigate the conservatism and reliability of the LPV control techniques based on the above GRID and QPAL. For a comparison, we also consider special cases of both techniques: the GRID approach using QAL such that $X(M)$ and $Y(M)$ are affine in M (GRID1); our QPAL approach using the same QAL as GRID1 (QPAL1). Note that for the QPAL

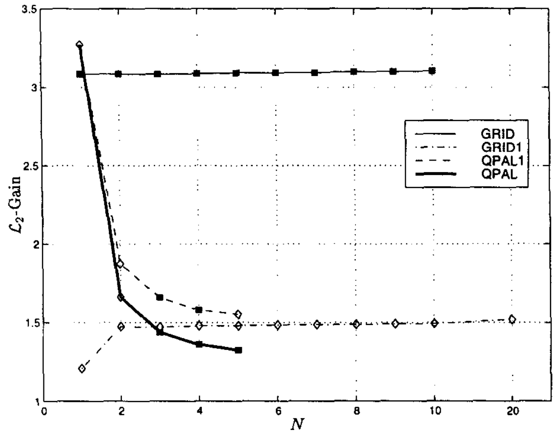

Fig. 5: $\mathcal{L}_2$-gains *vs.* N. Black □ indicates reliable results while white ◇ does unreliable results

and QPAL1 approach, we intentionally use the PALPV model (Eq. 4) with zero δ_l for $N = 1, 2$. We calculate $\mathcal{L}_2$-gains from the control techniques with several different partitions (N). Note that while the results are guaranteed values for the approximate model, they may not be guaranteed values for the original nonlinear missile dynamics. Therefore it is necessary to perform a post-analysis: to check eigenvalues of LMIs of Theorem 5.3.1 in [18] using the solutions (γ, $X(M)$ and $Y(M)$) on a dense post-analysis grid (50×50). The design and post-analysis results for four design techniques are plotted in Fig. 5. In this figure, the $\mathcal{L}_2$-gains verified by the post-analysis are plotted by black □; otherwise, they are plotted by white ◇. In Fig. 5, a smaller $\mathcal{L}_2$-gain indicates better performance of the controller for given N.

Fig. 5 shows that GRID is reliable, but yields overly conservative results. This result is consistent with [18]. This figure also shows that QPAL with $N \geq 3$ yields the best performance and also is reliable. As N increases, the performance is improved even further. This improvement is primarily due to the richness of PDLFs used for the synthesis. As expected, the results of QPAL with $N = 1, 2$ are unreliable because the PALPV model does not account for the modeling errors (note if $\delta_l(M, \theta)$ is included as a uncertainty, QPAL with $N = 1, 2$ can yield a reliable result, but the performance of the controller degrades). But QPAL with $N \geq 3$ yields good reliable performance. Fig. 5 shows that GRID1 produces unreliable results over quite dense grids (up to $N = 20$) which indicates that GRID1 is very sensitive to small modeling errors. A similar trend is also found in the QPAL1 approach (see the result of QPAL1 with $N = 5$). The sensitivity of the GRID1 technique is thus likely related to the type of PDLFs used in the synthesis (a popular heuristic approach is to select the PDLF for the grid technique so that it emulates the parameter-dependency of the nonlinear missile dynamics). However, our approach (QPAL) reduces the sensitivity of the technique by gridding the parameter space prior to the synthesis and using a very general PDLF.

5 Conclusions

This paper presents an improved LMI formulation for the $\mathcal{L}_2$-gain synthesis for an LPV system and its application to a missile autopilot design. The design process demonstrates that our technique yields less conservative and more reliable results in the missile autopilot design than the published approaches. These improvements are attributed to the use of a more accurate model and a more general class of PDLFs.

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