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Connectedness of complete block designs under an interference model

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Abstract We consider an experiment with fixed number of blocks, in which a response to a treatment can be affected by treatments from neighboring units. For such experiment the interference model with neighbor effects is studied. Under this model we study connectedness of binary complete block designs. Assuming the circular interference model with left-neighbor effects we give the condition for minimal number of blocks necessary to obtain connected design. For a specified class of binary, complete block designs, we show that all designs are connected. Further we present the sufficient and necessary conditions of connectedness of designs with arbitrary, fixed number of blocks.

Keywords Interference model · Information matrix · Connected design · Permutation matrix · Irreducible matrix · Circulant matrix

Mathematics Subject Classification (2000) 62K05 · 62K10

1 Introduction

If in the experiment the response to a treatment is affected by other treatments (for example in agricultural and horticultural experiments), an interference model is usually studied. It is worth observing that only experiments in which all treatment contrasts are

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estimable are interesting. Thus, in this paper we characterize connected designs with arbitrary, fixed number of blocks under an interference model with neighbor effects.

In the literature connectedness of designs under an interference model has not been discussed yet. Recently some results on optimality of binary designs under this model were published. These results concern mainly universal optimality of circular neighbor balanced designs and orthogonal arrays of type I under the fixed and mixed interference models, where the observations are correlated or not (see, e.g. Druilhet 1999; Filipiak and Markiewicz 2003, 2004, 2005, 2007). It is known, however, that for some combinations of design parameters the universally optimal designs cannot exist. In such a case efficiency of some designs (Filipiak and Różański 2004) or optimality with respect to the specified criteria (Filipiak et al. 2008) are considered.

This paper is organized as follows. In Sect. 2 we present some general definitions and notation. In Sect. 3 we give conditions for connectedness of complete designs with minimal number of blocks. We identify a specified class of designs which consists only of connected designs. At the end we present some necessary and sufficient conditions of connectedness of complete design with arbitrary, fixed number of blocks.

2 Definitions and notations

Let $\mathcal{D}_{t,b,k}$ be the set of designs with t treatments, b blocks and k experimental units per block. An interference model with left-neighbor effects associated with the design $d \in \mathcal{D}_{t,b,k}$ can be written as

$$\mathbf{y} = \mathbf{T}_d\tau + \mathbf{L}_d\lambda + (\mathbf{I}_b \otimes \mathbf{1}_k)\beta + \varepsilon, \tag{1}$$

where τ, λ and β are the vectors of treatment, left-neighbor and block effects, respectively. Here ε is a vector of random errors, $\varepsilon \sim \mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I}_{bk})$, where σ^2 is an unknown constant. The matrix $\mathbf{I}_b$ denotes the identity matrix of order b, $\mathbf{1}_k$ is the k-vector of ones and $\otimes$ denotes the Kronecker product.

Let $\mathbf{T}_{du}$ be the design matrix of treatment effects in the block u, $1 \le u \le b$. Further, define $\mathbf{T}_d = (\mathbf{T}'_{d1} : \cdots : \mathbf{T}'_{db})'$ as the design matrix of treatment effects. For each u we define $\mathbf{L}_{du} = \mathbf{H}_k\mathbf{T}_{du}$, where $\mathbf{H}_k$ is a $k \times k$ matrix of the form

$$\mathbf{H}_k = \begin{pmatrix} \mathbf{0}'_{k-1} & 1 \\ \mathbf{I}_{k-1} & \mathbf{0}_{k-1} \end{pmatrix}, \tag{2}$$

where $\mathbf{0}_n$ is the n-vector of zeros. Then, $\mathbf{L}_d = (\mathbf{I}_b \otimes \mathbf{H}_k)\mathbf{T}_d$ is the design matrix of left-neighbor effects. Model (1) with $\mathbf{H}_k$ and $\mathbf{L}_d$ defined above, is called *a circular interference model with left-neighbor effects*. This form of the matrix $\mathbf{H}_k$ follows from the assumption that each treatment has a left neighbor. This situation may occur if each block of a design has the form of a circle. If plots in blocks are arranged in linear form, we can obtain the effect of circularity by adding border plots at the beginning of each block, where the treatment at the border plot is the same as the treatment at the opposite end of the block (for more details see, e.g. Druilhet 1999). Border plots are not used for measuring the response variables.

The matrices $\mathbf{T}_d$ and $\mathbf{L}_d$ depend on the arrangement of treatments on plots, i.e. they change with the design. Thus, they are indexed by d.

Observe, that model (1) without left-neighbor effects, i.e.

$$\mathbf{y} = \mathbf{T}_d\tau + (\mathbf{I}_b \otimes \mathbf{1}_k)\,\beta + \varepsilon, \tag{3}$$

is the standard model of block experiments. Recall, that block designs from $\mathcal{D}_{t,b,k}$ are described by the incidence matrix $\mathbf{N}_d = \mathbf{T}'_d\,(\mathbf{I}_b \otimes \mathbf{1}_k) = (n_{d,ij})_{1\le i\le t,1\le j\le b}$, where $n_{d,ij}$ denotes the number of units in the jth block receiving ith treatment.

Under the interference model (1), the information matrix for the estimation of treatment effects, $\mathbf{C}_d$, can be expressed as

$$\mathbf{C}_d = \mathbf{T}'_d\mathbf{Q}_{I_b\otimes 1_k}\mathbf{T}_d - \mathbf{T}'_d\mathbf{Q}_{I_b\otimes 1_k}\mathbf{L}_d\left(\mathbf{L}'_d\mathbf{Q}_{I_b\otimes 1_k}\mathbf{L}_d\right)^{-}\mathbf{L}'_d\mathbf{Q}_{I_b\otimes 1_k}\mathbf{T}_d \tag{4}$$

(Markiewicz 1997), where $\mathbf{Q}_{I_b\otimes 1_k}$ is the orthogonal projector onto the orthocomplement of the column span of $\mathbf{I}_b \otimes \mathbf{1}_k$ and $\mathbf{A}^-$ denotes a generalized inverse of $\mathbf{A}$. It is easy to see that $\mathbf{Q}_{I_b\otimes 1_k} = \mathbf{I}_b \otimes \mathbf{E}_k$, where $\mathbf{E}_k = \mathbf{I}_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}'_k$. From the form of the matrix $\mathbf{E}_k$ and orthogonality of $\mathbf{H}_k$ it can be seen that $\mathbf{L}'_d\mathbf{Q}_{I_b\otimes 1_k}\mathbf{L}_d = \mathbf{T}'_d\mathbf{Q}_{I_b\otimes 1_k}\mathbf{T}_d$.

We are interested in determining connected designs, i.e. in testing the null hypothesis of equality of treatment effects, which is possible if and only if all functions of the type $\tau_i - \tau_j$, $i, j = 1, 2, \ldots, t$, $i \ne j$, are estimable.

Definition 1 A linear parametric function of treatment effects, $\ell'\tau$ is called a **contrast** if $\ell'\mathbf{1}_t = 0$. It is called an **elementary contrast** if ℓ has only two nonzero entries consisting of 1 and -1.

Theorem 1 (Raghavarao and Padgett 2005) *All elementary contrasts of treatment effects are estimable if and only if* rank $\mathbf{C}_d = t - 1$.

Definition 2 A block design is said to be **connected** if all elementary contrasts of treatment effects are estimable.

Theorem 1 and Definition 2 imply the following.

Corollary 1 *A block design d is connected if and only if* rank $\mathbf{C}_d = t - 1$.

Let $\mathbf{A}$ be nonnegative matrix of order n. We denote the eigenvalues of $\mathbf{A}$ by $\lambda_1(\mathbf{A}) \ge \lambda_2(\mathbf{A}) \ge \cdots \ge \lambda_{n-1}(\mathbf{A}) \ge \lambda_n(\mathbf{A}) \ge 0$. From the above Corollary it is easy to note that a design d is connected if $\lambda_{t-1}(\mathbf{C}_d) > 0$.

Observe, that a generalized inverse of the matrix in formula (4) depends on the design, i.e. it changes itself with the arrangement of treatments on experimental units. It makes the determination of connected design difficult. Therefore, in this paper we consider experiments in which each treatment occurs at most once in each block (binary designs) and with $t = k$ (complete designs), whilst b is arbitrary. The class of such designs we will denote by $\mathcal{B}_{t,b,t}$. The information matrix of $d \in \mathcal{B}_{t,b,t}$ has the form

$$\mathbf{C}_d = b\mathbf{E}_t - \frac{1}{b}\mathbf{K}_d\mathbf{K}'_d, \tag{5}$$

where $\mathbf{K}_d = \mathbf{T}_d'\mathbf{Q}_{I_b\otimes 1_t}\mathbf{L}_d = \mathbf{T}_d'\mathbf{L}_d - \frac{b}{t}\mathbf{1}_t\mathbf{1}_t'$. Since the vectors $\mathbf{T}_d\mathbf{1}_t$ and $\mathbf{L}_d\mathbf{1}_t$ are in the column space of $\mathbf{I}_b \otimes \mathbf{1}_t$, the matrix $\mathbf{K}_d$ has zero row and column sums. The (i, j)th element of the matrix $\mathbf{T}_d'\mathbf{L}_d$ denotes the number of occurrences of treatment i with treatment j as left neighbor in a design d. Therefore $\mathbf{T}_d'\mathbf{L}_d$ is called a *left-neighboring matrix* and it will be denoted by $\mathbf{S}_d$. It is easy to see that in the class $\mathcal{B}_{t,b,t}$ the diagonal entries of $\mathbf{S}_d$ are equal to 0 and the off-diagonal entries belong to the set $\{0, 1, \ldots, b\}$, such that the row and column sums are equal to b.

Observe, that the matrix $\mathbf{E}_t$ is symmetric, idempotent and $t-1$ of its eigenvalues are equal to 1. Moreover, $\mathbf{E}_t\mathbf{1}_t = \mathbf{0}_t$, $\mathbf{K}_d\mathbf{K}_d'\mathbf{1}_t = \mathbf{0}_t$ and matrices $\mathbf{E}_t$ and $\mathbf{K}_d\mathbf{K}_d'$ commute. It follows, that

$$\lambda_{t-1}(\mathbf{C}_d) = b - \frac{1}{b}\lambda_1\left(\mathbf{K}_d\mathbf{K}_d'\right). \tag{6}$$

We denote by $\mathcal{P}_n$ the class of permutation matrices of order n, and by $\widetilde{\mathcal{P}}_n \subset \mathcal{P}_n$ the subclass of derangement matrices, i.e. matrices with $p_{ii} \neq 0$, $i = 1, 2, \ldots, n$. By $\mathbb{C}$ we denote the set of complex numbers. We use the following definition.

Definition 3 (Horn and Johnson 1985) An $n \times n$ matrix $\mathbf{A}$ is said to be **reducible** if either of the following conditions is satisfied

1. $n = 1$ and $\mathbf{A} = 0$,
2. $n \geq 2$ and there is a permutation matrix $\mathbf{P} \in \mathcal{P}_n$ and an integer r with $1 \leq r \leq n-1$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \Theta & \mathbf{D} \end{bmatrix}$, where $\mathbf{B} \in \mathbb{C}^{r\times r}$, $\mathbf{D} \in \mathbb{C}^{(n-r)\times(n-r)}$, $\mathbf{C} \in \mathbb{C}^{r\times(n-r)}$ and $\Theta \in \mathbb{C}^{(n-r)\times r}$ is a zero matrix.

A matrix is called **irreducible** if it is not reducible.

Note that every irreducible permutation matrix $\mathbf{P} \in \mathcal{P}_n$ represents a cycle of length n ($\mathbf{P}$ is a full-cycle permutation matrix). Moreover, matrix $\mathbf{H}_n$ defined in (2) is a full-cycle permutation matrix.

It is worth observing, that the left-neighboring matrix can be expressed as $\mathbf{S}_d = \sum_{i=1}^{b}\widehat{\mathbf{P}}_i$, where all $\widehat{\mathbf{P}}_i$, $i = 1, 2, \ldots, b$, are full-cycle permutation matrices.

3 Results

3.1 Connected designs with minimal number of blocks

Note that if we add some blocks to a connected design, the extended design is also connected. Thus, in this section we determine the condition of minimal number of blocks necessary to construct connected design.

Theorem 2 *Let design $d \in \mathcal{B}_{t,b,t}$. If d is connected, then $b \geq 2$ for odd t and $b \geq 3$ if t is even.*

Proof Let $b = 1$. Without loss of generality let assume $\mathbf{S}_d = \mathbf{H}_t$. From (5) we have

$$\mathbf{C}_d = \mathbf{E}_t - \left(\mathbf{H}_t - \frac{1}{t}\mathbf{1}_t\mathbf{1}_t'\right)\left(\mathbf{H}_t - \frac{1}{t}\mathbf{1}_t\mathbf{1}_t'\right)' = \Theta_t,$$

where Θ_t is a $t \times t$ matrix of zeros. Thus, every design which contains only one block is disconnected.

Let $b = 2$. Without loss of generality let $\mathbf{S}_d = \mathbf{H}_t + \mathbf{P}\mathbf{H}_t\mathbf{P}'$, where $\mathbf{P} \in \widetilde{\mathcal{P}}_t$. From (5) we have

$$\begin{aligned}\mathbf{C}_d &= 2\mathbf{E}_t - \frac{1}{2}\left(\mathbf{H}_t + \mathbf{P}\mathbf{H}_t\mathbf{P}' - \frac{2}{t}\mathbf{1}_t\mathbf{1}_t'\right)\left(\mathbf{H}_t + \mathbf{P}\mathbf{H}_t\mathbf{P}' - \frac{2}{t}\mathbf{1}_t\mathbf{1}_t'\right)' \\ &= \mathbf{I}_t - \frac{1}{2}\left(\mathbf{H}_t\mathbf{P}\mathbf{H}_t'\mathbf{P}' + \mathbf{P}\mathbf{H}_t\mathbf{P}'\mathbf{H}_t'\right)\end{aligned}$$

and it can be observed that $\lambda_{t-1}(\mathbf{C}_d) \neq 0$ if and only if $\mathbf{H}_t\mathbf{P}\mathbf{H}_t'\mathbf{P}'$ is irreducible.

Let t be even. Since every cycle of even length is an odd permutation, the product of two odd permutations has to be even. The even permutation represents a cycle of odd length or the product of disjoint cycles. Since t is even, $\mathbf{H}_t \cdot \mathbf{P}\mathbf{H}_t'\mathbf{P}'$ has to be a product of disjoint cycles and hence it is reducible. Thus, every design $d \in \mathcal{B}_{t,2,t}$ with even t is disconnected.

Similarly, when t is odd we obtain the product of two even permutation which is also even. Thus, $\mathbf{H}_t \cdot \mathbf{P}\mathbf{H}_t'\mathbf{P}'$ is a cycle of odd length or the product of disjoint cycles. Since t is odd, there exists a matrix $\mathbf{P} \in \widetilde{\mathcal{P}}_t$ such that $\mathbf{H}_t\mathbf{P}\mathbf{H}_t'\mathbf{P}'$ is a cycle of odd length and the design with $\mathbf{S}_d = \mathbf{H}_t + \mathbf{P}\mathbf{H}_t\mathbf{P}'$ is connected.

Let $b = 3$. Since adding some blocks to connected design does not change the property of connectedness of extended design, it is enough to consider even t. We show that for even t there exists connected design $d \in \mathcal{B}_{t,3,t}$.

Let $\mathbf{S}_d = \mathbf{H}_t + \mathbf{H}_t' + \mathbf{G}$, where $\mathbf{G} = \mathbf{P}_1\mathbf{H}_t'\mathbf{P}_1'$ and

$$\mathbf{P}_1 = \begin{pmatrix} \mathbf{I}_{t-2} & \mathbf{0}_{t-2} & \mathbf{0}_{t-2} \\ \mathbf{0}_{t-2}' & 0 & 1 \\ \mathbf{0}_{t-2}' & 1 & 0 \end{pmatrix}.$$

From (5) we have

$$\mathbf{C}_d = 2\mathbf{I}_t - \frac{1}{3}\left(\mathbf{H}_t^2 + \mathbf{H}_t^{-2} + \mathbf{H}_t\mathbf{G}' + \mathbf{H}_t'\mathbf{G}' + \mathbf{G}\mathbf{H}_t' + \mathbf{G}\mathbf{H}_t\right).$$

Assume that d is disconnected. Thus, two eigenvalues of $\mathbf{C}_d$ have to be 0. It implies that there are two eigenvectors, $\mathbf{x}_1 \neq \mathbf{x}_2$, corresponding to zero eigenvalue. It is easy to see that one of these eigenvectors, say $\mathbf{x}_1$, is $\mathbf{1}_t$.

Observe, that the eigenvalue of $\mathbf{C}_d$ is zero if and only if

$$\lambda_1\left(\mathbf{H}_t^2 + \mathbf{H}_t^{-2} + \mathbf{H}_t\mathbf{G}' + \mathbf{H}_t'\mathbf{G}' + \mathbf{G}\mathbf{H}_t' + \mathbf{G}\mathbf{H}_t\right) = 6.$$

Since all components in this sum are permutation matrices, we obtain the eigenvalue equal to 6 if every matrix has the same eigenvector, different than $\mathbf{1}_t$, corresponding to the eigenvalue 1. It follows from John (1987) that the only eigenvector corresponding to the eigenvalue 1 of the matrix $\mathbf{H}_t^2$ is $\mathbf{x}_2 = (1, -1, 1, -1, \ldots, 1, -1)'$, which is not the eigenvector of $\mathbf{GH}_t$:

$$\mathbf{GH}_t\mathbf{x}_2 = (1, -1, 1, -1, \ldots, 1, -1, 1, 1, -1, -1)' \neq \lambda\,(\mathbf{GH}_t)\,\mathbf{x}_2.$$

Thus, $\lambda_{t-1}(\mathbf{C}_d) > 0$ and design d is connected. □

For $d \in \mathcal{B}_{t,b,t}$ and $p(t-1) < b < (p+1)(t-1)$, $p \in \mathbf{N} \cup \{0\}$, let

$$\mathcal{K}_{(b)} = \Big\{\mathbf{K}_d : \mathbf{K}_d = \textstyle\sum_{i=1}^{b-pt+p} \mathbf{P}_i + (p - \frac{b}{t})\mathbf{1}_t\mathbf{1}_t' - p\mathbf{I}_t, \\ \mathbf{P}_i, \mathbf{P}_i'\mathbf{P}_j \in \widetilde{\mathcal{P}}_t, \quad i \neq j, \quad i, j = 1, 2, \ldots, b - pt + p\Big\}.$$

For a given b and t the specified subclass of $\mathcal{K}_{(b)}$ was considered in Filipiak et al. (2008) and Filipiak and Różański (2005). We show that if the conditions of Theorem 2 are satisfied, then every design $d \in \mathcal{B}_{t,b,t}$ for which $\mathbf{K}_d \in \mathcal{K}_{(b)}$ is connected.

Theorem 3 *Let $d \in \mathcal{B}_{t,b,t}$ and b satisfies conditions from Theorem 2. If $\mathbf{K}_d \in \mathcal{K}_{(b)}$ then design d is connected.*

Proof From (6) it follows that the matrix $\mathbf{C}_d$ is of rank $t-1$ if and only if $\lambda_1\left(\mathbf{K}_d\mathbf{K}_d'\right) < b^2$. For $\mathbf{K}_d \in \mathcal{K}_{(b)}$ we have

$$\mathbf{K}_d\mathbf{K}_d' = \left(a + p^2\right)\mathbf{I}_t + \sum_{i=1}^{a}\sum_{\substack{j=1\\ j\neq i}}^{a} \mathbf{P}_i\mathbf{P}_j' - p\sum_{i=1}^{a}\left(\mathbf{P}_i + \mathbf{P}_i'\right) - \frac{(p-a)^2}{t}\mathbf{1}_t\mathbf{1}_t',$$

where $a = b - pt + p$, and

$$\lambda_1\left(\mathbf{K}_d\mathbf{K}_d'\right) \leq a + p^2 + a^2 - a + 2pa = (2p + b - pt)^2 = (b - p(t-2))^2 < b^2$$

for every $t > 2$. □

3.2 The conditions of connectedness of designs

It is known from the literature that in model (3) a block design is connected if and only if for a given any two treatments θ and ϕ and for a given m, there exists a chain of treatments $\theta = \theta_0, \theta_1, \ldots, \theta_m, \theta_{m+1} = \phi$ such that θ_i and θ_{i+1} occur together in a block for $i = 0, 1, \ldots, m$ (cf. Raghavarao and Padgett 2005). It should be remarked that a disconnected block design has a simple structure which is easy to recognize by utilizing the above chain definition of connectedness. However, there does not appear to be a simple way of checking for connectedness of, e.g. row–column designs. Nevertheless, to show connectedness of designs from $\mathcal{B}_{t,b,t}$ under model (1) we can formulate similar necessary and sufficient condition of connectedness. In such a situation the incidence matrix $\mathbf{N}_d$ in the information matrix of design under model (3) has to be replaced by left-neighboring matrix.

For every design $d \in \mathcal{D}_{t,b,k}$ we denote by $\tilde{d}$ a design associated with d, i.e. such that $\mathbf{N}_{\tilde{d}} = \mathbf{S}_d$. It is easy to see that $\tilde{d}$ has t treatments which are arranged in t blocks of size k_i, $i = 1, 2, \ldots, t$, $\sum_{i=1}^{t} k_i = t^2$. Moreover, the ith block of $\tilde{d}$ contains all treatments for which i is a left neighbor in d. Thus, if $d \in \mathcal{B}_{t,b,t}$, then $\tilde{d} \in \mathcal{D}_{t,t,b}$, and we can formulate the following.

Theorem 4 *A design $d \in \mathcal{B}_{t,b,t}$ is connected under model (1) if and only if a design $\tilde{d}$ associated with d is connected under model (3).*

Proof Let $d \in \mathcal{B}_{t,b,t}$. From (5) we have

$$\mathbf{C}_d = b\mathbf{I}_t - \frac{1}{b}\mathbf{S}_d\mathbf{S}_d',$$

and this matrix is also the information matrix of $\tilde{d}$ in model (3). Thus, connectedness of $\tilde{d}$ in model (3) is equivalent to connectedness of d in model (1). □

Example Let $t = 5$, $b = 3$. The designs d and $\tilde{d}$

$$d = \left(\begin{array}{c|ccccc} 5 & 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 & 4 \\ 2 & 1 & 5 & 4 & 3 & 2 \end{array}\right) \implies \tilde{d} = \begin{pmatrix} 2 & 2 & 5 \\ 1 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 5 \\ 1 & 4 & 4 \end{pmatrix}$$

are connected in models (1) and (3), respectively. □

For a specified classes of designs the algebraic conditions of connectedness can be formulated. Since the left-neighboring matrix of design from $\mathcal{B}_{t,b,t}$ is the sum of b full-cycle permutation matrices of order t, during the construction of designs (cf. Filipiak and Różański 2005) it is natural to consider circularity of this matrix.

Definition 4 A matrix $\mathbf{A} = \sum_{i=1}^{t} \alpha_i \mathbf{H}_t^i$, where α_i's are scalars, is called a **circulant matrix**.

Now we can prove the following theorem.

Theorem 5 *The design $d \in \mathcal{B}_{t,b,t}$ with circulant left-neighboring matrix, i.e.*

$$\mathbf{S}_d = \sum_{i=1}^{t} \alpha_i \mathbf{H}_t^i, \quad \textit{where} \quad \sum_{i=1}^{t} \alpha_i = b \quad \textit{and} \quad \alpha_i \in \mathbf{N} \cup \{0\}, \quad i = 1, 2, \ldots, t$$

is connected if and only if neither

(i) *for every pair of powers of $\mathbf{H}_t$, say i_1, i_2 ($i_1 \neq i_2$), such that $\alpha_{i_1}, \alpha_{i_2} \neq 0$, there exists $k \in \{1, 2, \ldots, t-1\}$ such that $\frac{k(i_1 - i_2)}{t}$ is an integer*
nor

(ii) $\mathbf{S}_d = b\mathbf{H}_t^i$.

Proof Assume that design d is disconnected. Then $\lambda_{t-1}(\mathbf{C}_d) = 0$. We need to determine matrix $\mathbf{K}_d$ with $\lambda_1(\mathbf{K}_d\mathbf{K}_d') = b^2$. It is easy to see that circularity of $\mathbf{S}_d$ implies circularity of $\mathbf{K}_d$. From the form of $\mathbf{K}_d$ we can write

$$\mathbf{K}_d\mathbf{K}_d' = \left(\sum_{i=1}^{t}\alpha_i\mathbf{H}_t^i - \frac{b}{t}\mathbf{1}_t\mathbf{1}_t'\right)\left(\sum_{i=1}^{t}\alpha_i\mathbf{H}_t^i - \frac{b}{t}\mathbf{1}_t\mathbf{1}_t'\right)'.$$

Now assume that for a given $i_1 \neq i_2$, $i_1, i_2 \in \{1, 2, \ldots, t\}$, there exist non-zero $\alpha_{i_1}, \alpha_{i_2}$. Since $\left(\mathbf{H}_t^i\right)' = \mathbf{H}_t^{-i}$, we obtain

$$\mathbf{K}_d\mathbf{K}_d' = \sum_{j=i+1}^{t}\sum_{i=1}^{t-1}\alpha_i\alpha_j\left(\mathbf{H}_t^{i-j} + \mathbf{H}_t^{j-i}\right) + \sum_{i=1}^{t}\alpha_i^2\mathbf{I}_t - \frac{b^2}{t}\mathbf{1}_t\mathbf{1}_t'.$$

For the circulant matrix $\mathbf{H}_t$ it is known (John 1987), that its eigenvalues are equal to the tth root from the unity. Thus, for $k = 1, 2, \ldots, t-1$

$$\mu_k(\mathbf{K}_d\mathbf{K}_d') = \sum_{i=1}^{t}\alpha_i^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t}\alpha_i\alpha_j\cos\frac{2k(j-i)\pi}{t},$$

where $\mu_k(\mathbf{A})$ are the unordered eigenvalues of $\mathbf{A}$, and

$$\lambda_1(\mathbf{K}_d\mathbf{K}_d') = \sum_{i=1}^{t}\alpha_i^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t}\alpha_i\alpha_j\max_{1\le k\le t}\left[\cos\frac{2k(j-i)\pi}{t}\right].$$

It is easy to see that the maximum is obtained if and only if $\cos\frac{2k(j-i)\pi}{t} = 1$. It holds if condition (i) is satisfied.

Now assume $\mathbf{S}_d = b\mathbf{H}_t^i$. Then $\mathbf{K}_d\mathbf{K}_d' = b^2\mathbf{I}_t - \frac{b^2}{t}\mathbf{1}_t\mathbf{1}_t'$ and each non-zero eigenvalue of $\mathbf{K}_d\mathbf{K}_d'$ is equal to b^2 and d is disconnected. □

Let consider designs from $\mathcal{B}_{t,b,t}$ with non-circulant left-neighboring matrix $\mathbf{S}_d$. In such a case we can prove only the sufficient condition of disconnectedness.

Theorem 6 *Let $d \in \mathcal{B}_{t,b,t}$. If at most one entry of $\mathbf{S}_d$ is maximal, i.e. equal to b, then design d is disconnected.*

Proof For a design d assume that in the left-neighboring matrix one entry is equal to b. Without loss of generality let assume that it is the $(1, 2)$th entry of $\mathbf{S}_d$. Since $\mathbf{K}_d\mathbf{1}_t = \mathbf{K}_d'\mathbf{1}_t = \mathbf{0}_t$, the first row and the second column of this matrix are $\frac{b}{t}(-1, t-1, -1, -1, \ldots, -1)$ and $\frac{b}{t}(t-1, -1, -1, \ldots, -1)'$, respectively, and hence the first row of $\mathbf{K}_d\mathbf{K}_d'$ is of the form: $\frac{b^2}{t^2}\left(t^2 - t, -t, \ldots, -t\right)$. To show that b^2 is the eigenvalue of $\mathbf{K}_d\mathbf{K}_d'$ it is enough to prove, that the matrix $\mathbf{K}_d\mathbf{K}_d' - b^2\mathbf{I}_t$ is singular. Observe, that the row and column sums of this matrix are $-b^2$ and the first row of this matrix is $-\frac{b^2}{t}\mathbf{1}_t'$.

If to the second row we add all remaining rows except the first one, we obtain the row of the form $\frac{b^2(1-t)}{t}\mathbf{1}_t'$, which is proportional to the first one and the determinant is 0.□

Caution! The converse is not true.

Maximal element in $\mathbf{S}_d$ means that treatment i is always preceded by treatment j. The consequence being that the effect of treatment i is confounded with the neighbor effect of treatment j. Thus, from Theorem 6 follows that if in the design d at least one pair of treatments occurs in each block, then d is disconnected.

Example Let $t = 5$, $b = 3$. The design

$$d = \left(\begin{array}{c|ccccc} 3 & 1 & 2 & 5 & 4 & 3 \\ 5 & 1 & 4 & 3 & 2 & 5 \\ 3 & 1 & 4 & 5 & 2 & 3 \end{array}\right)$$

has non-circulant left-neighboring matrix:

$$\mathbf{S}_d' = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Observe, that there is not any pair of treatments which occurs in every block, but design d is disconnected. □

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