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Incomplete self-similarity and fatigue-crack growth

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Abstract. The Paris power law, which relates fatigue-crack growth rates to the applied stress-intensity range, is an example of a scaling law with the inherent property of incomplete similarity. Previous considerations of dimensions and self-similarity have suggested that the assumed 'materials constants' in this law are also a function of specimen size. In this note, the question of the size-dependence of the Paris law is re-examined, and through comparison to a larger body of fatigue-crack growth data in steels, physical explanations why such scaling effects may exist are deduced.

Key words: Fatigue-crack growth, incomplete similarity, Paris law, scaling laws.

1. Introduction

Throughout science, engineering and economics, *scaling laws*, which describe power-law relationships between certain variables, reveal the important property of *self-similarity*, where a phenomenon reproduces itself on different time and/or space scales (Barenblatt, 1996, 2003). A notable example of such a scaling law is the well-known Paris law describing the relationship between the fatigue-crack growth rate per cycle, $\mathrm{d}a/\mathrm{d}N$, and the stress-intensity range, ΔK:

$$\mathrm{d}a/\mathrm{d}N = C(\Delta K)^m, \tag{1}$$

where $\Delta K = K_{\max} - K_{\min}$ ($K_{\max}$ and $K_{\min}$ are, respectively, the maximum and minimum stress intensities in the fatigue cycle), and C and m are experimentally determined scaling-law constants (Paris et al., 1961; Paris and Erdogan, 1963). This relationship has been verified over a wide range of growth rates in innumerable studies for the past 40 years on metallic, polymeric, ceramic, geological and even biological materials (e.g., Hertzberg and Manson, 1980; Ritchie, 1988; Suresh, 1998; Bažant and Planas, 1988; Nalla et al., 2005)[1], and provides the basis for the life prediction of structures and components using the fracture mechanics (or damage-tolerant) approach in terms of the applied stress ($\Delta\sigma$), initial (a_o) and final (a_c)

[1]The actual relationship between $\mathrm{d}a/\mathrm{d}N$ and ΔK in most instances is sigmoidal in shape, with Equation (1) describing the "linear" behavior over the mid-range of growth rates, typically from 10^{-9} to 10^{-6} m/cycle in most ductile materials. Growth rates tend to be less than those predicted by the Paris law at lower values of $\mathrm{d}a/\mathrm{d}N$ where a fatigue threshold is approached (below which cracking is considered to be dormant), and to be higher than the Paris law at higher $\mathrm{d}a/\mathrm{d}N$ values where instability (e.g., K_{Ic} or the limit load) is approached (Ritchie, 1979).

crack sizes, geometry, and properties of the material, e.g., the yield strength, σ_y, and fracture toughness, K_{Ic}.

Of particular significance to its use in life prediction is the nature of the self-similarity in the Paris law, the values of the constants C and m, and whether they depend upon specimen size. The similarity analysis of (Barenblatt and Botvina 1981, 1983), described in Section 2, suggests that the constant C should depend upon a characteristic specimen length h, and that exponent m should depend upon $\sqrt{h}$; the latter prediction was found to be consistent with the fatigue-crack growth data of Heiser and Mortimer (1972) on AISI 4340 steel, where h was identified with the specimen thickness. Since the use of $\mathrm{d}a/\mathrm{d}N$ vs. ΔK data from tests on small specimens to characterize crack-growth behavior of larger structures is almost implicit in the practice of fracture mechanics for engineering design and life prediction, such mathematical predictions that the growth rates of fatigue cracks may vary with specimen size are clearly of importance. It is therefore the objective of this note to re-examine such notions of self and incomplete similarities implied by the Paris law, and through comparison to a larger spectrum of fatigue-crack propagation rate data, to provide some physical interpretation as to why such scaling effects should exist.

2. Dimensional and similarity analysis

From a physical perspective, the Paris law in the form of Equation (1) provides little indication of the many factors that affect fatigue-crack propagation behavior. Clearly, the average growth rate depends upon a series of variables in addition to the stress-intensity range; these include (i) the nature of the loading, i.e., the load ratio, $R = K_{\min}/K_{\max}$, cyclic frequency ν and time t, (ii) materials properties, notably σ_y and K_{Ic}, and (iii) specimen size. Following the mathematical similarity approach of Barenblatt and Botvina (1972, 1981), where specimen size was defined in terms of a characteristic specimen length scale h, consideration of these principals variables in terms of dimensional analysis gives:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{\Delta K}{\sigma_y}\right)^2 \Phi\left(\frac{\Delta K}{K_{\text{Ic}},},\ R,\ Z,\ \nu t\right), \tag{2}$$

where Z is the basic similarity parameter, given by:

$$Z = \frac{\sigma_y\sqrt{h}}{K_{\text{Ic}}}. \tag{3}$$

In the absence of significant environmental effects such that the effect of νt is minimized, two possible modes of similarity are possible by considering the asymptotic relationships in Equations (2 and 3) for the mid-range of growth rates where $\Delta K/K_{\text{Ic}} \ll 1$:

- complete similarity ('similarity of the first kind' (Barenblatt, 2003)), when the limit of the function Φ at $\Delta K/K_{\text{Ic}} \to 0$ is finite and non-zero, and
- incomplete similarity ('similarity of the second kind' (Barenblatt, 2003)).

Note that one distinction between similarity of the first and second kind is that for complete similarity, all similarity parameters can be determined by dimensional analysis; this is not the case for incomplete similarity.

In the present example of the power-law relationship for fatigue-crack growth, notions of complete similarity simply give the scaling law for $\Delta K/K_{\rm Ic} \ll 1$ as:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{\Delta K}{\sigma_{\rm y}}\right)^2 \Phi(R, Z), \tag{4}$$

i.e., the Paris law in Equation (1) with a constant exponent of $m = 2$. Although consistent with most simple (perfectly plastic) models of cyclic crack advance in ductile materials via the formation of fatigue striations (Neumann, 1969; Pelloux, 1970), where growth rates are expected to be proportional to the crack-tip opening displacements and hence to scale with ΔK^2 (Neumann, 1969; Pelloux, 1970; Gu and Ritchie, 1999), Equation (4) represents a rather limited case. In reality, experimentally measured Paris law exponents typically vary between 2 and 4 for ductile materials, and can be considerably larger for brittle materials, i.e., approaching 10 in low toughness metals (Ritchie and Knott, 1973) and even higher in intermetallics and ceramics (Ritchie, 1999).

If incomplete similarity is assumed in the parameter $\Delta K/K_{\rm Ic} \ll 1$, then:

$$\Phi = \left(\frac{\Delta K}{K_{\rm Ic}}\right)^{\alpha} \Phi_1(R, Z), \quad \frac{\mathrm{d}a}{\mathrm{d}N} = \frac{(\Delta K)^{2+\alpha}}{\sigma_{\rm y}^2 K_{\rm Ic}^{\alpha}} \Phi_1(R, Z), \tag{5}$$

which now represents the Paris law (Equation (1)) with:

$$C = \frac{\Phi_1(R, Z)}{\sigma_{\rm y}^2 K_{\rm Ic}^{\alpha}}, \quad m = 2 + \alpha(R, Z), \tag{6}$$

where α is a function of R and Z. In this less restrictive form, it is apparent that considerations of similarity imply that both the C and m constants in the Paris law are not simply a function of the material properties and the nature of the applied loading, but also on specimen size, h, through the basic similarity parameter Z. Indeed, comparison with Heiser and Mortimer data (Heiser and Mortimer, 1972) on 4340 steel revealed such a relationship between m and Z, as predicted by Equations (5) and (6) (Barenblatt and Botvina, 1981). One aim of the present study is to explain such a result in terms of physical mechanisms.

3. Comparison with experimental data

Since in the original analysis of Barenblatt and Botvina (1981) only a single set of experimental data was considered, we first examine a large body of fatigue-crack growth rate results in order to interrogate relationships between the Paris exponent m and specimen dimensions. To facilitate this approach, we use the compilation of ambient-temperature data from the work of Ritchie and Knott (1973), based on experimental fatigue-crack propagation studies at ambient temperatures on a wide range of steels from (Carman and Katlin, 1966; Wei et al., 1967; Crooker et al., 1968; Miller, 1968; Bates et al., 1969; Crooker and Lange, 1970; Clark and Wessel, 1970; Evans et al., 1971; Ritchie and Knott, 1973). This collection of data represents steels with almost a fivefold variation in yield strength, from 433 to 2035 MPa, and includes several Ni–Cr low alloy steels (4340, En24, En30A, D6Ac), a 9Ni–4Co steel, Ni–Mo–V rotor steel, an H-11 tool steel, and a series of ultrahigh strength Ni-maraging steels. Furthermore, the variable R is nominally constant ($R \sim 0$) and any influence of νt is expected to be small.

Plotted in Figure 1 are the Paris exponents m from these data as a function of the basic similarity parameter $Z(=\sigma_y\sqrt{h}/K_{\mathrm{Ic}})$, where the characteristic specimen dimension h has been identified with the specimen width W and thickness B. Although these data represent a wide range of microstructures, orientations, specimen geometries and test frequencies, which contribute to the considerable scatter, it is clear that as first reported by Barenblatt and Botvina (1981), there is a definitive trend of increasing Paris exponents m with increasing specimen size, both in terms of the specimen width and thickness.

4. Physical interpretation

To provide some physical explanation as to why the Paris exponent m should increase with specimen size, or more specifically with increasing Z, we must examine the nature of the latter similarity parameter. Using linear-elastic fracture mechanics considerations, we note that the extent of local plasticity ahead of the crack tip, i.e., the plastic-zone size, r_{y}, scales with $K_{\mathrm{Ic}}^2/\sigma_{\mathrm{y}}^2$ (Irwin, 1960). It follows then that $Z \sim \sqrt{h/r_{\mathrm{y}}}$.

Considering first the case where the characteristic specimen dimension h is equated to the specimen width W, we note that according to ASTM Standard E-399 (ASTM E399-90, 2002) a state of small-scale yielding, i.e., a valid K-field at the crack tip, is reached when r_{y} is small (typically 1/15) compared to the specimen in-plane dimensions. This implies that small values of Z less than about 3 can be associated with a deviation from small-scale yielding, i.e., excessive local plasticity. Under such conditions, mechanisms of fatigue-crack advance pertaining to fully ductile materials would prevail, i.e., fatigue striation formation, via alternating or simultaneous shear, involving alternating blunting and re-sharpening of the crack tip during the fatigue cycle (Neumann, 1969; Pelloux, 1970). Since in the limit of fully plastic conditions, the (mode I) crack-growth rate per cycle is proportional to the cyclic crack-tip opening displacement (Δ CTOD) (Pelloux, 1970):

$$\frac{\mathrm{d}a}{\mathrm{d}N} \approx \beta\,\Delta\mathrm{CTOD} \propto \frac{\Delta K^2}{\sigma_y E'}, \qquad (7)$$

where the maximum value of β (at full recovery of the ΔCTOD on unloading) is 0.35 (Gu and Ritchie, 1999), and E' is equal to E (Young's modulus) in plane stress and $E/(1-\nu)^2$ in plane strain (ν is Poisson's ratio), we would expect that with decreasing values of Z, the Paris exponent should decrease to approach this lower (fully plastic) limit of $m = 2$, as shown by the experimental data in Figure 1.

On the other hand, where the characteristic specimen dimension h is equated to the specimen thickness B, we also note that according to ASTM Standard E-399

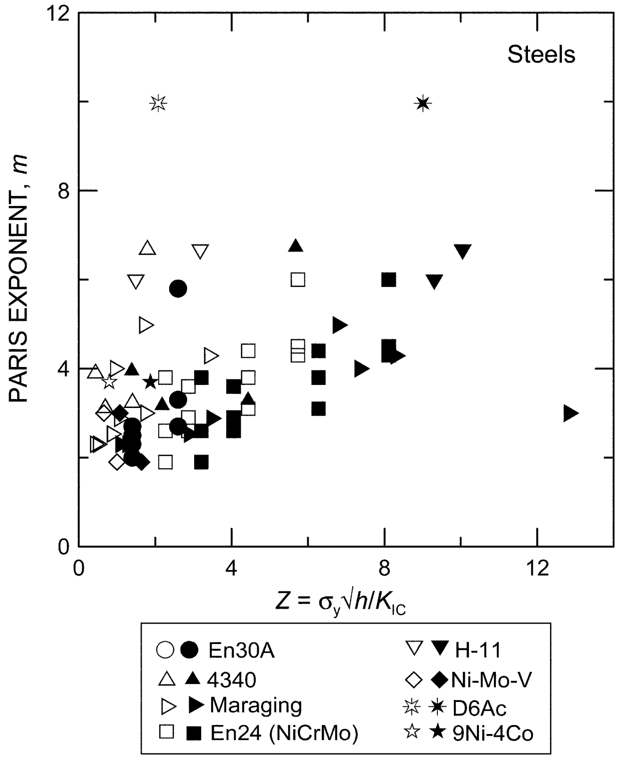

Figure 1. Plot of the Paris law exponent m for a range of steels as a function of the basic similarity parameter $Z = \sigma_y\sqrt{h}/K_{Ic}$, where h is a characteristic specimen dimension. In the diagram, h is identified both with the in-plane dimension of specimen width W (closed symbols) and with the out-of-plane specimen thickness B (open symbols). Data are taken from the compilation in Ritchie and Knott (1973), derived from the experimental fatigue-crack growth results for low to ultrahigh strength steels at ambient temperatures at $R \sim 0$ in Carman and Katlin (1966), Wei et al. (1967), Crooker et al. (1968), Miller (1968), Bates et al. (1969), Crooker and Lange (1970), Clark and Wessel (1970), Evans et al. (1971), Ritchie and Knott (1973).

(ASTM E399-90, 2002) a state of plane strain deformation is reached when r_y is small (typically 1/15) compared to B.[2] This implies that large values of Z greater than approximately 3 can be associated with plane-strain conditions being met, i.e., with full triaxial constraint at the crack tip. Furthermore, in general high Z ($= \sigma_y\sqrt{h}/K_{Ic}$) values are associated with high strength/low toughness materials. Under these conditions, more brittle mechanisms of fatigue-crack advance would be anticipated. Indeed, the premise of Ritchie and Knott's original study (Ritchie and Knott, 1973) was that as K_{max} in the fatigue cycle approaches K_{Ic}, additional modes of fracture, specifically motivated by higher peak and/or hydrostatic stresses, occur in additional to striation crack growth. Such *static modes*, i.e., transgranular cleavage, intergranular cracking and microvoid coalescence (dimples), were shown to significantly accelerate growth rates and specifically to result in higher values of the Paris exponent m. Their results showed an inverse relationship between m and K_{Ic} with the

[2]The E-399 Standard (Irwin, 1960) actually specifies that B, $(W - a)$ and $a > 2.5(K_{Ic}/\sigma_y)^2$ for plane strain and small-scale yielding conditions to exist. This is equivalent to r_y being approximately 1/13–1/15 of the out-of-plane and in-plane specimen dimensions, respectively.

high exponents ($m \geqslant 3$) occurring almost entirely in the lower toughness steels where K_{Ic} was less than $\sim 60\,\mathrm{MPa}\sqrt{\mathrm{m}}$. Thus, in physical terms, the increase in the Paris exponent m with increasing Z, shown in Figure 1 for $h = B$, is consistent with the onset of more brittle mechanisms of fatigue-crack advance (static modes), induced by the increasing triaxial constraint and limited local plasticity associated with high values of the similarity parameter Z (as $r_{\mathrm{y}} \ll h$).

5. Conclusions

The property of self-similarity associated with scaling laws has been considered in the context of the well known Paris power law – $\mathrm{d}a/\mathrm{d}N = C(\Delta K)^m$ – relating the fatigue-crack growth rates to the stress-intensity range. Although the constants C and m in the Paris law are generally assumed to depend only on material properties, dimensional and similarity analysis predicts that they should additionally be dependent on specimen size. Through comparison to the fatigue-crack growth properties of a wide range of low to high strength steels, the Paris exponent m is shown to indeed increase with increasing Z, the basic similarity parameter equal to $\sigma_y\sqrt{h}/K_{\mathrm{Ic}}$, where h is a characteristic specimen dimension. By identifying h respectively, with the specimen width W and thickness B, it is further shown that *in terms of physical mechanisms*, (i) low values of m ($m \to 2$) at small Z ($Z < \sim 3$) are consistent with extensive crack-tip plasticity which promotes crack advance via ductile striation formation (where $\mathrm{d}a/\mathrm{d}N \propto \Delta K^2$), whereas (ii) high values of m at larger Z result from increased triaxial constraint/plane-strain conditions at the crack tip that promote additional more brittle fatigue fracture mechanisms, i.e., static modes, leading to a much higher dependence of growth rates on the stress intensity.

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