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Some notes on using the homotopy perturbation method for solving time-dependent differential equations

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ABSTRACT

Although attempts have been made to solve time-dependent differential equations using homotopy perturbation method (HPM), none of the researchers have provided a universal homotopy equation. In this paper, going one step forward, we intend to make some guidelines for beginners who want to use the homotopy perturbation technique for solving their equations. These guidelines are based on the L part of the homotopy equation and the initial guess. Afterwards, for solving time-dependent differential equations, we suggest a universal L and v_0 in the homotopy equation. Examples assuring the efficiency and convenience of the suggested homotopy equation are comparatively presented.

1. Introduction

In recent years, the homotopy perturbation method (HPM), first proposed by Dr. Ji Huan He [1,2], has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields.

Dr. He used HPM to solve Lighthill equation [1], Duffing equation [3] and Blasius equation [4], and then the idea found its way in sciences and has been used to solve nonlinear wave equations [5], boundary value problems [6,7], quadratic Riccati differential equations [8], integral equations [9–11], Klein–Gordon and sine–Gordon equations [12,13], initial value problems [14,15], Schrödinger equation [16], Emden–Fowler type equations [17], nonlinear evolution equations [18], differential-difference equations [19], modified KdV equation [20] and many other problems. This wide variety of applications show the power of HPM in solving functional equations (although we know it has limitations, that will be mentioned later on).

Studying the method, we understand that the idea is straightforward, but everyone has solved his/her own problem, heuristically, using some tricks. Although this shows the flexibility of the method, a beginner confronts problems using it . In this paper, we intend to somehow generalize the idea and make some guidelines. These guidelines may have been discovered by other researchers but no one has presented them as general guidelines. After this general discussion, we restrict ourselves to the case of time-dependent differential equations and suggest a quite simple technique for using HPM. Comparatively speaking, even though we don't claim that the suggested technique is the best one, this is a reliable technique which one can simply use without having much experience and understanding of HPM.

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2. Basic ideas of homotopy perturbation method

For a good understanding of the homotopy perturbation method, the reader is referred to Dr. He's works [1,2], where more developments could be found in [21,22]. Also Liao's works [23,24] would be a good reference for this development, because this method is quite similar to the method proposed by Liao, known as homotopy analysis method (HAM).

To describe the basic ideas, consider the time-dependent differential equation in the following general form

$$A(y(r,t)) - f(r,t) = 0, \tag{1}$$

where A is a differential operator, $y(r,t)$ is an unknown function, r and t denote spatial and temporal independent variables, respectively, and $f(r,t)$ is a known analytic function. A, generally speaking, can be divided into two parts, L and N,

$$A = L + N, \tag{2}$$

where L is a simple part which is easy to handle and N contains the remaining parts of A. Using homotopy technique one can construct a homotopy $\phi(r,t;q)$ satisfying

$$H(\phi(r,t;q),q) = (1-q)\{L(\phi(r,t;q)) - L(v_0(r,t))\} + q\{A(\phi(r,t;q)) - f(r,t)\} = 0, \tag{3}$$

where $q \in [0,1]$ is an embedding parameter and $v_0(r,t)$ is an initial guess for Eq. (1), which satisfies initial/boundary condition(s). Eq. (3) is called *homotopy equation*. Equivalently it can be written as follows:

$$L(\phi(r,t;q)) - L(v_0(r,t)) + q\{N(\phi(r,t;q)) + L(v_0(r,t)) - f(r,t)\} = 0. \tag{4}$$

Clearly we have

$$q = 0 \Rightarrow H(\phi(r,t;0),0) = L(\phi(r,t;0)) - L(v_0(r,t)) = 0, \tag{5}$$

$$q = 1 \Rightarrow H(\phi(r,t;1),1) = A(\phi(r,t;1)) - f(r,t) = 0, \tag{6}$$

which the latter is actually Eq. (1) with solution $y(r,t)$. Eq. (5) has $v_0(r,t)$ as one of its solutions and in the case where L is assumed to be linear, $v_0(r,t)$ is the only solution. So we have

$$\phi(r,t;0) = v_0(r,t),$$
$$\phi(r,t;1) = y(r,t).$$

The changing process of q, from zero to unity, is just that of $\phi(r,t;q)$ from $v_0(r,t)$ to $y(r,t)$, this is called deformation. If the embedding parameter q ($0 \le q \le 1$) is considered as a "*small parameter*", applying the classic perturbation technique, we can naturally assume that the solution to Eqs. (5) and (6) can be given as a power series in q, i.e.

$$\phi(r,t;q) = u_0(r,t) + u_1(r,t)q + u_2(r,t)q^2 + \cdots. \tag{7}$$

Using (7) for $q = 1$, one has

$$y(r,t) = u_0(r,t) + u_1(r,t) + u_2(r,t) + \cdots \tag{8}$$

which is the approximate solution to Eq. (1) (see, e.g. [1,2]). In most cases the series (8) is a convergent one which leads to the exact solution of Eq. (1). One can take the closed form or truncate the series for obtaining approximate solutions. As this method is an iterative method, so the Banach's fixed point theorem can be applied for convergence study of the series (8). The interested reader can refer to [25].

3. Guidelines for choosing homotopy equation

In a homotopy equation, what we are mainly concerned about are the auxiliary operator L and the initial guess v_0. Once one chooses these parts, the homotopy equation is completely determined, because the remaining part is actually the original equation (see (4)) and we have less freedom to change it. Here we discuss some general rules that should be noted in choosing L and v_0.

3.1. *Discussion on* L

According to the steps of the homotopy perturbation procedure, L should be:

(i) "*Easy to handle*".

We mean that it must be chosen in such a way that one has no difficulty in subsequently solving systems of resulting equations. It should be noted that this condition doesn't restrict L to be linear. In some cases, as was done by He in [1] to solve the Lighthill equation, a nonlinear choice of L may be more suitable. But, it's strongly recommended for beginners to take a linear operator as L.

(ii) "*Closely related to the original equation*".

Strictly speaking, in constructing L, it's better to use some part of the original equation. We can see the effectiveness of this view in [17] where Chowdhury and Hashim have gained very good results with technically choosing the L part (later on we will show that they could choose other operators as L).

3.2. Discussion on initial guess

There is no universal technique for choosing the initial guess in iterative methods, but from previous works done on HPM and our own experiences, we can conclude the following facts:

(i) "*It should be obtained from the original equation*".

For example, it can be chosen to be the solution to some part of the original equation, or it can be chosen from initial/boundary conditions.

(ii) "*It should reduce complexity of the resulting equations*".

Although this condition only can be checked after solving some of the first few equations of the resulting system, this is the criteria that has been used by many authors when they encountered different choices as an initial guess.

4. The classic view on HPM

As done by many authors, in order to obtain a good approximation, one has to test different choices of L and v_0 and then choose the most suitable of all. This is the classic approach in using HPM and there is no general rule to choose L and v_0. Here we present two comparative examples to review this classic view.

Example 4.1. Consider the time-dependent Emden–Fowler equation

$$y_{xx} + \frac{2}{x}y_x - (6 + 4x^2 - \cos(t))y = y_t,$$

with the initial condition $y(x, 0) = e^{x^2}$, the boundary conditions $y(0, t) = e^{\sin(t)}$ and $y_x(0, t) = 0$. We solve this equation via HPM using different L and v_0.

(i) We choose $L\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{2}{x}\frac{\partial\phi}{\partial x}$ and $v_0(x, t) = y(0, t) = e^{\sin(t)}$, as used by Chowdhury and Hashim [17], this gives the homotopy equation as follows:

$$\phi_{xx} + \frac{2}{x}\phi_x - v_{0xx} - \frac{2}{x}v_{0x} + q\left\{v_{0xx} + \frac{2}{x}v_{0x} - (6 + 4x^2 - \cos(t))\phi - \phi_t\right\} = 0.$$

Using (7), then equating the terms with identical powers of q, we have the following system of equations

$$\begin{aligned}
&u_{0xx} + \frac{2}{x}u_{0x} - v_{0xx} - \frac{2}{x}v_{0x} = 0, && u_0(0, t) = e^{\sin(t)}, u_{0x}(0, t) = 0,\\
&u_{1xx} + \frac{2}{x}u_{1x} - v_{0xx} - \frac{2}{x}v_{0x} - (6 + 4x^2 - \cos(t))u_0 - u_{0t} = 0, && u_1(0, t) = 0, u_{1x}(0, t) = 0,\\
&u_{2xx} + \frac{2}{x}u_{2x} - (6 + 4x^2 - \cos(t))u_1 - u_{1t} = 0, && u_2(0, t) = 0, u_{2x}(0, t) = 0,\\
&\vdots && \vdots
\end{aligned}$$

Subsequently solving the above equations we have

$$\begin{aligned}
&u_0(x, t) = e^{\sin(t)},\\
&u_1(x, t) = e^{\sin(t)}\left\{x^2 + \frac{1}{5}x^4\right\},\\
&u_2(x, t) = e^{\sin(t)}\left\{\frac{3}{10}x^4 + \frac{13}{105}x^6 + \frac{1}{90}x^8\right\},\\
&u_3(x, t) = e^{\sin(t)}\left\{\frac{3}{70}x^6 + \frac{17}{360}x^8 + \frac{59}{11\,550}x^{10} + \frac{1}{3510}x^{12}\right\},\\
&\vdots
\end{aligned}$$

Finally, the approximate solution in a series form, according to (8) is

$$y(x, t) = e^{\sin(t)}\left\{1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right\},$$

leading to the closed form $y(x, t) = e^{\sin(t)+x^2}$, which is the exact solution.

(ii) We choose $L\phi = \frac{2}{x}\frac{\partial\phi}{\partial x}$ and $v_0(x, t) = 0$.

So we have the homotopy equation as follows:

$$\frac{2}{x}\phi_x + q\{\phi_{xx} - (6 + 4x^2 - \cos(t))\phi - \phi_t\} = 0.$$

Applying (7), then equating the terms with identical powers of q, we have the following system of equations

$$\begin{array}{ll} \frac{2}{x}u_{0x} = 0, & u_0(0,t) = e^{\sin(t)}, \\ \frac{2}{x}u_{1x} - (6+4x^2-\cos(t))u_0 - u_{0t} = 0, & u_1(0,t) = 0, \\ \frac{2}{x}u_{2x} - (6+4x^2-\cos(t))u_1 - u_{1t} = 0, & u_2(0,t) = 0, \\ \vdots & \vdots \end{array}$$

which yields to

$$\begin{aligned} u_0(x,t) &= e^{\sin(t)}, \\ u_1(x,t) &= e^{\sin(t)}\left\{\frac{3}{2}x^2 + \frac{1}{2}x^4\right\}, \\ u_2(x,t) &= e^{\sin(t)}\left\{\frac{-3}{4}x^2 + \frac{3}{8}x^4 + \frac{9}{12}x^6 + \frac{1}{8}x^8\right\}, \\ u_3(x,t) &= e^{\sin(t)}\left\{\frac{3}{8}x^2 - \frac{9}{8}x^4 - \frac{93}{48}x^6 - \frac{1}{16}x^8 + \frac{15}{80}x^{10} + \frac{1}{48}x^{12}\right\}, \\ &\vdots \end{aligned}$$

Employing (8) we have

$$y(x,t) = e^{\sin(t)}\left\{1 + 3\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right)x^2 + \left(\frac{1}{2} + \frac{3}{8} - \frac{9}{8} + \cdots\right)x^4 + \cdots\right\},$$

which yields $y(x,t) = e^{\sin(t)}\{1 + x^2 + \frac{x^4}{2!} + \cdots\}$, leading to the closed form $y(x,t) = e^{\sin(t)+x^2}$, which is also the exact solution.

(iii) Our choice is $L\phi = \frac{\partial\phi}{\partial t}$ and $v_0(x,t) = 0$.
We have the homotopy equation as follows:

$$\phi_t + q\left\{\phi_{xx} - \frac{2}{x}\phi_x + (6+4x^2-\cos(t))\phi\right\} = 0.$$

Using (7) and then equating the terms with identical powers of q, and then solving the resulting equations one has

$$\begin{aligned} u_0(x,t) &= e^{x^2}, \\ u_1(x,t) &= \sin(t)e^{x^2}, \\ u_2(x,t) &= \frac{1}{2}\sin^2(t)e^{x^2}, \\ u_2(x,t) &= \frac{1}{6}\sin^3(t)e^{x^2}, \\ &\vdots \end{aligned}$$

Again employing (8) the approximate solution is

$$y(x,t) = e^{x^2}\left\{1 + \sin(t) + \frac{\sin^2(t)}{2!} + \frac{\sin^3(t)}{3!} + \cdots\right\},$$

leading to the closed form $y(x,t) = e^{\sin(t)+x^2}$, which is the exact solution.

Example 4.2. Consider the Cauchy reaction-diffusion equation

$$y_t = y_{xx} - y,$$

with the initial condition $y(x,0) = e^{-x} + x$, the boundary conditions $y(0,t) = 1$ and $y_x(0,t) = e^{-t} - 1$. We solve this equation via HPM using different L and v_0.

(i) Our choice is $L\phi = \frac{\partial\phi}{\partial t}$ and $v_0(x,t) = 0$.
So we have the following homotopy equation

$$\phi_t + q\{-\phi_{xx} + \phi\} = 0.$$

Using (7), then equating the terms with identical powers of q, and then solving the resulting system of equations one has

$$u_0(x,t) = e^{-x} + x,$$
$$u_1(x,t) = -xt,$$
$$u_2(x,t) = x\frac{t^2}{2!},$$
$$u_3(x,t) = -x\frac{t^3}{3!},$$
$$\vdots$$

So we have the approximate solution

$$y(x,t) = e^{-x} + x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right),$$

yielding the closed form $y(x,t) = e^{-x} + xe^{-t}$, which is the exact solution.

(ii) We choose $L\phi = \frac{\partial\phi}{\partial t}$ and $v_0(x,t) = y(x,0) = e^{-x} + x$, note that here we set our initial guess to be the initial condition of the equation. In this case, we obtain the same terms as in (i).

(iii) We choose $L\phi = \frac{-\partial^2\phi}{\partial x^2}$ and $v_0(x,t) = 0$.
So we have the homotopy equation

$$-\phi_{xx} + q\{\phi_t + \phi\} = 0.$$

Using (7), then equating the terms with identical powers of q, and then subsequently solving the resulting equations we have

$$u_0(x,t) = 1 + x(e^{-t} - 1),$$
$$u_1(x,t) = \frac{x^2}{2!} - \frac{x^3}{3!},$$
$$u_2(x,t) = \frac{x^4}{4!} - \frac{x^5}{5!},$$
$$\vdots$$

Employing (8) leads to the exact solution

$$y(x,t) = xe^{-t} + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right) = xe^{-t} + e^{-x}.$$

(iv) We choose $L\phi = \frac{-\partial^2\phi}{\partial x^2}$ and $v_0(x,t) = y(0,t) + xy_x(0,t) = 1 + x(e^{-t} - 1)$, here we used the boundary conditions of the problem to construct an initial guess (see. e.g, [26]).
In this case we come exactly to the terms obtained in (iii).

These comparative examples show the efficiency and flexibility of HPM in solving equations. We should note that HPM has some limitations, for example when the equation under study contains terms like $\ln(y)$, $\sin(y)$, e^y, $\cosh(y)$, ... where y is the unknown function. In such cases, we have to use Taylor's expansion or some other approximations before applying HPM, so it needs some tricks to obtain an analytic or approximate solution (see, e.g. [13,27,28]).

5. Proposed choices for L and v_0

In time-dependent differential equations we propose to choose the L part using the highest order of derivative with respect to (only) t, i.e.

$$L\phi = \frac{\partial\phi}{\partial t}, \qquad L\phi = \frac{\partial^2\phi}{\partial t^2}, \ldots,$$

depending on the appearance of them in the equation under consideration.
Also, we suggest to put

$$v_0 = 0, \quad \text{i.e. the zero function.}$$

In time-dependent differential equations there always exist derivatives with respect to time which have simple forms like $y_t, y_{tt}, \ldots$ (not depending on the spatial variables of the equation). So according to discussion in Section 3.1, the L part can be chosen as $L\phi = \frac{\partial\phi}{\partial t}$ or $L\phi = \frac{\partial^2\phi}{\partial t^2}, \ldots$.

When we set the initial guess (v_0) as the *zero function*, applying (7), our first equation would be, e.g.

$$u_{0t} = 0 \quad \text{with the initial condition } u_0(x,0) = y(x,0).$$

This equation always has $u_0(x,t) = y(x,0)$ as its solution. Since $u_0(x,t)$ is the first term of our approximation, (8), so in this way we automatically allow the initial conditions of the problem to play their role and efficiently take part in construction of the solution. As an example, consider Cauchy reaction-diffusion equation, Example 4.2(i). If we choose the initial guess to be $v_0(x,t) = y(0,t) + y_x(0,t)$, as proposed in [26], we can't solve the resulting equations easily, so the HPM fails to solve the equation. Therefore, choosing $v_0 = 0$, in an indirect manner we use the rules suggested in Section 3.2. Besides, in this way, we escape from extra terms which may possibly destroy our analytic approximation (this comment is quite heuristic). In [3] Dr. He eliminates the secular terms heuristically and Liao in [24] gets rid of them by employing the *rule of coefficient ergodicity*. Our choice prevents their appearance.

6. Examples

6.1. Evolution equations

Ganji et al. [18] have solved three examples of evolution equations by HPM using $L\phi = \frac{\partial\phi}{\partial t}$ and, taking the initial condition as the initial guess. We present here these three examples and solve them by choosing $L\phi = \frac{\partial\phi}{\partial t}$ and $v_0(x,t) = 0$ as the initial guess.

Example 6.1.1. Consider the equation $y_t - y_{xxt} + (\frac{y^2}{2})_x = 0$ with the initial condition $y(x,0) = x$. We have the homotopy equation

$$\phi_t + q\left\{-\phi_{xxt} + \left(\frac{\phi^2}{2}\right)_x\right\} = 0.$$

Employing (7) and then equating the terms with identical powers of q, then solving the resulting equations one has

$$\begin{aligned} u_0(x,t) &= x,\\ u_1(x,t) &= -xt,\\ u_2(x,t) &= xt^2,\\ u_3(x,t) &= -xt^3,\\ &\vdots \end{aligned}$$

So we have the approximate solution as follows:

$$y(x,t) = x(1 - t + t^2 - t^3 + \cdots) = \frac{x}{1+t},$$

which has less complexity in comparison with the solution given in [18].

Example 6.1.2. Consider the equation $y_t + y_x = 2y_{xxt}$ with the initial condition $y(x,0) = \mathrm{e}^{-x}$. We have the following homotopy equation

$$\phi_t + q\{\phi_x - 2\phi_{xxt}\} = 0.$$

Using (7) and then equating the terms with identical powers of q, then solving the resulting system of equations one has

$$\begin{aligned} u_0(x,t) &= t\mathrm{e}^{-x},\\ u_1(x,t) &= \left(\frac{t^2}{2} + 2t\right)\mathrm{e}^{-x},\\ u_2(x,t) &= \left(\frac{t^3}{3!} + 2t^2 + 4t\right)\mathrm{e}^{-x},\\ u_3(x,t) &= \left(\frac{t^4}{4!} + t^3 + 6t^2 + 8t\right)\mathrm{e}^{-x},\\ &\vdots \end{aligned}$$

So the 7-term approximation is

$$\left(1 + 63\,t + \frac{129}{2}\,t^2 + \frac{37}{2}\,t^3 + \frac{49}{24}\,t^4 + \frac{11}{120}\,t^5 + \frac{1}{720}\,t^6\right)\mathrm{e}^{-x}$$

which is the same approximation obtained via variational iteration method (VIM) and HPM in [18].

Example 6.1.3. Consider the equation $y_t - y_{xxxx} = 0$ with the initial condition $y(x, 0) = \sin(x)$. We have the following homotopy equation

$$\phi_t + q\{-\phi_{xxxx}\} = 0.$$

Using (7), then equating the terms with identical powers of q, then solving the resulting system of equations one can see

$$\begin{aligned} u_0(x,t) &= \sin(x), \\ u_1(x,t) &= -t\sin(x), \\ u_2(x,t) &= \frac{t^2}{2!}\sin(x), \\ u_3(x,t) &= -\frac{t^3}{3!}\sin(x), \\ &\vdots \end{aligned}$$

So we have the approximate solution as follows:

$$y(x,t) = \sin(x)\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = \frac{\sin(x)}{1+t},$$

which is the exact solution and it is much easier to guess the closed form than that of [18].

6.2. Cauchy reaction-diffusion equations

Cauchy reaction-diffusion equations have been solved by various methods [26]. In Example 4.2, we solved one equation of this type. Here we present some other examples of this family and compare our approximations with solutions obtained via HAM in [26].

Example 6.2.1. We consider the equation $y_t = y_{xx} + 2ty$ with the initial condition $y(x, 0) = e^x$. We have the following homotopy equation

$$\phi_t + q\{-\phi_{xx} - 2t\phi\} = 0.$$

Applying (7) and then equating the terms with identical powers of q, then solving the resulting system of equations we have

$$\begin{aligned} u_0(x,t) &= e^x, \\ u_1(x,t) &= (t+t^2)e^x, \\ u_2(x,t) &= \frac{(t+t^2)^2}{2!}e^x, \\ u_3(x,t) &= \frac{(t+t^2)^3}{3!}e^x, \\ &\vdots \end{aligned}$$

So we have the approximate solution as follows:

$$y(x,t) = e^x\left(1 + t + t^2 + \frac{(t+t^2)^2}{2!} + \frac{(t+t^2)^3}{3!} + \cdots\right),$$

yielding the closed form $y(x,t) = e^{x+t+t^2}$, which is the exact solution as obtained via HAM in [26].

Example 6.2.2. We consider the equation $y_t = y_{xx} - (1 + 4x^2)y$ with the initial condition $y(x, 0) = e^{x^2}$. We have the following homotopy equation

$$\phi_t + q\{-\phi_{xx} + (1 + 4x^2)\phi\} = 0.$$

Using (7) and then equating the terms with identical powers of q, then solving the resulting system of equations one can see

$$\begin{aligned} u_0(x,t) &= e^{x^2}, \\ u_1(x,t) &= e^{x^2}t, \\ u_2(x,t) &= e^{x^2}\frac{t^2}{2!}, \end{aligned}$$

$$u_3(x,t) = e^{x^2}\frac{t^3}{3!},$$
$$\vdots$$

So we have the approximate solution as follows:

$$y(x,t) = e^{x^2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),$$

yielding the closed form $y(x,t) = e^{x^2+t}$, which is the exact solution as obtained via HAM in [26].

Example 6.2.3. Consider the equation $y_t = y_{xx} - (4x^2 - 2t + 2)y$ with the initial condition $y(x,0) = e^{x^2}$. We have the following homotopy equation

$$\phi_t + q\{-\phi_{xx} + (4x^2 - 2t + 2)\phi\} = 0.$$

Using (7) and then equating the terms with identical powers of q, then solving the resulting system of equations we have

$$u_0(x,t) = e^{x^2},$$
$$u_1(x,t) = e^{x^2}t^2,$$
$$u_2(x,t) = e^{x^2}\frac{t^4}{2!},$$
$$u_3(x,t) = e^{x^2}\frac{t^6}{3!},$$
$$\vdots$$

So we have the approximate solution as follows:

$$y(x,t) = e^{x^2}\left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots\right),$$

yielding the closed form $y(x,t) = e^{x^2+t^2}$, which is the exact solution as obtained via HAM in [26].

6.3. Emden–Fowler type equations

Chowdhury and Hashim have solved the time-dependent Emden–Fowler type equations via HPM in [17]. Also Sami Bataineh et al. [27] solved this type of equations using HAM. We presented one example of this family in Example 4.1. Here, comparatively, we give three more examples in question.

Example 6.3.1. Consider the equation $y_{xx} + \frac{2}{x}y_x - (5+4x^2)y = y_t + (6-5x^2-4x^4)$ with the initial condition $y(x,0) = x^2 + e^{x^2}$. Applying our method we have the following homotopy equation

$$\phi_t + q\left\{-\phi_{xx} - \frac{2}{x}\phi_x + (5 + 4x^2)\phi + (6 - 5x^2 - 4x^4)\right\} = 0.$$

Applying (7) and then equating the terms with identical powers of q, then solving the resulting equations we have

$$u_0(x,t) = x^2 + e^{x^2},$$
$$u_1(x,t) = e^{x^2}t,$$
$$u_2(x,t) = e^{x^2}\frac{t^2}{2!},$$
$$u_2(x,t) = e^{x^2}\frac{t^3}{3!},$$
$$\vdots$$

So according to (8), the approximate solution is

$$y(x,t) = x^2 + e^{x^2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),$$

leading to the closed form $y(x,t) = x^2 + e^{t+x^2}$, which is the exact solution and has less complexity in its terms in comparison with [17].

Example 6.3.2. Consider the equation $y_{xx} + \frac{2}{x}y_x - (5+4x^2)y = y_{tt} + (12x - 5x^3 - 4x^5)$ with the initial condition $y(x,0) = x^3 + e^{x^2}$. Applying our method we have the following homotopy equation

$$\phi_{tt} + q\left\{-\phi_{xx} - \frac{2}{x}\phi_x + (5+4x^2)\phi + (12 - 5x^3 - 4x^5)\right\} = 0.$$

Using (7) and then equating the terms with identical powers of q, then solving the resulting equations we have

$$u_0(x,t) = x^3 + (1-t)e^{x^2},$$
$$u_1(x,t) = e^{x^2}\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right),$$
$$u_2(x,t) = e^{x^2}\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right),$$
$$\vdots$$

So according to (8), the approximate solution is

$$y(x,t) = x^3 + e^{x^2}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots\right),$$

leading to the closed form $y(x,t) = x^3 + e^{x^2-t}$, which is also the exact solution. Our choice, in comparison with [17], has less complexity to guess the closed form.

Example 6.3.3. Consider the equation $y_{xx} + \frac{4}{x}y_x - (18+9x^4)y = y_{tt} - 2 - (18x + 9x^4)t^2$ with the initial conditions $y(x,0) = e^{x^3}$ and $y_t(x,0) = 0$.
In this example we use the *modified* HPM, proposed by Odibat in [29], so we have the *modified* homotopy equation as follows:

$$\phi_{tt} + q\left\{-\phi_{xx} - \frac{4}{x}\phi_x + (18x+9x^4)\phi - (18x+9x^4)t^2\right\} = 2.$$

Applying (7) and then equating the terms with identical powers of q, then solving the resulting equations we have

$$u_0(x,t) = t^2 + e^{x^3},$$
$$u_1(x,t) = 0,$$
$$u_2(x,t) = 0,$$
$$u_3(x,t) = 0,$$
$$\vdots$$

Employing (8), we simply have the exact solution

$$y(x,t) = t^2 + e^{x^3}.$$

6.4. Klein–Gordon equations

Klein–Gordon equations have been solved using HPM by Chowdhury and Hashim in [13], where they used a technical initial guess and obtained very good approximations. Also Odibat and Momani in [12] have solved this problems via HPM. Here, we test the efficiency of our choice on this family.

Example 6.4.1. We consider the linear Klein–Gordon equation $y_{tt} - y_{xx} = y$ with the initial conditions $y(x,0) = 1 + \sin(x)$ and $y_t(x,0) = 0$.
Our choice leads to the following homotopy equation

$$\phi_{tt} + q\{-\phi_{xx} - \phi\} = 0.$$

Using (7), then equating the terms with identical powers of q, then solving the resulting equations, we have

$$u_0(x,t) = 1+\sin(x),$$
$$u_1(x,t) = \frac{t^2}{2!},$$
$$u_2(x,t) = \frac{t^4}{4!},$$
$$u_3(x,t) = \frac{t^6}{6!},$$
$$\vdots$$

So according to (8), the approximate solution is

$$y(x,t) = \sin(x) + \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right),$$

which yields the closed form $y(x,t) = \sin(x) + \cosh(t)$. So we have the same terms as obtained in [13].

Example 6.4.2. We consider the nonlinear nonhomogeneous Klein–Gordon equation $y_{tt} - y_{xx} - 2y = -2\sin(x)\sin(t)$ with the initial conditions $y(x,0) = 0$ and $y_t(x,0) = \sin(x)$. Our choice leads to the following homotopy equation

$$\phi_{tt} + q\{-\phi_{xx} - 2\phi + 2\sin(x)\sin(t)\} = 0.$$

Applying (7), equating the terms with identical powers of q, then solving the resulting equations one can see

$$u_0(x,t) = t\sin(x),$$
$$u_1(x,t) = \sin(x)\left\{2\sin(t) - 2t + \frac{t^3}{6}\right\},$$
$$u_2(x,t) = \sin(x)\left\{-2\sin(t) + 2t - \frac{t^3}{3} + \frac{t^5}{120}\right\},$$
$$u_3(x,t) = \sin(x)\left\{+2\sin(t) - 2t + \frac{t^3}{3} - \frac{t^5}{60} + \frac{t^7}{7!}\right\},$$
$$u_4(x,t) = \sin(x)\left\{-2\sin(t) + 2t - \frac{t^3}{3} + \frac{t^5}{60} - \frac{t^7}{2\times 6!} + \frac{t^9}{9!}\right\},$$
$$\vdots$$

Using (8), for obtaining an approximate solution, some terms that appear in u_{2i-1} are canceled out with the terms appearing in u_{2i}, then an approximate solution is

$$y(x,t) = \sin(x)\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right\},$$

which implies the closed form $y(x,t) = \sin(x)\sin(t)$.
Chowdhury and Hashim in [13] have obtained the exact solution of the aforementioned equation in the first iteration. Our choice also gives a convergent series and this is worthy of note because we use a *general* approach.

Example 6.4.3. We consider the nonlinear nonhomogeneous Klein–Gordon equation $y_{tt} - y_{xx} + y^2 = -x\cos(t) + x^2\cos^2(t)$ with the initial conditions $y(x,0) = x$ and $y_t(x,0) = 0$, which has the exact solution $y(x,t) = x\cos(t)$. Our analytic approximation seems complicated but still converges to the exact solution.The 4-term approximation is

$$\begin{aligned}
y_{app4}(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)\\
&= \cos(t)\left\{\frac{1}{32}t^2\cos^2(t) + \frac{5}{192}t^4 - \frac{33}{256}\cos^2(t) - \frac{23}{256}t^2 + \frac{1}{256}\cos^4(t)\right.\\
&\quad \left. + \frac{1}{8} - \frac{7}{1440}t^6 - \frac{1}{16}t\cos(t)\sin(t)\right\}x^4\\
&\quad + \cos(t)\left\{\frac{-64}{9} + 2t\sin(t) + \frac{43}{6}\cos(t) + 2t^2 - \frac{1}{6}t^4 - \frac{1}{2}t^2\cos(t) - \frac{1}{18}\cos^3(t)\right\}x^3
\end{aligned}$$

Table 1
Absolute errors of a 4-term approximation of Example 6.4.3.

x_i	t_i	$\lvert y - y_{app4}\rvert$
0.1	0.1	3.749×10^{-12}
0.2	0.2	1.128×10^{-9}
0.3	0.3	3.175×10^{-8}
0.4	0.4	3.229×10^{-7}
0.5	0.5	1.781×10^{-6}

Table 2
Absolute errors of a 4-term approximation of Example 6.4.4.

x_i	t_i	$\lvert y - y_{app4}\rvert$
0.1	0.1	1.247×10^{-16}
0.2	0.2	2.044×10^{-12}
0.3	0.3	5.968×10^{-10}
0.4	0.4	3.349×10^{-8}
0.5	0.5	7.615×10^{-7}

$$
\begin{aligned}
&+ \cos(t)\left\{\frac{1}{4}\cos^2(t) + \frac{1}{4}t^2 - \frac{1}{12}t^4 - \frac{1}{4} + \cos(t) + \frac{1}{90}t^6\right\}x \\
&+ \cos(t)\left\{\frac{1}{8}\cos^2(t) - \frac{15}{8}t^2 - 4\cos(t) + \frac{1}{8}t^4 + \frac{31}{8}\right\}.
\end{aligned}
$$

In Table 1, we have computed the absolute errors for this approximation at some points which shows efficiency of our choice.

It seems that the error increases by increasing t and x. It is because we have used only 4 terms in our approximation. So it isn't a big problem. For obtaining more accurate results (with smaller error values) one should use more terms in his/her approximation.

Example 6.4.4. Consider the nonlinear nonhomogeneous Klein–Gordon equation $y_{tt} - y_{xx} + y^2 = 6xt(x^2 - t^2) + x^6t^6$ with initial conditions $y(x, 0) = 0$ and $y_t(x, 0) = 0$, which has the exact solution $y(x, t) = x^3t^3$. Our choice in this example yields a 4-term approximation as follows:

$$
\begin{aligned}
y_{app4} &= u_0 + u_1 + u_2 + u_3 \\
&= x^3t^3 + \frac{53}{4200}x^4t^{10} - \frac{13}{92\,400}x^2t^{12} - \frac{1}{959\,616}x^{12}t^{18} + \frac{1}{19\,600}x^7t^{15} - \frac{1}{4368}x^9t^{13}.
\end{aligned}
$$

In Table 2, we have computed the absolute errors for this approximation at some points which shows efficiency of our choice.

Again, we can use more terms in our approximation to get more accurate results for larger values of t and x. Moreover we should indicate that our choice in computing the error in points (x, t) with equal values of x and t is arbitrary and one can get very close results by choosing other (not equal) values.
Here, it is worth noting that the initial guess proposed by Chowdhury and Hashim in [13] may not be an efficient choice when used for other types of problems. For example when their initial guess is applied to time-dependent Emden–Fowler type equations [27], it yields a divergent series.

7. Conclusions

In this paper, we proposed some guidelines for beginners who intend to solve their problems using the homotopy perturbation method. In the sequel we comparatively reviewed procedures which are used by researchers, through two examples. Then we presented a simple way to choose L and v_0 when we use the homotopy perturbation method to solve time-dependent differential equations. In most cases, our simple choice yields exact an solution or at least very good approximations. Although there are examples that show our choice isn't as good as other choices, it still produces convergent series that makes it a reliable one in solving a wide class of functional equations.

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