Safety Science 82 (2016) 18–28

Contents lists available at ScienceDirect

Safety Science

journal homepage: www.elsevier.com/locate/ssci

# Aggregation of group fuzzy risk information in the railway risk decision making process

Min An [a,b,*], Yong Qin [b], Li Min Jia [b], Yao Chen [a]

[a] School of Civil Engineering, University of Birmingham, Birmingham B15 2TT, UK
[b] State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, China

ARTICLE INFO

*Article history:*
Received 9 January 2015
Received in revised form 10 August 2015
Accepted 21 August 2015

*Keywords:*
Railway risk assessment
Decision making
Expert and engineering judgement
Multiplicative preference relation
Fuzzy multiplicative consistency

ABSTRACT

Railway risk assessment is a hierarchical process where risk information obtained at lower levels may be used for risk assessment at higher levels. Fuzzy analytical hierarchy process (FAHP) is widely used in risk decision making process to solve imprecise hierarchical problems where the risk data are incomplete or there is a high level of uncertainty involved in the risk data, particularly, in the process of railway safety and risk decision making. However, the application of FAHP in risk decision making the risk analysts often face the circumstances where a large number of pairwise comparison matrices have to be established by expert knowledge and engineering judgements. There may be a lack of confidence that all comparisons associated with a railway system are completely justified in a rigorous way. This is particularly true when a complex railway system needs to be analysed or when subjective judgements should be involved. This paper presents a modified FAHP approach that employs fuzzy multiplicative consistency method for the establishment of pairwise comparison matrices in risk decision making analysis. The use of the proposed method yields a higher level of confidence that all of comparisons associated with the system are justified. In the meanwhile, the workload in determining the consistency of the judgements can be reduced significantly. A case example is used to demonstrate the proposed methodology. The results indicate that by using the proposed method, risks associated with a railway system can be assessed effectively and efficiently, and more reliable and accurate results can be obtained.


## 1. Introduction

Comparison with road transportation, railways are by far one of the safest means of ground transportation, especially for their passengers and employees. But comparison with airspace, there are some issues involved in both maintaining this position in reality and sustaining the public perception of railway safety excellence. The railway now finds itself in a situation where actual and perceived safeties are real issues, to be dealt with in a new public culture of rapid change, short-term pressures, and instant communications. The principal risks in the railway industry appear to be to people and property as a result of collision, derailment and fire. The concepts of design and construction for railway safety are introduced in the standards of EN 50126(1, 2, 3) (BS EN 50126-1, 1999; BS EN 50126-2, 2007; BS EN 50126-3, 2006), EN 50128 (BS EN 50128, 2009), and EN 50129 (BS EN 50129, 2003), which are widely applied to manage and control risks in the design and construction of railway systems. However, there are many possible causes, in operation and maintenance of vehicles and rail infrastructure, and also from outside the railway such as vandalism and road incidents. Specifically, in the modification and maintenance of plain line, the largest incidences are of derailments and vehicles fouling infrastructure such as station platforms. There are many chains of potential causes, and each involves several disciplines and work-groups. Incorporating safety aspects into the railway management and maintenance process can increase the level of safety (An et al., 2011, 2007; Bojadziev and Bojacziev, 1997; Chiclana et al., 2001). This shows the need for increased awareness and better safety management.

Railway risk analysis is to increase the level of safety to safeguard their assets, customers and employees while improving safety and reducing railway asset maintenance cost and environmental impacts. Any risk information produced from risk estimation phase may be used through the risk response phase to assist risk analysts, engineers and managers to make maintenance and future investment decision purposes. If risks are high, risk reduction measures must be applied or the maintenance work has to

* Corresponding author at: School of Civil Engineering, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK. Tel.: +44 121 414 5146; fax: +44 121 414 3698.

*E-mail address:* m.an@bham.ac.uk (M. An).

http://dx.doi.org/10.1016/j.ssci.2015.08.011
0925-7535/

be considered to reduce the occurrence probabilities or to control the possible consequences. If risks are negligible, no actions are required but the information produced needs to be recorded for audit purpose. However, the acceptable and unacceptable regions are usually divided by a transition region. Risks that fall in this transition region need to be reduced to as low as reasonably practicable (ALARP) (An et al., 2008, 2006; Railway Safety and Standard Board, 2007).

The purpose of railway risk assessment is to determine the risk likelihood and magnitude to assist with the decision-making. As discussed above, if risks are high, risk reduction measures must be applied or the maintenance work has to be considered to reduce the occurrence probabilities or to control the possible consequences. Many of the railway risk analysis techniques currently used are comparatively mature tools (An et al., 2007, 2006; Chen et al., 2007; Chen and An, 2011; Huang et al., 2007). The results of using these tools highly rely on the availability and accuracy of the risk data (An et al., 2011, 2007, 2008, 2006; Bojadziev and Bojacziev, 1997; Chiclana et al., 2001; Huang et al., 2007). However, railway risk analysts often face the circumstances where the risk data are incomplete or there is a high level of uncertainty involved in the risk data. This requires the involvement of expert knowledge and engineering judgement in the risk analysis process. Additionally, railway risk analysis is also a hierarchical process where risk information obtain at lower levels may be used for risk assessment at higher levels. In many circumstances, it may be extremely difficult to conduct probabilistic risk assessment to assess the occurrence of likelihood of hazards and the magnitudes of their possible consequences because of the uncertainty with risk data (An et al., 2006). Therefore, it is essential to develop new risk analysis methods to identify major hazards and assess the associated risks in an acceptable way in various environments where such mature tools cannot be effectively or efficiently applied (An et al., 2011, 2007, 2008, 2006; Bojadziev and Bojacziev, 1997; Chiclana et al., 2001; Railway Safety and Standard Board, 2007). The railway safety problem is appropriate for examination by fuzzy reasoning approach (FRA) combined with fuzzy analytical hierarchy process (FAHP). FRA method provides a useful tool for modelling risks and other risk parameters for risk analysis that involves the risks with incomplete or redundant safety information (An et al., 2011, 2007, 2008, 2006). The FRA allows imprecision or approximate information in risk assessment process (Berredo et al., 2005; Buckley, 1985; Bojadziev and Bojacziev, 1997; Chen et al., 2006; Dubois and Prade, 1980; BS EN 50126-2, 2007; Laarhoven and Pedrycz, 1983). However, because the contribution of each risk factor to the safety of a railway system is different, the weight of the contribution of each risk factor should be taken into consideration in order to represent its relative contribution to the risk level of the railway system. A FAHP technique is therefore required to be incorporated into the risk analysis to use its advantage in determining the relative importance of the risk factors so that the risk assessment can be progressed from hazardous event level to the identified hazard group level and finally to a railway system level. The FAHP is a very useful technique that has been applied in many fields of, for example, design and maintenance planning, reliability analysis, selecting a best alternative and resource allocations, etc. (An et al., 2011; Bojadziev and Bojacziev, 1997; Ekel et al., 2006; Fan et al., 2006; Gu and Zhu, 2006; Herrera et al., 2001; Herrera-Viedma et al., 2004; Leung and Cao, 2000). An advantage of the FAHP is its flexibility to be integrated with different techniques (An et al., 2006; Satty, 1980, 1994; Wang and Fan, 2007; Wang and Chen, 2008; Xu, 2004). The application of FAHP may solve the problems of risk information loss in the hierarchical process so that risk assessment can be carried out from hazardous event level to a railway system level (An et al., 2011, 2007, 2008, 2006). Both of these processes result in a set of probability distributions, which can be used not only to predict risk levels but also to design safety maintenance intervals. The use of these techniques is especially appropriate in the railway environment because of the volume of experience, which is still available from long-term employees. In order to show compliance with safety targets and to make future investment decisions, a railway risk assessment support system using FRA and FAHP has been developed. Details of fundamentals of a railway risk assessment support methodology will not be presented due to space constraints and the reader is referred to An et al. (2011, 2007) for details. This paper will focus on aggregation of group fuzzy risk information in the railway risk decision making process, particularly, to develop a methodology on the basis of fuzzy preference relations to deal with the consistency of the comparisons.

The application of FAHP in a risk decision making process is to determine the fuzzy priorities to produce weights of the contribution of risk factors to the safety of a railway system by conducting pairwise comparisons produced from a safety management team. In other words, it is based on preference relations, which the judgements are based on the expert experience and engineering knowledge to provide some degree of preference of any risk factor over another. However, when applying FAHP, the risk analysts often face the circumstances where a huge number of pairwise comparison matrices have to be established. Even if it is single pairwise comparison matrix, it still requires $n(n-1)/2$ judgements at a certain level with $n$ risk factors. With the number of risk factors increasing, the numbers of comparisons are increased rapidly. As a result, the judgements produced from safety management team will high likely become inconsistent. Therefore, consistency tests are required to avoid the misleading solutions. If a comparison matrix fails the consistency test, the risk analysts must request the safety management team to make the judgements again until the comparison matrix passes. However, perfect consistency is difficult to obtain in practice (Herrera et al., 2001; Herrera-Viedma et al., 2004), particularly, when measuring preferences on a set with a large number of risk factors. Consequently, the lack of consistency in decision making can lead to inconsistent conclusions. Additionally, the judgements are crisp values which are fuzzy numbers. The inconsistent crisp numbers may be far greater. Therefore, this work would be laborious and highly unrealistic. The literature review carried out by the authors indicates that some methods have been developed in the literature to use consistency tests to avoid inconsistency in risk analysis (An et al., 2006; Berredo et al., 2005; Buckley, 1985; Chen et al., 2007, 2006; Chen and An, 2011), but, however, these methods are very complex, particular, when the number of risk factors increases. It is particular by true when:

- thousands of risk factors (or hazard/failure modes) are identified within a system;
- the system consists of hundreds of sub-systems and components; and
- expert experience and engineering knowledge are involved in decision making process.

Therefore, numbers of comparisons will be increased rapidly with the numbers of the identified risk factors/sub-systems/components increased. There may be a lack of confidence that all comparisons associated with a railway system are completely justified in a rigorous way. Furthermore, too much workload is required in determining the consistency of the judgements.

To solve the above problems, a modified FAHP methodology has been proposed. In this method, the comparison matrix is established by using the additive transitivity property and consistency so that only $n-1$ comparison judgements are required at a level with $n$ risk factors, which a comparison matrix can be established

**Table 1**
FAHP estimation scheme.

| Qualitative descriptors | Description | Parameters of MFs (triangular) |
|---|---|---|
| Equal importance (EQ) | Two risk contributors contribute equally | (1, 1, 2) |
| Between equal and weak importance (BEW) | When compromise is needed | (1, 2, 3) |
| Weak importance (WI) | Experience and judgment slightly favour one risk contributor over another | (2, 3, 4) |
| Between weak and strong importance (BWS) | When compromise is needed | (3, 4, 5) |
| Strong importance (SI) | Experience and judgment strongly favour one risk contributor over another | (4, 5, 6) |
| Between strong and very strong importance (BSV) | When compromise is needed | (5, 6, 7) |
| Very strong importance (VI) | A risk contributor is favoured very strongly over the other | (6, 7, 8) |
| Between very strong and absolute importance (BVA) | When compromise is needed | (7, 8, 9) |
| Absolute importance (AI) | The evidence favouring one risk contributor over another is of the highest possible order of affirmation | (8, 9, 9) |

directly on the basis of such judgements by using multiplicative preference relation and then it is transformed into fuzzy preference relation to produce a fuzzy preference relation comparison matrix that can be used to calculate weights of the contribution of risk factors.

This paper presents the recent development of a modified FAHP method to improve the consistency of FAHP on the basis of multiplicative preference relations with trapezoidal fuzzy numbers in risk decision making process. The principle and algorithm of the proposed method are discussed in this paper, and two propositions are proposed, and proofed and validated, which can be used to calculate values of fuzzy reciprocal multiplicative preference matrix. A transformation function is then developed to transfer TranFNs preference relation decision matrix to fuzzy preference relation comparison matrix. The proposed method can be used to avoid the misleading conclusions to ensure the consistency of judgements and provide more reliable and accurate results. A case example of shunting at a railway depot is used to demonstrate the proposed methodology.

## 2. Application of FAHP in risk decision making

As stated earlier in Section 1, because the contribution of each risk factor to the overall risk level (RL) of a railway system is different, the weight of the contribution of each risk factor should be taken into consideration in order to represent its relative contribution to the RL of a railway system. Railway risk assessment is a hierarchical process where risk information obtained at lower levels may be used for risk assessment at higher levels. FAHP is widely used in risk decision making process to solve imprecise hierarchical problems where the risk data are incomplete or there is a high level of uncertainty involved in the risk data. The application of FAHP may also solve the problems of risk information loss in the hierarchical process in determining the relative importance of risk factors in the decision making process so that risk assessment can be progressed from hazardous event level to hazard group level, and finally to a railway system level. A FAHP is an important extension of the traditional AHP method (Satty, 1980, 1994; Wang and Chen, 2008), which uses a similar framework of AHP to conduct risk analysis but fuzzy ratios of relative importance replace crisp ratios to the existence of uncertainty in the risk assessment (An et al., 2011, 2006; Huang et al., 2007; Laarhoven and Pedrycz, 1983; Leung and Cao, 2000; Wang and Chen, 2008). An advantage of the FAHP is its flexibility to be integrated with different techniques. Therefore, a FAHP analysis leads to the generation of weight factors (WFs) for representing the primary hazardous events within each category. There are six steps to calculate WFs as described below (An et al., 2011, 2007; Buckley, 1985; Chen et al., 2007; Chen and An, 2011).

### 2.1. Step 1: Establish an estimation scheme

FAHP determines WFs by conducting pairwise comparison. The comparison is based on an estimation scheme, which lists intensity of importance using qualitative descriptors. Each qualitative descriptor has a corresponding triangular membership function (MF) that is employed to transfer expert knowledge and engineering judgments into a comparison matrix (An et al., 2011, 2007; Buckley, 1985; Chen et al., 2006; Laarhoven and Pedrycz, 1983). Table 1 describes qualitative descriptors and their corresponding triangular fuzzy numbers for railway risk analysis. Each grade is described by an important expression and a general intensity number. When two risk contributors are of equal importance, it is considered (1, 1, 2). Fuzzy number of (8, 9, 9) describes that one risk contributor is absolutely important than the other one. Fig. 1 shows triangular MFs (solid lines) with "equal importance" – (1, 1, 2), "weak importance" – (2, 3, 4), "strong importance" – (4, 5, 6), "very strong importance" – (6, 7, 8) and "absolute importance" – (8, 9, 9), respectively. The other triangular MFs (dash lines) describe the corresponding intermediate descriptors between them.

### 2.2. Step 2: Compare risk contributors

Suppose $n$ risk contributors, there are a total of $N = n(n-1)/2$ pairs need to be compared. For example, Fig. 2 illustrates a typical hierarchical risk assessment process of a railway depot system (An et al., 2011). "$HG_1$", "$HG_2$", ..., and "$HG_n$" are the risk contributors that contribute to overall RL of a railway depot system. Assume two risk contributors $HG_1$ and $HG_2$, if $HG_1$ is of very strong importance than $HG_2$, a fuzzy number of (6, 7, 8) is then assigned to $HG_1$ based on the estimation scheme as shown in Table 1. Obviously, risk contributor $HG_2$ has fuzzy number of (1/8, 1/7, 1/6). As the expert experience and engineering knowledge are often expressed in nature of language that describe the risks associated with a railway system. FAHP allows fuzzy ratios of relative importance to be used to the existence of uncertainty in risk assessment. The following classifications can be used in the comparison.

- A numerical value, e.g. "3"
- A linguistic term, e.g. "strong importance".
- A range, e.g. (2, 4), the scale is likely between 2 and 4.
- A fuzzy number, e.g. (2, 3, 4), the scale is between 2 and 4, most likely 3 or (2, 3, 4, 5), the scale is between 2 and 5, most likely between 4 and 5.
- 0, e.g. the two risk contributors cannot be compared at all.

### 2.3. Step 3: Convert comparison pairwise into UFNs

As described in steps 1 and 2, because the values of risk contributors are crisps, e.g. a numerical value, a range of numerical value,

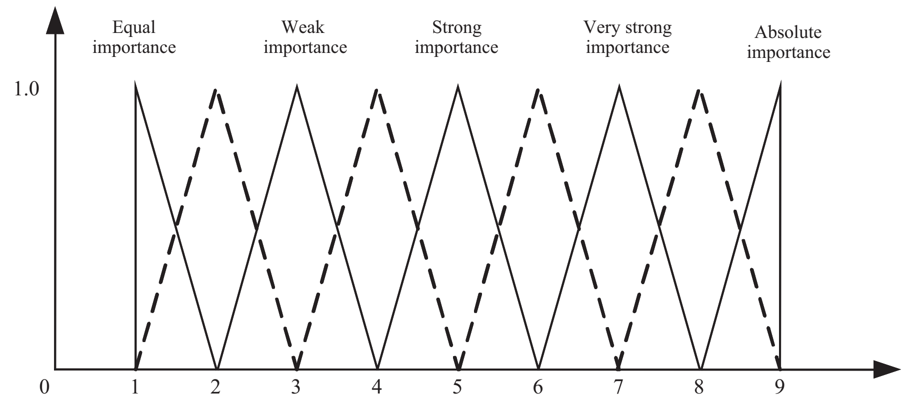


**Fig. 1.** MFs of the triangular fuzzy numbers.

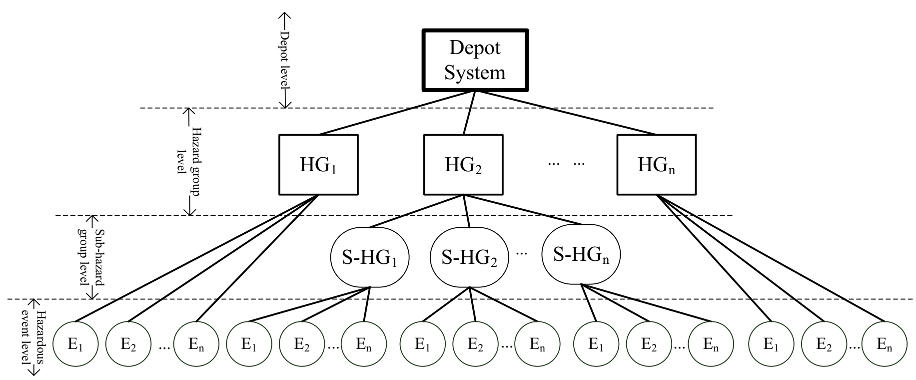


**Fig. 2.** A typical hierarchical risk assessment process.

a linguistic term or a fuzzy number, the uniform format number (UFN) is introduced to capture and convert expert and engineering subjective judgements for the composition of a final decision (An et al., 2011, 2007; Huang et al., 2007). An UFN can be defined as $A = \{a, b, c, d\}$, and its corresponding membership functions (MFs) indicates the degree of preference, which is defined as

$$\mu_A(x) = \begin{cases} (x-a)/(b-a), & x \in [a,b], \\ 1 & x \in [b,c] \\ (x-d)/(c-d), & x \in [c,d], \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

where four real numbers ($a$, $b$, $c$, and $d$) with satisfaction of the relationship $a \leqslant b \leqslant c \leqslant d$ determine the $x$-coordinates of the four corners of a trapezoidal MF. It should be noted that a numerical value, a range of numerical values, a fuzzy number and a linguistic term can be converted as a simplified UFN. Table 2 shows the possible inputs and its corresponding UFNs. A series of UFNs can be obtained to correspond to the scores and the scales of the defined risk contributors.

### 2.4. Step 4: Aggregate UFNs

Usually, there are a number of experts in the safety management team, their judgements may be different. Therefore, UFNs produced in Step 3 need to be aggregated into a group UFN for each risk constructor. The process is same as described in Section 2.3 at Step 3.

**Table 2**
Expert judgement and the corresponding UFNs.

| Description | Input values | Input type | UFNs |
|---|---|---|---|
| "...is *a*" | $\{a\}$ | A numerical value | $\{a, a, a, a\}$ |
| "...is between *a* and *b*" | $\{a, b\}$ | A range of number | $\{a, (a+b)/2, (a+b)/2, a\}$ |
| "...is between *a* and *c* and most likely to be *b*" | $\{a, b, c\}$ | Triangular fuzzy numbers | $\{a, b, b, a\}$ |
| "...is between *a* and *d* and most likely between *b* and *c*" | $\{a, b, c, d\}$ | Trapezoidal fuzzy numbers | $\{a, b, c, d\}$ |
| "...is *RARE*" | RARE | A linguistic term | RARE MF $\{a, b, c, d\}$ |

### 2.5. Step 5: Construct the fuzzy comparison matrix M

The aggregated UFN are then used to construct a comparison matrix. As shown in Fig. 2, suppose $E_1, E_2, \ldots, E_n$ are identified hazardous events in a hazard group $\text{HG}_n$, $m_{i,j}$ is the aggregated UFN representing the quantified judgement on $E_i$ comparing to $E_j$ and $E_i$ is more important than $E_j$. The pairwise comparison between $E_i$ and $E_j$ in the hazard group $\text{HG}_n$ thus yields a $n \times n$ matrix defined as

$$M = [m_{i,j}] = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \ldots & m_{n,n} \end{bmatrix} \quad i,j = 1, 2, \ldots, n \quad (2)$$

$m_{j,i} = 1/m_{i,j} = (1/a_{i,j}, 1/b_{i,j}, 1/c_{i,j}, 1/d_{i,j})$

where $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$ are the numbers of UFN $m_{ij}$.

### 2.6. Step 6: Calculate fuzzy weight

The WFs can be calculated by using geometric mean method (An et al., 2011, 2007; Saaty, 1994; Wang and Fan, 2007; Wang and Chen, 2008; Xu, 2004). The UFN geometric mean $\overline{M}_i$ of the $i$th row in the comparison matrix is defined as

$$\bar{M}_i = \{\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i\} = \left\{ \sqrt[n]{\prod_{j=1}^{n} a_{i,j}}, \sqrt[n]{\prod_{j=1}^{n} b_{i,j}}, \sqrt[n]{\prod_{j=1}^{n} c_{i,j}}, \sqrt[n]{\prod_{j=1}^{n} d_{i,j}} \right\} \tag{3}$$

$$W_i = \{a_i, b_i, c_i, d_i\} = \left\{ \frac{\bar{a}_i}{\sum_{j=1}^{n} \bar{d}_j}, \frac{\bar{b}_i}{\sum_{j=1}^{n} \bar{c}_j}, \frac{\bar{c}_i}{\sum_{j=1}^{n} \bar{b}_j}, \frac{\bar{d}_i}{\sum_{j=1}^{n} \bar{a}_j} \right\} \tag{4}$$

where $W_i$ is the fuzzy-WF of $E_i$.

### 2.7. Step 7: Defuzzification and normalisation

Because the outputs of geometric mean method are fuzzy WFs, a defuzzification is adopted to convert fuzzy WFs to the corresponding crisp WF (An et al., 2011; Saaty, 1994). Suppose a fuzzy WF of $w'_i$

$$w'_i = \frac{a_i + 2(b_i + c_i) + d_i}{6} \tag{5}$$

The WF of $E_i$ can be calculated by

$$WF_{E_i} = \frac{w'_i}{\sum_{i=1}^{n} w'_i} \tag{6}$$

### 2.8. Step 8: Calculate RLs of sub-hazard groups

Once the WFs of risk contributors are obtained, the overall RLs of sub-hazard groups can be calculated by synthesising of WF and RL of each hazardous event. As shown in Fig. 2, the RL of a sub-hazard group $S-HG_i$ is defined by An et al. (2011, 2007)

$$RL_{S-HG_i} = \sum_{i=1}^{n} RL_{E_i} WF_{E_i} \quad i = 1, 2, \ldots, n \tag{7}$$

where $RL_{E_i}$ and $WF_{E_i}$ are the RL and WF of $E_i$.

Similarly, $WF_{HG_i}$ of hazard groups can be obtained by repeating steps 1–7. The RLs of hazard groups and the overall RL of a railway system can be obtained by An et al. (2011, 2007, 2008, 2006)

$$RL_{HG_i} = \sum_{i=1}^{n} RL_{S-HG_i} WF_{S-HG_i} \quad i = 1, 2, \ldots, n \tag{8}$$

$$RL_{System} = \sum_{i=1}^{n} RL_{HG_i} WF_{HG_i} \quad i = 1, 2, \ldots, n \tag{9}$$

where $RL_{HG_i}$ and $WF_{HG_i}$ are the RL and WF of the $i$th hazard group $GH_i$, $RL_{S-HG_i}$ and $WF_{S-HG_i}$ are the RL and WF of the $i$th sub-hazard group and $RL_{System}$ is overall RL of the system.

As can be seen that the application of FAHP in risk decision making process, As described earlier in this paper, $n(n-1)/2$ judgements in a FAHP process need to be made in order to establish a comparison matrix with $n$ events. When hundreds of hazard events are identified and the system consists of a large number of sub-systems and components, in this case, they are hazardous events, sub-hazard groups and hazard groups, a large number of pairwise comparison matrices need to be established. There may be a lack of consistency test in the process. As a result, the judgements produced from safety management team may likely become inconsistent. There may also be a lack of confidence that all comparisons associated with a railway system are completely justified in a rigorous way. Therefore, fuzzy multiplicative consistency method may be needed (Herrera et al., 2001; Herrera-Viedma et al., 2004; Leung and Cao, 2000; Xu, 2004).

## 3. Consistent multiplicative preference relation

Multiplicative preference relations provide risk analysts with values presenting varying degrees of preference for first risk contributor over the second one (BS EN 50129, 2003). For a set of the identified hazard events at a particular level of a risk hierarchical decision making process, suppose a set of the identified hazard events is $H = \{E_1, E_2, \ldots, E_n\}$ and $n \geqslant 2$, which an expert in safety management team associates to every pair of events a value that reflects some degree of preference of the first event over the second one. Preference relation may be expressed either in multiplicative preference relation $M$ or in fuzzy preference relation $P$. Their definitions are discussed as follows (Chen and An, 2011; Herrera-Viedma et al., 2004).

A multiplicative preference relation $M$ on a set of hazard events $H$ is presented by a matrix $M \subset H \times H, M = (m_{i,j})$ where $m_{i,j}$ is interpreted as the preference intensity of two hazard events $E_i$ and $E_j$, i.e. it is interpreted as $E_i$ is $m_{i,j}$ times as important as $E_j$. The measurement of $m_{i,j}$ uses a ratio scale defined between 1 and 9 (Satty, 1980, 1994) as shown in Table 1, where $m_{i,j} = 1$ indicates the absence of a difference between $E_i$ and $E_j$, and $m_{i,j} = 9$ represents that $E_i$ is absolutely important than $E_j$. In this case, the preference relation matrix $M$ is usually assumed to be multiplicative reciprocal, i.e. $m_{i,j} \times m_{j,i} = 1, i,j \in \{1, 2, \ldots, n\}$. Therefore, if a comparison matrix $M$ is consistency, it has to satisfy $m_{i,j} \times m_{j,k} = m_{i,k}$ (Herrera et al., 2001; Herrera-Viedma et al., 2004).

**Definition 1.** A reciprocal multiplicative preference relation $M = (m_{i,j})$ is consistent if and only if $m_{i,j} \times m_{j,k} = m_{i,k}, i, j, k \in \{1, 2, \ldots, n\}$ and $i \leqslant j \leqslant k$.

Definition 1 clearly states that for checking the consistency of multiplicative preference relation it is only necessary to check the values of $m_{i,j}, m_{j,k}$ and $m_{i,k}$. As a consequence of this equivalent condition, consistent multiplicative preference relations can be constructed from a set of $n-1$ preference intensities (Chen and An, 2011; Herrera et al., 2001; Herrera-Viedma et al., 2004).

**Definition 2.** For a reciprocal multiplicative preference relation $M = (m_{i,j})$ $i \leqslant j \leqslant k$, the following statements are equivalent (Chen and An, 2011; Herrera et al., 2001; Herrera-Viedma et al., 2004):

$$m_{i,j} \times m_{j,k} = m_{i,k}, i \leqslant j \leqslant k \tag{10}$$

$$m_{i,j} = m_{i,i+1} \times m_{i+1,i+2} \times \cdots \times m_{j-1,j}, \quad i < j \tag{11}$$

Definition 2 indicates that a consistent multiplicative preference relation can be constructed from a set of $n-1$ preference data, i.e. $\{m_{1,2}, m_{2,3}, \ldots, m_{n-1,n}\}$ (Chen and An, 2011; Herrera et al., 2001; Herrera-Viedma et al., 2004; Satty, 1980). A pairwise comparison matrix with entries in the interval $[1/v, v], v > 0$ can then be established and the entries can be transformed into the interval $[1/9, 9]$ using a transformation function, i.e.

$$f : \left[\frac{1}{v}, v\right] \to \left[\frac{1}{9}, 9\right], \quad f(x) = x^{1/\log_9 v} \tag{12}$$

This transformation function will be discussed in Section 4. As described earlier in the paper, the experts in safety management team may have vague knowledge about the preference degree of one event over another, and cannot estimate their preferences with exact numerical values. It is more suitable to provide their preferences by means of linguistic variables rather than numerical ones. The disadvantage of the method as described above is that the values in consistent multiplicative preference relation matrix are crisp, which cannot capture imprecise judgements. In this study, therefore, trapezoidal fuzzy numbers should be introduced to develop transformation functions.

## 4. Fuzzy multiplicative consistency method

The application of fuzzy reasoning approach (FRA) in risk analysis may have the following advantages (An et al., 2011, 2007, 2008, 2006; Dubois and Prade, 1980; Ekel et al., 2006; Fan et al., 2006; Giachetti and Young, 1997; Gu and Zhu, 2006):

- the risk can be evaluated directly by using qualitative descriptions;
- it tolerant of imprecise data and ambiguous information; and
- it gives a more flexible structure for combining qualitative as well as quantitative information.

FRA focuses on qualitative descriptors in natural language and aims to provide fundamental approximate reasoning with imprecise propositions. A fuzzy number is defined on the universe $U$ as a convex and normalised fuzzy set (Buckley, 1985; Bojadziev and Bojacziev, 1997; Saaty, 1994; Wang and Fan, 2007; Wang and Chen, 2008; Xu, 2004), which can be converted into, for example, triangular fuzzy number, trapezoidal fuzzy number and bell-shaped fuzzy number, etc. However, trapezoidal fuzzy numbers are most widely used in the railway risk analysis because of their intuitive appeal and their perceived computational efficacy (An et al., 2006; Giachetti and Young, 1997).

### 4.1. Trapezoidal fuzzy numbers

A trapezoidal fuzzy number (TranFN) can be defined as $A = (a, b, c, d)$ and its corresponding fuzzy set $\tilde{A}$ is defined by Eq. (1) as described in Section 2.3. Fig. 3(A) shows a trapezoidal membership function (MF). If $b = c$, the fuzzy number becomes a triangular fuzzy number as shown in Fig. 3(B). A non-fuzzy number $A$ can then be expressed as $(a, a, a, a)$.

Suppose that two TranFNs are $B = (a_B, b_B, c_B, d_B)$ and $C = (a_C, b_C, c_C, d_C)$, the aggregation operators are defined by Chen et al. (2007, 2006), Chen and An (2011), Dubois and Prade (1980) and Gu and Zhu (2006)

$$B \oplus C = (a_B + a_C, b_B + b_C, c_B + c_C, d_B + d_C) \tag{13}$$

$$B \otimes C = (a_B \times a_C, b_B \times b_C, c_B \times c_C, d_B \times d_C) \tag{14}$$

$$B \text{Ø} C = (a_B/a_C, b_B/b_C, c_B/c_C, d_B/d_C) \tag{15}$$

$$B^k = ((a_B)^k, (b_B)^k, (c_B)^k, (d_B)^k), k > 0 \tag{16}$$

where $\oplus$ denotes fuzzy addition, $\otimes$ denotes fuzzy multiplication and Ø stands for fuzzy division.

### 4.2. Consistent fuzzy multiplicative preference relation

In this study, a fuzzy multiplicative consistency method is proposed in order to deal with inconsistency when constructing a comparison matrix in risk decision making process. In this method, the multiplicative preference relation matrix $M = (m_{i,j}) = (a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j})$ is constructed based on consistent multiplicative preference relation as described in Section 3.

The consistency of a fuzzy reciprocal matrix is defined as (Buckley, 1985):

**Definition 3.** A fuzzy matrix $M = (m_{i,j})$ is reciprocal if and only if $m_{i,j} = m_{j,i}^{-1}$.

**Definition 4.** A fuzzy matrix $M = (m_{i,j})$ is consistent if and only if $m_{i,j} \otimes m_{j,k} = m_{i,k}$.

On the basis of Definitions 3 and 4, the following two propositions are proposed in this study, which are proofed and validated. The Proposition 1 is used to calculate values of fuzzy reciprocal multiplicative preference matrix, and then Proposition 2 is applied to provide fundamentals in support this development. A transformation function $f : [1/v, v] \to [1/9, 9]$ is also developed to transfer TranFNs in the interval $[1/v, v]$ to $[1/9, 9]$. These are described as below.

**Proposition 1.** *Suppose that a set of events $H = (E_1, E_2, \ldots, E_n)$ associated with a fuzzy reciprocal multiplicative preference matrix $M = (m_{i,j}) = (a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j})$ with $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \in [1/9, 9]$, the following statements are equivalent:*

$$a_{i,j} = d_{j,i}^{-1}, \quad i,j \in (1, 2, \ldots, n) \tag{17}$$

$$b_{i,j} = c_{j,i}^{-1}, \quad i,j \in (1, 2, \ldots, n) \tag{18}$$

$$c_{i,j} = b_{j,i}^{-1}, \quad i,j \in (1, 2, \ldots, n) \tag{19}$$

$$d_{i,j} = a_{j,i}^{-1}, \quad i,j \in (1, 2, \ldots, n) \tag{20}$$

**Proof.** On the basis of Definition 3, $M = (m_{i,j})$ is a reciprocal fuzzy multiplicative preference matrix, i.e. $m_{i,j} = m_{j,i}^{-1} = 1\phi m_{j,i}$, $i,j \in (1, 2, \ldots, n)$.

By using Eq. (15)

$$m_{i,j} = (1, 1, 1, 1)\phi(a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i}), \quad i,j \in (1, 2, \ldots, n)$$

$$(a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}) = (1/d_{j,i}, 1/c_{j,i}, 1/b_{j,i}, 1/a_{j,i}) = (d_{j,i}^{-1}, c_{j,i}^{-1}, b_{j,i}^{-1}, a_{j,i}^{-1})$$

Therefore,

$$a_{i,j} = d_{j,i}^{-1}, b_{j,i} = c_{j,i}^{-1}, c_{i,j} = b_{j,i}^{-1}, d_{i,j} = a_{j,i}^{-1}, \quad i,j \in (1, 2, \ldots, n) \qquad \square$$

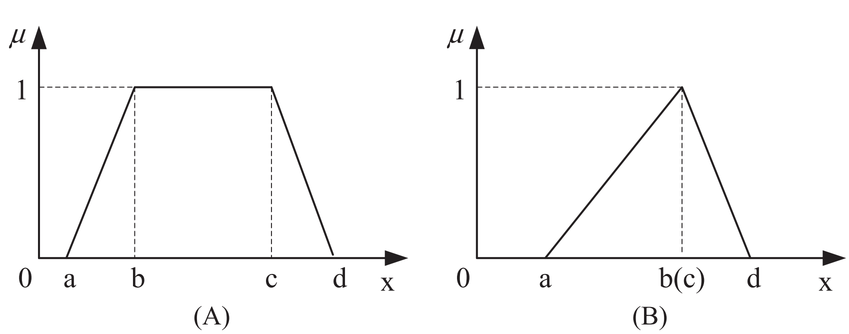


**Fig. 3.** Trapezoidal and triangular fuzzy numbers and corresponding MFs.

**Proposition 2.** *For a reciprocal fuzzy multiplicative preference relation $M = (m_{i,j}) = (a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j})$ to be consistent, the following statements are equivalent:*

$$a_{i,k} = a_{i,j} \times a_{j,k}, \quad i < j < k \tag{21}$$

$$b_{i,k} = b_{i,j} \times b_{j,k}, \quad i < j < k \tag{22}$$

$$c_{i,k} = c_{i,j} \times c_{j,k}, \quad i < j < k \tag{23}$$

$$d_{i,k} = d_{i,j} \times d_{j,k}, \quad i < j < k \tag{24}$$

$$a_{i,j} = a_{i,(i+1)} \times a_{(i+1),(i+2)} \times \cdots \times a_{(j-1),j}, \quad i < j \tag{25}$$

$$b_{i,j} = b_{i,(i+1)} \times b_{(i+1),(i+2)} \times \cdots \times b_{(j-1),j}, \quad i < j \tag{26}$$

$$c_{i,j} = a_{i,(i+1)} \times c_{(i+1),(i+2)} \times \cdots \times c_{(j-1),j}, \quad i < j \tag{27}$$

$$d_{i,j} = d_{i,(i+1)} \times d_{(i+1),(i+2)} \times \cdots \times d_{(j-1),j}, \quad i < j \tag{28}$$

**Proof.** On the basis of Definition 4, $M = (m_{i,j})$ is consistent then $m_{i,j} \otimes m_{j,k} = m_{i,k}$.

By using Eq. (14)

$$\begin{aligned} m_{i,j} \otimes m_{j,k} &= (a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}) \otimes (a_{j,k}, b_{j,k}, c_{j,k}, d_{j,k}) \\ &= (a_{i,j} \times a_{j,k}, b_{i,j} \times b_{j,k}, c_{i,j} \times c_{j,k}, d_{i,j} \times d_{j,k}) \\ &= (a_{i,k}, b_{i,k}, c_{i,k}, d_{i,k}) = m_{i,k} \end{aligned}$$

Therefore,

$$a_{i,k} = a_{i,j} \times a_{j,k}, \quad b_{i,k} = b_{i,j} \times b_{j,k}, \quad c_{i,k} = c_{i,j} \times c_{j,k},$$
$$d_{i,k} = d_{i,j} \times d_{j,k}, \quad i < j < k$$

The above expressions (21)–(24) are obtained and verified.

If $i < j$ and $k = j - i$, the expression (25) can be rewritten as

$$a_{i,j} = a_{i,(i+1)} \times a_{(i+1),(i+2)} \times \cdots \times a_{(i+k-1),(i+k)}$$

Mathematical induction is applied to proof expression (25) and assumptions are made as

If $k = 1$, then $a_{i,j} = a_{i,(i+1)}$.

If $k = n$, by using expression (21), then

$$\begin{aligned} a_{i,j} &= a_{i,(i+1)} \times a_{(i+1),(i+2)} \times \cdots \times a_{(i+n-1),(i+n)} \\ &= a_{i,(i+2)} \times a_{(i+2),(i+3)} \times \cdots \times a_{(i+n-2),(i+n-1)} \times a_{(i+n-1),(i+n)} \\ &= a_{i,(i++n-1)} \times a_{(i+n-1),(i+n)} = a_{i,(i+n)} = a_{i,j} \end{aligned}$$

If $k = n + 1$, by using expression (21), then

$$\begin{aligned} a_{i,j} &= a_{i,(i+2)} \times a_{(i+2),(i+3)} \times \cdots \times a_{(i+n-1),(i+n)} \times a_{(i+n),(i+n+1)} \\ &= a_{i,(i+n)} \times a_{(i+n),(i+n+1)} = a_{i,(i+n+1)} = a_{i,j} \end{aligned}$$

The hypothesis is proved to be true when $k = 1, n$ and $n + 1$, which completes proof of the expression (25). Similarly, expressions (26)–(28) can be verified. □

Proposition 2 indicates that a TranFNs preference relation decision matrix can be established with $n - 1$ preference TranFNs in the interval $[1/v, v] (v > 0)$. Because such a TranFNs preference relation decision matrix is not in the interval $[1/9, 9]$, therefore, a transformation function $f : [1/v, v] \to [1/9, 9]$ is developed to transfer TranFNs in the interval $[1/v, v]$ to Fuzzy preference relation comparison matrix in the interval $[1/9, 9]$:

$$f(v) = 9 \tag{29}$$

$$f(1/v) = 1/9 \tag{30}$$

$$f(x_a) \cdot f(y_d) = 1, \quad x_a, y_d \in [1/v, v] \text{ such that } x_a \cdot y_d = 1 \tag{31}$$

$$f(x_b) \cdot f(y_c) = 1, \quad x_b, y_c \in [1/v, v] \text{ such that } x_b \cdot y_c = 1 \tag{32}$$

$$f(x_c) \cdot f(y_b) = 1, \quad x_c, y_b \in [1/v, v] \text{ such that } x_c \cdot y_b = 1 \tag{33}$$

$$f(x_d) \cdot f(y_a) = 1, \quad x_d, y_a \in [1/v, v] \text{ such that } x_d \cdot y_a = 1 \tag{34}$$

$$f(x_a) \cdot f(y_a) = f(z_a), \quad x_a, y_a, z_a \in [1/v, v] \text{ such that } x_a \cdot y_a = z_a \tag{35}$$

$$f(x_b) \cdot f(y_b) = f(z_b), \quad x_b, y_b, z_b \in [1/v, v] \text{ such that } x_b \cdot y_b = z_b \tag{36}$$

$$f(x_c) \cdot f(y_c) = f(z_c), \quad x_c, y_c, z_c \in [1/v, v] \text{ such that } x_c \cdot y_c = z_c \tag{37}$$

$$f(x_d) \cdot f(y_d) = f(z_d), \quad x_d, y_d, z_d \in [1/v, v] \text{ such that } x_d \cdot y_d = z_d \tag{38}$$

It is well-known that the general solution of verifying expressions (29) and (30) has the following format

$$\begin{aligned} f(x_a) &= x_a^e, \quad \text{being } e \in R \\ f(x_b) &= x_b^e, \quad \text{being } e \in R \\ f(x_c) &= x_c^e, \quad \text{being } e \in R \\ f(x_d) &= x_d^e, \quad \text{being } e \in R \end{aligned}$$

According to Eq. (12), by taking logarithms the above expressions (29) and (30) can be rewritten as

$$f(x_a) = x_a^{1/\log_9^v}, \quad f(x_b) = x_b^{1/\log_9^v}, \quad f(x_c) = x_c^{1/\log_9^v}, \quad f(x_d) = x_d^{1/\log_9^v}$$

When $x_a \cdot y_d = 1$, $x_b \cdot y_c = 1$, $x_c \cdot y_b = 1$, $x_d \cdot y_a = 1$ the above expressions (31)–(34) can be verified as follows:

$$f(x_a) \cdot f(y_d) = x_a^{1/\log_9^v} \cdot y_d^{1/\log_9^v} = (x_a \cdot y_d)^{1/\log_9^v} = 1^{1/\log_9^v} = 1$$

$$f(x_b) \cdot f(y_c) = x_b^{1/\log_9^v} \cdot y_c^{1/\log_9^v} = (x_b \cdot y_c)^{1/\log_9^v} = 1^{1/\log_9^v} = 1$$

$$f(x_c) \cdot f(y_b) = x_c^{1/\log_9^v} \cdot y_b^{1/\log_9^v} = (x_c \cdot y_b)^{1/\log_9^v} = 1^{1/\log_9^v} = 1$$

$$f(x_d) \cdot f(y_a) = x_d^{1/\log_9^v} \cdot y_a^{1/\log_9^v} = (x_d \cdot y_a)^{1/\log_9^v} = 1^{1/\log_9^v} = 1$$

When $x_a \cdot y_a = z_a$, $x_b \cdot y_b = z_b$, $x_c \cdot y_c = z_c$, $x_d \cdot y_d = z_d$, the above expressions (35)–(38) can be verified.

$$f(x_a) \cdot f(y_a) = x_a^{1/\log_9^v} \cdot y_a^{1/\log_9^v} = (x_a \cdot y_a)^{1/\log_9^v} = z_a^{1/\log_9^v} = f(z_a)$$

$$f(x_b) \cdot f(y_b) = x_b^{1/\log_9^v} \cdot y_b^{1/\log_9^v} = (x_b \cdot y_b)^{1/\log_9^v} = z_b^{1/\log_9^v} = f(z_b)$$

$$f(x_c) \cdot f(y_c) = x_c^{1/\log_9^v} \cdot y_c^{1/\log_9^v} = (x_c \cdot y_c)^{1/\log_9^v} = z_c^{1/\log_9^v} = f(z_c)$$

$$f(x_d) \cdot f(y_d) = x_d^{1/\log_9^v} \cdot y_d^{1/\log_9^v} = (x_d \cdot y_d)^{1/\log_9^v} = z_d^{1/\log_9^v} = f(z_d)$$

Therefore, the following steps can be used to construct a consistent fuzzy multiplicative preference relation $M$ for a set of hazard events $H = (E_1, E_2, \ldots, E_n)$ and $n \geqslant 2$ on the basis of $n - 1$ TranFNs $\{m_{1,2}, m_{2,3}, \ldots, m_{(n-1),n}\}$.

(1) Let $X$ is the preference values

$$X = \{m_{i,j}, i < j, m_{i,j} \in \{m_{1,2}, m_{2,3}, \ldots, m_{(n-1),n}\}$$
$$m_{i,j} = (a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}\}$$
$$\left.\begin{aligned} a_{i,j} &= a_{i,(i+1)} \times a_{(i+1),(i+2)} \times \cdots \times a_{(j-1),j} \\ b_{i,j} &= b_{i,(i+1)} \times b_{(i+1),(i+2)} \times \cdots \times b_{(j-1),j} \\ c_{i,j} &= c_{i,(i+1)} \times c_{(i+1),(i+2)} \times \cdots \times c_{(j-1),j} \\ d_{i,j} &= d_{i,(i+1)} \times d_{(i+1),(i+2)} \times \cdots \times d_{(j-1),j} \end{aligned}\right. \quad i < j$$

(2) Calculate $v = \max(d_{i,j}), d_{i,j} \in m_{i,j}, m_{i,j} \in X$

(3) let $M' = \{m_{i,j}, m_{2,3}, \ldots, m_{n-1,n}\} \cup X \cup \{m_{i,j}, m_{2,3}, \ldots, m_{n-1,n}\}^{-1} X^{-1}$

(4) Construct consistent fuzzy multiplicative preference relation $M$ by using transformation function $M = f(M')$, where $f : [\frac{1}{v}, v] \to [\frac{1}{9}, 9]$, i.e.

$$\begin{aligned} &f(x_a) = x_a^{1/\log_9 v}, \quad f(x_b) = x_b^{1/\log_9 v}, \\ &f(x_c) = x_c^{1/\log_9 v}, \quad f(x_d) = x_d^{1/\log_9 v} \end{aligned} \tag{39}$$

As can be seen that by using the modified FAHP method in risk decision making process, if a railway system has $n$ identified events, only $n - 1$ comparisons are required by using estimate scheme as described in Section 2, and the results produced from pariwise comparison can then be used to produce $m_{i,j}$ of the fuzzy preference relations matrix by using a transformation function $f : [1/v, v] \to [1/9, 9]$. The proposed methodology can be applied directly into the process of the FAHP as described in Section 2. The application of the modified PAHP method is demonstrated by a case study in Section 5.

## 5. A case study: risk assessment of shunting at Waterloo depot

A case study on risk assessment of shunting at Waterloo depot is used to demonstrate the application of the proposed methodology. Waterloo depot is the one of largest depots in London Underground. The historical data of accidents and incidents have been recorded over the past 10 years. In this case, the historical accident and incident databases have been reviewed in the Waterloo depot. Ten hazard groups have been identified and defined as 'Derailment' $A_1$, 'Collision' $A_2$, 'Train fire' $A_3$, 'Electrocution' $A_4$, 'Slips/trips' $A_5$, 'Falls from height' $A_6$, 'Train strikes person' $A_7$, 'Platform train interface' $A_8$, 'Structural failure' $A_9$ and 'Health hazard' $A_{10}$ as shown in Table 7. Each hazard group consists of a number of hazardous events, for example, 'Derailment' hazard group includes typical outcome (minor injury) and worst-case scenario (major injury) which have been identified based on the pervious accidents and incidents such as track related faults including mechanical failure of track e.g., broken rail and fishplates; signalling related faults including mechanical failure of signals and points; rolling stock faults including mechanical failure of rolling stock e.g., brakes, axles and bogies; structure failure including collapsed drain or civil structure beneath track leading to derailment; object from train including object falls from train (e.g. motor) leading to derailment; human errors including human error causing derailment e.g. overspeeding, incorrect routing, etc. The outputs of risk assessment are RLs of hazard groups and the overall RL of shunting at Waterloo depot with risk scores located from 0 to 10 and risk categorised as 'Low', 'Possible', 'Substantial' and 'High' with a percentage belief. The RLs of hazard groups are calculated using the fuzzy reasoning approach based on the aggregation results of each hazardous event belonging to the particular hazard group. Details of risk assessment at hazardous event level will not be presented due to space constraints and the reader is referred to An et al. (2011, 2007) for details. This paper focuses on the application of the modified FAHP method to obtain the overall RL of shunting at Waterloo depot based on the aggregation of the RLs of each hazard group contribution.

**Table 3**
Pairwise comparison of hazard groups with respect to their relative importance to overall depot system.

| Comparison | Triangular fuzzy numbers | TranFNs |
|---|---|---|
| $A_1$ vs. $A_2 (m_{12})$ | [1.00, 1.00, 2.00] | [1.00, 1.00, 1.00, 2.00] |
| $A_2$ vs. $A_3 (m_{23})$ | [3.00, 4.00, 5.00] | [3.00, 4.00, 4.00, 5.00] |
| $A_3$ vs. $A_4 (m_{34})$ | [1.00, 2.00, 3.00] | [1.00, 2.00, 2.00, 3.00] |
| $A_4$ vs. $A_5 (m_{45})$ | [3.00, 4.00, 5.00] | [3.00, 4.00, 4.00, 5.00] |
| $A_5$ vs. $A_6 (m_{56})$ | [1.00, 1.00, 2.00] | [1.00, 1.00, 1.00, 2.00] |
| $A_6$ vs. $A_7 (m_{67})$ | [0.20, 0.25, 0.33] | [0.20, 0.25, 0.25, 0.33] |
| $A_7$ vs. $A_8 (m_{78})$ | [1.00, 2.00, 3.00] | [1.00, 2.00, 2.00, 3.00] |
| $A_8$ vs. $A_9 (m_{89})$ | [0.25, 0.33, 0.50] | [0.25, 0.33, 0.33, 0.50] |
| $A_9$ vs. $A_{10} (m_{9,10})$ | [2.00, 3.00, 4.00] | [2.00, 3.00, 3.00, 4.00] |

**Table 4**
Completed TranFNs preference relation decision matrix.

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | 1.00, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 2.00 | 3.00, 4.00, 4.00, 10.00 | 3.00, 8.00, 8.00, 30.00 | 9.00, 32.00, 32.00, 150.00 | 9.00, 32.00, 32.00, 300.00 | 1.80, 8.00, 8.00, 99.00 | 1.80, 16.00, 16.00, 297.00 | 0.45, 5.28, 5.28, 148.50 | 0.90, 15.80, 15.80, 594.00 |
| $A_2$ | 0.50, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 1.00 | 3.00, 4.00, 4.00, 5.00 | 3.00, 8.00, 8.00, 15.00 | 9.00, 32.00, 32.00, 75.00 | 9.00, 32.00, 32.00, 150.00 | 1.80, 8.00, 8.00, 49.50 | 1.80, 16.00, 16.00, 148.50 | 0.45, 5.28, 5.28, 74.25 | 0.90, 15.80, 15.80, 297.00 |
| $A_3$ | 0.10, 0.25, 0.25, 3.3e−1 | 0.20, 0.25, 0.25, 0.33 | 1.00, 1.00, 1.00, 1.00 | 1.00, 2.00, 2.00, 3.00 | 3.00, 8.00, 8.00, 15.00 | 3.00, 8.00, 8.00, 30.00 | 0.60, 2.00, 2.00, 9.90 | 0.60, 4.00, 4.00, 29.70 | 0.15, 1.32, 1.32, 14.85 | 0.30, 3.96, 3.96, 59.40 |
| $A_4$ | 3.3e−2, 0.125, 0.125, 3.3e−1 | 6.7e−2, 0.13, 0.13, 0.33 | 0.33, 0.50, 0.50, 1.00 | 1.00, 1.00, 1.00, 1.00 | 3.00, 4.00, 4.00, 5.00 | 3.00, 4.00, 4.00, 10.00 | 0.60, 1.00, 1.00, 3.30 | 0.60, 2.00, 2.00, 9.90 | 0.15, 0.66, 0.66, 4.95 | 0.30, 1.98, 1.98, 19.80 |
| $A_5$ | 6.7e−3, 3.1e−2, 3.1e−2, 0.11 | 1.3e−2, 3.1e−2, 3.1e−2, 0.11 | 6.7e−2, 0.13, 0.13, 0.33 | 0.20, 0.25, 0.25, 0.33 | 1.00, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 2.00 | 0.20, 0.25, 0.25, 0.66 | 0.20, 0.50, 0.50, 1.98 | 0.05, 0.17, 0.17, 0.99 | 0.10, 0.50, 0.50, 3.96 |
| $A_6$ | 3.3e−3, 3.1e−2, 3.1e−2, 0.11 | 6.7e−3, 3.1e−2, 3.1e−2, 0.11 | 3.3e−2, 0.13, 0.13, 0.33 | 0.10, 0.25, 0.25, 0.33 | 0.50, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 1.00 | 0.20, 0.25, 0.25, 0.33 | 0.20, 0.50, 0.50, 0.99 | 0.05, 0.17, 0.17, 0.50 | 0.10, 0.50, 0.50, 1.98 |
| $A_7$ | 0.01, 0.13, 0.13, 0.56 | 0.02, 0.13, 0.13, 0.56 | 0.10, 0.50, 0.50, 1.67 | 0.30, 1.00, 1.00, 1.67 | 1.52, 4.00, 4.00, 5.00 | 3.03, 4.00, 4.00, 5.00 | 1.00, 1.00, 1.00, 1.00 | 1.00, 2.00, 2.00, 3.00 | 0.25, 0.66, 0.66, 1.50 | 0.50, 1.98, 1.98, 6.00 |
| $A_8$ | 3.4e−3, 6.3e−2, 6.3e−2, 0.56 | 6.7e−3, 6.3e−2, 6.3e−2, 0.56 | 3.4e−2, 0.25, 0.25, 1.67 | 0.10, 0.50, 0.50, 1.67 | 0.50, 2.00, 2.00, 5.00 | 1.01, 2.00, 2.00, 5.00 | 0.33, 0.50, 0.50, 1.00 | 1.00, 1.00, 1.00, 1.00 | 0.25, 0.33, 0.33, 0.50 | 0.50, 0.99, 0.99, 2.00 |
| $A_9$ | 6.7e−3, 0.19, 0.19, 2.22 | 1.3e−2, 0.19, 0.19, 2.22 | 6.7e−2, 0.76, 0.76, 6.67 | 0.20, 1.52, 1.52, 6.67 | 1.01, 6.06, 6.06, 20.00 | 2.02, 6.06, 6.06, 20.00 | 0.67, 1.52, 1.52, 4.00 | 2.00, 3.03, 3.03, 4.00 | 1.00, 1.00, 1.00, 1.00 | 2.00, 3.00, 3.00, 4.00 |
| $A_{10}$ | 1.7e−3, 6.3e−2, 6.3e−2, 1.11 | 3.4e−3, 6.3e−2, 6.3e−2, 1.11 | 1.7e−2, 0.25, 0.25, 3.33 | 5.1e−2, 5.1e−2, 5.1e−2, 3.33 | 0.25, 2.02, 2.02, 10.00 | 0.51, 2.02, 2.02, 10.00 | 0.17, 0.51, 0.51, 2.00 | 0.50, 1.01, 1.01, 2.00 | 0.25, 0.33, 0.33, 0.50 | 1.00, 1.00, 1.00, 1.00 |

**Table 5**
Fuzzy preference relation comparison matrix by using the modified FAHP.

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | 1.00, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 2.00 | 3.00, 4.00, 4.00, 5.00 | 4.33, 5.67, 5.67, 7.00 | 7.33, 8.33, 8.33, 9.00 | 8.00, 9.00, 9.00, 9.00 | 2.00, 3.00, 3.00, 4.00 | 4.33, 5.33, 5.33, 6.33 | 6.00, 7.00, 7.00, 8.00 | 7.00, 8.00, 8.00, 9.00 |
| $A_2$ | 0.50, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 1.00 | 3.00, 4.00, 4.00, 5.00 | 3.67, 4.67, 4.67, 5.67 | 8.00, 9.00, 9.00, 9.00 | 7.33, 8.33, 8.33, 9.00 | 2.00, 3.00, 3.00, 4.00 | 5.67, 6.67, 6.67, 7.67 | 4.00, 5.00, 5.00, 6.00 | 7.00, 8.00, 8.00, 9.00 |
| $A_3$ | 0.20, 0.25, 0.25, 0.33 | 0.20, 0.25, 0.25, 0.33 | 1.00, 1.00, 1.00, 1.00 | 1.00, 2.00, 2.00, 3.00 | 4.67, 5.67, 5.67, 6.67 | 4.33, 5.33, 5.33, 6.33 | 1.00, 2.00, 2.00, 3.00 | 3.00, 4.00, 4.00, 5.00 | 3.00, 4.00, 4.00, 5.00 | 3.00, 4.00, 4.00, 5.00 |
| $A_4$ | 0.14, 0.18, 0.18, 0.23 | 0.18, 0.21, 0.21, 0.27 | 0.33, 0.50, 0.50, 1.00 | 1.00, 1.00, 1.00, 1.00 | 3.00, 4.00, 4.00, 5.00 | 3.00, 4.00, 4.00, 5.00 | 0.30, 0.43, 0.43, 0.75 | 1.00, 2.00, 2.00, 3.00 | 1.00, 1.00, 1.00, 2.00 | 2.00, 3.00, 3.00, 4.00 |
| $A_5$ | 0.11, 0.12, 0.12, 0.14 | 0.11, 0.11, 0.11, 0.13 | 0.15, 0.18, 0.18, 0.21 | 0.20, 0.25, 0.25, 0.33 | 1.00, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 2.00 | 0.25, 0.33, 0.33, 0.50 | 0.33, 0.50, 0.50, 1.00 | 0.20, 0.25, 0.25, 0.33 | 1.00, 2.00, 2.00, 3.00 |
| $A_6$ | 0.11, 0.11, 0.11, 0.13 | 0.11, 0.12, 0.12, 0.14 | 0.16, 0.19, 0.19, 0.23 | 0.20, 0.25, 0.25, 0.33 | 0.50, 1.00, 1.00, 1.00 | 1.00, 1.00, 1.00, 1.00 | 0.20, 0.25, 0.25, 0.33 | 0.33, 0.50, 0.50, 1.00 | 0.20, 0.25, 0.25, 0.33 | 1.00, 2.00, 2.00, 3.00 |
| $A_7$ | 2.00, 3.00, 3.00, 4.00 | 0.25, 0.33, 0.33, 0.50 | 0.33, 0.50, 0.50, 1.00 | 1.33, 2.33, 2.33, 3.33 | 2.00, 3.00, 3.00, 4.00 | 3.00, 4.00, 4.00, 5.00 | 1.00, 1.00, 1.00, 1.00 | 1.00, 2.00, 2.00, 3.00 | 1.00, 1.67, 1.67, 2.67 | 3.00, 4.00, 4.00, 5.00 |
| $A_8$ | 0.16, 0.19, 0.19, 0.23 | 0.13, 0.15, 0.15, 0.18 | 0.20, 0.25, 0.25, 0.33 | 0.33, 0.50, 0.50, 1.00 | 1.00, 2.00, 2.00, 3.00 | 1.00, 2.00, 2.00, 3.00 | 0.33, 0.50, 0.50, 1.00 | 1.00, 1.00, 1.00, 1.00 | 0.25, 0.33, 0.33, 0.50 | 0.33, 0.50, 0.50, 1.00 |
| $A_9$ | 0.13, 0.14, 0.14, 0.17 | 0.17, 0.20, 0.20, 0.25 | 0.20, 0.25, 0.25, 0.33 | 0.50, 1.00, 1.00, 1.00 | 3.00, 4.00, 4.00, 5.00 | 3.00, 4.00, 4.00, 5.00 | 0.37, 0.60, 0.60, 1.00 | 2.00, 3.00, 3.00, 4.00 | 1.00, 1.00, 1.00, 1.00 | 2.00, 3.00, 3.00, 4.00 |
| $A_{10}$ | 0.11, 0.13, 0.13, 0.14 | 0.11, 0.13, 0.13, 0.14 | 0.20, 0.25, 0.25, 0.33 | 0.25, 0.33, 0.33, 0.50 | 0.33, 0.50, 0.50, 1.00 | 0.33, 0.50, 0.50, 1.00 | 0.20, 0.25, 0.25, 0.33 | 1.00, 2.00, 2.00, 3.00 | 0.25, 0.33, 0.33, 0.50 | 1.00, 1.00, 1.00, 1.00 |

According to the estimation scheme as shown in Table 1 and Fig. 1, on the basis of expert and engineering judgements the triangular fuzzy numbers can be obtained. It should be noted that $n(n-1)/2$, in this case, $10 \times (10-1)/2 = 45$ comparisons are required in the FAHP process. But, however, by using the modified FAHP method, only $n-1$, in this case, $10-1=9$ comparisons need to be undertaken. As described in Section 4.1 these triangular fuzzy numbers are then converted into TranFNs as shown in Table 3, which will be used to establish fuzzy preference relation decision matrix.

As stated in Section 4.2, other $m_{i,j}$ in pairwise comparison matrix $M$ can be obtained on the basis of TranFNs by using Eqs. (17)–(20) of Proposition 1 and Eqs. (21)–(28) of Proposition 2 as shown in Table 4.

For example, $m_{2,9}$ and $m_{9,2}$ in Table 4 can be obtained

$$\begin{aligned} a_{2,9} &= a_{2,3} \times a_{3,4} \times a_{4,5} \times a_{5,6} \times a_{6,7} \times a_{7,8} \times a_{8,9} \\ &= 3 \times 1 \times 3 \times 1 \times 0.2 \times 1 \times 0.25 = 0.45 \\ b_{2,9} &= b_{2,3} \times b_{3,4} \times b_{4,5} \times b_{5,6} \times b_{6,7} \times b_{7,8} \times b_{8,9} \\ &= 4 \times 2 \times 4 \times 1 \times 0.25 \times 2 \times 0.33 = 5.28 \\ c_{2,9} &= c_{2,3} \times c_{3,4} \times c_{4,5} \times c_{5,6} \times c_{6,7} \times c_{7,8} \times c_{8,9} \\ &= 4 \times 2 \times 4 \times 1 \times 0.25 \times 2 \times 0.33 = 5.28 \\ d_{2,9} &= d_{2,3} \times d_{3,4} \times d_{4,5} \times d_{5,6} \times d_{6,7} \times d_{7,8} \times d_{8,9} \\ &= 5 \times 3 \times 5 \times 2 \times 0.33 \times 3 \times 0.5 = 74.25 \end{aligned}$$

**Table 6**
WFs of hazard groups.

| Alternatives | Fuzzy weights | Defuzzification | WFs (Modified FAHP) |
|---|---|---|---|
| $A_1$ | 0.07, 0.18, 0.18, 0.53 | 0.221 | 0.195 |
| $A_2$ | 0.07, 0.18, 0.18, 0.43 | 0.204 | 0.180 |
| $A_3$ | 0.05, 0.11, 0.11, 0.25 | 0.125 | 0.110 |
| $A_4$ | 0.04, 0.09, 0.09, 0.20 | 0.099 | 0.087 |
| $A_5$ | 0.03, 0.05, 0.05, 0.12 | 0.061 | 0.054 |
| $A_6$ | 0.02, 0.05, 0.05, 0.11 | 0.058 | 0.051 |
| $A_7$ | 0.04, 0.09, 0.09, 0.17 | 0.094 | 0.083 |
| $A_8$ | 0.03, 0.07, 0.07, 0.16 | 0.077 | 0.068 |
| $A_9$ | 0.04, 0.10, 0.10, 0.24 | 0.114 | 0.101 |
| $A_{10}$ | 0.02, 0.07, 0.07, 0.18 | 0.081 | 0.072 |

$$m_{2,9} = \{a_{2,9}, b_{2,9}, c_{2,9}, d_{2,9}\} = \{0.45, 5.28, 5.28, 74.25\}$$

then,

$$\begin{aligned} m_{9,2} &= 1/m_{2,9} = \{1/d_{2,9}, 1/c_{2,9}, 1/b_{2,9}, 1/c_{2,9}\} \\ &= \{0.013, 0.19, 0.19, 2.22\} \end{aligned}$$

However, it should be noted that $m_{2,9}$ and $m_{9,2}$ are not in the interval [1/9,9]. As described in Section 4.2, the transformation functions are needed to transfer TranFNs preference relation decision matrix to Fuzzy preference relation comparison matrix in the interval

**Table 7**
Risk levels of hazard groups.

| Operation | Index | Hazard groups | Risk score | Risk category | Risk contribution (%) |
|---|---|---|---|---|---|
| Shunting at Waterloo depot | $A_1$ | Derailment | 4.53 | Possible: 100% | 17.2 |
| | $A_2$ | Collision | 4.43 | Possible: 100% | 15.5 |
| | $A_3$ | Train fire | 5.79 | Possible: 100% | 12.4 |
| | $A_4$ | Electrocution | 6.22 | Possible: 78%<br>Substantial: 22% | 10.5 |
| | $A_5$ | Slips/trips | 5.78 | Possible: 100% | 6.1 |
| | $A_6$ | Falls from height | 4.28 | Possible: 100% | 4.2 |
| | $A_7$ | Train strikes person | 6.50 | Possible: 50%<br>Substantial: 50% | 10.5 |
| | $A_8$ | Platform train interface | 6.15 | Possible: 85%<br>Substantial: 15% | 8.1 |
| | $A_9$ | Structural failure | 4.39 | Possible: 100% | 8.6 |
| | $A_{10}$ | Health hazard | 5.00 | Possible: 100% | 7.0 |
| Overall RL | | | 5.19 | Possible:100% | |

[1/9,9]. In this case, the maximum of $v = \max(d_{i,j}) = d_{1,10} = 594$ and the following transformation functions are applied as

$$f(x_a) = x_a^{1/\log_9 594}, \quad f(x_b) = x_b^{1/\log_9 594},$$
$$f(x_c) = x_c^{1/\log_9 594}, \quad f(x_d) = x_d^{1/\log_9 594}$$

The fuzzy preference relation comparison matrix is obtained as shown in Table 5.

By using Eqs. (3)–(6), the WF of each hazard group are derived as shown in Table 6.

Finally, by using Eqs. (8) and (9), the RLs of hazard groups and their contributions to the overall RL of Shunting at Waterloo depot can be obtained as shown in Table 7. As can be seen that the overall RL of shunting at Hammersmith depot is 5.19 which indicates 'Possible' with a belief of 100%. It should be noted that, for example, in this case, the risk scores of derailment and collision are 4.53 and 4.43, respectively, with risk category 100% of possible. The case study just shows these two risks in Waterloo depot which demonstrates its current circumstances. However, in other depots/ systems, the risk scores may be higher or lower, this heavily depends on current situations of the depots/systems.

The results indicate that by using the modified FAHP method in the risk decision making process, the comparison matrix in this case can be established on the basis of consistency of fuzzy multiplicative preference relations. The amount of comparisons can be reduced significantly and human errors in the subjective judgments can be avoided, which provide more reliable and accurate results to decision makers to produce railway maintenance priorities and strategies.

## 6. Conclusions

The application of FAHP in risk decision making analysis often involves a large number of pairwise comparisons in the decision making process. There may be a lack of confidence that all comparisons associated with a railway system are completely justified in a rigorous way, particularly, when it is a complex railway system and subjective judgements should be involved. This study contributes significantly to the body of knowledge related to safety risk analysis and decision making in railway safety management, which can be summarised as:

(1) A modified FAHP approach in risk decision making has been proposed by introducing the fuzzy multiplicative preference relations to establish pairwise comparison matrix.
(2) Consistent multiplicative preference relation and fuzzy multiplicative consistency methods including definitions, propositions and proofs have been further developed, which can be used to calculate values of comparison matrices.
(3) A transformation function has been established, which is applied to transfer TranFNs preference relation decision matrix to fuzzy preference relation comparison matrix.
(4) The proposed method can reduce inconsistency in risk judgements significantly in dealing with uncertainty.
(5) It can handle qualitative and quantitative information in a consistent manner.
(6) It yields a higher level of confidence that all of comparisons associated with the system are justified, which can reduce human errors in the determining comparison significantly.

Comparing the traditional FAHP with the proposed method, the modified FAHP methodology can reduce a large amount of pairwise comparisons in the decision process significantly. For example, the illustrative case example involves 10 hazard groups which $45 - 9 = 36$ pairwise comparisons are reduced and only 9 pairwise comparisons are required while consistency is still satisfied. On the other hand, it will also reduce human errors in the risk decision making process so that more reliable and accurate results can be obtained. The proposed method not only improves the quality of risk analysis in imprecise or vague situations, but also solves the problems of consistency when applying FAHP method.

## Acknowledgements

The work described herein is part of a research project on "application of fuzzy reasoning methodology to railway safety risk assessment process" funded by Engineering and Physical Sciences Research Council (EPSRC, UK) under Grant No. GR/D066468, London Underground Ltd under Grant No. P/O:3100071997, and the State Key Laboratory of Rail Traffic Control and Safety of Beijing Jiaotong University under Contract No. RC2014K001 with technical support from Tube Lines Ltd, Metronnet SSL, Rail Safety & Standard Board (RSSB), Network Rail Ltd and Serco Assurance Ltd. Their support is gratefully acknowledged.

## References

An, M., Lin, W., Stirling, A., 2006. Fuzzy-reasoning-based approach to qualitative railway Risk assessment. Proc. Inst. Mech. Eng., Part F: J. Rail Rapid Transit 220, 153–167.
An, M., Huang, S., Baker, C.J., 2007. Railway risk assessment – the FRA and FAHP approaches: a case study of shunting at Waterloo depot. Proc. Inst. Mech. Eng., Part F: J. Rail Rapid Transit 221, 1–19.
An, M., Chen, Y., Baker, C.J., 2008. Development of an intelligent system for railway risk analysis. In: Proceeding of 3rd International Conference on System Safety, ISBN 0-479622-23-13.
An, M., Chen, Y., Baker, C.J., 2011. A fuzzy reasoning and fuzzy-analytical hierarchy process based approach to the process of railway risk information: a railway risk management system. Inf. Sci. 181, 3946–3966.
Berredo, R.C., Ekel, P.Y., Palhares, R.M., 2005. Fuzzy preference relations in models of decision making. Nonlin. Anal. 63, 735–741.

Bojadziev, G., Bojacziev, M., 1997. Fuzzy Logic for Business, Finance, and Management. World Scientific, Singapore.
BS EN 50126-1, 1999. Railway Applications – The Specification and Demonstration Reliability, Availability, Maintainability and Safety (RAMS) – Part 1: Basic Requirements and Generic Process. British Standards Institution (BSI), London.
BS EN 50126-2, 2007. Railway Applications – The Specification and Demonstration of Reliability, Availability, Maintainability and Safety (RAMS) – Part 3: Guide to the Application of EN 50126-1 for Safety. British Standards Institution (BSI), London.
BS EN 50126-3, 2006. Railway Applications – The Specification and Demonstration of Reliability, Availability, Maintainability and Safety (RAMS) – Part 3: Guide to the application of EN 50126-1 for Rolling Stock RAMS. British Standards Institution (BSI), London.
BS EN 50128, 2009. Railway Applications – Communication, Signalling and Processing Systems – Software for Railway Control and Protection Systems. British Standards Institution (BSI), London.
BS EN 50129, 2003. Railway Applications – Communication, Signalling and Processing Systems – Safety Related Electronic Systems for Signalling. British Standards Institution (BSI), London.
Buckley, J.J., 1985. Fuzzy hierarchical analysis. Fuzzy Sets Syst. 17 (3), 233–247.
Chen, Y., An, M., 2011. A modified fuzzy-AHP methodology in railway decision making process. In: Proceedings of the International Railway Engineering Conference, ISBN 0-947644-45-16.
Chen, C.T., Lin, C.T., Huang, S.F., 2006. A fuzzy approach for supplier evaluation and selection in supply chain management. Int. J. Product. Econ. 102 (2), 289–301.
Chen, Y., An, M., Huang, S., Baker, C.J., 2007. Application of FRA and FAHP approaches to railway maintenance safety risk assessment process. In: Proceedings of the International Railway Engineering Conference, ISBN 0-947644-61-10.
Chiclana, F., Herrera, F., Herrera-Viedma, E., 2001. Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations. Fuzzy Sets Syst. 122, 277–291.
Dubois, D., Prade, H., 1980. Fuzzy Sets and Systems: Theory and Applications. Academic Press, Inc., New York.
Ekel, P.Y., Silva, M.R., Neto, F.S., Palhares, R.M., 2006. Fuzzy preference modelling and its application to multiobjective decision making. Comp. Math. Appl. 52, 179–196.
Fan, Z.P., Ma, J., Jiang, Y.P., Sun, Y.H., Ma, L., 2006. A goal programming approach to group decision making based on multiplicative preference relations and fuzzy preference relations. Euro. J. Operat. Res. 174, 311–321.
Giachetti, R.E., Young, R.E., 1997. Analysis of the error in the standard approximation used for multiplication of triangular and trapezoidal fuzzy numbers and the development of a new approximation. Fuzzy Sets Syst. 91 (1), 1–13.
Gu, X., Zhu, Q., 2006. Fuzzy multi-attribute decision-making method based on eigenvector of fuzzy attribute evaluation space. Dec. Supp. Syst. 41 (2), 400–410.
Herrera, F., Herrera-Viedma, E., Chiclana, F., 2001. Multiperson decision-making based on multiplicative preference relations. Euro. J. Operat. Res. 129, 372–385.
Herrera-Viedma, E., Herrera, F., Chiclana, F., Luque, M., 2004. Some issues on consistency of fuzzy preference relations. Euro. J. Operat. Res. 154, 98–109.
Huang, S., An, M., Chen, Y., Baker, C.J., 2007. Railway safety risk assessment using FRA and fuzzy AHP approaches – a case study on risk analysis of shunting at Waterloo depot. In: Proceedings of the 2nd IET International Conference on System Safety. London, UK.
Laarhoven, P.J.M., Pedrycz, W., 1983. A fuzzy extension of Saaty's priority theory. Fuzzy Sets Syst. 11 (1–3), 229–241.
Leung, L.C., Cao, D., 2000. On consistency and ranking of alternatives in fuzzy AHP. Euro. J. Operat. Res. 124, 102–113.
Railway Safety and Standard Board, 2007. Guidance on the Preparation of Risk Assessment Within Railway Safety Cases. Railway Group Guidance Note – GE/GN8561, 1 June.
Saaty, T.L., 1994. Fundamentals of Decision Making and Priority Theory with the AHP. RWS Publications, Pittsburgh.
Satty, T.L., 1980. The Analytic Hierarchy Process. McGraw-Hill.
Wang, T.C., Chen, Y.H., 2008. Applying fuzzy linguistic preference relations to the improvement of consistency of fuzzy AHP. Inf. Sci. 178 (19), 3755–3765.
Wang, Y.M., Fan, Z.P., 2007. Fuzzy preference relations: aggregation and weight determination. Comp. Indust. Eng. 53, 163–172.
Xu, Z., 2004. A method based on linguistic aggregation operators for group decision making with linguistic preference relation. Inf. Sci. 166, 19–30.