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E. Hashemizadeh, A. Ebrahimzadeh

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An efficient numerical scheme to solve fractional diffusion-wave and fractional Klein-Gordon equations in fluid mechanics

E. Hashemizadeh[1,*], A. Ebrahimzadeh[2]

[1] Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran
[2] Assistant Professor, School of Basic Sciences, Farhangian University, Tehran, Iran

Abstract

The numerous applications of time fractional partial differential equations in different fields of science especially in fluid mechanics necessitate the presentation of an efficient numerical method to solve them. In this paper, Galerkin method and operational matrix of fractional Riemann-Liouville integration for shifted Legendre polynomials has been applied to solve these equations. Some definitions for fractional calculus along with some basic properties of shifted Legendre polynomials have also been put forth. When approximations are substituted into the fractional partial differential equations, a set of algebraic equations would be resulted. The convergence of the suggested method was also depicted. In the end, the linear time fractional Klein-Gordon equation, dissipative Klein-Gordon equations and diffusion-wave equations were utilized as three examples so as to study the performance of the numerical scheme.

Key words: fractional Klein-Gordon equation, fractional diffusion-wave equation, fractional dissipative Klein-Gordon equation, shifted Legendre polynomials, operational matrix

1 Introduction

In recent years, with the rapid development of nonlinear sciences, the theory of fractional differential equations have developed progressively and researchers have found that derivatives and integrals of non integer order are more suitable and accurate than integer-order equations for modeling some real world problems. These equations have attracted substantial attention of many investigator because they have practical applications in diverse areas of science and engineering such as bioengineering [1], anomalous transport [2], solid mechanics [3], continuum and statistical mechanics [4], nonlinear oscillation of earthquakes [5], economics[6], fluid dynamic [7], colored noise [8], viscoelastic damping [9] -[11] and modelling

*Corresponding author: hashemizadeh@kiau.ac.ir
†Ebrahimzadeh263@gmail.com

of an ultracapacitor [12] or the heating process [13], etc. Numerical solutions of these kind of fractional equations have been investigated by several authors [14]-[30]. This work has been concentrated on the following time-fractional partial differential equations with damping as:

$$\frac{\partial^\alpha \xi(x,t)}{\partial t^\alpha} + \rho \frac{\partial \xi(x,t)}{\partial t} + \Upsilon \xi(x,t) = \frac{\partial^2 \xi(x,t)}{\partial x^2} + \mathcal{F}(x,t), \quad 0 \le x \le 1,\ 0 \le t \le 1,\ 1 < \alpha \le 2, \tag{1}$$

with initial conditions

$$\xi(x,0) = \phi(x), \quad \frac{\partial \xi(x,0)}{\partial t} = \varphi(x), \tag{2}$$

and boundary conditions

$$\xi(0,t) = \zeta_1(t), \quad \xi(1,t) = \zeta_2(t). \tag{3}$$

Two significant cases of equations (1)-(3) were considered as follows:

- ◇ In equation (1), if one put $\Upsilon = \rho = 0$, then the fractional diffusion-wave equation will be obtained [31]-[34]. These equations can be used to model many of the universal electromagnetic, acoustic, and mechanical responses accurately [35, 36]. Equation (1) also is characterized as a telegraph equation which governs electrical transmission in a telegraph cable in the case $\alpha = 2$ [37]. Over the past few years, several numerical methods have been proposed for solving fractional diffusion-wave equations, for instance see [31]-[34] and [38]-[47].

- ◇ If one set $\Upsilon = 1$ in equation (1), then the linear time fractional Klein-Gordon equation will be obtained for $\rho = 0$ and the linear time fractional dissipative Klein-Gordon equation will be obtained for $\rho \neq 0$. These equations widely appear in fluid mechanics [48]. They also arise in modelling different phenomena, including the propagation of dislocations in crystals and the behavior of elementary particles [47], [49] and [50]. Fore more details about the numerical methods regarding these equations, see [51]-[54] and the refrences therein.

The properties of shifted Legendre polynomias along with their operational matrix of derivative and fractional Riemann-Liouville integration are utilized to reduce the equations (1)-(3) to a system of algebraic equations which can be solved easily.

The outline of this paper is as follows. In Section 2, some basic concepts for fractional calculus and shifted Legendre polynomials are expressed and the error estimate for function approximation with these bases is also given. The analysis of the proposed approach is introduced in Section 3. An upper error bound is presented in Section 4. In Section 5, the numerical results of solving the intended fractional differential problem with the proposed method for three test problems are reported. In this part, the linear time fractional Klein-Gordon equation, dissipative Klein-Gordon equations and diffusion-wave equations are considered as prototype examples. The discussion is wrapped up with the conclusion that appears in Section 6.

2 Background materials and preliminaries

2.1 The fractional derivative in the Caputo sense

There exist different definitions of fractional integration and differentiation [55], such as Grunwald-Letnikov, Riemann-Liouville, Caputo, Weyl, Marchaud, Riesz fractional derivatives, Nishimoto fractional operator and Jumaries definitions. The most important kinds of fractional derivatives are Caputo and Riemann-Liouville fractional derivatives. In this section, the essentials of the fractional calculus and shifted Legendre polynomials are reminded [55].

Definition 1. For every $\nu \in \mathcal{R}$ and $t > 0$, a real function $\xi(t)$, is said to be in the space C_ν if there exists a real number $p > \nu$ such that $\xi(t) = t^p \xi_1(t)$, where $\xi_1(t) \in C(0, \infty)$, and for $n \in N$ it is said to be in the space C_ν^n, if $\xi^{(n)} \in C_\nu$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ for a function $\xi(t) \in C_\nu$, $\nu \geq -1$, is defined as

$$I_t^\alpha \xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0, \quad t > 0, \quad I^0 \xi(t) = \xi(t). \tag{4}$$

$$I_t^\alpha \xi(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(x,s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0, \quad t > 0, \tag{5}$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 3. The Riemann-Liouville fractional derivative of $\xi(t)$ of order α is defined as

$$D_t^\alpha \xi(t) = \frac{d^n}{dt^n} I_t^{n-\alpha} \xi(t), \quad n-1 < \alpha \leq n, \quad n \in N \cup \{0\}, \tag{6}$$

in which $\xi(t) \in C_{-1}^n$ and $n \in N$.

Definition 4. The Caputo time fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{*t}^\alpha \xi(x,t) = \frac{\partial^\alpha \xi(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\partial^n \xi(x,s)}{\partial s^n} (t-s)^{n-\alpha-1} ds, & n-1 < \alpha < n, n \in N \cup \{0\}, \\ \frac{\partial^n \xi(x,t)}{\partial t^n}, & n = \alpha. \end{cases} \tag{7}$$

where n is the ceiling function of α.

The next theorem demonstrates the relation between a fractional derivative and a fractional integral.

Theorem 1. *Assume that the continuous function $\xi(t)$ has a fractional derivative of order α, then we have*

$$D_t^\alpha I_t^\beta \xi(t) = \begin{cases} I_t^{\beta-\alpha} \xi(t) & \alpha < \beta, \\ \xi(t) & \alpha = \beta, \\ D_t^{-\beta+\alpha} \xi(t) & \alpha > \beta, \end{cases} \tag{8}$$

$$I_t^\alpha D_{*t}^\alpha \xi(t) = \xi(t) - \sum_{i=0}^{n-1} \xi^{(k)}(0^+) \frac{t^k}{k!}, n-1 < \alpha \leq n, n \in N, \tag{9}$$

$$D_{*t}^\alpha I_t^\alpha \xi(t) = \begin{cases} \xi(t), & n-1 < \alpha \leq n, n \in N, \\ D_{*t}^\alpha I_t^\alpha \xi(t) + \xi(0), & 0 < \alpha < 1. \end{cases} \tag{10}$$

Remark. Fractional differentiation is a linear operation

$$D_{*t}^{\alpha}\left(\eta_1 \xi(t) + \eta_2 \chi(t)\right) = \eta_1 D_{*t}^{\alpha} \xi(t) + \eta_2 D_{*t}^{\alpha} \chi(t). \tag{11}$$

2.2 Properties of shifted Legendre polynomials

Legendre polynomials, $\tilde{\psi}_i(t)$, are orthogonal with respect to L^2 inner product on the interval $[-1, 1]$ with the weight function $\omega(t) = 1$. These polynomials are widely used because of their good properties in the approximation of functions. They are defined by the following recursive formula

$$\tilde{\psi}_{m+1}(t) = \frac{2m+1}{m+1} t \tilde{\psi}_m(t) - \frac{m}{m+1} \tilde{\psi}_{m-1}(t), \quad m = 1, 2, 3, \cdots, \tag{12}$$

$$\tilde{\psi}_0(t) = 1, \quad \tilde{\psi}_1(t) = t. \tag{13}$$

By a proper change of variable, we can define the so-called shifted Legendre polynomials on an arbitary interval $[a, b]$ as follows:

$$\tilde{\psi}_0(t) = 1, \quad \tilde{\psi}_1(t) = \frac{2(t-a) - h}{h}, \tag{14}$$

and for $m = 1, 2, 3, \cdots,$

$$\tilde{\psi}_{m+1}(t) = \frac{2m+1}{h(m+1)} (2(t-a) - h) \tilde{\psi}_m(t) - \frac{m}{m+1} \tilde{\psi}_{m-1}(t). \tag{15}$$

where $h = b - a$. By assuming that $\psi_m(t) = \sqrt{\frac{2m+1}{h}} \tilde{\psi}_m(t)$, we will have

$$\int_a^b \psi_i(t) \psi_j(t) dt = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{16}$$

The analytical form of the shifted Legendre polynomial of degree m, ψ, is given by [56]

$$\psi_m(t) = \sqrt{\frac{2m+1}{h}} \sum_{k=0}^{m} \frac{(-1)^{m+k} (m+k)! (t-a)^k}{(m-k)! (k!)^2 h^k}, \quad m = 0, \cdots, M. \tag{17}$$

Function approximation

We can expand a function $g(t)$ as

$$g(t) = \sum_{i=0}^{M} g_i \psi_i(t) = \mathcal{G}^T \psi(t), \tag{18}$$

in which

$$\mathcal{G} = [g_0, g_1, \cdots, g_M]^T, \quad \psi(t) = [\psi_0(t), \psi_1(t), \cdots, \psi_M(t)]^T, \tag{19}$$

and the coefficients g_i for $i = 0, 1, \cdots, M$ are given by

$$g_i = \langle g(t), \psi_i(t) \rangle. \tag{20}$$

Theorem 2. *[57] Let $g(t) \in H^k(-1,1)$ (Sobolev space) and $\sum_{i=0}^{M} g_i\psi_i(t)$ be the best approximation polynomial of $g(t)$ in L_2-norm. Then*

$$\|g(t) - \sum_{i=0}^{M} g_i\psi_i(t)\|_{L_2[-1,1]} \leq C_0 M^{-k}\|g(t)\|_{H^k(-1,1)},$$

where C_0 is a positive constant, which depend on the selected norm and is independent of $g(t)$ and m.

Remark 1. The computational interval can be transformed into an arbitrary interval $[a,b]$ via an affine transformation.

Remark 2. More general information about best approximation and related theorems can be found in [58].

A function $\xi(x,t)$ is approximated by

$$\xi(x,t) = \psi^T(x)\mathcal{X}\psi(t), \tag{21}$$

in which $\psi(t)$ is given in equation (19) and $\mathcal{X}$ is the $(M+1)\times(M+1)$ matrix. The elements of matrix $\mathcal{X}$ are obtained as follows:

$$\mathcal{X}_{ij} = \langle\psi_{i-1}(x), \langle\xi(x,t), \psi_{j-1}(t)\rangle\rangle = \int_0^1\int_0^1 \psi_{i-1}(x)\psi_{j-1}(t)\xi(x,t)dtdx, \quad 1\leq i\leq M+1,\ 1\leq j\leq M+1. \tag{22}$$

The Gauss–Legendre quadrature formula may be utilized to obtain the coefficients $\mathcal{X}_{ij}$

$$\mathcal{X}_{ij} \approx \sum_{k=0}^{M}\sum_{l=0}^{M} w_{1l}w_{2k}\psi_{i-1}(x_l)\psi_{j-1}(x_k)\xi(x_l,x_k). \tag{23}$$

In equation (23), x_l and x_k are the roots of Legendre polynomial $P_{M+1}(t)$ and the weights w_{1l} and w_{2k} are given by

$$w_{1i} = w_{2i} = \frac{2}{(1-x_i{}^2)[P'_{M+1}(x_i)]^2}, \quad 0\leq i\leq M.$$

Theorem 3. *[56] Suppose that function $\xi \in L^2[0,1]$ is approximated by g_M as follows*

$$g_M(t) = \sum_{j=0}^{M}\gamma_i\psi_i(t), \tag{24}$$

where

$$\gamma_i = \int_0^1 \psi_i(t)\xi(t)dt, \quad i = 0,1,\cdots,M. \tag{25}$$

Consider

$$S_M(\xi) = \int_0^1 [\xi(t) - g_M(t)]^2 dt, \tag{26}$$

then we have

$$\lim_{M\longrightarrow\infty} S_M(\xi) = 0. \tag{27}$$

Theorem 4. *[56] Let $\widehat{H} \subset L^2[0,1]$ and $\widehat{H} = span\{\psi_0,\psi_1,\cdots,\psi_M\}$ and $\xi(x,t)$ be an arbitrary function in $L^2[0,1]$. Hence by using Theorem 3, ξ has a best approximation, such that*

$$\forall\zeta(x,t) \quad : \|\xi(x,t) - \psi^T(x)\mathcal{X}\psi(t)\|_2 \leq \|\xi(x,t) - \zeta(x,t)\|_2, \tag{28}$$

where $\psi(x)$ and $\mathcal{X} = [\mathcal{X}]_{ij}$ are defined in equations (19) and (22).

Operational matrices

Recently, approaches relied on operational matrices have attracted a striking attention due to their agreeable attributes. There are several authors who have elaborated them to solve various kinds of equations (see for example [59] and the references therein). Among them, the operational matrix of integration and derivative are the most prominent matrices. The integral or derivative operator will be replaced by the related matrix so the main problem converted to a system of algebraic equations. The operational matrix of derivative of vector $\psi(x)$ is defined as

$$\psi^{'}(x) = D\psi(x), \tag{29}$$

where $D \in \mathcal{R}^{(M+1)(M+1)}$. Straightforward computations on (17) due to (29) demonstrate that each element of matrix D, d_{ij}, is given by

$$d_{ij} = \frac{1}{h}\sqrt{2i+1}\sqrt{2j+1}\sum_{k=0}^{i}\sum_{l=1}^{j}\frac{l(-1)^{i+k+l+j}(i+k)!(j+l)!}{(i-k)!(k!)^2(j-l)!(l!)^2(k+1)} \quad 1 \leq i \leq M+1, \ 1 \leq j \leq M+1. \tag{30}$$

By using Eq.(29), it is clear that

$$\frac{d^n\psi(x)}{dx^n} = D^n\psi(x), \tag{31}$$

where $n \in N$ denotes matrix power thus

$$D^{(n)} = D^n, \qquad n = 1, 2, \cdots \tag{32}$$

It sould be noted that one of the prominent advantages to apply the operational matrix of derivative is that there is no need to use any approximation to eliminate the differential part, so it is preferable to exert these matrices.

The operational matrix of the Riemann-Liouville fractional integration can be defined as

$$I_t^\alpha \psi \approx P^\alpha \psi. \tag{33}$$

Matrix P^α is constructed and defined in [56] as

$$P^\alpha = [P_{ij}], \ 1 \leq i, j \leq M+1, \tag{34}$$

where

$$Q_{ij} = \sqrt{(2i+1)(2j+1)}\sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{i+k+j+l}(i+k)!(j+l)!}{(i-k)!k!\Gamma(k+1+\alpha)(j-l)!(l!)^2(l+k+\alpha+1)}, \tag{35}$$

and

$$P_{ij} = Q_{i-1j-1}, \quad 1 \leq i, j \leq M+1. \tag{36}$$

3 Numerical method

In this section, the Galerkin method is expressed to solve problem (1)-(3). By using the operational matrices of derivative and fractional integration introduced in equations (29) and (33), along with Galerkin approach, the considered fractional differential equation is reduced to a system of algebraic equations. Firstly, we approximate the functions $\xi(x,t)$ and $\mathcal{F}(x,t)$ by Legendre polynomials

$$\xi(x,t) \approx \xi^{\mathcal{M}}(x,t) = \psi^T(x)\mathcal{X}\psi(t), \quad \mathcal{F}(x,t) \approx \mathcal{F}^{\mathcal{M}}(x,t) = \psi^T(x)F\psi(t), \tag{37}$$

in which the elements of matrices $\mathcal{X}$ and F are given in equation (22). Applying the Riemann-Liouville integral operator I_t^α to fractional differential equation (1) and using equation (9) together with initial conditions (2) leads to the following equation

$$\xi(x,t) - \widehat{G}(x,t) + \rho I_t^\alpha \frac{\partial \xi(x,t)}{\partial t} + \Upsilon I_t^\alpha \xi(x,t) = I_t^\alpha \frac{\partial^2 \xi(x,t)}{\partial x^2} + I_t^\alpha \mathcal{F}(x,t), \tag{38}$$

where $\widehat{G}(x,t) = \phi(x) + t\varphi(x)$. The function $\widehat{G}(x,t)$ is approximated as follows

$$\widehat{G}(x,t) = \psi^T(x)G\psi(t). \tag{39}$$

By substituting approximate functions (37) and (39) into equation (38) and using equations (29) and (33), this equation is transformed to a matrix equation

$$\psi^T(x)\left(\mathcal{X} + \rho\mathcal{X}DP^\alpha + \Upsilon\mathcal{X}P^\alpha - (D^2)^T\mathcal{X}P^\alpha\right)\psi(t) = \psi^T(x)\left(G + FP^\alpha\right)\psi(t) \tag{40}$$

By multiplying equations (40) in $\psi(x)$ from right side and integrating from 0 and 1, we get

$$\left(\mathcal{X} + \rho\mathcal{X}DP^\alpha + \Upsilon\mathcal{X}P^\alpha - (D^2)^T\mathcal{X}P^\alpha\right)\psi(t) = \left(G + FP^{(\alpha)}\right)\psi(t). \tag{41}$$

Now, by multiplying equations (41) in $\psi^T(t)$ and integrating from 0 to 1, we have

$$\mathcal{X} + \rho\mathcal{X}DP^\alpha + \Upsilon\mathcal{X}P^\alpha - (D^2)^T\mathcal{X}P^\alpha = G + FP^\alpha. \tag{42}$$

Finally, for solving the main problem we need to apply the boundary conditions in equqtion (3)

$$\psi^T(0)\mathcal{X}\psi(t) = \zeta_1(t) \tag{43}$$
$$\psi^T(1)\mathcal{X}\psi(t) = \zeta_2(t)$$

The functions $\zeta_1(t)$ and $\zeta_2(t)$ are approximated by using equation (18) as follows

$$\zeta_1(t) \approx \mathcal{M}_1^T\psi(t) \quad \zeta_2(t) \approx \mathcal{M}_2^T\psi(t) \tag{44}$$

By replacing equation (44) into (43), we obtain

$$\psi^T(0)\mathcal{X} = \mathcal{M}_1^T, \quad \psi^T(1)\mathcal{X} = \mathcal{M}_2^T, \tag{45}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are $(M+1)$ vectors and calculated in (19).
We omit the last $2(M+1)$ equations from equations (42) in order to have $(M+1)^2 - 2(M+1)$ equations. These equations added up toghether with equations in (45) so as to have the outcome of a linear system of equations with $(M+1)^2$ equations and unknowns.

Remark 1 We proceed by discussing the sparsity of the matrices D and D^2 as an important issue for increasing the computation speed.The number of nonzero matrix elements in D for odd values of M is $\frac{(M+1)^2}{4}$ and for even values of M is $\frac{M(M+2)}{4}$. These values in D^2 is $\frac{M^2-1}{4}$ and $\frac{M^2}{4}$ for odd and even values of M respectively. The graphs 1, 2, 3 and 4 are a visual representation of the values of elements in matix D and D^2 for $M = 5$ and $M = 6$. Black and white respectively show nonzero and zero elements of matrices. As the graphs 1, 2, 3 and 4 show matrices D and D^2 are lower triangular matrices.

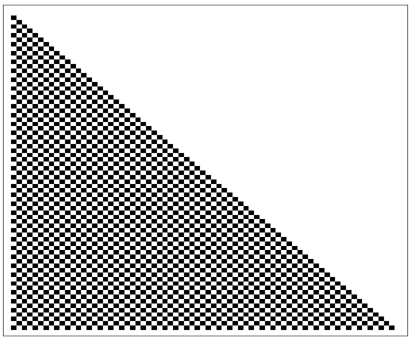


Figure 1:The structure of D for odd value of M

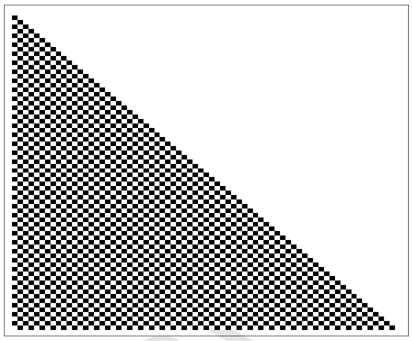


Figure 2: The structure of D for even value of M

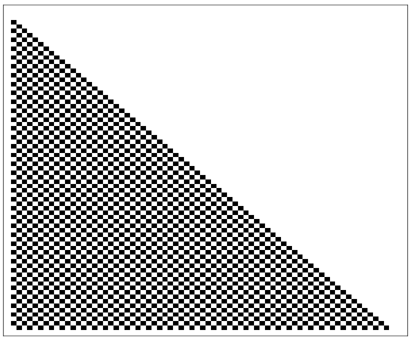


Figure 3:The structure of D^2 for odd value of M

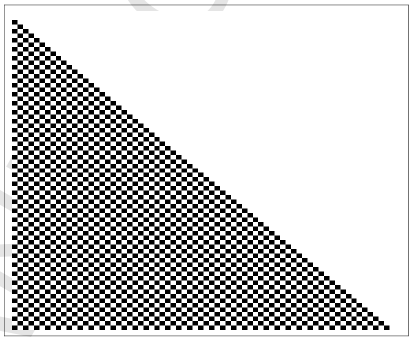


Figure 4:The structure of D^2 for even value of M

So one of advantages of the derivative operational matrix is that it allow us to solve the linear system (42) with much less computational cost.

Remark 2 The computational cost of our proposed method in section 3 is based on the counting of multiplication and summation operations according to [60]. The computational cost is mainly consisted of (i) construction of linear system of equation 42 and (ii) the linear system solver.

(i): In the first part, by considering of the sparsity of D and D^2, we have the following flops.

1. The calculation of long operations in term $\rho\mathcal{X}DP^{\alpha}$ for odd M is $\frac{(M+1)^2(M-1)}{4}+\frac{1}{2}(M+1)^2+M(M+1)^2+(M+1)^2$ and for even M is $\frac{M(M+1)(M+2)}{4}+M(M+1)^2+(M+1)^2$. For large M, the term M^3 is dominant one.

2. The long operations in $\Upsilon\mathcal{X}P^{\alpha}$ and FP^{α} is respectively $(M+1)^2(M+1)+(M+1)^2$ and $(M+1)(M+1)^2$. In this part for large M, the term M^3 is dominant.

3. The number of long operations in $(D^2)^T\mathcal{X}P^{\alpha}$ is $\frac{M(M+1)}{2}+\frac{M(M-2)(M+1)}{4}+(M+1)^3$ for even M and $\frac{(M+1)^2(M-1)}{8}+(M+1)^2(M+1)$ for odd M. In this case for large M, the term M^3 is dominant.

So, the linear system (42) can be obtained efficiently in $O(M^3)$ flops.

(ii): Gaussian elimination method which is used for solving linear system (42) is involved approximately $\frac{4}{3}(M+1)^3+\frac{1}{2}(M+1)^2$ long operations [60]. So, this equation is solved in $O(\frac{4}{3}M^3)$ flops.

4 Error Analysis

In this section, we present an upper error bound for the method. Assume that $\bar{\xi}(x,t)$ is a bivariate polynomial that interpolate $\xi(x,t)$ at points (x_i,t_j), we have from [61]

$$\xi(x,t)-\bar{\xi}(x,t)=\frac{\partial^{M+1}\xi(\zeta,t)}{\partial x^{M+1}(M+1)!}\Pi_{i=1}^{M+1}(x-x_i)+\frac{\partial^{M+1}\xi(x,\tau)}{\partial t^{M+1}(M+1)!}\Pi_{i=1}^{M+1}(t-t_j) \tag{46}$$

$$-\frac{\partial^{2M+2}\xi(\zeta^{'},\tau^{'})}{\partial x^{M+1}\partial t^{M+1}((M+1)!)^2}\Pi_{i=1}^{M+1}(x-x_i)\Pi_{j=1}^{M+1}(t-t_j),$$

where ζ, $\zeta^{'}$, τ and $\tau^{'}$ belong to interval $[0,1]$. Now if we assume that $\Omega=[0,1]\times[0,1]$ and

$$\overline{K}=\max\left\{\max_{(x,t)\in\Omega}\frac{\partial^{M+1}\xi(x,t)}{\partial x^{M+1}},\max_{(x,t)\in\Omega}\frac{\partial^{M+1}\xi(x,t)}{\partial t^{M+1}},\max_{(x,t)\in\Omega}\frac{\partial^{M+1}\xi(x,t)}{\partial x^{M+1}\partial t^{M+1}}\right\}, \tag{47}$$

as a result, we obtain

$$\left|\xi(x,t)-\bar{\xi}(x,t)\right|\leq\frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right). \tag{48}$$

Proposition 1. [62]Let $\xi(x,t)$ be a sufficiently smooth function on $L^2[0,1]$ that approximated by Legendre polynomials as $\xi(x,t)\simeq\psi^T(x)\mathcal{X}\psi(t)$, then the upper bound to estimate the error is as

$$\left\|\xi(x,t)-\psi^T(x)\mathcal{X}\psi(t)\right\|_2\leq\frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right). \tag{49}$$

By utilizing proposition 1, the following result is obtained.

Theorem 5. *Let $\xi(x,t)$ be the exact solution of fractional differential equation (1)-(3) and $\xi^{\mathcal{M}}(x,t)=\phi^T(x)\mathcal{X}\phi(t)$ be its approximation obtained by the method presented in section 3, then*

$$\left\|\xi(x,t)-\psi^T(x)\mathcal{X}\psi(t)\right\|_2\leqslant\frac{5\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right). \tag{50}$$

where $\overline{K}$ is given in (47) and M is the maximum order of Legendre polynomials.

Proof. It is evident that the exact solution $\xi(x,t)$ of fractional differential equation (1)-(3) satisfies in equation (38). Also, using approximations (37) and (39), we have

$$\psi^T(x)\mathcal{X}\psi(t)+\rho\psi^T(x)\mathcal{X}DP^{\alpha}\psi(t)+\Upsilon\psi^T(x)\mathcal{X}P^{\alpha}\psi(t)-\psi^T(x)(D^2)^T\mathcal{X}P^{\alpha}\psi(t)$$

$$=\psi^T(x)G\psi(t)+\psi^T(x)FP^{\alpha}\psi(t). \tag{51}$$

Substracting equation (51) from(38) yields

$$\left\|\xi(x,t)-\psi^T(x)\mathcal{X}\psi(t)\right\|_2\leq\|\overline{G}(x,t)-\psi^T(x)G\psi(t)\|_2+|\rho|\|I_t^{\alpha}\frac{\partial\xi(x,t)}{\partial t}-\psi^T(x)\mathcal{X}DP^{\alpha}\psi(t)\|_2 \tag{52}$$

$$+|\Upsilon|\|I_t^{\alpha}\xi(x,t)-\psi^T(x)\mathcal{X}P^{\alpha}\psi(t)\|_2+\|I_t^{\alpha}\frac{\partial^2\xi(x,t)}{\partial x^2}-\psi^T(x)(D^2)^T\mathcal{X}P^{\alpha}\psi(t)\|_2$$

$$+\|I_t^{\alpha}\mathcal{F}(x,t)-\psi^T(x)FP^{\alpha}\psi(t)\|_2$$

By neglecting the error of the operational matrix of fractional integration and using prepositon 1, the following estimates will be obtained

$$\left\|g(x,t)-\psi^T(x)G\psi(t)\right\|_2 \le \frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right), \tag{53}$$

$$\left\|I_t^\alpha \frac{\partial \xi(x,t)}{\partial t}-\psi^T(x)\mathcal{X}DP^\alpha\psi(t)\right\|_2 \le \frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right), \tag{54}$$

$$\left\|I_t^\alpha \frac{\partial \xi(x,t)}{\partial t}-\psi^T(x)\mathcal{X}DP^\alpha\psi(t)\right\|_2 \le \frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right), \tag{55}$$

$$\left\|I_t^\alpha \frac{\partial^2 \xi(x,t)}{\partial x^2}-\psi^T(x)(D^2)^T\mathcal{X}P^\alpha\psi(t)\right\|_2 \le \frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right), \tag{56}$$

$$\left\|I_t^\alpha \mathcal{F}(x,t)-\psi^T(x)FP^\alpha\psi(t)\right\|_2 \le \frac{\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right). \tag{57}$$

Finally, using equations (53)-(57) we obtain

$$\left\|\xi(x,t)-\psi^T(x)\mathcal{X}\psi(t)\right\|_2 \leqslant \frac{\overline{K}(|\rho|+|\Upsilon|+3)}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right). \tag{58}$$

By assuming that $|\rho| \leqslant 1$ and $|\Upsilon| \leqslant 1$, we obtain

$$\left\|\xi(x,t)-\psi^T(x)\mathcal{X}\psi(t)\right\|_2 \leqslant \frac{5\overline{K}}{2^M(M+1)!}\left(2+\frac{1}{2^2(M+1)!}\right). \tag{59}$$

For sufficiently large M, this error bound tends to zero. □

5 Numerical experiments

In this section, the proposed scheme is implemented to construct solutions for variants of fractional differential equations. The results demonstrate the effectiveness of the presented method. Assume that E^ξ shows absolute error.

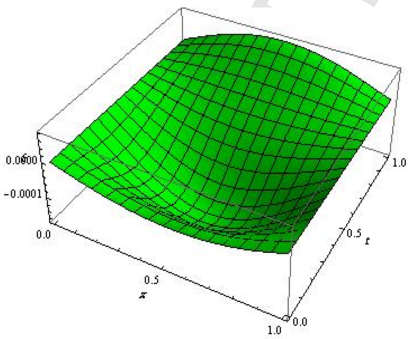


Figure 5: E^ξ for $\alpha = 1.5$ and $M = 6$

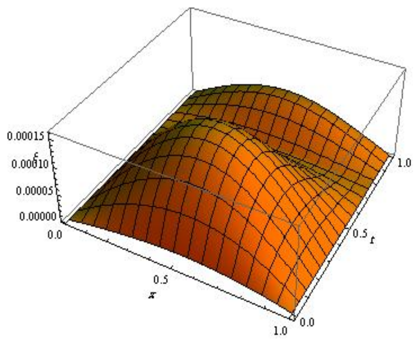


Figure 6: E^ξ for $\alpha = 1.5$ and $M = 7$

Example 1

As the fist example, consider the following linear time fractional Klein-Gordon equation[63]

$$\frac{\partial^\alpha \xi(x,t)}{\partial t^\alpha} + \xi(x,t) = \frac{\partial^2 \xi(x,t)}{\partial t^2} + \mathcal{F}(x,t). \tag{60}$$

with boundary conditions

$$\xi(0,x) = 0, \quad \xi(1,x) = 0, \tag{61}$$

and initial conditions

$$\xi(x,0) = 0, \quad \frac{\partial \xi(x,0)}{\partial t} = 0, \tag{62}$$

and

$$\mathcal{F}(x,t) = \frac{2t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)}(e-e^x)\sin(x) + t^2(2e-e^x)\sin(x) + 2t^2e^x\cos(x). \tag{63}$$

$\xi(x,t) = t^2(e-e^x)\sin(x)$ is the exact solution of this problem. This equation is solved for $M = 6$ and $M = 7$ and different values of α. The numerical results are given in Table 1 and in Figures 5 and 6. The exact and approximate solutions are shown in Figures 7 and 8.

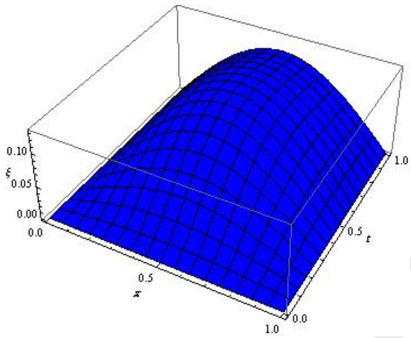


Figure 7: Exact solution for example 1

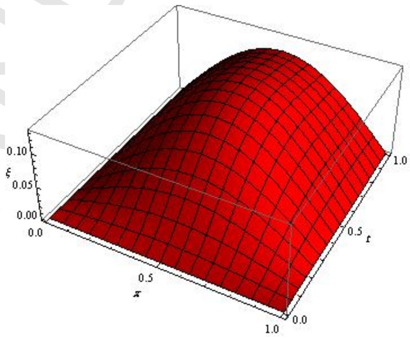


Figure 8: Approximate solution for example 1

Example 2

Consider the fractional dissipative Klein-Gordon equation [63]

$$\frac{\partial^\alpha \xi(x,t)}{\partial t^\alpha} + \frac{\partial \xi(x,t)}{\partial t} + \xi(x,t) = \frac{\partial^2 \xi(x,t)}{\partial x^2} + \mathcal{F}(x,t).$$

with initial condition

$$\xi(0,t) = 0, \quad \frac{\partial \xi}{\partial t}(0,t) = 0,$$

and boundary condition

$$\xi(x,0) = 0, \quad \xi(x,1) = 0,$$

and

$$\mathcal{F}(x,t) = \frac{2t^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} x\sin(x-1) + 2tx\sin(x-1) + t^2x\sin(x-1) - t^2(2cos(x-1)) - x\sin(x-1)).$$

The exact solution of this problem is $\xi(x,t) = t^2 x\sin(x-1)$. Table 2 represents the results of using propounded approach for $M = 5$ and various values of α. The numerical results of this example are compared with those in [63] in table 3.

			$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.9$
t	x	ξ_{Exact}	$\xi_{Approximate}$	$\xi_{Approximate}$	$\xi_{Approximate}$
0.2	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	0.0141936	0.0142381	0.0142909	0.0142717
	0.5	0.0205110	0.0205748	0.0206538	0.0206252
	0.75	0.0163943	0.0164398	0.0164997	0.0164811
	1	0.0000000	0.0000000	0.0000000	0.0000000
0.5	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	0.0887102	0.0887181	0.0887920	0.0888142
	0.5	0.1281940	0.1282040	0.1283060	0.1283400
	0.75	0.1024640	0.1024720	0.1025410	0.1025650
	1	0.0000000	0.0000000	0.0000000	0.0000000
0.8	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	0.2270980	0.2270890	0.2270860	0.2271240
	0.5	0.3281760	0.3281640	0.328160	0.3282090
	0.75	0.2623080	0.2623000	0.2622980	0.2623320
	1	0.0000000	0.0000000	0.0000000	0.0000000

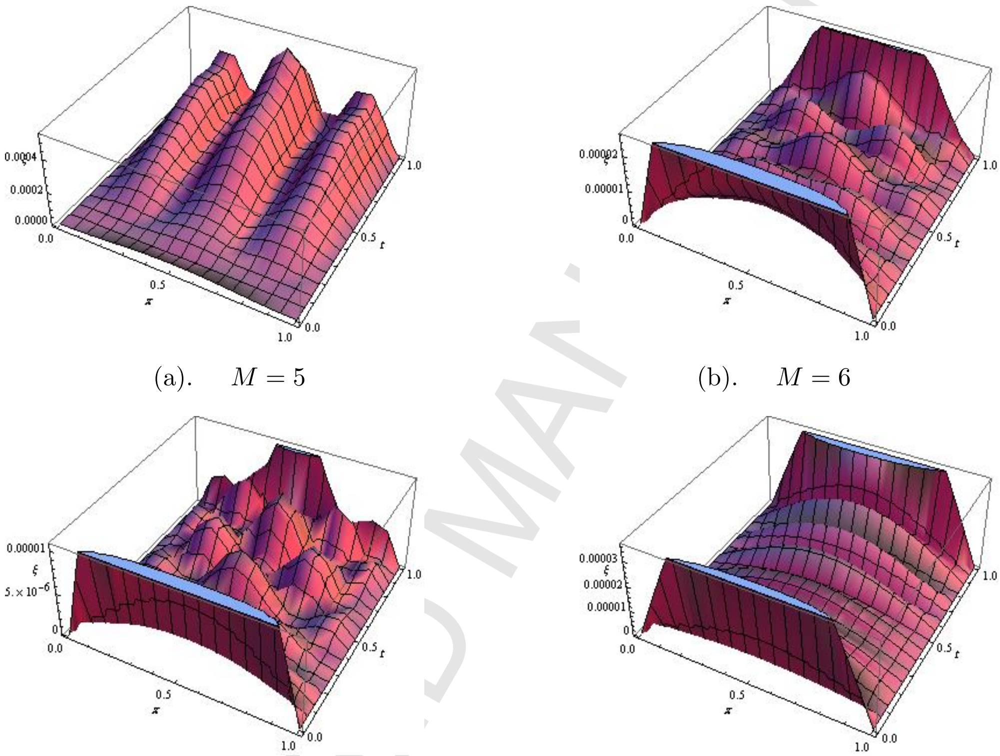
Table 1: Results of example 1 for $M = 7$

			$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 1.9$
t	x	ξ_{Exact}	$\xi_{Approximate}$	$\xi_{Approximate}$	$\xi_{Approximate}$
0.2	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	−0.0068164	−0.0068386	−0.0068570	−0.0068493
	0.5	−0.0095885	−0.0096193	−0.0096462	−0.0096356
	0.75	−0.0074221	−0.0074448	−0.0074646	−0.0074572
	1	0.0000000	0.0000000	0.0000000	0.0000000
0.5	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	−0.0426024	−0.0426158	−0.0426445	−0.0426395
	0.5	−0.0599282	−0.0599404	−0.0599806	−0.0599808
	0.75	−0.0463882	−0.0464011	−0.0464294	−0.0464247
	1	0.0000000	0.0000000	0.0000000	0.0000000
0.8	0	0.000000	0.000000	0.000000	0.000000
	0.25	−0.109062	−0.109069	−0.109067	−0.109073
	0.5	−0.153416	−0.153409	−0.153405	−0.153431
	0.75	−0.118754	−0.118761	−0.118758	−0.118764
	1	0.000000	0.000000	0.000000	0.000000

Table 2: Results of example 2 for $M = 5$

(x,t)	$\xi_{approximate}$[63]	$\xi_{approximate}$	ξ_{Exact}
$(\frac{1}{8},1)$	-0.095847	-0.0959423	-0.095942
$(\frac{2}{8},1)$	-0.170240	-0.170409	-0.170409
$(\frac{3}{8},1)$	-0.219193	-0.219410	-0.219411
$(\frac{4}{8},1)$	-0.239474	-0.239712	-0.239712

Table 3: Comparision of numerical results for $\alpha = 1.25$



(a). $M = 5$ (b). $M = 6$

(c). $M = 7$ (d). $M = 8$

Figure. 9 Absolute Errors E^{ξ} for $\alpha = 1.9$ and different values of M.

Example 3

As the last example, consider the following fractional diffusion-wave equation [31] and [34]

$$\frac{\partial^\alpha \xi}{\partial t^\alpha} = \frac{\partial^2 \xi}{\partial x^2} + \sin(\pi x), \quad 0 \leq x \leq 1, 0 \leq t \leq 1, \quad 1 < \alpha \leq 2.$$

with the initial and boundary conditions

$$\xi(x,0) = 0, \quad \frac{\partial \xi(x,0)}{\partial t} = 0, \quad 0 \leq x \leq 1,$$

		$\alpha = 1.25$		$\alpha = 1.75$	
t	x	ξ_{Exact}	$\xi_{Approximate}$	ξ_{Exact}	$\xi_{Approximate}$
0.2	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	0.0542538	0.0543029	0.0242284	0.0242339
	0.5	0.0767264	0.0767959	0.0342641	0.0342719
	0.75	0.0542538	0.0543029	0.0242284	0.0242339
	1	0.0000000	0.0000000	0.0000000	0.0000000
0.5	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	0.0789221	0.0789479	0.0864843	0.0864867
	0.5	0.1116130	0.1116490	0.1223070	0.1223110
	0.75	0.0789221	0.0789479	0.0864843	0.0864867
	1	0.0000000	0.0000000	0.0000000	0.0000000
0.8	0	0.0000000	0.0000000	0.0000000	0.0000000
	0.25	0.0761343	0.0761451	0.1140370	0.1140380
	0.5	0.1076700	0.1076850	0.1612720	0.1612730
	0.75	0.0761343	0.0761451	0.1140370	0.1140380
	1	0.0000000	0.0000000	0.0000000	0.0000000

Table 4: Results of example 3 for $M = 6$

$$\xi(0,t) = 0 \quad \xi(1,t) = 0, \quad 0 \leq t \leq 1.$$

In this case, the exact solution of this problem is as follows:

$$\xi(x,t) = \frac{1}{\pi^2}(1 - Q_\alpha(-\pi^2 t^\alpha))\sin(\pi x),$$

in which

$$Q_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Table 4 exhibits the results of this example for $M = 6$ and various values of α. Absolute errors of this example for $\alpha = 1.9$ and different values of M are shown in Figure 9. As the figures show the absolute error decreases with the increase in the order of Legendre polynomials. The results of solving this example by the proposed method and comparing with the exact solution and the methods in [31] and [34] are given in Table 5.

6 Concluding remarks

Fractional differential equations have found applications in many different fields. In this paper, the operational matrices of derivative and fractional Riemann-Liouville integration with Legendre polynomials

(x,t)	$\xi_{approximate}$[31]	$\xi_{approximate}$[34]	$\xi_{approximate}$	ξ_{Exact}
$(\frac{1}{8},1)$	4.3724×10^{-2}	4.34346×10^{-2}	4.32696×10^{-2}	4.32436×10^{-2}
$(\frac{2}{8},1)$	8.0607×10^{-2}	8.02566×10^{-2}	7.99037×10^{-2}	7.99519×10^{-2}
$(\frac{3}{8},1)$	1.0532×10^{-1}	1.04860×10^{-1}	1.04462×10^{-1}	1.04399×10^{-1}
$(\frac{4}{8},1)$	1.1400×10^{-1}	1.13500×10^{-1}	1.13069×10^{-1}	1.13001×10^{-1}

Table 5: Comparision of numerical results for $\alpha = 1.5$

have been successfully applied to compute approximate solutions of some fractional partial differential equations. As test examples, the linear time fractional Klein-Gordan equation, dissipative Klein-Gordan equation and diffusion-wave equation were considered. The numerical results demonstrated that the presented scheme provide approximate solutions in an acceptable agreement with exact solutions. Moreover, results indicated that the propounded approach leads to a better approximation as the order of Legendre polynomials increases. For future works this work can also be generalized and verified for more complicated linear, nonlinear or high dimensions problems. It is worth mentioning that the numerical solutions were obtained using Mathematica 11 software.

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- Applications of fractional partial differential equations appears in fluid mechanics.
- The Klein-Gordon and diffusion-wave are two important kinds of these equations.
- The computational cost of the presented numerical method is derived.
- This approach based on the operational matrix of derivative which is sparse and cause the implementation run faster.
- Comparison with other existed method show the superiority of the presented numerical method.