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Ranking DMUs by using the upper and lower bounds of the normalized efficiency in data envelopment analysis

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ABSTRACT

In data envelopment analysis, the existing methods for measuring the relative efficiencies of decision making units (DMUs) are to compare DMUs relative to the best or the worst of all DMUs. In this paper, we consider both the best DMU and the worst DMU as the reference DMUs and propose the normalized efficiency. Further, from the optimistic and pessimistic viewpoints, we construct two DEA models to obtain the upper and lower bounds of the normalized efficiency and then achieve an interval efficiency evaluation to rank all DMUs completely. Finally, two examples are presented to illustrate the performance of the interval efficiency evaluation.

1. Introduction

Data envelopment analysis (DEA) has been proved to be an effective approach for measuring the performance of a group of decision making units (DMUs) with multiple inputs and multiple outputs. For a DMU, the CCR efficiency, developed by Charnels, Cooper and Rhodes (1978), is achieved by maximizing the ratio of the weighted sum of its outputs to that of its inputs under the constraint that the ratio should not exceed one for every DMU. Accordingly, the CCR efficiency is regarded as the best relative efficiency.

The CCR efficiency evaluation can classify all DMUs into two groups, namely CCR efficient units and CCR inefficient units, but cannot rank all the DMUs completely. In recent years, a variety of DEA methods have been proposed to rank the performance of all DMUs. Adler, Friedman, and Sinuany-Stern (2002) divided these ranking methods into six areas. Cross-efficiency evaluation and super efficiency evaluation are the first and the second areas. Cross-efficiency evaluation, first proposed by Sexton, Silkman, and Hogan (1986), requests each DMU not only to be self-evaluated but also to be peer-evaluated. Specifically, based on the CCR model, a DMU determines a set of weights to evaluate the other DMUs. Yet, due to the non-uniqueness of the CCR optimal weights, the secondary goals have to be proposed to deal with the non-uniqueness issue. Doyle and Green (1994, 1995a) constructed several aggressive or benevolent cross-efficiency models. For more contributions to the cross-efficiency evaluation, readers are referred to the literature (Chen 2002; Contreras 2012; Liang, Wu, Cook, & Zhu, 2008a, 2008b; Oral, Amin, & Oukil, 2015; Wang & Chin 2010; Wu, Sun, & Liang, 2012; Yang, Ang, Xia, & Yang, 2012; Jeong and OK 2013; Hong and Jeong 2017).

Andersen and Petersen (1993) considered a reference technology spanned by all the other DMUs except the evaluated DMU and then achieved the super efficiency evaluation. Indeed, when DMUs are evaluated, the reference technology is crucial. Doyle and Green (1995b) pointed out three reference points which occur naturally in everyday comparison, namely comparison relative to the best, to the average or to the worst of the rest. For each of these reference points of comparison, they presented two DEA models to obtain the best performance and the worst performance of each DMU, respectively, and then constructed the upper and lower bound evaluation.

The traditional DEA models are usually built to achieve the best performance of DMUs from the optimistic viewpoint. Accordingly, the maximum ratio of the weighted sum of outputs to the weighted sum of inputs under some constraints is called the best relative efficiency or the optimistic efficiency. In fact, the worst relative efficiency or the pessimistic efficiency of a DMU, namely the minimum ratio of the weighted sum of outputs to that of inputs under some constraints, should still be paid enough attention to. The best relative efficiency and the worst relative efficiency measure the two kinds of extreme performances of a DMU. It is easily biased if considering only the best relative efficiency or the worst relative efficiency while neglecting the other one.

Recently, many pessimistic DEA models have been presented to obtain the pessimistic efficiency of a DMU. For example, Entani, Maeda, and Tanaka (2002) proposed a pessimistic DEA model to minimize the efficiency of each DMU under the constraint that the maximum

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efficiency among all DMUs is one, and achieved an interval efficiency evaluation from the optimistic and pessimistic viewpoints. Wang, Chin, and Yang (2007) proposed the worst relative efficiency, which is measured by minimizing the efficiency of each DMU within the efficiency range of greater than or equal to one, and then achieved a geometric average efficiency evaluation based on the worst relative efficiency as well as the CCR efficiency. Additionally, Wang and Yang (2007) introduced a virtual anti-ideal DMU to obtain a lower bound, which is further improved by Azizi and Jahed (2011), and developed the upper and lower bound evaluation under the constraint that the efficiency of each DMU is not greater than 1 and not less than the lower bound. Toloo and Tichý (2015) not only proposed the multiplier form of selecting model to obtain the maximum efficiency, but also applied the envelopment form to achieve the maximum discrimination between efficient units, which is a kind of the pessimistic efficiency. Additionally, for DMUs with imprecise data, Despotis and Smirlis (2002) achieved the upper and lower bounds for efficiency scores of DMUs. Further, Azizi, Kordrostami, and Amirteimoori (2015) and Toloo, Keshavarz, and Hatami-Marbini (2017) both presented optimistic and pessimistic perspectives for obtaining efficiency evaluations. For DMUs with random inputs and random outputs, Liu, Wang, and Lyu (2017) developed two stochastic DEA models to obtain the upper and lower bounds of the quantile efficiency and achieved an interval efficiency evaluation.

Up to now, DEA models usually evaluate a DMU by comparing it relative to another DMU, such as the best or the worst of all DMUs. In everyday comparison, a superior object must be far from the worst object as well as being close to the best. Stimulated by this idea, we consider both the best DMU and the worst DMU as the reference DMUs and propose a new relative efficiency, namely the normalized efficiency. Further, from the optimistic viewpoint and pessimistic viewpoint respectively, we construct two DEA models to obtain the upper and lower bounds of the normalized efficiency and then achieve an interval efficiency evaluation for all the DMUs.

The rest of the paper is organized as follows: Section 2 briefly reviews the different formulations of DEA models. The best normalized efficiency evaluation model and the worst normalized efficiency evaluation model are proposed and achieved in Section 3 and Section 4, respectively. Based on the two models, the interval evaluation is developed in Section 5. Section 6 presents two examples to illustrate the proposed approach. Concluding remarks are offered in Section 7.

2. Different formulations of DEA models

Suppose there are n DMUs with m inputs and s outputs to be evaluated. Let $x_{ij} > 0\,(i = 1, 2, ..., m)$ and $y_{rj} > 0\,(r = 1, 2, ..., s)$ be the input and output values of DMU$_j$ $(j = 1, 2, ..., n)$. In vector notation:

$$\boldsymbol{x}_j = (x_{1j}, x_{2j}, ..., x_{mj})^T,\ \ \boldsymbol{y}_j = (y_{1j}, y_{2j}, ..., y_{sj})^T,\ \ j = 1, 2, ..., n.$$

For any evaluated DMU$_k$ $(k = 1, 2, ..., n)$ and the given weights of the m inputs and s outputs $\boldsymbol{v} = (v_1, v_2, ..., v_m)^T$, $\boldsymbol{u} = (u_1, u_2, ..., u_s)^T$, the ratio of the weighted sum of the outputs to that of the inputs, namely $\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k$ is considered as the efficiency of DMU$_k$. The following CCR model, proposed by Charnels et al. (1978), seeks a set of weights $\boldsymbol{u}$, $\boldsymbol{v}$ to maximize the efficiency of DMU$_k$ under a set of constraints:

$$\begin{aligned} \overline{E}_k^{(1)} = &\max_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0} \frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k} \\ \text{s.t.}\ \ &\frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} \leqslant 1,\ \ j = 1, 2, ..., n. \end{aligned} \tag{1}$$

The CCR model is constructed from the optimistic viewpoint because it aims to maximize the efficiency of DMU$_k$ under a set of constraints. Considering the constraint $\boldsymbol{v}^T\boldsymbol{x}_k = 1$, Charnels et al. (1978) converted model (1) into a linear programming model as below:

$$\begin{aligned} \overline{E}_k^{(1)} = &\max \boldsymbol{u}^T\boldsymbol{y}_k \\ \text{s.t.}\ \ &\boldsymbol{v}^T\boldsymbol{x}_k = 1, \\ &\boldsymbol{u}^T\boldsymbol{y}_j - \boldsymbol{v}^T\boldsymbol{x}_j \leqslant 0, j = 1, 2, ..., n, \\ &\boldsymbol{u} \geqslant 0, \boldsymbol{v} \geqslant 0. \end{aligned} \tag{2}$$

If the optimal value of model (2), namely the CCR efficiency of DMU$_k$, is one, DMU$_k$ is called CCR weakly efficient. Now, denote by $S_{\max}$ the set of the index numbers of all the CCR weakly efficient DMUs. Precisely speaking, DMU$_k$ is CCR efficient if its CCR efficiency is 1 and there exists an optimal solution $(\boldsymbol{v}^*, \boldsymbol{u}^*)$ with $\boldsymbol{v}^* > 0$, $\boldsymbol{u}^* > 0$.

Moreover, Cooper, Thompson, and Thrall (1996) pointed out that CCR model is equivalent to the following model:

$$\max_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0} \frac{\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k}{\max_j \boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j}. \tag{3}$$

In model (3), the efficiency of DMU$_k$ is compared relative to the maximum efficiency of all DMUs. Model (3) seeks a set of input and output weights to maximize the ratio of the efficiency of DMU$_k$ to the maximum efficiency. Therefore, the CCR efficiency is regarded as the best relative efficiency. Meanwhile, a CCR weakly efficient DMU implies that there must be a set of input and output weights to satisfy that its efficiency is ranked first among all DMUs.

On the contrary, Entani et al. (2002) considered the minimization problem of model (3) and proposed a pessimistic DEA model as below:

$$\underline{E}_k^{(1)} = \min_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0} \frac{\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k}{\max_j \boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j} \tag{4}$$

To solve model (4), Entani et al. (2002) derived the following equivalent model:

$$\min_{i, r} \frac{y_{ik}/x_{rk}}{\max_j y_{ij}/x_{rj}}. \tag{5}$$

Based on models (1) and (4), Entani et al. (2002) developed the interval efficiency evaluation. To avoid the injustice caused by only one strategy, it is a good idea to evaluate all DMUs with interval efficiencies obtained from the optimistic and pessimistic viewpoints.

Further, Wang et al. (2007) modified the constraints of model (1) and proposed the worst relative efficiency evaluation model as below:

$$\begin{aligned} \underline{E}_k^{(2)} = &\min_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0} \frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k} \\ \text{s.t.}\ \ &\frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} \geqslant 1, j = 1, 2, ..., n. \end{aligned} \tag{6}$$

Considering that the constraint $\boldsymbol{v}^T\boldsymbol{x}_k = 1$, similarly, Wang et al. (2007) converted model (6) to the following linear programming model and achieved the worst relative efficiency evaluation:

$$\begin{aligned} \underline{E}_k^{(2)} = &\min \boldsymbol{u}^T\boldsymbol{y}_k \\ \text{s.t.}\ \ &\boldsymbol{v}^T\boldsymbol{x}_k = 1, \\ &\boldsymbol{u}^T\boldsymbol{y}_j - \boldsymbol{v}^T\boldsymbol{x}_j \geqslant 0, j = 1, 2, ..., n, \\ &\boldsymbol{u} \geqslant 0, \boldsymbol{v} \geqslant 0. \end{aligned} \tag{7}$$

If the optimal value of model (7) is one, Wang et al. (2007) called that DMU$_k$ is inefficient. Now, denote by $S_{\min}$ the set of the index numbers of all the inefficient DMUs. Further, it is easy to prove that model (6) and model (7) are equivalent to the following model:

$$\min_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0} \frac{\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k}{\min_j \boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j}. \tag{8}$$

From the pessimistic point of view, model (8) evaluates the efficiency of DMU$_k$ relative to the minimum efficiency of all DMU. Based on model (8), an inefficient DMU indicates that there must exist a set of weights such that its efficiency is ranked last among all DMUs.

Modifying model (8) to a maximization problem, we have a new optimistic DEA model as follows:

$$\overline{E}_k^{(2)} = \max_{\mathbf{u} \geqslant 0,\, \mathbf{v} \geqslant 0} \frac{\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k}{\min_j \boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j}. \tag{9}$$

Theorem 1. *Model (9) is equivalent to the following model:*

$$\max_{i,\, r} \frac{y_{ik}/x_{rk}}{\min_j y_{ij}/x_{rj}}. \tag{10}$$

The proof of Theorem 1 is shown in Appendix A. Based on model (10), we can obtain the optimistic efficiency $\overline{E}_k^{(2)}$ easily rather than solving linear programming problems.

In brief, models (1), (4), (6) and (9) all achieve the relative efficiency evaluation. Specifically, models (1) and (4) evaluate DMU$_k$ relative to the best of all the DMUs, while models (6) and (9) evaluate DMU$_k$ relative to the worst of all the DMUs. Additionally, models (1) and (9) aim to maximize the relative efficiency from the optimistic viewpoint, while models (4) and (6) minimize the relative efficiency from the pessimistic viewpoint.

3. The best normalized efficiency evaluation

In everyday comparison, the reference point is very important, and various reference points will change the result of comparison. For example, a father may be very pleased with his daughter if she gets 90 score in a test since her score is very close to the highest score, 95, in the class. Further, she tells her father that the lowest score in the class is 89, and then the father may change his evaluation of his daughter in the test. Naturally, he wishes his daughter's score to be far away from the lowest score as well as to be close to the highest.

In the traditional DEA models, a DMU is always compared relative to another particular DMU, such as the best or the worst of all DMUs. To evaluate DMUs comprehensively, we apply the idea of data normalization and consider both the best DMU and the worst DMU as the reference DMUs. It is noted that normalization is a method used to standardize the data with different ranges and is usually performed during the data preprocessing step.

For a set of output and input weights $\boldsymbol{u}$, $\boldsymbol{v}$, the following expression is the normalized form of the efficiency of DMU$_k$ and is referred as to the normalized efficiency of DMU$_k$:

$$\frac{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}{\max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}.$$

The above normalized efficiency reveals the relative position of DMU$_k$ relative to the best DMU and the worst DMU. For example, for a set of output and input weights $\boldsymbol{u}$, $\boldsymbol{v}$, if the normalized efficiency of DMU$_k$ is 0.5, then we know its efficiency $\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k$ is right in the middle of the maximum efficiency and the minimum efficiency of all the DMUs. Specially, if the normalized efficiency of DMU$_k$ is zero, then $\boldsymbol{u}^T\boldsymbol{y}_k/\boldsymbol{v}^T\boldsymbol{x}_k$ is the minimum among all the DMUs.

From the optimistic viewpoint, we propose the following model to maximize the normalized efficiency of DMU$_k$:

$$\theta_k^{\max} = \max_{\boldsymbol{u} \geqslant 0,\, \boldsymbol{v} \geqslant 0} \frac{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}{\max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}. \tag{11}$$

The optimal value of model (11) is called the best normalized efficiency of DMU$_k$, which indicates the best relative efficiency of DMU$_k$ in the interval formed by the best DMU and the worst DMU. It is certain that the best normalized efficiency is in the interval [0, 1].

It is worthy pointing out that the significance of the best normalized efficiency is different from the CCR efficiency. For example, if the CCR efficiency of a DMU is 0.5, then we know that 0.5 is its best relative efficiency compared to the best DMU, which means its weighted output-input ratio is right the half of the maximum ratio of all the DMUs based on the optimal weights. If the best normalized efficiency of DMU$_k$ is 0.5, then there must exist a set of optimal weights $\boldsymbol{u}^*$, $\boldsymbol{v}^*$ satisfying that the weighted output-input ratio of DMU$_k$ $(\boldsymbol{u}^*)^T\boldsymbol{y}_k/(\boldsymbol{v}^*)^T\boldsymbol{x}_k$ is right in the middle of the maximum ratio and minimum ratio of all DMUs, namely the mean of maximum ratio and minimum ratio.

Further, the best normalized efficiency has close relations to the CCR efficiency. First, it is easy to see that the best normalized efficiency of DMU$_k$ is 1 if and only if its CCR efficiency is 1. Second, if the CCR efficiency of a DMU is less than 1, we can easily have that its best normalized efficiency is less than its CCR efficiency.

Now, in order to obtain the best normalized efficiency of an arbitrary DMU, we resort to the following programming problem with a given parameter $i \in \{1, 2, \ldots, n\}$:

$$\theta_{ki}^{\max} = \max_{\boldsymbol{u} \geqslant 0,\, \boldsymbol{v} \geqslant 0} \frac{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k} - \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}{\max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}$$
$$\text{s.t.} \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i} \leqslant \frac{\boldsymbol{u}^T\boldsymbol{y}_l}{\boldsymbol{v}^T\boldsymbol{x}_l},\ l = 1, 2, \ldots, n; l \neq i. \tag{12}$$

The above model aims to seek a set of input and output weights to maximize the normalized efficiency of DMU$_k$ under the constraint that the efficiency of DMU$_i$ is just the worst of all the DMUs. Further, assume that the worst relative efficiency of DMU$_i$ determined by model (7) is just 1, namely $i \in S_{\min}$. This means that there must exist a set of input and output weights such that the efficiency of DMU$_i$ is ranked last among all the DMUs, and then the feasible region of model (12) is not empty. Hence, calculating model (12) for all $i \in S_{\min}$, we can obtain the best normalized efficiency of DMU$_k$ as follows:

$$\theta_k^{\max} = \max_{i \in S_{\min}} \{\theta_{ki}^{\max}\}. \tag{13}$$

To solve model (12), we reduce it to the following model:

$$\begin{aligned} \theta_{ki}^{\max} = \max\ & \frac{\boldsymbol{u}^T\boldsymbol{y}_k - 1}{b - 1} \\ \text{s.t.}\ & \boldsymbol{v}^T\boldsymbol{x}_k = 1, \\ & \boldsymbol{u}^T\boldsymbol{y}_i - \boldsymbol{v}^T\boldsymbol{x}_i = 0, \\ & -\boldsymbol{u}^T\boldsymbol{y}_l + \boldsymbol{v}^T\boldsymbol{x}_l \leqslant 0,\ l = 1, 2, \ldots, n; l \neq i, \\ & \boldsymbol{u}^T\boldsymbol{y}_j - b\boldsymbol{v}^T\boldsymbol{x}_j \leqslant 0,\ j = 1, 2, \ldots, n; j \neq i, \\ & \boldsymbol{u} \geqslant 0,\ \boldsymbol{v} \geqslant 0,\ b_1 \leqslant b \leqslant b_2. \end{aligned} \tag{14}$$

Here, $b_1 = \max_k\{\underline{E}_k^{(2)}\}$, $b_2 = \max_k\{\overline{E}_k^{(2)}\}$, where $\underline{E}_k^{(2)}$, $\overline{E}_k^{(2)}$ are the optimal values of models (7) and (9), respectively.

Theorem 2. *Models (12) and (14) are equivalent.*

The proof of Theorem 2 is shown in Appendix A.

In model (14), if the variable b is a given value, then model (14) will be a linear programming model. Let $f_{ki}(b)$ be the optimal value of the following linear model:

$$\begin{aligned} f_{ki}(b) = \max\ & \frac{\boldsymbol{u}^T\boldsymbol{y}_k - 1}{b - 1} \\ \text{s.t.}\ & \boldsymbol{v}^T\boldsymbol{x}_k = 1, \\ & \boldsymbol{u}^T\boldsymbol{y}_i - \boldsymbol{v}^T\boldsymbol{x}_i = 0, \\ & -\boldsymbol{u}^T\boldsymbol{y}_l + \boldsymbol{v}^T\boldsymbol{x}_l \leqslant 0,\ l = 1, 2, \ldots, n; l \neq i, \\ & \boldsymbol{u}^T\boldsymbol{y}_j - b\boldsymbol{v}^T\boldsymbol{x}_j \leqslant 0,\ j = 1, 2, \ldots, n; j \neq i, \\ & \boldsymbol{u} \geqslant 0,\ \boldsymbol{v} \geqslant 0, \end{aligned} \tag{15}$$

where b is a parameter.

Evidently, model (14) can be expressed as below:

$$\theta_{ki}^{\max} = \max_{b_1 \leqslant b \leqslant b_2} f_{ki}(b) \tag{16}$$

Then we can use the one-dimensional search method to obtain the global optimal value of model (16). Accordingly, we achieve the best normalized efficiency evaluation based on models (13), (15) and (16). Basically, for the best normalized efficiency evaluation approach, the complete algorithm is summarized by the following steps:

Step 1: For each DMU_k, $k = 1, 2, \ldots, n$, compute $\overline{E}_k^{(1)}$ by solving the linear program model (2). If $\overline{E}_k^{(1)} = 1$, then the best normalized efficiency of DMU_k is 1. Otherwise, go to steps 2–4.
Step 2: For each DMU_k, $k = 1, 2, \ldots, n$, compute $\underline{E}_k^{(2)}$, $\overline{E}_k^{(2)}$ by solving model (7) and model (9). Then we obtain $S_{\min} = \{k|\underline{E}_k^{(2)} = 1\}$, $b_1 = \max_k\{\underline{E}_k^{(2)}\}$ and $b_2 = \max_k\{\overline{E}_k^{(2)}\}$.
Step 3: For $i \in S_{\min}$, solve $\theta_{ki}^{\max} = \max_{b_1 \leqslant b \leqslant b_2} f_{ki}(b)$ by using one-dimensional search method, where $f_{ki}(b)$ is obtained by solving linear model (15).
Step 4: The best normalized efficiency of DMU_k is obtained by formula (13).

4. The worst normalized efficiency evaluation

The best normalized efficiency evaluation is developed from the optimistic viewpoint. On the other hand, from the pessimistic viewpoint, we convert model (11) to the following minimization problem:

$$\theta_k^{\min} = \min_{\boldsymbol{u} \geqslant 0, \boldsymbol{v} \geqslant 0} \frac{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}{\max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}. \tag{17}$$

Model (17) intends to seek a set of input and output weights to minimize the normalized efficiency of DMU_k. Then the optimal value of model (17) is referred to as the worst normalized efficiency of DMU_k. Like the best normalized efficiency, the worst normalized efficiency is always in the interval [0,1] too. Specially, for DMU_k, if its worst relative efficiency calculated by model (7) is 1, then there must exist a set of input and output weights such that its efficiency is the minimum of all DMUs, and its worst normalized efficiency is certainly 0. Otherwise, if the worst relative efficiency of DMU_k calculated by model (7) is greater than 1, then it is sure that the worst normalized efficiency of DMU_k is greater than 0.

Note that model (17) is equivalent to the following model:

$$\theta_k^{\min} = \min_{\boldsymbol{u} \geqslant 0, \boldsymbol{v} \geqslant 0} 1 - \frac{\max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}}{\max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}. \tag{18}$$

To achieve the worst normalized efficiency evaluation, we develop the following model to minimize the normalized efficiency of DMU_k under the constraint that the efficiency of DMU_j is just the maximum among all the DMUs:

$$\theta_{kj}^{\min} = \min_{\boldsymbol{u} \geqslant 0, \boldsymbol{v} \geqslant 0} 1 - \frac{\frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}}{\frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j} - \min_i \frac{\boldsymbol{u}^T\boldsymbol{y}_i}{\boldsymbol{v}^T\boldsymbol{x}_i}}$$
$$\text{s.t.} \frac{\boldsymbol{u}^T\boldsymbol{y}_l}{\boldsymbol{v}^T\boldsymbol{x}_l} \leqslant \frac{\boldsymbol{u}^T\boldsymbol{y}_j}{\boldsymbol{v}^T\boldsymbol{x}_j}, \ l = 1, 2, \ldots, n; l \neq j. \tag{19}$$

As stated above, a CCR weakly efficient DMU means that there must exist a set of input and output weights such that its efficiency is ranked first among all the DMUs. To ensure that the feasible region of model (19) is not empty, it is sufficient and necessary that DMU_j is CCR weakly efficient. Hence, calculating model (19) for all $j \in S_{\max}$, we can express the worst normalized efficiency of DMU_k as follows:

$$\theta_k^{\min} = \min_{j \in S_{\max}} \{\theta_{kj}^{\min}\}. \tag{20}$$

Theorem 3. *Model (19) is equivalent to the following model:*

$$\begin{aligned} \theta_{kj}^{\min} &= \min 1 - \frac{1 - \boldsymbol{u}^T\boldsymbol{y}_k}{1 - a} \\ s.t.\ & \boldsymbol{v}^T\boldsymbol{x}_k = 1, \\ & \boldsymbol{u}^T\boldsymbol{y}_j - \boldsymbol{v}^T\boldsymbol{x}_j = 0, \\ & \boldsymbol{u}^T\boldsymbol{y}_l - \boldsymbol{v}^T\boldsymbol{x}_l \leqslant 0, \ l = 1, 2, \ldots, n; l \neq j, \\ & -\boldsymbol{u}^T\boldsymbol{y}_i + a \cdot \boldsymbol{v}^T\boldsymbol{x}_i \leqslant 0, \ i = 1, 2, \ldots, n; i \neq j, \\ & \boldsymbol{u} \geqslant 0, \boldsymbol{v} \geqslant 0, a_1 \leqslant a \leqslant a_2. \end{aligned} \tag{21}$$

Here, $a_1 = \min_k\{\underline{E}_k^{(1)}\}$, $a_2 = \min_k\{\overline{E}_k^{(1)}\}$, where $\underline{E}_k^{(1)}$, $\overline{E}_k^{(1)}$ are the optimal values of models (4) and (1) respectively.

The proof of Theorem 3 is shown in Appendix A. Just because of the decision variable a, model (21) is a nonlinear programming model. Like model (14), yet we can still use a linear programming and one-dimensional research method to obtain the global optimal value of model (21). For the worst normalized efficiency evaluation approach, we summary the complete algorithm as below:

Step 1: For each DMU_k, $k = 1, 2, \ldots, n$, compute $\underline{E}_k^{(2)}$ by solving the linear model (7). If $\underline{E}_k^{(2)} = 1$, then the worst normalized efficiency of DMU_k is 0. Otherwise, go to steps 2–4.
Step 2: For each DMU_k, $k = 1, 2, \ldots, n$, compute $\underline{E}_k^{(1)}$, $\overline{E}_k^{(1)}$ by model (5) and model (2). Then we obtain $S_{\max} = \{k|\overline{E}_k^{(1)} = 1\}$, $a_1 = \min_k\{\underline{E}_k^{(1)}\}$, and $a_2 = \min_k\{\overline{E}_k^{(1)}\}$.
Step 3: For $j \in S_{\max}$, solve model (21) based on one-dimensional search method and obtain $\theta_{kj}^{\min}$.
Step 4: Compute the worst normalized efficiency of DMU_k by formula (20).

Specially, if all the DMUs are measured by only one input and only one output, we have the following theorem about their best normalized efficiency evaluation and the worst normalized efficiency evaluation:

Theorem 4. *If $r = 1$ and $m = 1$, then the best normalized efficiency of each DMU is equal to its worst normalized efficiency, and is expressed as follows:*

$$\theta_k^{\max} = \theta_k^{\min} = \frac{\frac{y_k}{x_k} - \min_i \frac{y_i}{x_i}}{\max_j \frac{y_j}{x_j} - \min_i \frac{y_i}{x_i}}.$$

5. The interval efficiency evaluation

The worst normalized efficiency and the best normalized efficiency are the lower bound and the upper bound of the normalized efficiency of a DMU respectively. This means, $[\theta_k^{\min}, \theta_k^{\max}]$ is the range of the normalized efficiency of DMU_k. In order to make full use of the interval efficiency and have an overall assessment of the performance of all DMUs, we utilize Hurwicz criterion approach, which was introduced by Wang and Yang (2007), to rank interval efficiencies and calculate the weighted average value of the upper and lower bounds to rank DMUs.

Definition 1. Let α be a parameter in [0, 1] and $A_k = [\theta_k^{\min}, \theta_k^{\max}]$ be the interval efficiency of DMU_k, where $\theta_k^{\min}, \theta_k^{\max}$ are the worst normalized efficiency and the best normalized efficiency respectively. Then the Hurwicz index value of A_k is defined as:

$$\theta_k^H = \alpha\theta_k^{\max} + (1-\alpha)\theta_k^{\min}, \ k = 1, 2, \ldots, n. \tag{22}$$

Property 1. *If $\theta_i^{\max} > \theta_j^{\max}$ and $\theta_i^{\min} > \theta_j^{\min}$, then $\theta_i^H > \theta_j^H$ with* $\alpha \in [0, 1]$.

Specially, when α = 1 or 0, the Hurwicz index value becomes the best normalized efficiency or the worst normalized efficiency respectively. Here the parameter α can be considered as the assessor's level of optimism. If $\alpha = 0.5$, then the assessor is optimistic. On the contrary, the assessor is pessimistic when. If $\alpha = 0.5$, the assessor is completely neutral, and the Hurwicz index value is the average of the best normalized efficiency and the worst normalized efficiency. It is suggested that the parameter α is set to 0.5 if the assessor's level of optimism is unknown in the application of our approach.

Let $A_i = [\theta_i^{\min}, \theta_i^{\max}]$, $A_j = [\theta_j^{\min}, \theta_j^{\max}]$ be the interval efficiencies of DMU_i and DMU_j. For the given $\alpha \in [0, 1]$, if $\theta_i^H > \theta_j^H$, then we call DMU_i is superior to DMU_j, and denote $DMU_i \succ DMU_j$ and $A_i \succ A_j$. Based on Definition 1, the following property 1 is easily obtained.

Property 2. *Suppose A_j is included in A_i. That is, $\theta_i^{\max} \geqslant \theta_j^{\max}$, $\theta_i^{\min} \leqslant \theta_i^{\min}$ and there is at least an inequality in the two formulas. Let $\alpha_0 = \frac{\theta_j^{\min} - \theta_i^{\min}}{(\theta_i^{\max} - \theta_i^{\min}) - (\theta_j^{\max} - \theta_j^{\min})}$. Then, $\theta_i^H > \theta_j^H$ if and only if the parameter α is in $(\alpha_0, 1]$; $\theta_i^H < \theta_j^H$ if $\alpha \in [0, \alpha_0)$; otherwise, $\theta_i^H = \theta_j^H$ if α is just α_0.*

Proof. Based on Definition 1, the inequality $\theta_i^H > \theta_j^H$ means

$$\alpha \cdot \theta_i^{\max} + (1-\alpha) \cdot \theta_i^{\min} > \alpha \cdot \theta_j^{\max} + (1-\alpha) \cdot \theta_j^{\min},$$

which is equivalent to

$$\alpha\,[(\theta_i^{\max} - \theta_i^{\min}) - (\theta_j^{\max} - \theta_j^{\min})] > \theta_j^{\min} - \theta_i^{\min}.$$

If $\theta_i^{\max} \leqslant \theta_i^{\max}$, $\theta_i^{\min} \leqslant \theta_i^{\min}$ and there is at least an inequality in the two formulas, it is easy to have that $(\theta_i^{\max} - \theta_i^{\min}) - (\theta_j^{\max} - \theta_j^{\min}) > 0$ and $\alpha_0 = \frac{\theta_j^{\min} - \theta_i^{\min}}{(\theta_i^{\max} - \theta_i^{\min}) - (\theta_j^{\max} - \theta_j^{\min})} \in [0, 1]$. Hence, $\theta_i^H > \theta_j^H$ if and only if $\alpha \in (\alpha_0, 1]$. Similarly, we have that $\theta_i^H < \theta_j^H$ if and only if $\alpha \in [0, \alpha_0)$. Additionally, it is obvious that $\theta_i^H = \theta_j^H$ if α is just α_0. □

From Property 2, we have that the comparison of two sets of interval efficiencies will be affected by the parameter α. Further, Wang and Yang (2007) presented the following theorem to show the sensitivity analysis to α.

Theorem 5. *Let $A_i = [\theta_i^{\min}, \theta_i^{\max}]$ $(i = 1, 2, ..., n)$ be a set of interval efficiencies. For a given level of optimism, α_0, if the ranking is $A_{i_1} \succ A_{i_2} \succ ... \succ A_{i_n}$, then there exists an interval for level of optimism, α, which is determined by $(\alpha_L, \alpha_R) \cap [0, 1]$, where*

$$\alpha_L = \max_j \left\{ \frac{\theta_{i_{j+1}}^{\min} - \theta_{i_j}^{\min}}{(\theta_{i_j}^{\max} - \theta_{i_j}^{\min}) - (\theta_{i_{j+1}}^{\max} - \theta_{i_{j+1}}^{\min})} \,|\, (\theta_{i_j}^{\max} - \theta_{i_j}^{\min}) - (\theta_{i_{j+1}}^{\max} - \theta_{i_{j+1}}^{\min}) > 0 \right\},$$

$$\alpha_R = \min_j \left\{ \frac{\theta_{i_{j+1}}^{\min} - \theta_{i_j}^{\min}}{(\theta_{i_j}^{\max} - \theta_{i_j}^{\min}) - (\theta_{i_{j+1}}^{\max} - \theta_{i_{j+1}}^{\min})} \,|\, (\theta_{i_j}^{\max} - \theta_{i_j}^{\min}) - (\theta_{i_{j+1}}^{\max} - \theta_{i_{j+1}}^{\min}) < 0 \right\}.$$

When α varies within the above interval, the ranking among the interval efficiencies remains unchanged.

6. Numerical examples

Example 1. Consider a numerical example in Table 1, where five DMUs are evaluated in light of one input and two outputs.

Table 2 reports four kinds of interval efficiency evaluation results. The second column shows Entani et al.'s evaluation results (Entani et al., 2002) where the upper bound of each DMU is its CCR efficiency and the lower bound is calculated by model (5). Obviously, the maximum efficiency is at most one. Specially, the CCR efficiencies of DMU_4 and DMU_5 are both one. In fact, the two DMUs are CCR weakly efficient. It is worthy pointing out that the minimum efficiency of all the DMUs by the CCR model and model (5) varies with different examples. In this example, the minimum efficiency is the lower bound efficiency of DMU_1, namely 0.1333.

Table 1
Data for 5 DMUs with one input and two outputs.

DMU	x	y_1	y_2
1	1	2	7
2	1	4	6
3	1	5	15
4	1	9	16
5	1	15	7

Table 2
The interval evaluation results of 5 DMUs.

DMU	Entani et al.'s models	Model (7) and model (9)	Wang and Yang's models	Our normalized efficiency models
1	[0.1333, 0.4375]	[1.0000, 1.1667]	[0.3750, 0.4375]	[0.0000, 0.1000]
2	[0.2667, 0.4068]	[1.0000, 2.0000]	[0.3750, 0.3952]	[0.0000, 0.1538]
3	[0.3333, 0.9375]	[2.1875, 2.5000]	[0.8203, 0.9375]	[0.2308, 0.9000]
4	[0.6000, 1.0000]	[2.5625, 4.5000]	[0.9609, 1.0000]	[0.5385, 1.0000]
5	[0.4375, 1.0000]	[1.1667, 7.5000]	[0.4375, 0.7944]	[0.1000, 1.0000]

The interval efficiencies shown in the third column are achieved by models (7) and (9). Because models (7) and (9) both measure the efficiency of a DMU relative to the worst DMU, these efficiency evaluation results of the five DMUs are at least one. Similarly, the maximum efficiency of all the DMUs by models (7) and (9) differs in different examples. In this example, the maximum efficiency is 7.5, which is the upper bound efficiency of DMU_5.

The fourth column reports Wang and Yang's evaluation results (Wang and Yang, 2007). Wang and Yang (2007) introduced a virtual anti-ideal DMU to obtain a lower bound and then developed the interval efficiency evaluation of each DMU under the constraint that the efficiency of each DMU is not greater than 1 and not less than the lower bound. In this example, the input of the virtual anti-ideal DMU is still 1, and each output of the virtual anti-ideal DMU is the minimum of the five DMUs, namely 2 and 6, respectively. Then the lower bound is the CCR efficiency of the virtual anti-ideal DMU, namely 0.375.

It is well known that the CCR efficiency of a DMU will remain unchanged even if the inputs and outputs are multiplied or divided by the same nonnegative value. Yet Wang and Yang's evaluation does not possess this property. For example, if the input and the outputs of DMU_1 are divided by 2, then the output vector of the virtual anti-ideal will be modified as $(1, 3.5)^T$. Accordingly, the evaluation results of some DMUs achieved by Wang and Yang (2007) will be changed. For example, the interval efficiency evaluation of DMU_2 will be adjusted to [0.2188, 0.4068] from [0.3750, 0.3952].

The interval normalized efficiencies instituted by the worst normalized efficiency and the best normalized efficiency are shown in the fifth column. It is seen that the best normalized efficiencies of the CCR efficient DMUs, namely DMU_4 and DMU_5, are both 1 too, and the best normalized efficiencies of the other DMUs are less than their CCR efficiencies respectively. On the other hand, the worst normalized efficiencies of DMU_1 and DMU_2 are both 0, which means that there must exist a set of weights to make the efficiency of DMU_1 (or DMU_2) minimum of the five DMUs. In fact, the efficiency scores calculated by model (4) are also 1 and rank the last among the five DMUs, as can be seen in the third column of Table 2. It is worthy pointing out that the minimum and the maximum of the interval normalized efficiencies are certainly 0 and 1, respectively.

Now, we compare the CCR efficiency evaluation and the best normalized efficiency evaluation of the 5 DMUs. The best normalized efficiency of each DMU is smaller than its CCR efficiency except two CCR weakly efficient DMUs, namely DMU_4 and DMU_5. Additionally, in the second column of Table 2, the CCR efficiencies of DMU_1 and DMU_2 are 0.4375 and 0.4068 respectively. Then DMU_2 lags behind DMU_1 based on the CCR efficiency evaluation. It is interesting that the best normalized efficiency evaluation changes the rank orders of the two DMUs. Specifically, the best normalized efficiency of DMU_1 is 0.100 and

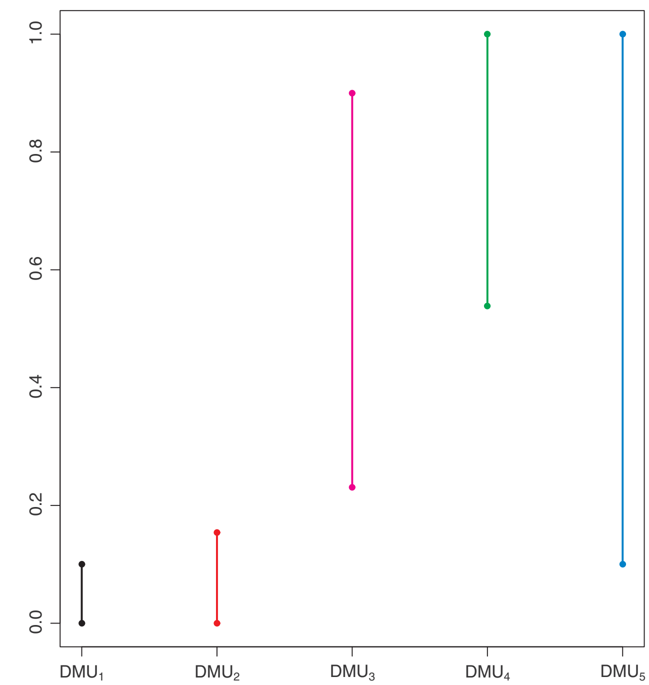

Fig. 1. The interval efficiencies of example 1.

smaller than that of DMU_2, namely 0.1538.

For each DMU, the worst normalized efficiency and the best normalized efficiency constitute an efficiency interval, which is graphically portrayed in Fig. 1. Obviously, the interval efficiency of DMU_4 is at the top of Fig. 1, and the interval efficiencies of DMU_1 and DMU_2 are both at the bottom of Fig. 1. Additionally, DMU_3 and DMU_5 have the large range of interval efficiencies. Specifically, for DMU_5, although its best normalized efficiency is 1, yet its worst normalized efficiency is just 0.1, which means that there is a set of weights making the efficiency of DMU_5 very close to the minimum of all DMUs. In fact, DMU_5 has the peculiar outputs. Specifically, the first output of DMU_5 is the maximum of all DMUs, whereas its second output is the second lowest. Thus, from the optimistic viewpoint, the best normalized efficiency model makes the highest evaluation for DMU_5. While the worst normalized efficiency model makes a low evaluation for it from the pessimistic viewpoint.

To rank all the DMUs, we consider interval efficiencies $A_k = [\theta_k^{\min}, \ \theta_k^{\max}]$, $i = 1, 2, ..., 5$, and calculate their Hurwicz index values with $\alpha = 0.25, \ 0.5, \ 0.75$, which are reported in Table 3. From it, we see that the ranking of the 5 DMUs is $DMU_4 \succ DMU_3 \succ DMU_5 \succ DMU_2 \succ DMU_1$ when $\alpha = 0.25, \ 0.5$. In fact, based on Theorem 3, the ranking remains unchanged if $\alpha \in (0, \ 0.5667)$, which means that the assessor is pessimistic or neutral. Yet, if the assessor is optimistic and α is set in $(0.5667, \ 1)$, then the ranking orders of DMU_3 and DMU_5 will exchange, which is seen in the fourth column of Table 3.

Example 2. Now, we evaluate the scientific research performance of 12 key universities of science and engineering in China. This data set is from an investigation of the science and technology work of China's universities in 2016, and is shown in Table 4, where the inputs are the number of academic and teaching staff (x_1) and the expenditure on science and technology (x_2) measured in thousand RMB, and the outputs are the numbers of the published scientific and technological works (y_1), academic papers published in foreign and national periodicals (y_2), intellectual property rights (y_3), and the amount of technology transfer (y_4) measured in thousand RMB.

Table 3
Hurwicz index values for 5 DMUs.

DMU	Hurwicz index values		
	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
1	0.0250 (5)	0.0500 (5)	0.0075 (5)
2	0.0385 (4)	0.0769 (4)	0.1154 (4)
3	0.3981 (2)	0.5654 (2)	0.7327 (3)
4	0.6538 (1)	0.7693 (1)	0.8846 (1)
5	0.3250 (3)	0.5500 (3)	0.7750 (2)

Table 4
Data for 12 universities of China.

DMU	University	x_1	x_2	y_1	y_2	y_3	y_4
1	Tsinghua University	5506	3,636,283	50	4976	48	500,525
2	Beihang University	2189	2,150,830	23	3536	28	1553
3	Beijing Institute of Technology	2413	2,102,809	28	2677	12	163,858
4	Tianjin University	2794	2,179,348	2	5751	13	14,925
5	Dalian University of Technology	2818	1,254,213	33	1878	8	1250
6	Northeastern University	2303	1,524,399	76	2062	13	44,000
7	Harbin Institute of Technology	3679	2,190,971	29	6892	65	12,480
8	Tongji University	5988	2,472,129	14	3542	30	2940
9	University of Science and Technology of China	2442	1,966,802	9	3665	2	35,860
10	South China University of Technology	2836	1,199,259	9	2780	4	45,069
11	University of Electronic Science and Technology of China	2257	1,149,589	22	2489	12	6134
12	Northwestern Polytechnical University	2249	1,473,336	11	311	14	10,670

For the scientific research performance of the 12 universities in China, four kinds of interval efficiency evaluation results are reported in Table 5. The best normalized efficiency and the worst normalized efficiency are the upper and lower bounds of our interval normalized efficiency respectively, which are shown in the fifth column of Table 5. It is seen that the maximum upper bound and the minimum lower bound of the interval normalized efficiencies of all DMUs are 1 and 0 respectively. Specifically, the best normalized efficiencies of DMU_1, DMU_3, DMU_4, DMU_6 and DMU_7 are 1. It is noted that the CCR efficiencies of the 5 DMUs are also 1, which is seen from the second column of Table 5. On the other hand, the worst normalized efficiencies of DMU_2, DMU_4, DMU_5, DMU_8, DMU_9, DMU_{10} and DMU_{12} are zero. For each of the seven DMUs, this means there must exist a set of weights to satisfy its efficiency to be the minimum of all DMUs.

For each interval efficiency evaluation, the lower and upper bounds are achieved from the pessimistic viewpoint and the optimistic viewpoint respectively. Thus, there may be significant differences between them for some DMUs. For example, based on each of the four kinds of interval efficiency evaluations shown in Table 5, the lower and upper bounds of DMU_4 differ greatly. Specifically, its CCR efficiency is 1, yet the efficiency achieved by model (5) is just 0.0184. The efficiencies achieved by model (7) and model (9) are 1 and 14.8849 respectively. It is more interesting that its worst and best normalized efficiencies are 0 and 1 respectively.

As stated above, the optimistic and pessimistic models are achieved from the optimistic viewpoint and the pessimistic viewpoint respectively. In our views, it is more reliable to evaluate and rank DMUs based on the optimistic efficiency and the pessimistic efficiency simultaneously. We calculate the Hurwicz index values of the four kinds of interval efficiencies with $\alpha = 0.5$, namely the average of the upper and lower bounds of interval efficiencies, and then obtain the ranking orders of the 12 DMUs, which are shown in Table 6.

In Table 6, three kinds of evaluation results except Wang and Yang' models put DMU_1 first. However, for some DMUs, there are differences among the ranking results of the four kinds of evaluation methods. To further analyze the averages of the four kinds of interval efficiency

Table 5
The interval efficiency evaluation results of 12 universities.

DMU	Entani et al.'s models	Model (7) and model (9)	Wang and Yang's models	Normalized efficiency models
1	[0.2752, 1.0000]	[1.8585, 204.9371]	[0.0558, 1.0000]	[0.2040, 1.0000]
2	[0.0052, 0.9285]	[1.0000, 15.6181]	[0.0300, 0.9285]	[0.0000, 0.9158]
3	[0.1924, 1.0000]	[1.1566, 153.0881]	[0.0347, 1.0000]	[0.0475, 1.0000]
4	[0.0184, 1.0000]	[1.0000, 14.8849]	[0.0300, 1.0000]	[0.0000, 1.0000]
5	[0.0049, 0.7343]	[1.0000, 28.6708]	[0.0300, 0.7343]	[0.0000, 0.6866]
6	[0.2097, 1.0000]	[1.4059, 54.3265]	[0.0422, 1.0000]	[0.1357, 1.0000]
7	[0.0373, 1.0000]	[2.3562, 29.1748]	[0.0707, 1.0000]	[0.0326, 1.0000]
8	[0.0054, 0.4555]	[1.0000, 11.9339]	[0.0300, 0.4555]	[0.0000, 0.4163]
9	[0.0343, 0.8231]	[1.0000, 33.1052]	[0.0300, 0.8231]	[0.0000, 0.7911]
10	[0.0798, 0.8765]	[1.0000, 52.0475]	[0.0300, 0.8765]	[0.0000, 0.8634]
11	[0.0299, 0.8177]	[1.3822, 20.8534]	[0.0415, 0.8177]	[0.0251, 0.7850]
12	[0.0522, 0.4099]	[1.0000, 10.6956]	[0.0300, 0.4099]	[0.0000, 0.3209]

Table 6
Hurwicz index values of four kinds of interval efficiencies with $\alpha = 0.5$.

DMU	Entani et al.'s models	Model (7) and model (9)	Wang and Yang's models	Our normalized efficiency models
1	0.6376 (1)	103.3978 (1)	0.5279 (2)	0.6020 (1)
2	0.4669 (7)	8.3090 (9)	0.4793 (6)	0.4579 (6)
3	0.5962 (3)	77.1223 (2)	0.5174 (4)	0.5238 (3)
4	0.5092 (5)	7.9425 (10)	0.5150 (5)	0.5000 (5)
5	0.3696 (10)	14.8354 (7)	0.3821 (10)	0.3433 (10)
6	0.6048 (2)	27.8662 (3)	0.5211 (3)	0.5679 (2)
7	0.5187 (4)	15.7655 (6)	0.5353 (1)	0.5163 (4)
8	0.2304 (12)	6.4669 (11)	0.2427 (11)	0.2082 (11)
9	0.4287 (8)	17.0526 (5)	0.4265 (9)	0.3956 (9)
10	0.4781 (6)	26.5237 (4)	0.4532 (7)	0.4317 (7)
11	0.4238 (9)	11.1178 (8)	0.4296 (8)	0.4051 (8)
12	0.2311 (11)	5.8478 (12)	0.2200 (12)	0.1605 (12)

Table 7
Spearman's rank correlation coefficients of four kinds of evaluation results.

	Entani et al.'s models	Model (7) and Model (9)	Wang and Yang's models	Our models
Entani et al.'s models	1.0000	0.7902	0.9371	0.9790
Model (7) and model (9)	0.7902	1.0000	0.6573	0.7413
Wang and Yang's models	0.9371	0.6573	1.0000	0.9580
Our models	0.9790	0.7413	0.9580	1.0000

evaluation, we calculate Spearman's rank correlation coefficients for the four kinds of evaluation results and show them in Table 7. Obviously, the rank correlation coefficients among Entani et al.'s evaluation, Wang and Yang's and ours are all greater than 0.9, which means that the three kinds of evaluation results are very similar in the example. Yet, they are very different from the average efficiency evaluation determined by models (7) and (9). Additionally, Entani et al.'s evaluation and ours are the most similar because their rank correlation coefficient is the maximum among the correlation matrix, namely 0.9790. In fact, based the two kinds of evaluations, the top 5 DMUs are the same, namely $DMU_1 \succ DMU_6 \succ DMU_3 \succ DMU_7 \succ DMU_4$, which is seen in Table 6.

For Entani et al.'s interval efficiencies and ours, we further calculate Hurwicz index values of the two kinds of interval efficiencies with $\alpha = 0.25,\ 0.75$ and obtain the ranking orders of DMUs, which are reported in Table 8. It is seen that the ranking orders of some DMUs change when the parameter α is adjusted to 0.25 or 0.75 from 0.5 based on Hurwicz index values of Entani et al.'s interval efficiencies. For example, DMU_{12} ranks 11th when $\alpha = 0.5$, while it ranks the last when $\alpha = 0.75$. By contrary, based on Hurwicz index values of our normalized interval efficiencies, the ranking is stable in the example because the ranking orders of all DMUs are the same when $\alpha = 0.25,\ 0.5,\ 0.75$. In fact, based on Theorem 3, the ranking will remain unchanged if $\alpha \in (0,\ 1)$.

7. Conclusion

The CCR model measures the best relative efficiency of a DMU by maximizing the ratio of the weighted outputs to the weighted inputs under the constraint that the maximum ratio of all DMUs is 1. Hence, all DMUs are compared relative to the best DMU. Considering that a superior object must be far from the worst object as well as being close to the best, we evaluate DMUs relative to the worst DMU as well as the best DMU. Then, we propose the normalized efficiency and achieve the best normalized efficiency and the worst normalized efficiency respectively. Further, we build the interval efficiency evaluation and rank all DMUs completely based on the upper and lower bounds of the

Table 8
Hurwicz index values of two kinds of interval efficiencies for 12 universities.

DMU	Entani et al.'s models			Normalized efficiency models		
	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
1	0.4564 (1)	0.6376 (1)	0.8188 (1)	0.4030 (1)	0.6020 (1)	0.8010 (1)
2	0.2361 (7)	0.4669 (7)	0.6977 (6)	0.2290 (6)	0.4579 (6)	0.6869 (6)
3	0.3943 (3)	0.5962 (3)	0.7981 (3)	0.2856 (3)	0.5238 (3)	0.7619 (3)
4	0.2638 (6)	0.5092 (5)	0.7546 (5)	0.2500 (5)	0.5000 (5)	0.7500 (5)
5	0.1872 (10)	0.3696 (10)	0.5519 (10)	0.1717 (10)	0.3433 (10)	0.5150 (10)
6	0.4073 (2)	0.6048 (2)	0.8024 (2)	0.3518 (2)	0.5679 (2)	0.7839 (2)
7	0.2780 (5)	0.5187 (4)	0.7593 (4)	0.2745 (4)	0.5163 (4)	0.7582 (4)
8	0.1179 (12)	0.2304 (12)	0.3430 (11)	0.1041 (11)	0.2082 (11)	0.3122 (11)
9	0.2315 (8)	0.4287 (8)	0.6259 (8)	0.1978 (9)	0.3956 (9)	0.5933 (9)
10	0.2790 (4)	0.4781 (6)	0.6773 (7)	0.2158 (7)	0.4317 (7)	0.6475 (7)
11	0.2268 (9)	0.4238 (9)	0.6208 (9)	0.2151 (8)	0.4051 (8)	0.5950 (8)
12	0.1416 (11)	0.2311 (11)	0.3205 (12)	0.0802 (12)	0.1604 (12)	0.2406 (12)

normalized efficiency.

Compared with the traditional DEA methods, our evaluation method has some attractive advantages. First of all, for the given input and output weights, the normalized efficiency is obtained by normalizing the output-input ratio of a DMU, and it measures the position of a DMU in the interval built by the best DMU and the worst DMU. Hence, it is fairer for all DMUs to compare the normalized efficiencies. Second, from the optimistic viewpoint and the pessimistic viewpoint, we achieve the upper and lower bounds of the normalized efficiency respectively. It is more comprehensive to evaluate DMUs by considering the best normalized efficiency and the worst normalized efficiency simultaneously. Last but not least, based on the interval efficiency, we can rank all DMUs with the assessor's given level of optimism. Although the assessor's level of optimism could affect the ranking of all DMUs, yet there exists an interval of the parameter α such that the ranking is stable.

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Appendix A

Proof of Theorem 1. If the weight vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are both confined that only one element is 1 and the other elements are all 0, then model (9) will be reduced to model (10). Considering model (9) and model (10) are both the maximization problems, we have that the optimal value of model (9), namely $\overline{E}_k^{(2)}$, is not smaller than the optimal value of model (10). Suppose E_k^* is the optimal value of model (10). Then $\overline{E}_k^{(2)} \geqslant E_k^*$.

Note that model (9) and model (10) are equivalent to the following models respectively:

$$\overline{E}_k^{(2)} = \max_{\boldsymbol{u} \geqslant 0,\, \boldsymbol{v} \geqslant 0} \max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}\cdot\frac{\boldsymbol{v}^T\boldsymbol{x}_j}{\boldsymbol{u}^T\boldsymbol{y}_j} = \max_{\boldsymbol{u} \geqslant 0,\, \boldsymbol{v} \geqslant 0} \max_j \frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{u}^T\boldsymbol{y}_j}\cdot\frac{\boldsymbol{v}^T\boldsymbol{x}_j}{\boldsymbol{v}^T\boldsymbol{x}_k}, \tag{23}$$

$$E_k^* = \max_{i,\, r}\max_j \frac{y_{ik}}{x_{rk}}\cdot\frac{x_{rj}}{y_{ij}} = \max_{i,\, r}\max_j \frac{y_{ik}}{y_{ij}}\cdot\frac{x_{rj}}{x_{rk}} \tag{24}$$

Suppose $\boldsymbol{u}^* = (u_1{}^*, u_2{}^*, \ldots, u_s{}^*)^T$, $\boldsymbol{v}^* = (v_1{}^*, v_2{}^*, \ldots, v_m{}^*)^T$, j^* are the optimal solutions of model (23). Then,

$$\overline{E}_k^{(2)} = \frac{(\boldsymbol{u}^*)^T\boldsymbol{y}_k}{(\boldsymbol{u}^*)^T\boldsymbol{y}_{j^*}}\cdot\frac{(\boldsymbol{v}^*)^T\boldsymbol{x}_{j^*}}{(\boldsymbol{v}^*)^T\boldsymbol{x}_k}.$$

Here, it is obvious that

$$\frac{(\boldsymbol{u}^*)^T\boldsymbol{y}_k}{(\boldsymbol{u}^*)^T\boldsymbol{y}_{j^*}} = \frac{\sum_{i=1}^{s} u_i{}^* y_{ik}}{\sum_{i=1}^{s} u_i{}^* y_{ij^*}} \leqslant \max_i \frac{u_i{}^* y_{ik}}{u_i{}^* y_{ij^*}} = \max_i \frac{y_{ik}}{y_{ij^*}},$$

$$\frac{(\boldsymbol{v}^*)^T\boldsymbol{x}_{j^*}}{(\boldsymbol{v}^*)^T\boldsymbol{x}_k} = \frac{\sum_{r=1}^{s} v_r{}^* x_{rj^*}}{\sum_{r=1}^{s} v_r{}^* x_{rk}} \leqslant \max_r \frac{v_r{}^* x_{rj^*}}{v_r{}^* x_{rk}} = \max_r \frac{x_{rj^*}}{x_{rk}}.$$

Then, we have that $\overline{E}_k^{(2)} \leqslant \max_i \frac{y_{ik}}{y_{ij*}}\cdot\max_r \frac{x_{rj*}}{x_{rk}} = \max_{i,\, r} \frac{y_{ik}}{y_{ij*}}\cdot\frac{x_{rj*}}{x_{rk}} \leqslant \max_{i,\, r}\max_j \frac{y_{ik}}{y_{ij}}\frac{x_{rj}}{x_{rk}} = E_k^*$.

Hence, $\overline{E}_k^{(2)} = E_k^*$, and then model (9) and model (10) are equivalent. □

Proof of Theorem 2. Considering the constraint $\boldsymbol{u}^T\boldsymbol{y}_i/\boldsymbol{v}^T\boldsymbol{x}_i = 1$ and $\boldsymbol{v}^T\boldsymbol{x}_k = 1$, easily we have that model (12) is equivalent to the following model:

$$\begin{aligned} \theta_{ki}^{\max} &= \max \frac{\boldsymbol{u}^T\boldsymbol{y}_k - 1}{b-1} \\ \text{s.t. } & \boldsymbol{v}^T\boldsymbol{x}_k = 1, \\ & \boldsymbol{u}^T\boldsymbol{y}_i - \boldsymbol{v}^T\boldsymbol{x}_i = 0, \\ & -\boldsymbol{u}^T\boldsymbol{y}_l + \boldsymbol{v}^T\boldsymbol{x}_l \leqslant 0,\ l = 1, 2, \ldots, n; l \neq i, \\ & \max_j \boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j = b, \\ & \boldsymbol{u} \geqslant 0,\ \boldsymbol{v} \geqslant 0. \end{aligned} \tag{25}$$

Based on model (8) and model (9), we know that

$$\max_k\{\underline{E}_k^{(2)}\} = \max_k\left\{\min_{\mathbf{u} \geqslant 0,\, \mathbf{v} \geqslant 0}\left\{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}\,\middle|\,\min_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}\right\} = \min_{\mathbf{u} \geqslant 0,\, \mathbf{v} \geqslant 0}\left\{\max_k \frac{\mathbf{u}^T\mathbf{y}_k}{\mathbf{v}^T\mathbf{x}_k}\,\middle|\,\min_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}$$

and

$$\max_k\{\overline{E}_k^{(2)}\} = \max_k\left\{\max_{\mathbf{u} \geqslant 0,\, \mathbf{v} \geqslant 0}\left\{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}\,\middle|\,\min_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}\right\} = \max_{\mathbf{u} \geqslant 0,\, \mathbf{v} \geqslant 0}\left\{\max_k \frac{\mathbf{u}^T\mathbf{y}_k}{\mathbf{v}^T\mathbf{x}_k}\,\middle|\,\min_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}$$

Let $b_1 = \max_k\{\underline{E}_k^{(2)}\}$, $b_2 = \max_k\{\overline{E}_k^{(2)}\}$. Obviously, $1 \leqslant b_1 \leqslant b_2$. Then we can add the constraint $b_1 \leqslant b \leqslant b_2$ in model (25). Additionally, the difference of model (14) and model (25) mainly lies in the fourth constraint. Considering the second constraint of model (14) and $1 \leqslant b_1 \leqslant b$, we have that the fourth constraint of model (14) is equivalent to the following term:

$\max_j \boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j \leqslant b$.

Let θ^* be the optimal value of model (14) and $\boldsymbol{u}^*$, $\boldsymbol{v}^*$, b^* be the optimal solutions. Assume $\widetilde{b} = \max_j (\boldsymbol{u}^*)^T\boldsymbol{y}_j/(\boldsymbol{v}^*)^T\boldsymbol{x}_j < b^*$. It follows that $\boldsymbol{u}^*$, $\boldsymbol{v}^*$, $\widetilde{b}$ is feasible to model (14) and $\theta^* = \frac{(\boldsymbol{u}^*)^T\boldsymbol{y}_k - 1}{b^* - 1} < \frac{(\boldsymbol{u}^*)^T\boldsymbol{y}_k - 1}{\widetilde{b} - 1}$, which contradicts the knowledge that θ^* is the optimal value of model (14). Therefore, we have that $\max_j (\boldsymbol{u}^*)^T\boldsymbol{y}_j/(\boldsymbol{v}^*)^T\boldsymbol{x}_j = b^*$, and model (14) is equivalent to model (25) and model (12). □

Proof of Theorem 3. Considering the constraint that $\boldsymbol{v}^T\boldsymbol{x}_k = 1$ and $\boldsymbol{u}^T\boldsymbol{y}_j/\boldsymbol{v}^T\boldsymbol{x}_j = 1$, we convert model (19) into the following model:

$$\begin{aligned}
&\theta_{kj}^{\min} = \min 1 - \frac{1 - \boldsymbol{u}^T\boldsymbol{y}_k}{1 - a}\\
&\text{s.t.}\ \ \boldsymbol{v}^T\boldsymbol{x}_k = 1,\\
&\quad \boldsymbol{u}^T\boldsymbol{y}_j - \boldsymbol{v}^T\boldsymbol{x}_j = 0,\\
&\quad \boldsymbol{u}^T\boldsymbol{y}_l - \boldsymbol{v}^T\boldsymbol{x}_l \leqslant 0,\ \ l = 1, 2, \ldots, n; l \neq j,\\
&\quad \min_i \boldsymbol{u}^T\boldsymbol{y}_i/\boldsymbol{v}^T\boldsymbol{x}_i = a,\\
&\quad \boldsymbol{u} \geqslant 0,\ \boldsymbol{v} \geqslant 0.
\end{aligned} \tag{26}$$

Based on model (4) and model (1), we know that

$$\min_k\{\underline{E}_k^{(1)}\} = \min_k\left\{\min_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0}\left\{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}\middle| \max_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}\right\} = \min_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0}\left\{\min_k \frac{\mathbf{u}^T\mathbf{y}_k}{\mathbf{v}^T\mathbf{x}_k}\middle| \max_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\},$$

and

$$\min_k\{\overline{E}_k^{(1)}\} = \min_k\left\{\max_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0}\left\{\frac{\boldsymbol{u}^T\boldsymbol{y}_k}{\boldsymbol{v}^T\boldsymbol{x}_k}\middle| \max_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}\right\} = \max_{\mathbf{u} \geqslant 0, \mathbf{v} \geqslant 0}\left\{\min_k \frac{\mathbf{u}^T\mathbf{y}_k}{\mathbf{v}^T\mathbf{x}_k}\middle| \max_l \frac{\mathbf{u}^T\mathbf{y}_l}{\mathbf{v}^T\mathbf{x}_l} = 1\right\}$$

Let $a_1 = \min_k\{\underline{E}_k^{(1)}\}$, $a_2 = \min_k\{\overline{E}_k^{(1)}\}$. Then we can add the constraint $a_1 \leqslant a \leqslant a_2$ in model (26). Additionally, noting that the objective function of model (26) will increase with decreasing the value of a, we can modify the fourth constraint of model (26) as $\boldsymbol{u}^T\boldsymbol{y}_i/\boldsymbol{v}^T\boldsymbol{x}_i \geqslant a$. That is, model (26) is equivalent to model (21). Hence, model (19) and model (21) are equivalent. □

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