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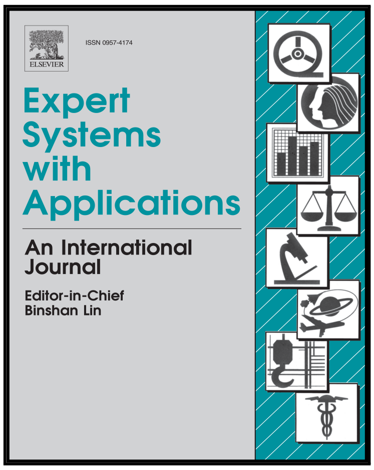

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Highlights

- We introduce the new hybrid model called hesitant N-soft sets
- We investigate their properties and de
ne fundamental operations on them
- The concept of hesitant N-soft set is illustrated with real life examples
- This model provides additional justi
cation of the notion of hesitancy

Group Decision-Making Methods Based on Hesitant N-Soft Sets

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Abstract

In this article, we introduce a new hybrid model called hesitant N-soft sets by a suitable combination of hesitancy with N-soft sets, a model that extends N-soft sets. Our novel concept is illustrated with real life examples. Moreover, we investigate some useful properties of hesitant N-soft sets and construct fundamental operations on them. We describe potential applications of hesitant N-soft sets in group decision-making, and finally we present some group decision-making methods as algorithms.

Keywords: Soft set; N-soft set; hesitant N-soft set; hesitant fuzzy set; ordered grades; decision-making.

2010 Mathematics Subject Classification: 03E05, 03B52, 94D05

1 Introduction

Data related to most of our practical life problems including medical science, engineering, economics and environmental sciences among others, are imprecise and their corresponding solutions require the use of mathematical conventions based on imprecision and uncertainty. We cannot use traditional mathematical tools to overcome uncertainties existing in these problems.

Consequently and in order to handle such uncertainties, a number of theories have been introduced including fuzzy set theory (Zadeh, 1965) and its extensions (Bustince et al., 2016, 2008), probability, rough set theory (Greco, Matarazzo & Slowinski, 2001, 2002; Liu, Qin & Martínez, 2018; Pawlak, 1982), et cetera. Merigó, Gil-Lafuente & Yager (2015) and Blanco-Mesa, Merigó Gil-Lafuente (2017) are updated overviews of fuzzy research and fuzzy decision making with respective biblio-metric indicators. Anyhow all of these theories have their immanent difficulties (Paternain et al., 2012), a drawback that motivated Molodtsov (1999) to introduce the idea of soft sets as a new mathematical tool to tackle some of their difficulties. Soft set theory has significant use in game theory, smoothness of functions, medicine, operational research and probability theory (Alcantud & Santos-García, 2017; Molodtsov, 1999, 2004). Their algebraic analysis and applications developed rapidly. Maji, Biswas, & Roy (2003) presented some basic algebraic operations on soft sets and provide an analytical approach to theory of soft sets. Ali et al. (2009) suggested some different operations for soft sets and developed the idea of complement of soft set. They showed that certain De Morgan's laws are valid in soft sets. Maji, Biswas, & Roy (2002) discussed the use of soft sets in decision making problems. It is observed that fuzzy sets, soft sets and rough sets are conveniently related notions. Maji, Biswas, & Roy (2001) combined soft sets with other mathematical structures and introduced an hybrid model called fuzzy soft sets, which is the natural fuzzy generalization of soft sets. They investigated many useful results related to this model. Optimization in this setting has been recently studied in Alcantud (2015, 2016a), Alcantud & Mathew (2017) and Liu, Qin & Pei (2017), see also Khameneh & Kiliçman (2018) for an updated survey. Afterwards Majumdar & Samanta (2010) revised the definition of fuzzy soft set and proposed the concept of generalized fuzzy soft sets based on Maji, Biswas, & Roy (2003).

From latest surveys of hybrid soft set models, it is apparent that most of the researchers in soft-set-inspired models worked on binary evaluation (either 0 or 1) or else, real numbers between 0 and 1 (Fatimah et al., 2018b; Ma, Li, & Zhang, 2017; Zou & Xiao, 2008, among others). But in our daily life problems we mostly find out data with non-binary evaluations. In social judgement systems, Alcantud & Laruelle (2014) investigated ternary voting system and explained their use in real situations. Non-binary estimations are also expected in rating or ranking positions. Examples existing in real life systems show that rankings can be expressed in the form of number of dots and stars (like 'one big dot,' 'one star', 'two stars', 'three stars', ... in hotel classifications), or also by means of numbers (as the scores of exams). Further, Herawan & Deris (2009) indicated n binary-valued information system in soft sets where each of parameter has its own rankings, as compared to rating orders described in Chen et al. (2013). Instead of ratings as assessment, Ali et al. (2015) organized rating system among the elements of soft sets parameters. Motivated by these concerns, Fatimah et al. (2018a) proposed the idea of an extended soft set model (whose founding notion is N-soft set) and elaborated on this notion in order to describe the importance of ordered grades in actually existing problems. N-soft sets generate the idea of a multinary parameterized characterization of the universe of objects, because it depends on a finite fixed number of ordered grades.

From another position, hesitant fuzzy sets (HFSs) constitute a generalization of fuzzy sets that are developing rapidly with their extensions and applications to many fields. HFSs (Torra, 2010; Torra & Narukawa, 2009) are more suitable for dealing with the situations where decision makers have hesitancy in providing their assessments over objects, or also when we merge the opinions of various experts into one single input. Indeed in many decision making cases, experts are usually hesitant and/or irresolute which prevents them from producing unique appraisals (Sun et al., 2018; Xia & Xu, 2011; Zhu, Xu, & Xu, 2014). The concept is amenable to successful hybridization with other approaches to uncertainty and vagueness (Alcantud & Giarlotta, 2018; Liao et al., 2015; Ngan, 2017; Wang, Li, & Chen, 2014). The ideas of uncertainty/hesitancy in multi-criteria group decision making had also been dealt with in the evidential reasoning framework (Ngan, 2015; Xu, 2012; Yang & Xu, 2002). For other notations, terminologies and applications not mentioned in this paper, the reader is referred to (Akram, Ali & Alshehri, 2017; Feng, Akram, Davvaz, & Fotea , 2014; Roy & Maji, 2007).

In this article we are inspired by both hesitancy and N-soft sets. We introduce a novel model called hesitant N-soft sets that emerges from the hybridization of N-soft sets and hesitancy. We present this model in view of the importance of grades in real world situations. We intend to account for the situations when either the decision-makers have hesitancy in providing their multinary evaluations of the objects, or the practitioner incorporates different assessments provided by multiple decision makers. Therefore, for the purpose of the modelization of group decision-making problems the new model provides more flexibility by connecting the variability of two very different designs.

The organization of this research article is as follows. We introduce our new hybrid model and its basic operations in section 2. We illustrate this novel concept with real life examples. Moreover, we investigate its relationships with existing models like N-soft sets, soft sets and incomplete soft sets. In section 3 we describe potential applications of hesitant N-soft sets. We also present our proposed methods as algorithms. In section 4 we conclude.

2 Hesitant N-soft sets and their operations

In this Section, we introduce our new model and relate it with existing models in the literature. It is based on the following concept model N-soft set from Fatimah et al. (2018a):

Definition 2.1. (Fatimah et al., 2018a) Let O be a universe of objects and P the set of attributes, $T \subseteq P$. Let $G = \{0, 1, 2, \cdots, N-1\}$ be the set of ordered grades where $N \in \{2, 3, \ldots\}$. A triple (F, T, N) is called an *N-soft set* on O if F is mapping from T to $2^{O \times G}$, with the property that for each $t \in T$ and $o \in O$ there exists a unique $(o, g_t) \in O \times G$ such that $(o, g_t) \in F(t)$, $g_t \in G$.

When $N = 2$, we obtain a soft set as defined in Molodtsov (1999, 2004). Therefore, the model above generalizes soft sets.

In Fatimah et al. (2018a) several real examples prove that this concept is both useful and applicable in decision making. In a parameterizations of the universe of the alternatives it gives their approximate description, which is not binary like the case of ordinary soft sets, but multinary.

The next subsection defines and explains an extension of this concept where hesitancy is allowed. We also present two convenient representations, namely, a tabular and a functional representation of our concept. Afterwards, subsection 2.2 defines scores associated with an important component of our model and subsection 2.3 states some fundamental operations for this new setting.

2.1 The concept of hesitant N-soft sets

Definition 2.2 below introduces a novel model that emerges from the hybridization of N-soft sets and hesitancy which is the key contribution of hesitant fuzzy sets (Torra, 2010; Alcantud & Torra, 2018). Afterwards, we proceed to explain its intuitive interpretation and suggest that tabular and functional representations simplify their implementation. A streamlined example shows that the combination of models that we propose is natural in standard decision-making situations (see Example 2.3 below).

Henceforth, we denote by $\mathcal{F}(X)$ the set of all fuzzy sets on X, and we denote by $\mathcal{P}(X)$ the set of all subsets of X. $\mathcal{P}^*(X)$ stands for the set of all non-empty subsets of X.

Definition 2.2. Let O be a universe of objects and P the set of attributes, $T \subseteq P$. Let $G = \{0, 1, 2, \cdots, N-1\}$ be the set of ordered grades where $N \in \{2, 3, \ldots\}$. A triple (H, T, N) is called a *hesitant N-soft set* (henceforth HNSS) on O if H is a mapping $H : T \to 2^{O \times G}$ such that for each $t \in T$ and $o \in O$ there exists at least one pair $(o, g_t) \in O \times G$ such that $(o, g_t) \in H(t)$, $g_t \in G$.

We recall that in mathematical terms, the expression $S \in 2^{O \times G}$ is equivalent to $S \subseteq O \times G$. Therefore, we can also regard H as a mapping $H : T \to \mathcal{P}(O \times G)$ with the property that for each $t \in T$ and $o \in O$ there exists at least one pair $(o, g_t) \in O \times G$ such that $(o, g_t) \in H(t)$, $g_t \in G$.

According to Definition 2.2, with each attribute the mapping H assigns a non-empty collection of pairs formed by objects and possible grades. We deduce the following particular specifications:

(i) In the case where $N = 2$, HNSS is equivalent to an incomplete soft set, i.e., a soft set where some of the information is missing (which is symbolized by *). This fact deserves a separate remark (cf., Remark 1 below).

(ii) In the case where for each $t \in T$ and $o \in O$, H always assigns *exactly* one pair $(o, g_t) \in O \times G$ such that $(o, g_t) \in H(t)$, $g_t \in G$, is an N-soft set.

(iii) Therefore, the case where in addition to condition (ii) we have $N = 2$, is a soft set.

The following streamlined example illustrates the formal Definition 2.2 while introducing a convenient tabular representation for HNSSs. In passing it provides a very natural justification for the assumption of hesitation in grades:

Example 2.3. In a company, the selection of an employee based on check mark rankings and ratings awarded by a selection panel formed by director, manager and HR. To simplify the explanation let $O = \{o_1, o_2\}$ be the universe of candidates appearing in an interview of the company, and let P be the set of attributes "evaluation of candidates by the selection panel". The subset $T \subseteq P$ is such that $T = \{t_1, t_2, t_3\}$ in our example. A hesitant 4-soft set (H4SS) can be obtained from Table 1, where

- Three check marks represent 'Very Good',
- Two check marks represent 'Good',
- One check mark represents 'Normal',
- Box represents 'Poor'.

This graded evaluation by check marks can easily identified with numbers as $G = \{0, 1, 2, 3\}$, where

- 0 serves as "$\square$",
- 1 serves as "$\checkmark$",
- 2 serves as "$\checkmark\checkmark$",
- 3 serves as "$\checkmark\checkmark\checkmark$".

The information extracted from the three members of the panel on related data is described in Table 1, which summarizes the three 4-soft sets provided by the members. And then the tabular representation of the H4SS that results from their fusion is easily given by Table 2. In each cell, all the possible grades assigned by each of the panel members are incorporated to the hesitant multinary approximation of the universe of candidates.

	O/T	t_1	t_2	t_3
Member 1	o_1	$\square$	$\checkmark\checkmark$	$\checkmark\checkmark\checkmark$
	o_2	$\checkmark\checkmark$	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark$
Member 2	o_1	$\checkmark$	$\square$	$\checkmark\checkmark\checkmark$
	o_2	$\checkmark\checkmark$	$\checkmark\checkmark$	$\checkmark$
Member 3	o_1	$\square$	$\square$	$\checkmark\checkmark\checkmark$
	o_2	$\checkmark\checkmark$	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark\checkmark$

Table 1: Information extracted from three members of the panel on related data. It is naturally transformed into three respective 4-soft sets.

$(H, T, 4)$	t_1	t_2	t_3
o_1	$\{0, 1\}$	$\{0, 2\}$	$\{3\}$
o_2	$\{2\}$	$\{2, 3\}$	$\{1, 2, 3\}$

Table 2: Tabular representation of the hesitant 4-soft set resulting from the fusion of information in Table 1 (i.e., of the three 4-soft sets).

Therefore, the following H4SS (cf., Definition 2.2) is defined:

$$H(t_1) = \{(o_1, 0), (o_1, 1), (o_2, 2)\} \in 2^{O \times G},$$
$$H(t_2) = \{(o_1, 0), (o_1, 2), (o_2, 2), (o_2, 3)\} \in 2^{O \times G},$$
$$H(t_3) = \{(o_1, 3), (o_2, 1), (o_2, 2), (o_2, 3)\} \in 2^{O \times G}.$$

This model captures the information that the company should use in order to select one of the candidates. In short, a H4SS is needed in order to present all the information gathered from the 3 sources.

Thus, we have shown in Example 2.3 that hesitation in grades arises in a natural way, when we incorporate the information from various sources and each of them parameterizes the universe in a multinary categorization (that is, as an N-soft set).

Remark 1. In the literature on soft set theory, hesitancy had been associated with hesitant fuzzy soft sets (Wang, Li, & Chen, 2014) but not with soft sets. The reason of such absence is that when one hesitates in a soft set in a non-trivial manner, the consequence is that full uncertainty is allowed (for any given object and attribute, if we hesitate then we admit that either the object verifies or it does not verify the attribute). That case is the same as incomplete information, which has produced its own strand of literature: cf., Alcantud & Santos-García (2017) and the references therein. However, hesitancy is a characteristic of N-soft sets in its own right, which emphasizes the formal advantages of N-soft sets with respect to ordinary soft sets (i.e., the case $N = 2$). This will be apparent in Example 2.4 below.

Example 2.4. Let us see how a HNSS where $N = 2$ is equivalent to an incomplete soft set. Intuitively, we just need to replace the only possible true hesitancy $\{0, 1\}$ by the lack of information symbol $*$ as follows:

$(H, T, 2)$	t_1	t_2	t_3
o_1	$\{0, 1\}$	$\{0\}$	$\{0\}$
o_2	$\{1\}$	$\{0\}$	$\{0, 1\}$

Table 3: A hesitant 2-soft set.

	t_1	t_2	t_3
o_1	$*$	0	0
o_2	1	0	$*$

Table 4: Tabular representation of the incomplete soft set resulting from Table 3.

The intuitive interpretations given above for Example 2.3, hint that a tabular representation permits to display the information in any HNSS over a finite set of alternatives when the set of attributes is finite. Table 5 below shows such tabular representation.

(H,T,N)	t_1	t_2	$\ldots\ldots$	t_q
o_1	$\{\eta_{11}^1, \eta_{11}^2, \ldots, \eta_{11}^{l(11)}\}$	$\{\eta_{12}^1, \eta_{12}^2, \ldots, \eta_{12}^{l(12)}\}$	$\ldots\ldots$	$\{\eta_{1q}^1, \eta_{1q}^2, \ldots, \eta_{1q}^{l(1q)}\}$
$\vdots$				
o_p	$\{\eta_{p1}^1, \eta_{p1}^2, \ldots, \eta_{p1}^{l(p1)}\}$	$\{\eta_{p2}^1, \eta_{p2}^2, \ldots, \eta_{p2}^{l(p2)}\}$	$\ldots\ldots$	$\{\eta_{pq}^1, \eta_{pq}^2, \ldots, \eta_{pq}^{l(pq)}\}$

Table 5: Tabular representation of a general HNSS when both O and T are finite. Number $l(ij)$ means the length or cardinality of the subset of grades that appears in position i, j of the tabular representation.

Therefore in Table 5, the presence of element $\eta_{ij}^k \in G$ in cell (i, j) is equivalent to the fact $(o_i, \eta_{ij}^k) \in H(t_j)$.

It is now easy to state that the following function provides an alternative and convenient representation of HNSSs:

Definition 2.5. If (H, T, N) is an HNSS, then its *functional representation* is the mapping $\eta : O \times T \longrightarrow \mathcal{P}^*(G)$ such that for all $t \in T$ and $o \in O$,

$$\eta(o, t) = \{g \in G : (o, g) \in H(t)\}.$$

Table 6 below displays this alternative general representation and links it with its tabular representation explained above, when the standard finiteness restrictions hold.

(H,T,N)	t_1	$\ldots\ldots$	t_q
o_1	$\eta(o_1, t_1) = \{\eta_{11}^1, \ldots, \eta_{11}^{l(11)}\}$	$\ldots\ldots$	$\eta(o_1, t_q) = \{\eta_{1q}^1, \ldots, \eta_{1q}^{l(1q)}\}$
$\vdots$			
o_p	$\eta(o_p, t_1) = \{\eta_{p1}^1, \ldots, \eta_{p1}^{l(p1)}\}$	$\ldots\ldots$	$\eta(o_p, t_q) = \{\eta_{pq}^1, \ldots, \eta_{pq}^{l(pq)}\}$

Table 6: Functional representation of a general hesitant N-soft set when both O and T are finite.

One can easily retrieve the original formulation of the HNSS in Definition 2.2, both from its tabular and its functional representations.

2.2 Hesitant N-tuples and their scores

For convenience, the sets h with the property $\varnothing \neq h \subseteq G = \{0, 1, 2, \cdots, N-1\}$ will be called *hesitant N-tuples* (also, HNT). Alternatively, HNTs are elements from $\mathcal{P}^*(G)$. Therefore, a tabular representation of a HNSS is a matrix whose cells are HNTs (provided that the finiteness assumptions hold). For example, in Table 5, all the cells are HNTs. Similarly, the functional representation of a HNSS associates a HNT to each (o, t) as displayed in Table 6.

Henceforth, we assume that all HNTs are written in such way that $h = \{\eta_{ij}^1, \eta_{ij}^2, \ldots, \eta_{ij}^{l(ij)}\}$ verifies $\eta_{ij}^1 < \eta_{ij}^2 < \ldots < \eta_{ij}^{l(ij)}$. The length of such h is obviously $l(ij) \geqslant 1$, which in turn verifies $1 \leqslant l(ij) \leqslant N$.

Scores for hesitant fuzzy elements are an important tool in the analysis and applications of hesitant fuzzy sets, see for example, Alcantud, de Andrés, & Torrecillas (2016), Farhadinia (2013, 2014), and Xu & Xia (2011). Therefore it seems crucial to formulate a similar technique in our setting. The following definition is the key part of the analysis:

Definition 2.6. A score for hesitant N-tuples is a mapping $s\colon \mathcal{P}^*(G) \longrightarrow \mathbb{R}^+$, where $G = \{0, 1, 2, \cdots, N-1\}$, with the following properties:

1. $s(\{0\}) = 0$, $s(\{1\}) = 1$ and $s(\{N-1\}) = N-1$;
2. *Boundedness*: for all $h \in \mathcal{P}^*(G)$, we have $s(\min(h)) \leqslant s(h) \leqslant s(\max(h))$.

The following expressions define examples of scores for hesitant N-tuples: let $h = \{\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(l)}\} \in \mathcal{P}^*(G)$ where $G = \{0, 1, 2, \cdots, N-1\}$, then

1. Min score: $s_m(h) = \eta^{(1)}$;
2. Max score: $s_M(h) = \eta^{(l)}$;
3. Arithmetic score: $s_a(h) = (\eta^{(1)} + \eta^{(2)} + \ldots + \eta^{(l)})/l$;
4. Geometric score: $s_g(h) = (\eta^{(1)} \cdot \eta^{(2)} \cdot \ldots \cdot \eta^{(l)})^{1/l}$.

The next example shows the application of these scores:

Example 2.7. The application of the scores on hesitant 4-tuples above produces the following results:
For $h = \{1, 3\}$: $s_m(h) = 1$, $s_M(h) = 3$, $s_a(h) = \frac{1+3}{2} = 2$, $s_g(h) = \sqrt{1 \cdot 3} = \sqrt{3}$.
For $h = \{0, 2, 3\}$: $s_m(h) = 0$, $s_M(h) = 3$, $s_a(h) = \frac{0+2+3}{3} = 1.66$, $s_g(h) = \sqrt[3]{0 \cdot 2 \cdot 3} = 0$.

2.3 Basic operations for hesitant N-soft sets

We proceed to define some fundamental algebraic operations in the framework that we have posed in subsection 2.1. Equality can be defined in the following obvious sense:

Definition 2.8. Let (H, T, N) and (H', T', N') be two HNSSs on a universe O, then (H, T, N) and (H', T', N') are said to be *equal* if and only if $H = H'$, $T = T'$ and $N = N'$.

Definition 2.9. Let (H, T, N) be a HNSS on a universe O. Every constituent HNT (dependent of the object o_i and the attribute t_j) is such that $h_{ij} = \{\eta_{ij}^1, \eta_{ij}^2, \ldots, \eta_{ij}^{l(ij)}\}$ verifies $\eta_{ij}^1 < \eta_{ij}^2 < \ldots < \eta_{ij}^{l(ij)}$ without loss of generality.

Lower and upper bounds for HNSS can be defined by routine application of the following operators on HNTs:

1. lower bound of h_{ij}: $h^-_{ij} = \min\{\eta^1_{ij}, \eta^2_{ij}, \ldots, \eta^{l(ij)}_{ij}\} = \eta^1_{ij}$,
2. upper bound of h_{ij}: $h^+_{ij} = \max\{\eta^1_{ij}, \eta^2_{ij}, \ldots, \eta^{l(ij)}_{ij}\} = \eta^{l(ij)}_{ij}$,
3. α-lower bound of h_{ij}: $h^-_{ij,\alpha} = \{\eta^k_{ij} \in h | \eta^k_{ij} \leq \alpha\}$,
4. α-upper bound of h_{ij}: $h^+_{ij,\alpha} = \{\eta^k_{ij} \in h | \eta^k_{ij} \geq \alpha\}$.

Complements are a standard tool to investigate soft sets, fuzzy sets, and related notions. We have the following concept:

Definition 2.10. If (H, T, N) is an HNSS, and its functional representation is given by the mapping η, then its *complement* is (H^c, T, N) whose functional representation is defined by the mapping $\eta^c : O \times T \longrightarrow \mathcal{P}(G)$ such that for all $t \in T$ and $o \in O$,

$$\eta^c(o, t) = G - \eta(o, t).$$

Nonetheless the concept of weak complement of N-soft sets in Definition 12 of Fatimah et al. (2018a) can also be imported to our model. Many N-soft sets can act as weak complements of a fixed of N-soft set. The same happens in the hesitant N-soft set case according to the following formalization:

Definition 2.11. If (H, T, N) is an HNSS, and its functional representation is given by the mapping η, then a *weak complement* of (H, T, N) is any triple (H^w, T, N) whose functional representation is defined by a mapping $\eta^w : O \times T \longrightarrow \mathcal{P}(G)$ that verifies $\eta^w(o, t) \cap \eta(o, t) = \varnothing$ for all $t \in T$ and $o \in O$.

Of course, the complement of a HNSS is a weak complement of it, but not conversely. But we can also define other concrete examples of weak complements. Let us fix (H, T, N), an HNSS whose functional representation is given by η.

(1) The *top weak complement* of (H, T, N) is $(H^>, T, N)$ whose functional representation is

$$\eta^>(o, t) = \begin{cases} \max \eta^c(o, t), & \text{if} \quad \eta^c(o, t) \neq \varnothing, \\ \varnothing, & \text{otherwise.} \end{cases}$$

(2) The *bottom weak complement* of (H, T, N) is $(H^<, T, N)$ whose functional representation is

$$\eta^<(o, t) = \begin{cases} \min \eta^c(o, t), & \text{if} \quad \eta^c(o, t) \neq \varnothing, \\ \varnothing, & \text{otherwise.} \end{cases}$$

It should be noted that η^c, $\eta^>$ and $\eta^<$ are not HNSSs.

The following example illustrates the notions of complementarity that we have defined above:

Example 2.12. A weak complement of the $H4SS$ in Example 2.3 is defined by Table 7. Furthermore the complement, top weak complement, and bottom weak complement of this 4-soft set are defined by Tables 8, 9, and 10.

$(H^w, T, 4)$	t_1	t_2	t_3
o_1	$\{2\}$	$\{1, 3\}$	$\{1, 2\}$
o_2	$\{0, 3\}$	$\{1\}$	$\{0\}$

Table 7: Tabular representation of a weak complement of the H4SS in Example 2.3.

$(H^c, T, 4)$	t_1	t_2	t_3
o_1	$\{2, 3\}$	$\{1, 3\}$	$\{0, 1, 2\}$
o_2	$\{0, 1, 3\}$	$\{0, 1\}$	$\{0\}$

Table 8: Tabular representation of the complement of the H4SS in Example 2.3.

$(H^>, T, 4)$	t_1	t_2	t_3
o_1	$\{3\}$	$\{3\}$	$\{2\}$
o_2	$\{3\}$	$\{1\}$	$\{0\}$

Table 9: Tabular representation of the top weak complement of the H4SS in Example 2.3.

$(H^<, T, 4)$	t_1	t_2	t_3
o_1	$\{2\}$	$\{1\}$	$\{0\}$
o_2	$\{0\}$	$\{0\}$	$\{0\}$

Table 10: Tabular representation of the bottom weak complement of the H4SS in Example 2.3.

We proceed to define suitable notions of intersection and union of HNSSs:

Definition 2.13. If (H_1, T, N_1) and (H_2, K, N_2) are two HNSSs on the same universe of objects O, and their functional representations are given by the mappings η_1 and η_2, then their *restricted intersection* $(H_1, T, N_1) \cap_{\mathcal{R}} (H_2, K, N_2)$ is $(J, T \cap K, \min(N_1, N_2))$ whose functional representation is $\eta_{RI} : O \times (T \cap K) \longrightarrow \mathcal{P}^*(G)$ such that $\forall\ t_j \in T \cap K$ and $o_i \in O$,

$$\eta_{ij}^k \in \eta_{RI}(o_i, t_j) \Leftrightarrow \eta_{ij}^k \in h_1 \cup h_2 \text{ and } \eta_{ij}^k \leq \min\{h_1^+, h_2^+\},$$
$$\text{where } \ h_1 = \eta_1(o_i, t_j) \ \text{ and } \ h_2 = \eta_2(o_i, t_j).$$

Definition 2.14. If (H_1, T, N_1) and (H_2, K, N_2) are two HNSSs on the same universe of objects O, and their functional representations are given by the mappings η_1 and η_2, then their *extended intersection* $(H_1, T, N_1) \cap_{\mathcal{E}} (H_2, K, N_2)$ is $(E, T \cup K, \max(N_1, N_2))$ whose functional representation is $\eta_{EI} : O \times (T \cup K) \longrightarrow \mathcal{P}^*(G)$ such that $\forall\ t_j \in T \cup K$ and $o_i \in O$,

$$\eta_{EI}(o_i, t_j) = \begin{cases} \eta_1(o_i, t_j), & if \ \ t_j \in T - K, \\ \eta_2(o_i, t_j), & if \ \ t_j \in K - T, \\ \eta_{RI}(o_i, t_j), & \text{when} \ \ t_j \in T \cap K. \end{cases}$$

Example 2.15. Consider $(H_1, T, 4)$ and $(H_2, K, 5)$, the H4SS and H5SS on the common set of objects $\{o_1, o_2, o_3\}$ respectively given by Tables 11 and 12.

$(H_1, T, 4)$	t_1	t_2	t_3
o_1	$\{0,1\}$	$\{2,3\}$	$\{2\}$
o_2	$\{0,1,3\}$	$\{0,2\}$	$\{3\}$
o_3	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$

Table 11: Tabular representation of H4SS in Example 2.15.

$(H_2, K, 5)$	t_2	t_3	c	d
o_1	$\{1,3,4\}$	$\{0,1,2\}$	$\{0\}$	$\{0,3\}$
o_2	$\{0,1\}$	$\{3,4\}$	$\{1,4\}$	$\{2\}$
o_3	$\{2,3\}$	$\{4\}$	$\{2,3,4\}$	$\{0,4\}$

Table 12: Tabular representation of H5SS in Example 2.15.

The restricted intersection and extended intersection of these HNSSs are given by Table 13 and Table 14.

$(H_{RI}, T \cap K, 4)$	t_2	t_3
o_1	$\{1,2,3\}$	$\{0,1,2\}$
o_2	$\{0,1\}$	$\{3\}$
o_3	$\{1,2,3\}$	$\{1,2,3\}$

Table 13: Tabular representation of restricted intersection in Example 2.15.

$(H_{EI}, T \cup K, 5)$	t_1	t_2	t_3	c	d
o_1	$\{0,1\}$	$\{1,2,3\}$	$\{0,1,2\}$	$\{0\}$	$\{0,3\}$
o_2	$\{0,1,3\}$	$\{0,1\}$	$\{3\}$	$\{1,4\}$	$\{2\}$
o_3	$\{2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{0,4\}$

Table 14: Tabular representation of extended intersection in Example 2.15.

Definition 2.16. If (H_1, T, N_1) and (H_2, K, N_2) are two HNSSs on the same universe of objects O, and their functional representations are given by the mappings η_1 and η_2, then their *restricted union* $(H_1, T, N_1) \cup_{\mathcal{R}} (H_2, K, N_2)$ is $(M, T \cap K, \max(N_1, N_2))$ whose functional representation is $\eta_{RU} : O \times (T \cap K) \longrightarrow \mathcal{P}^*(G)$ such that $\forall\ t_j \in T \cap K$ and $o_i \in O$,

$$\eta_{ij}^k \in \eta_{RU}(o_i, t_j) \Leftrightarrow \eta_{ij}^k \in h_1 \cup h_2 \text{ and } \eta_{ij}^k \geq \max\{h_1^-, h_2^-\},$$
$$\text{where } \ h_1 = \eta_1(o_i, t_j) \text{ and } \ h_2 = \eta_2(o_i, t_j).$$

Definition 2.17. If (H_1, T, N_1) and (H_2, K, N_2) are two HNSSs on the same universe of objects O, and their functional representations are given by the mappings η_1 and η_2, then their *extended union* $(H_1, T, N_1) \cup_{\mathcal{E}} (H_2, K, N_2)$ is $(P, T \cup K, \max(N_1, N_2))$ whose functional representation is $\eta_{EU} : O \times (T \cup K) \longrightarrow \mathcal{P}^*(G)$ such that $\forall\ t_j \in T \cup K$ and $o_i \in O$,

$$\eta_{EU}(o_i, t_j) = \begin{cases} \eta_1(o_i, t_j), & \text{when} \quad t_j \in T - K, \\ \eta_2(o_i, t_j), & \text{when} \quad t_j \in K - T, \\ \eta_{RU}(o_i, t_j), & \text{when} \quad t_j \in T \cap K. \end{cases}$$

Example 2.18. In the situation of Example 2.15, the restricted union and extended union of the HNSSs are respectively given by Tables 15 and 16.

$(H_{RU}, T \cap K, 5)$	t_2	t_3
o_1	$\{2, 3, 4\}$	$\{2\}$
o_2	$\{0, 1, 2\}$	$\{3, 4\}$
o_3	$\{2, 3\}$	$\{4\}$

Table 15: Tabular representation of the restricted union of the HNSSs in Example 2.15.

$(H_{EU}, T \cup K, 5)$	t_1	t_2	t_3	c	d
o_1	$\{0, 1\}$	$\{2, 3, 4\}$	$\{2\}$	$\{0\}$	$\{0, 3\}$
o_2	$\{0, 1, 3\}$	$\{0, 1, 2\}$	$\{3, 4\}$	$\{1, 4\}$	$\{2\}$
o_3	$\{2, 3\}$	$\{2, 3\}$	$\{4\}$	$\{2, 3, 4\}$	$\{0, 4\}$

Table 16: Tabular representation of the extended union of the HNSSs in Example 2.15.

The concept of HNSS can be related to both N-soft sets and soft sets. Section 2.1 explained that HNSSs contain N-soft sets, incomplete soft sets, and of course soft sets as particular cases. We proceed to produce more relationships, and for that purpose let us fix the following setting: O denotes a universe of objects and (H, T, N) is a HNSS.

In order to derive N-soft sets from (H, T, N) we can use the following definition:

Definition 2.19. Let $0 < A < N$ be a threshold. An N-soft set over O associated with (H, T, N) and A is (H^A, T) defined by: for each $t_j \in T$, $H^A(t_j) \in \mathcal{P}(O \times G)$ is such that

$$H^A(t_j) = \begin{cases} (o_i, \max \eta_{ij}^k), & \text{if} \quad (o_i, \eta_{ij}^k) \in H(t_j) \text{ and } s_a(\eta(o_i, t_j)) \geq A, \\ (o_i, 0), & \text{otherwise.} \end{cases}$$

In particular, we say that (H^1, T) is the *bottom N-soft set* associated with (H, T, N) and (H^{N-1}, T) is the *top N-soft set* associated with (H, T, N).

We can also design a flexible way to associate soft sets with HNSSs:

Definition 2.20. Let $0 < \alpha < N$ be a threshold. A soft set over O associated with (H, T, N) and α is (H^α, T) defined by: for each $t_j \in T$, $H^\alpha(t_j) \in \mathcal{P}(O \times G)$ is such that

$$H^\alpha(t_j) = \begin{cases} (o_i, 1), & \text{if} \quad (o_i, \eta_{ij}^k) \in H(t_j) \text{ and } s_a(\eta(o_i, t_j)) \geq \alpha, \\ (o_i, 0), & \text{otherwise.} \end{cases}$$

Example 2.21. Consider the H4SS $(H_1, T, 4)$, represented by Table 11. From Definition 2.19, the range of a threshold is $0 < A < 4$. The possible 4-soft sets associated with feasible thresholds $1, 2, 3$ and $(H_1, T, 4)$ are given by Tables 17, 18 and 19 respectively.

(H^1, T)	t_1	t_2	t_3
o_1	0	3	2
o_2	3	2	3
o_3	3	3	3

Table 17

(H^2, T)	t_1	t_2	t_3
o_1	0	3	2
o_2	0	0	3
o_3	3	3	3

Table 18

(H^3, T)	t_1	t_2	t_3
o_1	0	0	0
o_2	0	0	3
o_3	0	0	0

Table 19

3 Decision making and applications

This section explains the group decision-making (GDM) mechanisms that operate on the model that we have described in previous sections. GDM is a process in which a set of experts combine their skills to decide the best alternative from a set of feasible alternatives under a particular situation. Consequently, we now define respective algorithms for problems that are characterized by HNSSs. Although there exist numerous fusion methods such as average method, weighted average method et cetera (Kuncheva, 2004), in the current context we propose choice values and weighted choice values in order to calculate the final decision because they are a keystone in well-established mechanisms from the soft sets and N-soft sets literature. Then we design two decision making procedures that are flexible in the sense that they enable the practitioner to reflect his specific priorities with regard to the attributes, when the situation is hesitant. The first procedure is flexible in the sense that it allows the practitioner to select a score. The second one is a modification of it that also incorporates adjustability in terms of weights for the attributes.

Algorithm 1 - The algorithm of choice values of HNSSs.

1: Select a score s for HNTs (e.g., geometric or arithmetic score).
2: Input $O = \{o_1, o_2, \cdots, o_p\}$ as a universe of objects, and $T = \{t_1, t_2, \cdots, t_q\}$ as a set of attributes.
3: Input $G = \{0, 1, 2, \cdots, N-1\}$, $N \in \{2, 3, \cdots\}$, for each $o_i \in O$, $t_j \in T$, there exists $g_{ij} \in G$, according to each expert.
4: Compute the HNSS (h, T, N).
5: Compute scores $s(h_i(o))$ of HNTs, $\forall\ o_i \in O$.
6: Compute $\varrho_i = \sum_{j=1}^{q} s(h_{ij})$, $\forall\ o_i \in O$.
7: Compute all the indices l for which $\varrho_l = \max_{i=1,2,\cdots,p} \varrho_i$.

Any of the alternatives for which $\varrho_l = \max_{i=1,2,\cdots,p} \varrho_i$ can be chosen.

Algorithm 2 - The algorithm of weighted choice values of HNSSs.

1: Select a score s for HNTs (e.g., geometric or arithmetic score).
2: Input $O = \{o_1, o_2, \cdots, o_p\}$ as a universe of objects, and $T = \{t_1, t_2, \cdots, t_q\}$ as a set of attributes.
3: Input $G = \{0, 1, 2, \cdots, N-1\}$, $N \in \{2, 3, \cdots\}$, for each $o_i \in O$, $t_j \in T$, there exists $g_{ij} \in G$, according to each expert.
4: Compute HNSS (h, T, N).
5: Compute scores $s(h_i(o))$ of HNTs, $\forall\ o_i \in O$.
6: Select weights for each attribute from $(0, 1]$.
7: Compute $\varrho_i^w = \sum_{j=1}^{q} w_j \times s(h_{ij})$, $\forall\ o_i \in O$.
8: Compute all the indices l for which $\varrho_l^w = \max_{i=1,2,\cdots,p} \varrho_i^w$.

Any of the alternatives for which $\varrho_l^w = \max_{i=1,2,\cdots,p} \varrho_i^w$ can be chosen.

Admittedly, Algorithms 1 and 2 are very similar, the only difference being the application of a weighting vector in the second case. Weights are an important element in the broad field of decisions under hesitancy: for a short sample of recent references, see Xu & Zhang (2013); Xu et al. (2016); Zhang et al. (2016). Hence we have opted for keeping two separate formulations in order to acknowledge the role of weights and simplify the identification of which procedure to apply in each situation.

In order to prove the importance and feasibility of these algorithms, we proceed to apply them to real situations that are fully developed. The scores that are selected for this exercise are the geometric and arithmetic scores.

1. Recovery order of patients for muscle weakness

Muscle weakness or myasthenia is a lack of muscle strength. It may have many causes, and it can be divided into conditions that are either perceived or true muscle weakness. True muscle weakness is considered as a primary symptom of a variety of skeletal muscle diseases, including inflammatory myopathy and muscular dystrophy. It occurs in neuromuscular junction disorders such as myasthenia gravis. Another cause of muscle weakness is low levels of potassium and other type of electrolytes within muscle cells. It can be temporary or long-lasting. It is not an easy task for doctors to check and compare the recovery of patients, especially when more than one doctor are involved in a case and they hesitate to make a unanimous diagnosis.

Hesitation in grades arises in a natural way when we incorporate the information from various sources and each of them parameterizes the universe in a multinary categorization. Gradings and check mark ratings by different external sources are very preferable in making a group decision. In order to check the recovery of patients for muscle weakness one typically finds different gradings for the same disease from different classification and external resources including ICD-10, ICD-9-CM, Diseases DB, and MeSH among others. For this hesitant situation we use the concept of HNSS.

Let $P_t = \{p_1, p_2, p_3, p_4\}$ be the universe of patients, and let P be the set of attributes "evaluation of patients by different doctors" using external sources. The subset $T \subseteq P$ is such that $T = \{t_1, t_2, t_3, t_4\}$. A H6SS can be obtained from Table 20, where the severity of muscle weakness can be classified by three doctors into different "check mark ratings" based on the following criteria (see Example 2.3 for a more explicit description):

- Five check marks represent "Normal strength", and they are represented by 5.
- Four check marks represent "Movement against external resistance with less strength than usual", and they are represented by 4.
- Three check marks represent "Movement against gravity, but not against added resistance", and they are represented by 3.
- Two check marks represent "Movement at the joint with gravity eliminated", and they are represented by 2.
- One check mark represents "Trace of contraction, but no movement at the joint", and they are represented by 1.
- Box represents "No contraction or muscle movement", and they are represented by 0.

This graded evaluation uses $G = \{0, 1, 2, 3, 4, 5\}$. [1] The information extracted from the three doctors on related data is described in Table 20, which summarizes the three H6SSs provided by the doctors. And then the tabular representation of the H6SSs that results

[1] As explained in the entry "Muscle weakness" in Wikipedia https://en.wikipedia.org/wiki/Muscle_weakness (retrieved June 26, 2018).

from their fusion is easily given by Table 21. In each cell, all the possible grades assigned by each of the doctors are incorporated to the hesitant multinary approximation of the universe of patients.

	P_t/T	t_1	t_2	t_3	t_4
Doctor 1	p_1	✓✓	✓✓✓	□	✓✓✓✓
	p_2	✓✓✓	✓✓	✓✓✓	✓✓
	p_3	□	✓✓✓✓	✓✓✓✓✓	✓✓✓
	p_4	✓✓✓✓	□	✓✓	✓
Doctor 2	p_1	✓	✓✓✓	✓	✓✓✓
	p_2	✓✓✓	✓✓	✓✓	✓✓✓
	p_3	✓✓	✓✓✓	✓✓✓✓	✓✓✓
	p_4	✓✓✓✓✓	✓	✓✓	✓
Doctor 3	p_1	✓✓	✓✓	✓	✓✓✓✓
	p_2	✓✓	✓✓	✓✓✓✓	✓
	p_3	✓✓✓	✓✓✓	✓✓✓	✓✓✓✓✓
	p_4	✓✓✓	✓✓	✓	✓✓✓

Table 20: Information extracted from three doctors on related data. It is naturally transformed into three respective H6SSs.

$(H, T, 6)$	t_1	t_2	t_3	t_4
p_1	$\{1, 2\}$	$\{2, 3\}$	$\{0, 1\}$	$\{3, 4\}$
p_2	$\{2, 3\}$	$\{2\}$	$\{2, 3, 4\}$	$\{1, 2, 3\}$
p_3	$\{0, 2, 3\}$	$\{3, 4\}$	$\{3, 4, 5\}$	$\{3, 5\}$
p_4	$\{3, 4, 5\}$	$\{0, 1, 2\}$	$\{1, 2\}$	$\{1, 3\}$

Table 21: Tabular representation of H6SS.

Table 21 provides the whole information about the opinion of the doctors in the form of a H6SS. Now for disclosing their common decision we need to calculate the scores of the H6Ts, which are given by Table 22. The geometric scores are selected for this purpose.

	t_1	t_2	t_3	t_5
p_1	1.41	2.45	0	3.46
p_2	2.45	2	2.88	1.82
p_3	0	3.46	3.91	3.87
p_4	3.91	0	1.5	2

Table 22: Tabular representation of geometric scores $s_g(h_{ij})$ of the H6Ts in Table 21.

In order to make a final decision, we use the two algorithms of group decision-making that respectively appeal to choice values and weighted choice values.

- **Choice values for geometric scores of H6SS.**

  The choice values of patients $p_i \in P_t$ for geometric scores can be calculated as

$$\varrho_i = \sum_{j=1}^{q} s_g(h_{ij}), \quad i = 1, 2, \cdots, p.$$

  and they are displayed in Table 23.

	t_1	t_2	t_3	t_4	ϱ_i
p_1	1.41	2.45	0	3.46	7.32
p_2	2.45	2	2.88	1.82	9.15
p_3	0	3.46	3.91	3.87	11.24
p_4	3.91	0	1.5	2	7.91

Table 23: Tabular representation of choice values for geometric scores of the H6Ts in Table 21.

  Finally from Table 23, the recovery order of four different patients for muscle weakness is $p_3 > p_2 > p_4 > p_1$.

- **Weighted choice values for geometric scores of H6SS**

  It may happen that all the external resources do not have the same worth for the doctors, and they prefer to assign weights according to their common point of views. Thus the joint opinion of these doctors produces a weight $w_j \in (0, 1]$ for each external resource. The weights assigned by the doctors are normalized, i.e., $\sum\limits_{j=1}^{q} w_j = 1$.

  The weighted choice values ϱ_i^w of patient $p_i \in P_t$ for geometric scores is defined as

$$\varrho_i^w = \sum_{j=1}^{q} (w_j \times s_g(h_{ij})), \quad i = 1, 2, \cdots, p.$$

  Suppose that the doctors assign the weights $w_1 = 0.3, w_2 = 0.2, w_3 = 0.2, w_4 = 0.3$ to each external source. Then we get the weighted table displayed in Table 24 below.

	$t_1 \cdot w_1$	$t_2 \cdot w_2$	$t_3 \cdot w_3$	$t_4 \cdot w_4$	ϱ_i^w
p_1	0.42	0.49	0	1.04	1.95
p_2	0.74	0.4	0.58	0.55	2.27
p_3	0	0.69	0.78	1.16	2.63
p_4	1.17	0	0.4	0.6	2.17

Table 24: Tabular representation of weighted choice values for geometric scores of the H6Ts in Table 21.

From Table 24, it is easy to see that we have the same decision as described in Table 23, i.e., $p_3 > p_2 > p_4 > p_1$.

2. Severity of hepatic encephalopathy in its victims

Hepatic encephalopathy is a set of symptoms examined in patients with cirrhosis. Hepatic encephalopathy is determined as a spectrum of neuropsychiatric abnormalities with liver dysfunction, after exclusion of brain disease. It may be sudden or gradual. It is characterized by personality changes, intellectual impairment, and a depressed level of consciousness. Other symptoms may include changes in mood, movement problems, or changes in personality. In the most severe stages it can result in a coma. An important prerequisite for the syndrome is diversion of portal blood into the systemic circulation through portosystemic collateral vessels. An important assignment for experts is to assess the severity of hepatic encephalopathy, and according to the stage of hepatic encephalopathy, which dose and treatment is suitable for the patient. To handle such a situation we use the concept of HNSS. We consider $V = \{v_1, v_2, v_3\}$ the universe of victims of hepatic encephalopathy, and P be the set of attributes "evaluation of victims by various experts" using set of symptoms of hepatic encephalopathy including level of consciousness, personality and intellect neurologic signs, electroencephaiogram and abnormalities. The subset $T \subseteq P$ is such that $T = \{t_1, t_2, t_3, t_4\}$. An H5SS can be obtained from Table 25, where the severity of hepatic encephalopathy can be classified into different "check mark ratings" based on the following criteria:

- Four check marks represent "Coma with or without response to painful stimuli", and they are represented by 4.
- Three check marks represent "Somnolent but can be aroused", and they are represented by 3.
- Two check marks represent "Lethargy or apathy, minimal disorientation, inappropriate behavior, obvious personality changes", and they are represented by 2.
- One check mark represents "Trivial lack of awareness, euphoria or anxiety, shortened attention span", and they are represented by 1.

- Box represents "No abnormality detected", and they are represented by 0.

This graded evaluation uses $G = \{0, 1, 2, 3, 4\}$, which corresponds with the West Haven Criteria (Cash et al., 2010) as introduced in the World Congress of Gastroenterology 1998 (Ferenci et al., 2002). The information extracted from the four experts on related data is described in Table 25. Then the tabular representation of the H5SSs that results from their fusion is easily given by Table 26.

	V/T	t_1	t_2	t_3	t_4
Expert 1	v_1	✓	□	✓✓✓	✓✓
	v_2	✓✓	✓✓✓	✓	✓
	v_3	✓✓✓	✓✓✓✓	✓✓✓✓	✓✓✓
Expert 2	v_1	✓✓	✓	✓✓	✓
	v_2	✓✓	✓✓✓	✓	□
	v_3	✓✓	✓✓✓	✓✓✓✓	✓✓✓
Expert 3	v_1	✓	□	✓✓✓	✓✓
	v_2	✓✓✓	✓✓✓	✓✓	✓✓
	v_3	✓✓✓	✓✓	✓✓✓✓	✓✓✓
Expert 4	v_1	□	✓	✓✓	✓✓✓
	v_2	✓	✓✓✓	□	✓✓✓
	v_3	✓✓✓✓	✓	✓✓✓✓	✓✓✓

Table 25: Information extracted from experts on related data.

$(H, T, 5)$	t_1	t_2	t_3	t_4
v_1	$\{0, 1, 2\}$	$\{0, 1\}$	$\{2, 3\}$	$\{1, 2, 3\}$
v_2	$\{1, 2, 3\}$	$\{3\}$	$\{0, 1, 2\}$	$\{0, 1, 2, 3\}$
v_3	$\{2, 3, 4\}$	$\{1, 2, 3, 4\}$	$\{4\}$	$\{3\}$

Table 26: Tabular representation of the H5SS derived from Table 26.

Table 26 provides the information about the opinions of the experts in the form of a H5SS. Now for their joint decision we observe Table 26, and calculated the arithmetic scores of the H5Ts involved in Table 27.

	t_1	t_2	t_3	t_4
v_1	1	0.5	2.5	2
v_2	2	3	1	1.5
v_3	3	2.5	4	3

Table 27: Tabular representation of arithmetic scores $s_a(h_{ij})$ of the H5Ts in Table 26.

Similarly, for the purpose of making a final decision, we use the two algorithms of group decision-making that respectively appeal to choice values and weighted choice values.

- **Choice values for arithmetic scores of H5SS.**

  The choice values ϱ_i for arithmetic scores of hepatic encephalopathy victims $v_i \in V$ are defined as

$$\varrho_i = \sum_{j=1}^{q} s_a(h_{ij}), \quad i = 1, 2, \cdots, p$$

  and they are displayed in Table 28.

	t_1	t_2	t_3	t_4	ϱ_i
v_1	1	0.5	2.5	2	6
v_2	2	3	1	1.5	7.5
v_3	3	2.5	4	3	12.5

Table 28: Tabular representation of choice values for the arithmetic scores of the H5Ts in Table 26.

  Finally from Table 28, the severity of hepatic encephalopathy in its victims is as $v_3 > v_2 > v_1$.

- **Weighted choice values for arithmetic scores of H5SS**

  The weighted choice values ϱ_i^w for arithmetic scores of hepatic encephalopathy victims $v_i \in V$ are defined as

$$\varrho_i^w = \sum_{j=1}^{q} (w_j \times s_a(h_{ij})), \quad i = 1, 2, \cdots, p.$$

  Suppose that the weights $w_j \in (0, 1]$ assigned by the experts to the symptoms according to their common point of views are as follows:

$$w_1 = 0.4, w_2 = 0.3, w_3 = 0.1, w_4 = 0.2.$$

  Table 29 shows the corresponding weighted values.

	$t_1 \cdot w_1$	$t_2 \cdot w_2$	$t_3 \cdot w_3$	$t_4 \cdot w_4$	ϱ_i^w
v_1	0.4	0.15	0.25	0.4	1.2
v_2	0.8	0.9	0.1	0.3	2.1
v_3	1.2	0.75	0.4	0.6	2.6

Table 29: Tabular representation of weighted choice values for the arithmetic scores of the H5Ts in Table 26.

From Table 29, it is easy to see that we reach the same decision as described after Table 28, i.e., $v_3 > v_2 > v_1$.

3. Health check mark ratings

For the comparison of products within the same category, the health check mark rating system provides key information to consumers. Further, in the nutrition information panel on food packaging the nutrient information is also available. Since 2014, retailers and food manufacturers have been implementing the health check mark rating system. Check mark ratings displayed on the front of food packages helps the customer to compare similar products. Some electrical appliances may display the energy icon only, allowing you to compare the energy content of different products. Health check mark ratings can appear on packs in two general ways. The first one shows just the check mark rating of the product, the second one can show the check mark rating plus additional specific nutrient content of the product. Under the system, packaged foods are given a check mark rating based on their nutritional profile, which includes

1. Energy (kilojoules).
2. Risk nutrients-saturated fat, sodium (salt) and sugars.
3. Positive nutrients-dietary fibre, protein and the proportion of fruits, vegetables, nuts and legumes.

For such a health check mark ratings, we use the concept of $HNSS$ because when more than one food taster are testing the packaged foods they have different opinion and grading for same product. Let $D = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ be the universe of products which have to be compared, and P be the set of nutritional profile act as attributes "evaluation of products by food taster". The subset $T \subseteq P$ is such that $T = \{t_1, t_2, t_3\}$. A hesitant 5-soft set can be obtained from Table 30, where

- Four check marks represent 'Excellent quality', and they are represented by 4.
- Three check marks represent 'Very Good quality', and they are represented by 3.
- Two check marks represent 'Good quality', and they are represented by 2.

- One check mark represents ‘Normal quality’, and they are represented by 1.
- Box represents ‘Poor quality’, and they are represented by 0.

Hence we use the ordered grades $G = \{0, 1, 2, 3, 4\}$. The information extracted from food taster on related data is described in Table 30. Then the tabular representation of the H5SSs that results from their fusion is easily given by Table 31.

	D/T	t_1	t_2	t_3
Food taster 1	d_1	$\square$	$\checkmark\checkmark$	$\square$
	d_2	$\checkmark\checkmark$	$\checkmark$	$\checkmark\checkmark$
	d_3	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark\checkmark$	$\checkmark$
	d_4	$\checkmark\checkmark\checkmark\checkmark$	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark\checkmark$
	d_5	$\checkmark\checkmark$	$\checkmark$	$\checkmark\checkmark$
	d_6	$\checkmark$	$\checkmark\checkmark\checkmark\checkmark$	$\checkmark$
Food taster 2	d_1	$\checkmark\checkmark$	$\checkmark$	$\checkmark$
	d_2	$\checkmark\checkmark$	$\checkmark\checkmark$	$\square$
	d_3	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark$
	d_4	$\checkmark\checkmark\checkmark$	$\checkmark$	$\checkmark\checkmark\checkmark$
	d_5	$\checkmark$	$\checkmark$	$\checkmark\checkmark\checkmark\checkmark$
	d_6	$\checkmark\checkmark$	$\checkmark\checkmark\checkmark$	$\checkmark\checkmark$

Table 30: Information extracted from food tasters on related data.

$(H, T, 5)$	t_1	t_2	t_3
d_1	$\{0, 2\}$	$\{1, 2\}$	$\{0, 1\}$
d_2	$\{2\}$	$\{1, 2\}$	$\{0, 2\}$
d_3	$\{3\}$	$\{3\}$	$\{1, 2\}$
d_4	$\{3, 4\}$	$\{1, 3\}$	$\{3\}$
d_5	$\{1, 2\}$	$\{1\}$	$\{2, 4\}$
d_6	$\{1, 2\}$	$\{3, 4\}$	$\{1, 2\}$

Table 31: Tabular representation of H5SS.

Table 31 contains the information about the opinion of food tasters in the form of a H5SS. Now for their common decision, both the arithmetic scores and geometric scores of H5Ts are calculated in Tables 32 and 33.

	t_1	t_2	t_3
d_1	1	1.5	0.5
d_2	2	1.5	1
d_3	3	3	1.5
d_4	3.5	2	3
d_5	1.5	1	3
d_6	1.5	3.5	1.5

Table 32: Tabular representation of the arithmetic scores $s_a(h_{ij})$ of the H5Ts in Table 31.

	t_1	t_2	t_3
d_1	0	1.41	0
d_2	2	1.41	0
d_3	3	3	1.41
d_4	3.46	2	3
d_5	1.41	1	2.83
d_6	1.41	3.46	1.41

Table 33: Tabular representation of the geometric scores $s_g(h_{ij})$ of the H5Ts in Table 31.

For the purpose of the final group decision and comparison of arithmetic and geometric scores, choice values are calculated respectively in Tables 34 and 35 below.

- **Choice values for arithmetic scores of H5SS**
  The choice value for arithmetic scores of products are calculated in Table 34.

	t_1	t_2	t_3	ϱ_i
d_1	1	1.5	0.5	3
d_2	2	1.5	1	4.5
d_3	3	3	1.5	7.5
d_4	3.5	2	3	8.5
d_5	1.5	1	3	5.5
d_6	1.5	3.5	1.5	6.5

Table 34: Tabular representation of choice values $\varrho_i = \sum_{j=1}^{q} s_a(h_{ij})$, $i = 1, 2, \cdots, p$, for arithmetic scores of H5Ts.

According to the choice values for arithmetic scores, the prioritization ratings of products are $d_4 > d_3 > d_6 > d_5 > d_2 > d_1$.

- **Choice values for geometric scores of H5SS**
  The choice values for geometric scores of products are calculated in Table 35.

	t_1	t_2	t_3	ϱ_i
d_1	0	1.41	0	1.41
d_2	2	1.41	0	3.41
d_3	3	3	1.41	7.41
d_4	3.46	2	3	8.46
d_5	1.41	1	2.83	5.24
d_6	1.41	3.46	1.41	6.28

Table 35: Tabular representation of choice values $\varrho_i = \sum_{j=1}^{q} s_g(h_{ij})$, $i = 1, 2, \cdots, p$, for geometric scores of H5Ts.

Similarly, the choice values for geometric scores of products has same prioritization ratings $d_4 > d_3 > d_6 > d_5 > d_2 > d_1$ as compared to choice values for arithmetic scores.

Remark 2. We can use any approach for scores calculations, because prioritization ratings remains same. The verdict from both methods of decision-making has varied, which justifies the fact that the weights associated with the attributes can affect the decision according to the opinions of decision-makers. Therefore, they are a crucial ingredient in the decision-making procedure.

4 Conclusion

N-soft sets are the extended and applicable version of soft sets that can deal with both binary and non-binary evaluations. That model is widely used to make decisions in general real situations, and examples were provided in the founding reference Fatimah et al. (2018a). An interesting feature of its design is that contrary to the case of soft sets, the introduction of multinary categorizations is naturally compatible with hesitation.

However the model by N-soft sets is unable to make decisions when data collection produces hesitancy. In this research article we have introduced a new extended model of N-soft sets as an answer for this drawback, which is the natural hybridization of N-soft sets and hesitancy. In addition, the case $N = 2$ reduces to incomplete soft sets. Hesitant data may come from sources like hesitation on the side of the decision-maker, or the combination of the evaluations provided by various decision makers. The fundamental concept that we introduced is called hesitant N-soft set or simply HNSS. We have put forward real life examples that adopt the format of HNSSs. A related concept is hesitant N-tuple, which appears as an inherent element of the practical description of HNSSs. We have introduced scores of hesitant N-tuples in order to have new tools for prioritization. Particularly, scores are adequate for implementation of GDM algorithms based on HNSSs. We have set forth an algorithm for making decisions that can account for parameters with unequal weights. Moreover, we have investigated the properties and basic operations of HNSSs, as well as their interaction with less flexible models.

In the future, we expect to extend our research work on related hybrid models with the construction of (1) N-soft mF rough graphs, (2) N-soft rough mF graphs, (3) Hesitant N-soft graphs, (4) Hesitant N-soft hypergraphs, and (5) Hesitant Pythagorean fuzzy graphs

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