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Extending fuzzy logics with many hedges

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Abstract

Fuzzy logic aims at modeling logical reasoning with vague or imprecise statements, which may contain linguistic hedges. In fact, many hedges, e.g., *very*, *highly*, *rather*, and *slightly*, can be used simultaneously to express different levels of emphasis. Moreover, each hedge might have a dual one, e.g., *slightly* can be seen as a dual hedge of *very*. Thus, it is necessary to extend systems of fuzzy logic with multiple hedges. This work proposes two axiomatizations for multiple hedges as an expansion of a core fuzzy logic. In one axiomatization, hedges do not have any dual one while in the other, each hedge can have its own dual one. It is shown that the proposed logics not only cover a large class of hedge functions but also have all completeness properties as the underlying logic w.r.t. the class of their chains as well as distinguished subclasses of their chains, including standard completeness. The axiomatizations are also extended to the first-order level. Furthermore, we present a method to build linguistic fuzzy logics based on the axiomatizations and a hedge algebra, whose corresponding algebras use a linguistic truth domain taken from the hedge algebra, for representing and reasoning with linguistically-expressed human knowledge, where truth values of vague sentences are given in linguistic terms.

Keywords: Mathematical fuzzy logic; t-Norm based logics; Axiomatization; Hedge; Completeness; Hedge algebra

1. Introduction

Previous works on adding hedges to logical systems of mathematical fuzzy logic (MFL) [9] include those by Hájek [18], Vychodil [37], and Esteva et al. [14]. In MFL, hedges are called *truth-stressing* or *truth-depressing* depending on whether they strengthen or weaken the meaning of the proposition. The intuitive interpretation of a truth-stressing (resp., truth-depressing) hedge on a chain of truth values is a subdiagonal (resp., superdiagonal) non-decreasing function preserving 0 and 1. Such functions are called *hedge functions*. Hájek [18] introduces an axiomatization of a truth-stressing hedge vt as an expansion of Basic Logic (BL) [17], and the resulting logic is called BL_{vt}. Vychodil [37] extends BL_{vt} to a logic $\mathrm{BL}_{vt,st}$ with a truth-depressing hedge *st* dual to *vt*. The logics are shown to be algebraizable and enjoy completeness w.r.t. the classes of their chains, but are not proved to enjoy standard completeness in

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general. Moreover, Hájek and Vychodil's axiomatizations do not cover a large class of hedge functions [14]. Hence, Esteva et al. [14] propose weaker axiomatizations over any core fuzzy logic for a truth-stressing hedge or/and a truth-depressing one, which do not impose any more constraints on hedge functions, and the axiomatizations are proved to enjoy standard completeness.

This work proposes two axiomatizations over any propositional core fuzzy logic for multiple truth-stressing and truth-depressing hedges, one for non-dual hedges and the other for dual ones. The axiomatizations not only cover a large class of hedge functions but also have all completeness properties of the underlying core fuzzy logic w.r.t. the class of their chains and distinguished subclasses of their chains, including standard completeness. The axiomatizations are also extended to the first-order level. Moreover, we show how to build linguistic fuzzy logics based on the axiomatizations and hedge algebras [32,33] for representing and reasoning with linguistically-expressed human knowledge.

The remainder of the paper is organized as follows. Section 2 gives an overview of MFL, previous axiomatizations for hedges, linguistic truth domains and operations on them. Section 3 presents an axiomatization for multiple non-dual hedges while Section 4 provides an axiomatization for multiple dual ones. Section 5 shows how to build linguistic fuzzy logics. Section 6 extends the axiomatizations to the first-order level. Section 7 discuses related work. Finally, Section 8 concludes the paper.

2. Preliminaries

2.1. Preliminaries on mathematical fuzzy logic

Let L be a logic in a language $\mathcal{L}$, a set of connectives with finite arity. A truth constant $\overline{r}$ is a special formula whose truth value under each evaluation is r. Formulae are built from propositional variables and truth constants using connectives in $\mathcal{L}$. Each evaluation e of propositional variables by truth values uniquely extends to an evaluation $e(\varphi)$ of all formulae φ using truth functions of connectives, which are operations of an algebra used to interpret the language. A formula φ is called an *1-tautology* if $e(\varphi) = 1$ for every evaluation e. A number of 1-tautology formulae are taken as *axioms* of the logic. A *theory* is a set of formulae called *special axioms*. An evaluation e is called a *model* of a theory T if $e(\varphi) = 1$ for all φ in T. A *proof* in a theory T is a sequence $\varphi_1, \dots, \varphi_n$ of formulae such that each element of the sequence is either an axiom of the logic or an element of T or follows from some preceding elements of the sequence using the deduction rule(s) of the logic. The last member φ of a proof in T is called a *provable* formula, denoted $T \vdash_L \varphi$. If $T = \emptyset$, it is said that φ is *provable* in the logic [17,9].

It is said that L is a *Rasiowa-implicative logic* [35,10] if there is a binary (either primitive or definable by a formula) connective $\rightarrow$ in its language such that:

(R) $\vdash_L \varphi \rightarrow \varphi$, (MP) $\varphi, \varphi \rightarrow \psi \vdash_L \psi$,

(W) $\varphi \vdash_L \psi \rightarrow \varphi$, (T) $\varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_L \varphi \rightarrow \chi$,

(sCng) $\varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash_L c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \rightarrow c(\chi_1, \dots \chi_i, \psi, \dots, \chi_n)$

for each n-ary $c \in \mathcal{L}$ and each $i < n$.

Every finitary Rasiowa-implicative logic L is algebraizable and its equivalent algebraic semantics, a class of L-algebras, is a quasivariety [5], denoted $\mathbb{L}$. The algebraic semantics enjoys the following completeness.

Theorem 1 *(Strong completeness). [9] For every set $T \cup \{\varphi\}$ of formulae, $T \vdash_L \varphi$ iff for every $\mathbf{A} \in \mathbb{L}$ and every $\mathbf{A}$-model e of T, $e(\varphi) = 1$.*

Every L-algebra $\mathbf{A}$ is endowed with a *preorder* relation $\leq$ by setting, for every $a, b \in A$, $a \leq b$ iff $a \Rightarrow b = 1$, where $\Rightarrow$ is the truth function of $\rightarrow$. $\mathbf{A}$ is called an L-*chain* if $\leq$ is a total order, i.e., A is linearly ordered. L is called a *semilinear* logic iff it is strongly complete w.r.t. the class of L-chains or, equivalently, if every L-algebra is representable as subdirect product of L-chains [3,10].

In the literature, most logical systems referred to as *fuzzy logics* are, indeed, a finitary Rasiowa-implicative semilinear logic. They belong to a large class of systems which are axiomatic expansions of MTL (*monoidal t-norm based logic*) satisfying (sCng) for any possible new connective [13]. Such systems are called *core fuzzy logics*. Well-known

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examples of them are BL, G (Gödel logic), Ł (Łukasiewicz logic) [18], and MTL [13]. The language of MTL is defined from a countable set of propositional variables $p_1, p_2, \ldots$, three connectives &, $\to$, $\wedge$, and the truth constant $\overline{0}$. Further definable connectives are:

$$\begin{aligned} \neg\varphi &\equiv \varphi \to \overline{0}, \quad & \varphi \vee \psi &\equiv ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi), \\ \overline{1} &\equiv \neg\overline{0}, \quad & \varphi \leftrightarrow \psi &\equiv (\varphi \to \psi) \wedge (\psi \to \varphi). \end{aligned}$$

The only deduction rule of MTL (and of the core fuzzy logics) is (MP).

Lemma 1. *[13] The following formulae are provable in MTL and core fuzzy logics:*

$$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi) \tag{1}$$

$$\varphi \to (\neg\varphi \to \psi), \varphi \to \neg\neg\varphi, (\varphi \& \neg\varphi) \to \overline{0} \tag{2}$$

$$(\varphi \to \psi) \vee (\psi \to \varphi) \tag{3}$$

The connective $\vee$ defined above is called a *disjunction*. Given a disjunction $\vee$ and a finitary inference rule (R): $\Gamma \vdash \varphi$ (axioms are seen as rules with $\Gamma = \emptyset$), the $\vee$-*form* of (R), denoted $(R^{\vee})$, is defined as the rule $\Gamma \vee p \vdash \varphi \vee p$, where p is an arbitrary propositional variable not appearing in $\Gamma \cup \{\varphi\}$. Let L_1 be a logic with a disjunction $\vee$ and L_2 be an expansion of L_1 by a set of finitary rules $\mathcal{R}$. Then, $\vee$ is a disjunction in L_2 iff $(R^{\vee})$ holds in L_2 for each $(R) \in \mathcal{R}$. Moreover, if L is a finitary Rasiowa-implicative logic with a disjunction $\vee$ satisfying (3), L is semilinear [11].

Core fuzzy logics are semilinear and, thus, strongly complete w.r.t. the class of their chains. This completeness is sometimes refined to distinguished subclasses of chains as follows.

Definition 1. [8] Let L be a core fuzzy logic and $\mathbb{K}$ a class of L-chains. It is said that L has the (finite) strong $\mathbb{K}$-completeness, (F)S$\mathbb{K}$C for short, if for every (finite) set of formulae T and every formula φ, it holds that $T \vdash_L \varphi$ iff $e(\varphi) = 1$ for every L-algebra $\mathbf{A} \in \mathbb{K}$ and each $\mathbf{A}$-model e of T. It is said that L has the $\mathbb{K}$-completeness, $\mathbb{K}$C for short, when the equivalence is true for $T = \emptyset$.

Clearly, the S$\mathbb{K}$C implies the FS$\mathbb{K}$C, and the FS$\mathbb{K}$C implies the $\mathbb{K}$C. When $\mathbb{K}$ is the class of all chains whose support is the unit interval [0, 1] with the usual ordering, the (F)S$\mathbb{K}$C can be called the (finite) strong *standard* completeness, (F)SSC for short. The (F)S$\mathbb{K}$C have traditionally been proved by showing an embeddability property as follows.

Theorem 2. *[8] Let L be a core fuzzy logic and $\mathbb{K}$ a class of L-chains. Then, (i) L has the S$\mathbb{K}$C iff every countable L-chain is embeddable into some member of $\mathbb{K}$; (ii) if the language of L is finite, L has the FS$\mathbb{K}$C iff every countable L-chain is* partially *embeddable into some member of $\mathbb{K}$, i.e., for every finite partial of a countable L-chain, there is a one-to-one mapping preserving the operations into some member of $\mathbb{K}$.*

G enjoys SSC and S$\mathbb{K}$C for any class $\mathbb{K}$ of G-chains, and Ł enjoys FSSC and FS$\mathbb{K}$C for any class $\mathbb{K}$ of MV-chains [9]. Let $*$, $\cap$, $\cup$, $-$ denote truth functions of &, $\wedge$, $\vee$, $\neg$, respectively. For BL, G and Ł, $\cap$ and $\cup$ are min and max, respectively.

2.2. Hájek, Vychodil, and Esteva's axiomatizations

The logic BL_{vt} of Hájek [18] is an expansion of BL with a new connective vt, for a truth-stressing hedge, and the following axioms:

$$\text{(VT1) } vt\varphi \to \varphi, \ \text{(VT2) } vt(\varphi \to \psi) \to (vt\varphi \to vt\psi), \ \text{(VT3) } vt(\varphi \vee \psi) \to (vt\varphi \vee vt\psi),$$

and the necessitation deduction rule: (NEC) from φ infer $vt\varphi$.

Axiom (VT2) is indeed the well-known modal axiom K in modal logics. BL_{vt}-algebras are expansions of BL-algebras with a new unary operator. BL_{vt} is complete w.r.t. the class of BL_{vt}-chains. This completeness is extended to any axiomatic extension of BL such as G and Ł, but standard completeness is left open, except for G.

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Vychodil [37] extends BL_{vt} to $\mathrm{BL}_{vt,st}$ with a new truth-depressing hedge st, dual to vt, by the following additional axioms:

$$\text{(ST1)}\; \varphi \to st\varphi, \quad \text{(ST2)}\; st\varphi \to \neg vt\neg\varphi, \quad \text{(ST3)}\; vt(\varphi \to \psi) \to (st\varphi \to st\psi).$$

$\mathrm{BL}_{vt,st}$ is complete w.r.t. the class of its chains, but standard completeness is not mentioned. Vychodil also introduces two slightly different axiomatizations (systems I and II) over BL for only *st*. More precisely, system I is composed of Axiom (ST1) and the following additional axioms:

$$\text{(ST4)}\; \neg st(\bar{0}), \quad \text{(ST5)}\; st(\varphi \to \psi) \to (st\varphi \to st\psi),$$

while system II consists of (ST1), (ST4) and the below axiom:

$$\text{(ST6)} \quad (\varphi \to \psi) \to (st\varphi \to st\psi).$$

In general, Hájek and Vychodil's axiomatizations do not cover a large class of hedge functions due to the presence of (VT2), (ST3), (ST5) or (ST6), which put a lot of constraints on the hedge functions. In particular, for Ł, the only truth-depressing hedge function satisfying the conditions is the identity (see Example 1 of Esteva et al. [14]).

To overcome the drawback, Esteva et al. [14] propose weaker axiomatizations over any core fuzzy logic for a truth-stressing hedge s or/and a truth-depressing one d. Let L be a core fuzzy logic, $\mathrm{L}_{s,d}$ is an expansion of L with two new connectives s and d by the following additional axioms:

$$\text{(VTL1)}\; s\varphi \to \varphi, \quad \text{(VTL2)}\; s\bar{1}, \quad \text{(STL1)}\; \varphi \to d\varphi, \quad \text{(STL2)}\; \neg d\bar{0},$$

and the following additional deduction rule:

$$\text{(MON)} \quad \text{from } (\varphi \to \psi) \vee \chi \text{ infer } (h\varphi \to h\psi) \vee \chi \text{ for } h \in \{s, d\}.$$

We can have the axiomatizations L_s and L_d for s and d alone, respectively. The axiomatizations allow a very large class of hedge functions, which are only required to be subdiagonal (resp., superdiagonal) non-decreasing functions preserving 0 and 1 for the truth-stressing (resp., truth-depressing) hedge on a chain of truth values. It is proved that if L enjoys one of the completeness properties in Definition 1, including standard completeness, so does $\mathrm{L}_{s,d}$.

2.3. Linguistic truth domains and operations

2.3.1. Linguistic truth domains

Let V, H, R, S, T, and F stand for *very*, *highly*, *rather*, *slightly*, *true*, and *false*, respectively. In hedge algebra (HA) theory [32,33], values of the linguistic variable *Truth*, e.g., *VT* and *VSF*, can be regarded as being generated from a set of primary terms $\mathcal{G} = \{F, T\}$ using hedges from a set $\mathcal{H} = \{V, S, \dots\}$ as unary operations. There exists a natural ordering among the values, e.g., $ST < T$. Thus, a term domain of *Truth* is a partially ordered set (poset) and can be characterized by an HA $\underline{X} = (\mathcal{X}, \mathcal{G}, \mathcal{H}, \leq)$, where $\mathcal{X}$ is a term set, $\mathcal{G}$ is a set of primary terms, $\mathcal{H}$ is a set of hedges, and $\leq$ is a *semantic order relation* (SOR) on $\mathcal{X}$ preserving the natural ordering of the values.

There are natural semantic properties of hedges and terms. Hedges either increase or decrease the meaning of terms, i.e., $\forall h \in \mathcal{H}, \forall x \in \mathcal{X},$ either $hx \geq x$ or $hx \leq x$. It is denoted by $h \geq k$ if a hedge h modifies terms more than or equal to another hedge k, i.e., $\forall x \in \mathcal{X}$, $hx \leq kx \leq x$ or $x \leq kx \leq hx$. Since $\mathcal{H}$ and $\mathcal{X}$ are disjoint, the same notation $\leq$ can be used for different order relations on $\mathcal{H}$ and $\mathcal{X}$ without confusion. A hedge has a semantic effect on others. If h strengthens the degree of modification of k, i.e., $\forall x \in \mathcal{X}$, $hkx \leq kx \leq x$ or $x \leq kx \leq hkx$, then h is *positive* w.r.t. k. If h weakens the degree of modification of k, i.e., $\forall x \in \mathcal{X}$, $kx \leq hkx \leq x$ or $x \leq hkx \leq kx$, then h is *negative* w.r.t. k. Let $\mathcal{H}(u)$ denote the set of all terms generated from u using hedges, i.e., $\mathcal{H}(u) = \{\sigma u | \sigma \in \mathcal{H}^*\}$, where $\mathcal{H}^*$ is the set of all strings of symbols in $\mathcal{H}$ including the empty one. An important semantic property of hedges, called *semantic heredity*, is that hedges change the meaning of a term, but somewhat preserve its original meaning. Thus, if $hx \leq kx$, then $\forall y \in \mathcal{H}(hx)$ and $\forall z \in \mathcal{H}(kx)$, we have $y \leq z$.

For *Truth*, primary terms $F < T$ are denoted by c^- and c^+, respectively. $\mathcal{H}$ can be divided into two disjoint subsets $\mathcal{H}^+$ and $\mathcal{H}^-$ by $\mathcal{H}^+ = \{h | hc^+ > c^+\}$ and $\mathcal{H}^- = \{h | hc^+ < c^+\}$. For example, $\mathcal{H} = \{V, H, R, S\}$ is decomposed into $\mathcal{H}^+ = \{V, H\}$ and $\mathcal{H}^- = \{R, S\}$. In fact, $\mathcal{H}^+$ and $\mathcal{H}^-$ are the sets of truth-stressing and truth-depressing hedges, respectively. Hedges in each of $\mathcal{H}^+$ and $\mathcal{H}^-$ may be comparable. So, $\mathcal{H}^+$ and $\mathcal{H}^-$ become posets. Let $I \notin \mathcal{H}$ be an artificial hedge, called the *identity*, defined by $\forall x \in \mathcal{X}$, $Ix = x$. I is the least element in each of $\mathcal{H}^+ \cup \{I\}$ and $\mathcal{H}^- \cup \{I\}$. An HA is said to be *linear* if both $\mathcal{H}^+$ and $\mathcal{H}^-$ are linearly ordered. The term domain $\mathcal{X}$ of a linear HA is also linearly ordered. In this work, we restrict ourselves to linear HAs.

A *linguistic truth domain* (LTD) taken from a linear HA $\underline{X} = (\mathcal{X}, \{c^-, c^+\}, \mathcal{H}, \leq)$ is the linearly ordered set $\overline{X} = \mathcal{X} \cup \{0, W, 1\}$, where 0 (*AbsolutelyFalse*), W (the *middle truth value*), and 1 (*AbsolutelyTrue*) are respectively the least, the neutral and the greatest elements of $\overline{X}$, and $\forall x \in \{0, W, 1\}$, $\forall h \in \mathcal{H}$, $hx = x$ [28,27].

2.3.2. Truth functions of hedge connectives

An *extended order relation* $\leq_e$ on $\mathcal{H} \cup \{I\}$ is defined by [27,28]: $\forall h, k \in \mathcal{H} \cup \{I\}$, $h \leq_e k$ if one of the following conditions is satisfied: (i) $h \in \mathcal{H}^-, k \in \mathcal{H}^+$; (ii) $h, k \in \mathcal{H}^+ \cup \{I\}$ and $h \leq k$; (iii) $h, k \in \mathcal{H}^- \cup \{I\}$ and $h \geq k$. It is denoted by $h <_e k$ if $h \leq_e k$ and $h \neq k$.

Definition 2. [27] Let $\underline{X} = (\mathcal{X}, \{c^+, c^-\}, \mathcal{H}, \leq)$ be a linear HA. *Truth functions* $h^\bullet : \overline{X} \to \overline{X}$ of all $h \in \mathcal{H} \cup \{I\}$ satisfy the following:

$$\forall x \in \{0, W, 1\}, h^\bullet(x) = x \tag{4}$$

$$\forall x \in \overline{X}, I^\bullet(x) = x \tag{5}$$

$$h^\bullet(hc^+) = c^+ \tag{6}$$

$$\text{if } x \geq y, h^\bullet(x) \geq h^\bullet(y) \tag{7}$$

$$\forall k \in \mathcal{H} \cup \{I\} \text{ such that } h \leq_e k, h^\bullet(x) \geq k^\bullet(x) \tag{8}$$

Condition (4) says that the truth functions preserve 0 and 1 while by (7), they are non-decreasing. Since for all truth-stressing hedges $h \in \mathcal{H}^+$, we have $h \geq_e I$, thus by (8), $h^\bullet(x) \leq I^\bullet(x) = x$, i.e., $h^\bullet$ is subdiagonal. Analogously, truth functions of all truth-depressing hedges are superdiagonal. Condition (6) ensures that if the truth value of the statement "Lucia is *young*" is *very true*, then that of the statement "Lucia is *very young*" is *true*. This result is also obtained when the fuzzy predicates and truth values are expressed by fuzzy sets [39,4]. It is shown in [28] that truth functions of hedges always exist.

2.3.3. Gödel operations

Gödel t-norm, its residuum, and negation are defined on an LTD $\overline{X}$ as follows:

$$x * y = \min(x, y), \quad x \Rightarrow y = \begin{cases} 1 & \text{if } y \geq x \\ y & \text{otherwise,} \end{cases} \quad -x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.3.4. Łukasiewicz operations

To have well-defined operations, we consider only finitely many truth values. An *l-limit* HA, where l is a positive integer, is a linear HA in which all terms have a length of at most $l + 1$. An LTD taken from an l-limit HA is finite [28,27]. Łukasiewicz t-norm, its residuum can be defined on a finite LTD $\overline{X} = \{v_0, \ldots, v_n\}$ with $v_0 \leq v_1 \leq \cdots \leq v_n$ as follows:

$$v_i * v_j = \begin{cases} v_{i+j-n} & \text{if } i + j - n > 0 \\ v_0 & \text{otherwise,} \end{cases} \qquad v_i \Rightarrow v_j = \begin{cases} v_n & \text{if } i \leq j \\ v_{n+j-i} & \text{otherwise.} \end{cases}$$

The negation is defined by: given $x = \sigma c$, where $\sigma \in \mathcal{H}^*$ and $c \in \{c^+, c^-\}$, $y = -x$, if $y = \sigma c'$ and $\{c, c'\} = \{c^+, c^-\}$, e.g., $Vc^+ = -Vc^-$, $Vc^- = -Vc^+$.

3. Fuzzy logics with multiple hedges

This section proposes an axiomatization over any core fuzzy logic for multiple hedges, in which hedges do not have a dual one. A hedge may modify the meaning of propositions more than another [38,32], so in the axiomatization, there should be axioms expressing their comparative modification strength. To ease the presentation, let s_0, d_0 denote the *identity* connective, i.e., $\forall \varphi$, $\varphi \equiv s_0\varphi \equiv d_0\varphi$, and their truth functions $s_0^\bullet$ and $d_0^\bullet$ are the identity.

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Definition 3. Let L be a core fuzzy logic. A logic $\mathrm{L}^{p,q}_{s,d}$, where p, q are positive integers, is an expansion of L with new unary connectives $s_1, ..., s_p$ (for truth-stressers) and $d_1, ..., d_q$ (for truth-depressers) by the following additional axioms, for $i = \overline{1, p}$ and $j = \overline{1, q}$:

$$(S_i) \quad s_i\varphi \to s_{i-1}\varphi, \quad (S_{p+1}) \quad s_p\overline{1}, \quad (D_j) \quad d_{j-1}\varphi \to d_j\varphi, \quad (D_{q+1}) \quad \neg d_q\overline{0},$$

and the following additional deduction rule:

$$(\mathrm{DR}_h) \text{ from } (\varphi \to \psi) \vee \chi \text{ infer } (h\varphi \to h\psi) \vee \chi, \text{ for every hedge } h.$$

Axiom (S_i) (resp., axiom (D_j)) expresses that s_i (resp., d_j) modifies meaning of propositions more than s_{i-1} (resp., d_{j-1}), e.g., V (resp., S) modifies truth more than H (resp., R) since $T < HT < VT$ (resp., $ST < RT < T$). Rule (DR_h) coincides with Rule (MON) for each $h \in \{s_1, ..., s_p, d_1, ..., d_q\}$. Using the same examples as in [14], it can be shown that our axiomatization cannot be readily simplified, e.g., by replacing Rule (DR_h) with a simpler rule: either *from* $\varphi \to \psi$ *infer* $h\varphi \to h\psi$ or *from* φ *infer* $h\varphi$.

Lemma 2. *The following deductions are valid, for* $i = 1, ..., p$ *and* $j = 1, ..., q$*:*

$$\vdash_{L^{p,q}_{s,d}} s_i\varphi \to \varphi \tag{9}$$
$$\vdash_{L^{p,q}_{s,d}} \varphi \to d_j\varphi \tag{10}$$
$$\vdash_{L^{p,q}_{s,d}} \neg s_i\overline{0} \tag{11}$$
$$\vdash_{L^{p,q}_{s,d}} s_i\overline{1} \tag{12}$$
$$\vdash_{L^{p,q}_{s,d}} d_j\overline{1} \tag{13}$$
$$\vdash_{L^{p,q}_{s,d}} \neg d_j\overline{0} \tag{14}$$
$$\varphi \to \psi \vdash_{L^{p,q}_{s,d}} s_i\varphi \to s_i\psi \tag{15}$$
$$\psi \vdash_{L^{p,q}_{s,d}} s_i\psi \tag{16}$$
$$\varphi \to \psi \vdash_{L^{p,q}_{s,d}} d_j\varphi \to d_j\psi \tag{17}$$
$$d_j\varphi, \varphi \to \psi \vdash_{L^{p,q}_{s,d}} d_j\psi \tag{18}$$
$$s_i\varphi, \varphi \to \psi \vdash_{L^{p,q}_{s,d}} s_i\psi \tag{19}$$
$$s_i\varphi, s_i(\varphi \to \psi) \vdash_{L^{p,q}_{s,d}} s_i\psi \tag{20}$$

Proof. (9) follows from $(S_i), ..., (S_1)$, and (T). (10) follows from $(D_1), ..., (D_j)$, and (T). (11) follows immediately from (9) by taking $\varphi = \overline{0}$. (12) $\vdash_{L^{p,q}_{s,d}} s_p\varphi \to s_i\varphi$ (for $i < p$, by $(S_p), ..., (S_{i+1})$ and (T)); $\vdash_{L^{p,q}_{s,d}} s_p\overline{1} \to s_i\overline{1}$ (by taking $\varphi = \overline{1}$); $\vdash_{L^{p,q}_{s,d}} s_i\overline{1}$ (by (S_{p+1}) and (MP)). (13) follows immediately from (10) by taking $\varphi = \overline{1}$. (14) $\vdash_{L^{p,q}_{s,d}} d_j\varphi \to d_q\varphi$ (for $j < q$, by $(D_{j+1}), ..., (D_q)$, and (T)); $\vdash_{L^{p,q}_{s,d}} \neg d_q\varphi \to \neg d_j\varphi$ (by (1) and (MP)); $\vdash_{L^{p,q}_{s,d}} \neg d_q\overline{0} \to \neg d_j\overline{0}$ (by taking $\varphi = \overline{0}$); $\vdash_{L^{p,q}_{s,d}} \neg d_j\overline{0}$ (by (D_{q+1}) and (MP)). (15) follows directly from (DR_h) by taking $\chi = \overline{0}$, $h = s_i$. (16) follows from (15) by taking $\varphi = \overline{1}$ and using (12). (17) follows directly from (DR_h) by taking $\chi = \overline{0}$, $h = d_j$. (18) follows immediately from (17) and (MP). (19) follows immediately from (15) and (MP). (20) follows from (9), (MP) and (19). □

Note that (9), (12), (10), and (14) coincide with Axioms (VTL1), (VTL2), (STL1), and (STL2) in the axiomatization of Esteva et al., respectively. Properties (11)–(14) imply that truth functions of s_i and d_j preserve 0 and 1. Properties (16) and (20) are the necessitation deduction rule and a deduction version of the K-like axiom used in Hájek's and Vychodil's axiomatizations, respectively. Properties (18) and (19) are a stronger version of (MP): if φ implies ψ, then *very* (resp., *slightly*) φ implies *very* (resp., *slightly*) ψ.

Since (15) and (17) express that (sCng) is satisfied for each s_i and d_j, $L^{p,q}_{s,d}$ is a finitary Rasiowa-implicative logic, and its equivalent algebraic semantics is the class of $L^{p,q}_{s,d}$-algebras.

Definition 4. An algebra $\mathbf{A} = \langle A, *, \Rightarrow, \cap, \cup, 0, 1, s_1^\bullet, ..., s_p^\bullet, d_1^\bullet, ..., d_q^\bullet \rangle$ of type $\langle 2, 2, 2, 2, 0, 0, 1, ..., 1 \rangle$ is an $L_{s,d}^{p,q}$-algebra if it is an L-algebra expanded by unary operators $s_i^\bullet, d_j^\bullet : A \to A$ that satisfy, for all $x, y, z \in A$, $i = \overline{1, p}$ and $j = \overline{1, q}$,

$$s_i^\bullet(x) \le s_{i-1}^\bullet(x) \tag{21}$$

$$s_p^\bullet(1) = 1 \tag{22}$$

$$d_j^\bullet(x) \ge d_{j-1}^\bullet(x) \tag{23}$$

$$d_q^\bullet(0) = 0 \tag{24}$$

$$\text{if } (x \Rightarrow y) \cup z = 1 \text{ then } (s_i^\bullet(x) \Rightarrow s_i^\bullet(y)) \cup z = 1 \tag{25}$$

$$\text{if } (x \Rightarrow y) \cup z = 1 \text{ then } (d_j^\bullet(x) \Rightarrow d_j^\bullet(y)) \cup z = 1 \tag{26}$$

where $s_i^\bullet$ and $d_j^\bullet$ are truth functions of connectives s_i and d_j, respectively.

By (21), $s_i^\bullet(x) \le s_0^\bullet(x) = x$, i.e., $s_i^\bullet$ is subdiagonal. By (23), $d_j^\bullet$ is superdiagonal. In a chain of truth values, the quasiequations (25) and (26) turn out to be equivalently expressed by: if $x \Rightarrow y = 1$, then $s_i^\bullet(x) \Rightarrow s_i^\bullet(y) = 1$ and $d_j^\bullet(x) \Rightarrow d_j^\bullet(y) = 1$, respectively, i.e., $s_i^\bullet$ and $d_j^\bullet$ are non-decreasing. Obviously, if $\langle A, *, \Rightarrow, \cap, \cup, 0, 1 \rangle$ is an L-chain, and $s_i^\bullet, d_j^\bullet$ satisfy (21)–(26), the expanded structure $\left\langle A, *, \Rightarrow, \cap, \cup, 0, 1, s_1^\bullet, ..., s_p^\bullet, d_1^\bullet, ..., d_q^\bullet \right\rangle$ is an $\mathrm{L}_{s,d}^{p,q}$-chain.

Theorem 3. *Let L be a core fuzzy logic, $\mathbb{K}$ a class of L-chains, and $\mathbb{K}_{s,d}^{p,q}$ the class of the $L_{s,d}^{p,q}$-chains whose $s_1^\bullet, ..., s_p^\bullet, d_1^\bullet, ..., d_q^\bullet$-free reducts are in $\mathbb{K}$. Then: (i) $L_{s,d}^{p,q}$ is a conservative expansion of L; (ii) $L_{s,d}^{p,q}$ is strongly complete w.r.t. the class of all $L_{s,d}^{p,q}$-chains, i.e., $L_{s,d}^{p,q}$ is semilinear; (iii) L has the FSSC, FS$\mathbb{K}$C, SSC, and S$\mathbb{K}$C iff $L_{s,d}^{p,q}$ has the FSSC, FS$\mathbb{K}$C, SSC, and S$\mathbb{K}$C, respectively.*

Proof. (i) Let $\mathcal{L}$ be the language of L. We show that, for every set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$-formulae, $\Gamma \vdash_{L_{s,d}^{p,q}} \varphi$ iff $\Gamma \vdash_L \varphi$. Obviously, if $\Gamma \vdash_L \varphi$ then $\Gamma \vdash_{L_{s,d}^{p,q}} \varphi$. If $\Gamma \nvdash_L \varphi$, there is an L-chain $\mathbf{A}$ and an $\mathbf{A}$-evaluation e such that e is $\mathbf{A}$-model of Γ and $e(\varphi) \neq 1$. $\mathbf{A}$ can be expanded to an $\mathrm{L}_{s,d}^{p,q}$-chain $\mathbf{A}'$ by defining $s_i^\bullet(1) = 1$; $\forall a \in A \setminus \{1\}$, $s_i^\bullet(a) = 0$; $d_j^\bullet(0) = 0$; and $\forall a \in A \setminus \{0\}$, $d_j^\bullet(a) = 1$, for all $i = \overline{1, p}$ and $j = \overline{1, q}$. It can be easily checked that $s_i^\bullet$ and $d_j^\bullet$ satisfy (21)–(26). Thus, in the expanded language, we have $\Gamma \nvdash_{L_{s,d}^{p,q}} \varphi$. (ii) Since Rule (DR$_h$) is in and closed under $\vee$-forms, the connective $\vee$ remains a disjunction in the logic $\mathrm{L}_{s,d}^{p,q}$. Moreover, since Property (3) is already valid in L, $\mathrm{L}_{s,d}^{p,q}$ is also semilinear. Therefore, $\mathrm{L}_{s,d}^{p,q}$ is complete w.r.t. the semantics of all $\mathrm{L}_{s,d}^{p,q}$-chains. (iii) We prove for the case of the SSC, and the others can be done analogously. Since $\mathrm{L}_{s,d}^{p,q}$ is a conservative expansion of L, if $\mathrm{L}_{s,d}^{p,q}$ has the SSC, so does L. Assume that L has the SSC. We show that any countable $\mathrm{L}_{s,d}^{p,q}$-chain $\mathbf{A}$ can be embedded into a standard $\mathrm{L}_{s,d}^{p,q}$-chain. By Theorem 2, the $s_1^\bullet, ..., s_p^\bullet, d_1^\bullet, ..., d_q^\bullet$-free reduct of $\mathbf{A}$ can be embedded into a standard L-chain $\mathbf{B} = \langle [0, 1], *, \Rightarrow, \cap, \cup, 0, 1 \rangle$ by an one-to-one mapping f preserving $*, \Rightarrow, \le, 0$ and 1. Let $s_i' : [0, 1] \to [0, 1]$ be defined as follows: for all $x \in [0, 1]$, $s_0'(x) = x$ and for all $i = \overline{1, p}$, $s_i'(x) = \sup\{f(s_i(x_k)) | x_k \in A, f(x_k) \le x\}$. It is not difficult to show that for all $i = \overline{1, p}$, s_i' is non-decreasing, $s_i'(f(x_k)) = f(s_i(x_k))$ for all $x_k \in A$, and $s_i'(x) \le s_{i-1}'(x)$ for all $x \in [0, 1]$, i.e., all s_i' satisfy (21) and (25). Similarly, by defining $d_j' : [0, 1] \to [0, 1]$ such that $d_0'(x) = x$ and for all $j = \overline{1, q}$, $d_j'(x) = \sup\{f(d_j(x_k)) | x_k \in A, f(x_k) \le x\}$, we have all d_j' satisfy (23), (26), and $d_j'(f(x_k)) = f(d_j(x_k))$ for all $x_k \in A$. Hence, $\mathbf{B}$ expanded by all s_i' and d_j' is a standard $\mathrm{L}_{s,d}^{p,q}$-chain where $\mathbf{A}$ is embedded. □

4. Fuzzy logics with multiple dual hedges

It can be observed that each hedge can have a dual one, e.g., *slightly* and *rather* can be seen as a dual hedge of *very* and *highly*, respectively. Thus, there might be additional axioms expressing dual relations of hedges. In the following axiomatization, each hedge can have its own dual one.

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Definition 5. Let L be a core fuzzy logic. A logic $L^{2n}_{s,d}$, where n is a positive integer, is an expansion of L with new unary connectives $s_1, \ldots, s_n$ (for truth-stressers) and $d_1, \ldots, d_n$ (for truth-depressers) by the axioms (S_i), (S_{n+1}), (D_i), (SD^{dh}_i) $d_i\varphi \to \neg s_i \neg\varphi$, for $i = \overline{1,n}$, and the deduction rule (DR_h).

The logic $L^{2n}_{s,d}$ is L expanded by $2n$ hedges, where hedges are divided into pairs of dual ones. This axiomatization differs from the one for multiple hedges without duality in that (D_{q+1}) is replaced with (SD^{dh}_i), expressing a dual relation between hedges s_i and d_i. Axiom (SD^{dh}_i) coincides with (ST2) in Vychodil's axiomatization. For the case of *very*, *slightly*, and $\varphi = short$, it means "*slightly short* implies *not very tall*". Interestingly, as shown in Lemma 3 below, $s_i\varphi \to \neg d_i \neg\varphi$ can be derived from this axiomatization. Hence, "*very short*" implies "*not slightly tall*" as well.

One could consider other similar dual relations such as $\neg s_i\neg\varphi \to d_i\varphi$, $\neg s_i\neg\varphi \leftrightarrow d_i\varphi$, $\neg s_i\varphi \to d_i\neg\varphi$, $s_i\neg\varphi \to \neg d_i\varphi$, $\neg d_i\varphi \to s_i\neg\varphi$, $d_i\neg\varphi \to \neg s_i\varphi$, $\neg d_i\neg\varphi \to s_i\varphi$, and $\neg d_i\neg\varphi \leftrightarrow s_i\varphi$. There are a number of researchers, to some extent, mentioning this kind of relationship. Cock et al. [12], Kerre and Cock [22] discuss a possible connection between a truth-stressing (therein called *intensifying*) hedge *very* and a truth-depressing (therein called *weakening*) hedge *more or less* as equality of "*not very* φ" and "*more or less not* φ" while Bolinger [6] describes the meaning of "*not overly bright*" as "*rather underly bright, rather stupid*". Hersh and Caramazza [21] prove experimentally that "*not very tall*" is occasionally interpreted as "*sort of short*" ("*more or less short*"). Also, in Example 3 of Le et al. [28], we put $h^\bullet_r(x) = -h^\bullet_{-r}(-x)$, where if $h^\bullet_r$ is the truth function of a truth-stressing (resp., truth-depressing) hedge h_r then $h^\bullet_{-r}$ is that of a truth-depressing (resp., truth-stressing) hedge h_{-r}.

It can be seen that in a case when there is one truth-stressing (resp., truth-depressing) hedge without a dual one, we just add the axioms expressing its relations to the existing truth-stressing (resp., truth-depressing) hedges according to their comparative modification strength.

Since in the proof of Lemma 2, (D_{q+1}) is not used for any items except (14), all items in Lemma 2 except (14) are valid for $\vdash_{L^{2n}_{s,d}}$. In fact, (14) also holds for $\vdash_{L^{2n}_{s,d}}$ as follows.

Lemma 3. *The following deductions are valid, for $i = 1, \ldots, n$:*

$$\vdash_{L^{2n}_{s,d}} \neg d_i\overline{0} \tag{27}$$
$$\vdash_{L^{2n}_{s,d}} \neg s_{i-1}\neg\varphi \to \neg s_i\neg\varphi \tag{28}$$
$$\vdash_{L^{2n}_{s,d}} s_i\varphi \to \neg d_i\neg\varphi \tag{29}$$

Proof. (27) $\vdash_{L^{2n}_{s,d}} d_i\overline{0} \to \neg s_i\neg\overline{0}$ (by (SD^{dh}_i) with $\varphi = \overline{0}$); $\vdash_{L^{2n}_{s,d}} d_i\overline{0} \to \neg s_i\overline{1}$; $\vdash_{L^{2n}_{s,d}} \neg\neg s_i\overline{1} \to \neg d_i\overline{0}$ (by (1) and (MP)); $\vdash_{L^{2n}_{s,d}} s_i\overline{1} \to \neg d_i\overline{0}$ (by (2) and (T)); $\vdash_{L^{2n}_{s,d}} \neg d_i\overline{0}$ (by (12) and (MP)). (28) $\vdash_{L^{2n}_{s,d}} s_i\neg\varphi \to s_{i-1}\neg\varphi$ (by (S^{dh}_i)); $\vdash_{L^{2n}_{s,d}} \neg s_{i-1}\neg\varphi \to \neg s_i\neg\varphi$ (by (1) and (MP)). (29) $\vdash_{L^{2n}_{s,d}} d_i\neg\varphi \to \neg s_i\neg\neg\varphi$ (by (S^{dh}_i)); $\vdash_{L^{2n}_{s,d}} \neg\neg s_i\neg\neg\varphi \to \neg d_i\neg\varphi$ (by (1) and (MP)); $\vdash_{L^{2n}_{s,d}} s_i\neg\neg\varphi \to \neg d_i\neg\varphi$ (by (2) and (T)); $\vdash_{L^{2n}_{s,d}} s_i\varphi \to \neg d_i\neg\varphi$ (by (2), (15) and (T)). □

$L^{2n}_{s,d}$ is also a finitary Rasiowa-implicative logic, and its equivalent algebraic semantics is the class of $L^{2n}_{s,d}$-algebras.

Definition 6. An algebra $\mathbf{A} = \langle A, *, \Rightarrow, \cap, \cup, 0, 1, s^\bullet_1, \ldots, s^\bullet_n, d^\bullet_1, \ldots, d^\bullet_n\rangle$ of type $\langle 2, 2, 2, 2, 0, 0, 1, \ldots, 1\rangle$ is an $L^{2n}_{s,d}$-algebra if it is an L-algebra expanded by unary operators $s^\bullet_i, d^\bullet_i : A \to A$, for $i = \overline{1,n}$, that satisfy (21)–(23) and (25)–(26) (note that $p = q = n$) and

$$d^\bullet_i(x) \le -s^\bullet_i(-x). \tag{30}$$

Property (21) says that $s^\bullet_i(x)$ and x are lower and upper boundaries of $s^\bullet_{i-1}(x)$ for all $i = \overline{2,n}$, respectively. Also, (23) and (30) state that $d^\bullet_{i-1}(x)$ and $-s^\bullet_i(-x)$ are lower and upper boundaries of $d^\bullet_i(x)$, respectively. Thus, given a non-decreasing, subdiagonal hedge function $s^\bullet_n$ preserving 0 and 1, boundaries for the other hedge functions $s^\bullet_{n-1}, \ldots, s^\bullet_1, d^\bullet_1, \ldots, d^\bullet_n$ are one by one determined by preceding functions in the sequence as in Table 1. With such boundaries, the functions are only further required to be non-decreasing. In fact, given a hedge function of any hedge, boundaries for the other hedge functions can be determined in a similar manner.

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Table 1
Boundaries of hedge functions.

Hedge function	Lower boundary	Upper boundary
$s_{n-1}^\bullet(x)$	$s_n^\bullet(x)$	x
...		
$s_1^\bullet(x)$	$s_2^\bullet(x)$	x
$d_1^\bullet(x)$	x	$-s_1^\bullet(-x)$
...		
$d_n^\bullet(x)$	$d_{n-1}^\bullet(x)$	$-s_n^\bullet(-x)$

Theorem 4. *Let L be a core fuzzy logic, $\mathbb{K}$ a class of L-chains, and $\mathbb{K}_{s,d}^{2n}$ the class of the $L_{s,d}^{2n}$-chains whose $s_1^\bullet, ..., s_n^\bullet, d_1^\bullet, ..., d_n^\bullet$-free reducts are in $\mathbb{K}$. Then: (i) $L_{s,d}^{2n}$ is a conservative expansion of L; (ii) $L_{s,d}^{2n}$ is strongly complete w.r.t. the class of all $L_{s,d}^{2n}$-chains, i.e., $L_{s,d}^{2n}$ is semilinear; (iii) L has the FSSC, FS$\mathbb{K}$C, SSC, and S$\mathbb{K}$C iff $L_{s,d}^{2n}$ has the FSSC, FS$\mathbb{K}$C, SSC, and S$\mathbb{K}$C, respectively.*

Proof. Proofs of (i) and (ii) are respectively the same as those of items (i) and (ii) of Theorem 3. For (iii), let s_i' and d_i' be defined as in item (iii) of Theorem 3. It remains to prove that they satisfy (30). We have $\forall x_k \in A,\ f(d_i^\bullet(x_k)) \leq f(-s_i^\bullet(-x_k)) = f(s_i^\bullet(x_k \Rightarrow 0) \Rightarrow 0) = f(s_i^\bullet(x_k \Rightarrow 0)) \Rightarrow f(0) = f(s_i^\bullet(x_k \Rightarrow 0)) \Rightarrow 0 =^{(*)} s_i'(f(x_k \Rightarrow 0)) \Rightarrow 0 = s_i'(f(x_k) \Rightarrow 0) \Rightarrow 0 = -s_i'(-f(x_k))$ ((∗) holds since $(x_k \Rightarrow 0) \in A$). Hence, $d_i'(x) = \sup\{f(d_i^\bullet(x_k)) | x_k \in A,\ f(x_k) \leq x\} \leq \sup\{-s_i'(-f(x_k)) | x_k \in A,\ f(x_k) \leq x\} \leq -s_i'(-x)$ since all $-s_i'(-f(x_k)) \leq -s_i'(-x)$. □

5. Linguistic fuzzy logics with multiple hedges

Let L be a core fuzzy logic. Given a linear HA $\underline{X}$, we can build a linguistic fuzzy logic with multiple hedges, denoted L^{lh}, and a linguistic fuzzy logic with multiple dual hedges, denoted L^{ldh}, based on L and the HA for representing and reasoning with linguistically-expressed human knowledge. The algebras of the logics utilize an LTD $\overline{X}$ taken from the HA.

For instance, let $\underline{X}$ be the HA $(\mathcal{X}, \{c^-, c^+\}, \mathcal{H} = \{V, H, R, S\}, \leq)$. $\underline{X}$ is a linear HA since in $\mathcal{H}^+$, we have $H < V$, and in $\mathcal{H}^-$, $R < S$. Hence, $S <_e R <_e I <_e H <_e V$. By (8), we have $\forall x \in \overline{X}$, $V^\bullet(x) \leq H^\bullet(x) \leq x \leq R^\bullet(x) \leq S^\bullet(x)$. This is in accordance with fuzzy-set-based interpretations of hedges [38], which satisfy the *semantic entailment* [23]:

x is *very* $A \Rightarrow x$ is *highly* $A \Rightarrow x$ is $A \Rightarrow x$ is *rather* $A \Rightarrow x$ is *slightly* A

where A is a fuzzy predicate.

The logic L^{lh} is an expansion of L with new unary connectives V, H, R, S by the following additional axioms:

$$\begin{array}{llllll} (S_1^{lh}) & H\varphi \to \varphi, & (S_2^{lh}) & V\varphi \to H\varphi, & (S_3^{lh}) & V\overline{1}, \\ (D_1^{lh}) & \varphi \to R\varphi, & (D_2^{lh}) & R\varphi \to S\varphi, & (D_3^{lh}) & \neg S\overline{0}, \end{array}$$

and the following additional deduction rule:

$$(DR^{lh}) \text{ from } (\varphi \to \psi) \vee \chi \text{ infer } (h\varphi \to h\psi) \vee \chi, \text{ for each } h \in \{V, H, R, S\}.$$

The equivalent algebraic semantics of L^{lh} is the class of L^{lh}-algebras. An L^{lh}*-algebra* is an L-algebra expanded by unary non-decreasing operators $V^\bullet, H^\bullet, R^\bullet, S^\bullet : \overline{X} \to \overline{X}$ satisfying, $\forall x \in \overline{X}$,

$$\begin{array}{lll} H^\bullet(x) \leq x, & V^\bullet(x) \leq H^\bullet(x), & V^\bullet(1) = 1, \\ R^\bullet(x) \geq x, & S^\bullet(x) \geq R^\bullet(x), & S^\bullet(0) = 0. \end{array}$$

Similarly, the logic L^{ldh} is an expansion of L by Axioms (S_1^{lh}), (S_2^{lh}), (S_3^{lh}), (D_1^{lh}), (D_2^{lh}), (SD_1^{ldh}) $R\varphi \to \neg H\neg\varphi$, (SD_2^{ldh}) $S\varphi \to \neg V\neg\varphi$, and the deduction rule (DR^{lh}). The equivalent algebraic semantics of L^{ldh} is the class of L^{ldh}-algebras. The definition of an L^{ldh}*-algebra* can be obtained from that of an L^{lh}-algebra by removing Condition $S^\bullet(0) = 0$ and adding $S^\bullet(x) \leq -V^\bullet(-x)$, $R^\bullet(x) \leq -H^\bullet(-x)$. Note that there always exist hedge functions satisfying all the above conditions. For example, the hedge functions given in Example 3 of Le et al. [28] (therein called *inverse mappings of hedges*) for a general linear HA. The HA $\underline{X}$ given here can be obtained from that general linear HA by putting $p = q = 2$, $h_{-2} = S$, $h_{-1} = R$, $h_0 = I$, $h_1 = H$, and $h_2 = V$.

Obviously, Theorem 3 (resp., Theorem 4) holds for L^{lh} (resp., L^{ldh}).

In particular, given Gödel and Łukasiewicz operations respectively defined in Subsections 2.3.3 and 2.3.4, we can have linguistic fuzzy logics based on G or Ł.

6. First-order fuzzy logics with multiple hedges

In order to save space, we do not recall several notions and results of first-order core fuzzy logics here. For more detail, the reader is referred to Cintula et al. [9]. Given a propositional core fuzzy logic L, the language $\mathcal{PL}$ of the first-order core fuzzy logic L∀ is built from the propositional language $\mathcal{L}$ of L by extending it with a non-empty set of predicate symbols $Pred$, a set of function symbols $Func$ (disjoint with $Pred$), a set of object variables *Var*, and two quantifiers ∃ and ∀. For first-order core fuzzy logics, in order to obtain completeness w.r.t. linearly ordered algebras for semilinear logics, several additional axioms are added, e.g., Axiom (∀3) in the sequel. Hence, it is usual to restrict the semantics to chains only. The axioms for L∀ are obtained from those of L by substituting formulae of $\mathcal{PL}$ for propositional variables and adding the following axioms for quantifiers:

(∀1) $(\forall x)\varphi(x) \to \varphi(t)$, where the term t is substitutable for x in φ

(∃1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ

(∀2) $(\forall x)(\psi \to \varphi) \to (\psi \to (\forall x)\varphi)$, where x is not free in ψ

(∃2) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$, where x is not free in ψ

(∀3) $(\forall x)(\psi \vee \varphi) \to (\psi \vee (\forall x)\varphi)$, where x is not free in ψ

The deduction rules of L∀ are (MP) and *generalization*: (*Gen*) From φ infer $(\forall x)\varphi$.

The syntax and semantics of first-order core fuzzy logics are bounded together by the following completeness theorem.

Theorem 5. *[8] For any first-order core fuzzy logic L∀ with a predicate language $\mathcal{PL}$, any $\mathcal{PL}$-theory T, and any $\mathcal{PL}$-formula φ, the following are equivalent: (a) $T \vdash_{L\forall} \varphi$; (b) $\langle \mathbf{A}, \mathbf{M}\rangle \models \varphi$ for each model $\langle \mathbf{A}, \mathbf{M}\rangle$ of T with $\mathbf{A}$ being a countable L-chain.*

The notions S𝕂C, FS𝕂C, and 𝕂C can be defined similarly to the propositional case. Theorem 5 states that every first-order core fuzzy logic enjoys the S𝕂C, where 𝕂 is the class of all countable chains.

Definition 7 *(First-order fuzzy logic with multiple non-dual hedges).* Given a first-order core fuzzy logic L∀, let $\mathrm{L}^{p,q}_{s,d}\forall$ be the expansion of L∀ with new unary connectives $s_1, ..., s_p, d_1, ..., d_q$, axioms (S_i), (S_{p+1}), (D_j), (D_{q+1}), and the deduction rule (DR_h), for $i = \overline{1, p}$ and $j = \overline{1, q}$.

$\mathrm{L}^{p,q}_{s,d}\forall$ can be obtained by expanding $\mathrm{L}^{p,q}_{s,d}$ to the first-order level as above.

Theorem 6. *$L^{p,q}_{s,d}\forall$ is a conservative expansion of L∀, i.e., for every set $T \cup \varphi$ of $\mathcal{PL}$-formulae, $T \vdash_{L^{p,q}_{s,d}\forall} \varphi$ iff $T \vdash_{L\forall} \varphi$, where $\mathcal{PL}$ is the language of L∀.*

Proof. The proof is similar to that of item (i) of Theorem 3. □

Let $\mathbf{A}, \mathbf{B}$ be two algebras of the same type with (defined) lattice operations sup, inf. It is said that an embedding $f : \mathbf{A} \to \mathbf{B}$ is a *σ-embedding* if $f(\sup C) = \sup f[C]$ (whenever $\sup C$ exists) and $f(\inf D) = \inf f[D]$ (whenever $\inf D$ exists) for each countable $C, D \subseteq A$, i.e., f preserves existing suprema and infima. A usual way to prove S𝕂C is to show that every non-trivial countable L-chain can be σ-embedded into some chain of 𝕂. In this case, it is said that L has the *𝕂-σ-embedding property*. This is a *sufficient*, but in general *not necessary*, condition for the S𝕂C [8].

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Theorem 7 *(Strong standard completeness). Let L be a core fuzzy logic, $\mathbb{K}$ the class of all* standard *L-chains, and $\mathbb{K}^{p,q}_{s,d}$ the class of all* standard *$L^{p,q}_{s,d}$-chains whose $s^\bullet_1, ..., s^\bullet_p, d^\bullet_1, ..., d^\bullet_q$-free reducts are in $\mathbb{K}$. Then: (i) If L has the $\mathbb{K}$-σ-embedding property, then $L^{p,q}_{s,d}\forall$ has the $S\mathbb{K}^{p,q}_{s,d}C$, i.e., SSC. (ii) If L does not have $\mathbb{K}$-embedding property, then $L^{p,q}_{s,d}\forall$ does not have the $S\mathbb{K}^{p,q}_{s,d}C$.*

Proof. (i) It suffices to prove that any non-trivial countable $\mathrm{L}^{p,q}_{s,d}$-chain **A** can be σ-embedded into some standard chain of $\mathbb{K}^{p,q}_{s,d}$. The proof is similar to that of item (iii) of Theorem 3 taking into account that the mapping f preserves suprema, infima, and thus the order relation. (ii) Since L does not have the $\mathbb{K}$-embedding property, L$\forall$ does not enjoy the S$\mathbb{K}$C. Moreover, since $\mathrm{L}^{p,q}_{s,d}\forall$ is a conservative expansion of L$\forall$, $\mathrm{L}^{p,q}_{s,d}\forall$ does not have the $\mathrm{S}\mathbb{K}^{p,q}_{s,d}$C either. □

Theorem 7 can be generalized to arbitrary classes of L-chains and their $s^\bullet_1, ..., s^\bullet_p, d^\bullet_1, ..., d^\bullet_q$-expansions as follows.

Corollary 1 *(Strong $\mathbb{K}$-completeness). Let L be a core fuzzy logic, $\mathbb{K}$ a class of L-chains, and $\mathbb{K}^{p,q}_{s,d}$ the class of the $L^{p,q}_{s,d}$-chains whose $s^\bullet_1, ..., s^\bullet_p, d^\bullet_1, ..., d^\bullet_q$-free reducts are in $\mathbb{K}$. Then: (i) If L has the $\mathbb{K}$-σ-embedding property, then $L^{p,q}_{s,d}\forall$ has the $S\mathbb{K}^{p,q}_{s,d}C$. (ii) If L does not have $\mathbb{K}$-embedding property, then $L^{p,q}_{s,d}\forall$ does not have the $S\mathbb{K}^{p,q}_{s,d}C$.*

Definition 8 *(First-order fuzzy logic with multiple dual hedges).* Given a first-order core fuzzy logic L$\forall$, let $\mathrm{L}^{2n}_{s,d}\forall$ be the expansion of L$\forall$ with new unary connectives $s_1, ..., s_n, d_1, ..., d_n$, axioms (S_i), (S_{n+1}), (D_i), (SD^{dh}_i), and the deduction rule (DR$_h$), for $i = \overline{1, n}$.

Similar to the case of $\mathrm{L}^{p,q}_{s,d}\forall$, the following theorem can be proved.

Theorem 8. *Let L be a core fuzzy logic, $\mathbb{K}$ a class of L-chains, and $\mathbb{K}^{2n}_{s,d}$ the class of the $L^{2n}_{s,d}$-chains whose $s^\bullet_1, ..., s^\bullet_n, d^\bullet_1, ..., d^\bullet_n$-free reducts are in $\mathbb{K}$. Then: (i) The logic $L^{2n}_{s,d}\forall$ is a conservative expansion of the logic L$\forall$. (ii) If L has the $\mathbb{K}$-σ-embedding property, then $L^{2n}_{s,d}\forall$ has the $S\mathbb{K}^{2n}_{s,d}C$. (iii) If L does not have $\mathbb{K}$-embedding property, then $L^{2n}_{s,d}\forall$ does not have the $S\mathbb{K}^{2n}_{s,d}C$.*

In the case that $\mathbb{K}$ is the class of all standard L-chains, we have the corresponding result for the SSC of $\mathrm{L}^{2n}_{s,d}\forall$.

7. Related work

In addition to [18], Hájek has other works on adding hedges to logical systems of MFL. In [20], the authors show that the *Yashin strong future tense operator*, a closure operator satisfying Axiom K, can be interpreted as a hedge over G. In [19], two unary connectives L (for truth-stresser) and U (for truth-depresser), which are idempotent w.r.t. the monoidal operation, are added to BL$_\Delta$ (BL expanded with the projection operator Δ) to obtain the logic $\mathrm{BL}^!_{LU}$.

In [2,15,16], G is expanded with a monotone unary operator $\circ$ interpreted as monotone functions $f : [0, 1] \to [0, 1]$ preserving 1, but not necessarily preserving 0. The proposed logics in [15] can be regarded as an expansion of G with a truth-stresser since $\circ$ satisfies a stronger version of Axiom K, namely, $\circ(\varphi \to \psi) \leftrightarrow (\circ\varphi \to \circ\psi)$, and the necessitation rule.

Another work on logics with truth-stressers is by Ciabattoni et al. [7]. In the study, a unary modality s, which is an operator satisfying Axiom K and the necessitation rule, is added to extensions L of the logic MTL, e.g., MTL, BL, SMTL, and IMTL, by the following axiomatizations:

$$\text{L-KT}^r = \text{L} + (\text{VT1}) + (\text{VT2}) + (\text{VT3}) + (\text{NEC})$$
$$\text{L-S4}^r = \text{L-KT}^r + (\text{VT4})\ s(\varphi) \to s(s(\varphi))$$

Proof systems for the new logics are developed, and their algebraic and completeness properties are studied. Indeed, the logic BL-KT^r is Hájek's BL$_{vt}$. For the logic L-S4^r, s becomes a closure operator. The authors also prove standard completeness of the logics L-S4^r for different underlying logics L.

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Montagna [31] studies the system obtained by adding to a fuzzy logic L a unary connective called *storage operator* acting as an idempotent truth-stresser closed over its image.

Antoni et al. [1] study the relationship between heterogeneous concept lattices and concept lattices with heterogeneous hedges, where hedges are interpreted as a unary operator which is subdiagonal and closed over its image, satisfies axiom K, preserves 1, but not necessarily preserves 0.

In [25], we expand fuzzy logic in a narrow sense with graded syntax [34] with multiple hedges, in which every hedge h satisfies the following axiom, among others:

$$(\varphi \to \psi) \to (h\varphi \to h\psi),$$

which is indeed Axiom (ST6) and stronger than the rule (MON). The resulting logics are proved to also have the Pavelka-style completeness.

In spite of the significant contributions of the above papers, hedge functions that are either a closure operator or an idempotent operator, or satisfy Axiom K have a very limited behavior and can represent for some special cases of truth-stressing hedges [14].

In [28,27], fuzzy logic programming [36,30] is extended with hedge connectives to obtain fuzzy linguistic logic programming for representing and reasoning with linguistically-expressed human knowledge. The framework uses an LTD taken from a linear HA. It turns out that the truth functions of hedges therein are required to satisfy all conditions of a hedge function according to Esteva et al. [14]. The papers [29,24,26] contain preliminary results which are presented in complete form in the present work.

8. Conclusion

This paper proposes two axiomatizations for multiple hedges as an expansion over a core fuzzy logic. In one axiomatization, hedges do not have any dual relations while in the other, each hedge can have its own dual one. These axiomatizations can be seen as an expansion of those of Esteva et al. [14], which are an expansion of a core fuzzy logic with a truth-stressing hedge and/or a truth-depressing one. It is shown that the proposed logics not only cover a large class of hedge functions but also have all completeness properties of the underlying logic w.r.t. the class of its chains and distinguished subclasses of its chains, including standard completeness. The axiomatizations are also extended to the first-order level. Moreover, we show how to build linguistic fuzzy logics based on the axiomatizations and a linear hedge algebra for representing and reasoning with linguistically-expressed human knowledge, where truth values of vague sentences are given in linguistic terms, and many hedges are often used simultaneously to express different levels of emphasis.

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