Nonlinear Analysis 171 (2018) 32–40

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Superposition principle on the viscosity solutions of infinity Laplace equations

Guanghao Hong*, Xiaomeng Feng

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, PR China

ARTICLE INFO

Article history:
Received 31 August 2017
Accepted 21 January 2018
Communicated by Enzo Mitidieri

MSC:
primary 35J25
35J70
secondary 49N60

Keywords:
Superposition principle
Infinity Laplace equation
Viscosity solution

ABSTRACT

We consider the sum of the solutions of two infinity Laplace equations in disjoint variables. We prove that the superposed function is a viscosity solution of the infinity Laplace equation in the extension domains with the sum of inhomogeneous terms if one of the solutions is in the sense of viscosity and the other is in the classical sense. We also construct a counterexample to show that the conclusion may not be true if both of the solutions are merely in the viscosity sense.

1. Introduction

The infinity Laplace equation

$$\triangle_\infty u(x) := \sum_{1\leq i,j\leq n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

was introduced by G. Aronsson [1] in the 1960s. R. Jensen [10] proved the equivalence of the infinity Laplace equation and the absolutely minimizing Lipschitz extension problem. He also proved the existence and uniqueness of the viscosity solution to the Dirichlet problem:

$$\triangle_\infty u = 0 \text{ in } \Omega,\ \ u = g \text{ on } \partial\Omega$$

for any bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in C(\partial\Omega)$. Crandall–Evans–Gariepy [2] introduced the property of comparison with cones and proved that it is a characteristic property of infinity harmonic functions. The interior regularity for infinity harmonic functions was achieved by Evans, Savin and Smart in [4,14] and [3]. The boundary regularity was studied by Wang–Yu [15], Hong [6,8] and Hong-Liu [9].

* Corresponding author.
E-mail addresses: ghhongmath@xjtu.edu.cn (G. Hong), iamme1208@stu.xjtu.edu.cn (X. Feng).

https://doi.org/10.1016/j.na.2018.01.011
0362-546X/

The inhomogeneous infinity Laplace equation:

$$\triangle_\infty u = f \quad in \ \Omega \tag{1}$$

was introduced by Lu-Wang [13]. Lindgren [11] proved that the blow-ups are linear if $f \in C(\Omega) \cap L^\infty(\Omega)$ and u is everywhere differentiable if $f \in C^1(\Omega) \cap L^\infty(\Omega)$. Hong [7] proved the boundary differentiability of u at a differentiable boundary point and Feng–Hong [5] studied the slope estimate and boundary differentiability of u on the convex domains.

In [11], Lindgren constructed an extension

$$\tilde{u}(x_1, \dots, x_{n+2}) = u(x_1, \dots, x_n) + 5x_{n+1} + C|x_{n+2}|^{\frac{4}{3}}$$

and used the conclusion that if $\triangle_\infty u = f$ in $\mathbb{R}^n$ in the viscosity sense then $\triangle_\infty \tilde{u} = f + \frac{2^6}{3^4}C$ in $\mathbb{R}^{n+2}$ in the viscosity sense without a proof. Both of the papers [7] and [5] used the same extension and conclusion. The purpose of the extension is to make the slope function strictly positive and the inhomogeneous term bounded away from 0. The conclusion seems obvious but we will see it is not so. In the book [12](Page 58), Lindqvist also used the similar extension and conclusion. The author gave a very short proof of the conclusion in the footnote, but we do not think the proof is strict enough. The last sentence of the proof says "The desired inequality follows", but we cannot see why the inequality follows from the proceeding deduction. The argument does not involve an analysis on the second order derivatives, the counterexample in this paper indicates that one should not prove the conclusion without going deep into the analysis on the second order derivatives.

In this note, we will give a strict proof of the above mentioned conclusion and provide a counterexample to show that the things are not that simple. We begin by recalling the definition of viscosity solution.

Definition 1. Let $u \in C(\Omega)$, $x_0 \in \Omega$ and $K \in \mathbb{R}$. We say $\triangle_\infty u(x_0) \geq K$ in the viscosity sense if $\triangle_\infty \varphi(x_0) \geq K$ whenever $\varphi \in C^2(D)$ for some neighborhood D of x_0, $\varphi(x_0) = u(x_0)$ and $\varphi(x) \geq u(x)$ in D. We have the similar definition for $\triangle_\infty u(x_0) \leq K$ in the viscosity sense and we say $\triangle_\infty u(x_0) = K$ in the viscosity sense if u satisfies both $\triangle_\infty u(x_0) \geq K$ and $\triangle_\infty u(x_0) \leq K$ in the viscosity sense.

We will prove the following main conclusion in Section 2.

Theorem 1. *Let $x_0' \in \Omega' \subset \mathbb{R}^k$, $x_0'' \in \Omega'' \subset \mathbb{R}^{n-k}$, $\Omega := \Omega' \times \Omega'' \subset \mathbb{R}^n$ and $x_0 = (x_0', x_0'') \in \Omega$. Assume that $v \in C(\Omega')$ satisfies*

$$\triangle_\infty v(x_0') \geq K'$$

in the viscosity sense and $w \in C(\Omega'')$ is second order differentiable at x_0'' and satisfies

$$\triangle_\infty w(x_0'') \geq K''$$

in the classical sense. Let $u(x) := v(x') + w(x'')$ for $x = (x', x'') \in \Omega$. Then $u \in C(\Omega)$ satisfies

$$\triangle_\infty u(x_0) \geq K' + K''$$

in the viscosity sense.

In Section 3, we will give a counterexample to show that if $\triangle_\infty w(x_0'') \geq K''$ is also merely true in the viscosity sense then we may not have $\triangle_\infty u(x_0) \geq K' + K''$ in the viscosity sense.

2. Proof of Theorem 1

Proof of Theorem 1. Without loss of generality, we may assume that

$$x_0' = 0 \in B' := B_1^k(0) \subset \Omega',\ v(0) = 0,$$

$$x_0'' = 0 \in B'' := B_1^{n-k}(0) \subset \Omega'',\ w(0) = 0,$$

$$x_0 = 0 \in B := B_1^n(0) \subset B' \times B'' \subset \Omega,\ u(0) = v(0) + w(0) = 0.$$

For $\Phi \in C^2(B)$ satisfying

$$\Phi(0) = u(0) = 0 \quad \text{and} \quad \Phi(x) \geq u(x) \ \text{ in } B,$$

we have $D\Phi(0) = pe_1 + qe_{k+1}$ by doing some coordinate transformations. We want to show that

$$\triangle_\infty \Phi(0) = D\Phi(0) \cdot D^2\Phi(0) D\Phi(0) = p^2\tilde{a} + q^2\tilde{b} + 2pq\tilde{c} \geq K' + K''$$

where $\tilde{a} = \Phi_{11}(0)$, $\tilde{b} = \Phi_{k+1,k+1}(0)$ and $\tilde{c} = \Phi_{1,k+1}(0) = \Phi_{k+1,1}(0)$.

Clearly, $w(x'')$ is second order differentiable at 0 and $Dw(0) = qe_{k+1}$. We denote $b := w_{k+1,k+1}(0)$. It is easy to see

$$b \leq \tilde{b}.$$

We denote

$$a_e := \overline{\lim_{t \to 0}} \frac{v(te) - pe_1 \cdot te}{\frac{1}{2}t^2}$$

and

$$a := \overline{\lim_{e \to e_1}} a_e$$

where $e \in span\{e_1, \ldots, e_k\}$ and $|e| = 1$. For any $\varepsilon > 0$, there exists $\theta_0 > 0$ such that

$$a_e \leq a + \frac{\varepsilon}{2}$$

for any

$$e \in C_{\theta_0} := \{e \in S^{k-1} : d_{S^{k-1}}(e, e_1) \leq \theta_0\}.$$

From the definition of a_e, there exists $t_e > 0$ such that

$$v(te) \leq pe_1 \cdot te + \frac{1}{2}(a_e + \frac{\varepsilon}{2})t^2 \quad \text{for all } t \in (-t_e, t_e).$$

From the continuity of v and the compactness of C_{θ_0}, we have

$$t_0 := \inf_{e \in C_{\theta_0}} \{t_e\} > 0.$$

Therefore, for any $t \in (-t_0, t_0)$ and $e \in C_{\theta_0}$,

$$v(te) \leq pe_1 \cdot te + \frac{1}{2}(a + \varepsilon)t^2. \tag{2}$$

It is clear that

$$a \leq \tilde{a}.$$

In the following we show that

$$\begin{aligned}\triangle_\infty \Phi(0) &= p^2\tilde{a} + q^2\tilde{b} + 2pq\tilde{c} \\ &= p^2 a + q^2 b + p^2(\tilde{a} - a) + q^2(\tilde{b} - b) + 2pq\tilde{c} \\ &\geq K' + K''.\end{aligned}$$

Clearly,

$$\triangle_\infty w(0) = q^2 b \geq K''.$$

Let

$$[D^2\Phi(0)]_{k\times k} := D^2\Phi(0)\big|_{span\{e_1,\ldots,e_k\}}$$

and Λ denotes the maximum eigenvalue of $[D^2\Phi(0)]_{k\times k}$. For arbitrary $\varepsilon > 0$, let

$$\varphi_\varepsilon(x') := pe_1 \cdot x' + \frac{1}{2}(a+\varepsilon)x_1^2 + \frac{1}{2}M(|x_2|^2 + \cdots + |x_k|^2) \tag{3}$$

where $M > \frac{\Lambda - a}{tan^2\theta_0}$. Then

$$\varphi_\varepsilon(x') \geq v(x') \text{ in } B_{t_0}(0) \setminus C_{\theta_0} \times (-t_0, t_0).$$

Combine (2) and (3), we must have

$$\varphi_\varepsilon(x') \geq v(x') \quad \text{in } C_{\theta_0} \times (-t_0, t_0).$$

Therefore,

$$\varphi_\varepsilon(x') \geq v(x') \text{ in } B_{t_0}(0).$$

According to the definition of the viscosity subsolution, it follows that

$$\triangle_\infty \varphi_\varepsilon(0) = pe_1 \cdot D^2\varphi_\varepsilon(0)pe_1 = p^2(a+\varepsilon) \geq K'.$$

Sending $\varepsilon \to 0$, we have

$$p^2 a \geq K'.$$

It remains to show

$$p^2(\tilde{a} - a) + q^2(\tilde{b} - b) + 2pq\tilde{c} \geq 0,$$

that is

$$A = \begin{pmatrix} \tilde{a} - a & \tilde{c} \\ \tilde{c} & \tilde{b} - b \end{pmatrix} \tag{4}$$

is a 2×2 nonnegative definite matrix. We prove (4) by contradiction. If A has a negative eigenvalue $-\lambda_0 < 0$, then there exists $\delta > 0$ such that for all symmetric matrix B satisfying $|B - A| < \delta$, B has a negative eigenvalue $-\lambda < -\frac{\lambda_0}{2} < 0$. By the definition of a, we have for any $0 < \varepsilon < \frac{\lambda_0}{4}$ and any $\theta > 0$, there exist $e \in C_\theta$ and a sequence $\{t_j\}$ with $t_j \to 0$, such that

$$v(t_j e) > pe_1 \cdot t_j e + \frac{1}{2}(a - \varepsilon)t_j^2. \tag{5}$$

Define

$$\bar{v}(t) := v(te), \qquad \bar{w}(s) := w(se_{k+1}),$$

$$\bar{u}(t,s) := \bar{v}(t) + \bar{w}(s) = u(te + se_{k+1}),$$

$$\bar{\Phi}(t,s) := \Phi(te + se_{k+1}).$$

By (5), we obtain

$$\bar{v}(t_j) > pe_1 \cdot et_j + \frac{1}{2}(a-\varepsilon)t_j^2.$$

Clearly,

$$\bar{\Phi}(0,0) = \bar{u}(0,0), \qquad \bar{\Phi}(t,s) \geq \bar{u}(t,s),$$

$$\bar{\Phi}(0,0) = (pe_1 \cdot e, q),$$

$$D^2\bar{\Phi}(0,0) = \begin{pmatrix} \Phi_{ee}(0) & \Phi_{e,k+1}(0) \\ \Phi_{k+1,e}(0) & \Phi_{k+1,k+1}(0) \end{pmatrix} = \begin{pmatrix} \Phi_{ee}(0) & \Phi_{e,k+1}(0) \\ \Phi_{k+1,e}(0) & \tilde{b} \end{pmatrix}.$$

We denote

$$B = \begin{pmatrix} \Phi_{ee}(0) - a & \Phi_{e,k+1}(0) \\ \Phi_{k+1,e}(0) & \tilde{b} - b \end{pmatrix}.$$

Let θ be sufficiently small, then $|e - e_1|$ is very small. It follows that

$$|B - A| < \delta.$$

Then B has a negative eigenvalue $-\lambda < -\frac{\lambda_0}{2} < 0$ and there exists $\xi = (\xi^1, \xi^2)$, $|\xi| = 1$ such that

$$B\xi = -\lambda\xi \quad \text{and} \quad \xi \cdot B\xi = -\lambda < 0.$$

Notice that $\tilde{b} - b \geq 0$, then $\xi \neq (0,1)$. Set

$$s_j := \frac{\xi^2}{\xi^1} t_j \ \text{ and } \ \zeta_j := (t_j, s_j).$$

It follows that

$$\zeta_j \cdot B\zeta_j = -\lambda|\zeta_j|^2 < -\frac{\lambda_0}{2}|\zeta_j|^2 < 0.$$

For small enough t_j and $0 < \varepsilon < \frac{\lambda_0}{4} < \frac{\lambda}{2}$,

$$\begin{aligned}
\bar{\Phi}(t_j, s_j) &= \bar{\Phi}(0,0) + D\bar{\Phi}(0,0)\cdot\zeta_j + \frac{1}{2}\zeta_j \cdot D^2\bar{\Phi}(0,0)\zeta_j + o(|\zeta_j|^2) \\
&= 0 + pe_1 \cdot et_j + qs_j + \frac{1}{2}at_j^2 + \frac{1}{2}bs_j^2 + \frac{1}{2}\zeta_j \cdot B\zeta_j + o(|\zeta_j|^2) \\
&< \bar{v}(t_j) + \bar{w}(s_j) + \frac{1}{2}\varepsilon t_j^2 - \frac{1}{2}\lambda|\zeta_j|^2 + o(|\zeta_j|^2) \\
&\leq \bar{u}(t_j, s_j) + (\varepsilon - \frac{1}{2}\lambda)|\zeta_j|^2 \\
&< \bar{u}(t_j, s_j)
\end{aligned}$$

which is a contradiction to $\bar{\Phi}(t,s) \geq \bar{u}(t,s)$. In the third line of the above inequality, we used that $\bar{w}(s_j) + o(|s_j|^2) \geq qs_j + \frac{1}{2}bs_j^2$ for small enough s_j. This is true because w is second order differentiable. If we only assume that w solves the inequality in the viscosity sense, then $\bar{w}(s_j) + o(|s_j|^2) \geq qs_j + \frac{1}{2}bs_j^2$ is only true for a special choice of the sequence $\{s_j\}$. But here $\{s_j\}$ are decided by $\{t_j\}$ (which is a special choice for v) and ξ so cannot meet a specified requirement of w simultaneously. This is why we must require one of v and w to be second order differentiable. The counterexample in the next section exposes this fact from the opposite angle.

Hence A is a nonnegative definite matrix. Theorem 1 is established. □

Remark 1. Similarly, if $v \in C(\Omega')$ satisfies

$$\triangle_\infty v(x_0') \le K'$$

in the viscosity sense and $w \in C(\Omega'')$ is second order differentiable at x_0'' and satisfies

$$\triangle_\infty w(x_0'') \le K''$$

in the classical sense. Then $u = v + w \in C(\Omega)$ satisfies

$$\triangle_\infty u(x_0) \le K' + K''$$

in the viscosity sense.

Theorem 1 concerns the conclusion on the pointwise situation, whereas we now focus on the whole domain. It is easy to get the following corollary.

Corollary 1. *Let $x' \in \Omega'$, $x'' \in \Omega''$, $x = (x', x'') \in \Omega := \Omega' \times \Omega''$. Assume that $v(x') \in C(\Omega')$ is a viscosity solution of*

$$\triangle_\infty v(x') = f(x') \qquad \text{in } \Omega'$$

and $w(x'') \in C^2(\Omega'')$ is a classical solution of

$$\triangle_\infty w(x'') = g(x'') \qquad \text{in } \Omega''.$$

Then $u(x) = v(x') + w(x'') \in C(\Omega)$ is a viscosity solution of

$$\triangle_\infty u(x) = f(x') + g(x'') \qquad \text{in } \Omega.$$

3. A counterexample

The following example shows that Theorem 1 cannot in general be extended to merely assume $w \in C(\Omega'')$ satisfying $\triangle_\infty w(x_0'') \ge K''$ in the viscosity sense. The counterexample is in dimension 2.

Let

$$v(x) = x + \frac{1}{2}x^2\tau(x) \quad \text{and} \quad w(y) = y + \frac{1}{2}y^2\tau(3y)$$

where $\tau(x) \in L^\infty(-1,1) \cap C^\infty((-1,1)/\{0\})$, $\tau(0) = 0$, $\tau(-x) = \tau(x)$ and on $(0,1)$ $\tau(x)$ is given by

$$\begin{cases} \tau(x) = 1, & if \quad x = \dfrac{1.2}{2^{k+1}} \\ 1 > \tau(x) > -2, & if \quad \dfrac{1.2}{2^{k+1}} < x < \dfrac{1.2+\varepsilon}{2^{k+1}} \\ \tau(x) = -2, & if \quad \dfrac{1.2+\varepsilon}{2^{k+1}} \le x \le \dfrac{1.2-\varepsilon}{2^{k}} \\ -2 < \tau(x) < 1, & if \quad \dfrac{1.2-\varepsilon}{2^{k}} < x < \dfrac{1.2}{2^{k}} \end{cases}$$

with small enough $\varepsilon \ll 0.1$ and $k = 0, 1, 2, \ldots$. Note that any point $x \in (0,1)$ belongs to one of the four kinds of intervals for some k exactly. It is easy to check that

$$\overline{\lim_{x\to 0}} \frac{v(x) - x}{\frac{1}{2}x^2} = \overline{\lim_{x\to 0}}\, \tau(x) = 1,$$

and

$$\overline{\lim_{y\to 0}} \frac{w(y) - y}{\frac{1}{2}y^2} = \overline{\lim_{y\to 0}}\, \tau(3y) = 1.$$

Thus, for any $\phi \in C^2(-1,1)$ satisfying

$$\phi(0) = v(0) \quad \text{and} \quad \phi(x) \geq v(x),$$

we have $\phi'(0) = v'(0) = 1$ and

$$\triangle_\infty \phi(0) = \phi'(0)\phi''(0)\phi'(0) \geq 1.$$

Similarly, for any $\psi \in C^2(-1,1)$ satisfying

$$\psi(0) = w(0) \quad \text{and} \quad \psi(y) \geq w(y),$$

we have $\psi'(0) = w'(0) = 1$ and

$$\triangle_\infty \psi(0) = \psi'(0)\psi''(0)\psi'(0) \geq 1.$$

Hence $v(x)$ and $w(y)$ satisfy

$$\triangle_\infty v(0) \geq 1 \ \text{ and } \ \triangle_\infty w(0) \geq 1$$

in the viscosity sense respectively.

Let

$$u(x,y) := v(x) + w(y).$$

In the following we will show that u does not satisfy

$$\triangle_\infty u(0,0) \geq 1 + 1 = 2$$

in the viscosity sense.

Define

$$C = \left\{ (x,y) : \frac{y}{x} \in (\frac{9}{10}, \frac{10}{9}) \right\}.$$

We claim that for all $(x,y) \in C$,

$$\text{either} \ \ \tau(x) = -2 \ \text{ or } \ \tau(3y) = -2.$$

Fix $(x,y) \in C$. In fact, if $\tau(x) \neq -2$, then there exists $l \in \mathbf{Z}$ such that

$$x \in \left(\frac{1.2 - \varepsilon}{2^l}, \frac{1.2 + \varepsilon}{2^l} \right).$$

It follows that

$$3y \in \left(\frac{27}{10} \cdot \frac{1.2 - \varepsilon}{2^l}, \frac{10}{3} \cdot \frac{1.2 + \varepsilon}{2^l} \right) \subset \left(\frac{1.2 + \varepsilon}{2^{l-1}}, \frac{1.2 - \varepsilon}{2^{l-2}} \right),$$

i.e.

$$\tau(3y) = -2.$$

Therefore,

$$\overline{\lim_{\substack{(x,y) \in C \\ (x,y) \to (0,0)}}} \frac{u(x,y) - Du(0,0) \cdot (x,y)}{\frac{1}{2}|(x,y)|^2} = \overline{\lim_{\substack{(x,y) \in C \\ (x,y) \to (0,0)}}} \frac{x^2\tau(x) + y^2\tau(3y)}{x^2 + y^2} < -\frac{1}{3}.$$

Let

$$\Phi(x,y) = x + y - \frac{1}{12}(x+y)^2 + M(x-y)^2$$

with $M := M(\varepsilon)$ large enough. Using the same argument as (3), we have

$$\Phi(x,y) \geq u(x,y).$$

Note that $\Phi(0,0) = u(0,0)$ and $D\Phi(0,0) = Du(0,0) = (1,1)$. Then we have

$$\begin{aligned} \triangle_\infty \Phi(0,0) &= D\Phi(0,0) \cdot D^2\Phi(0,0) D\Phi(0,0) \\ &= (1,1) \begin{pmatrix} -\frac{1}{6} + 2M & -\frac{1}{6} - 2M \\ -\frac{1}{6} - 2M & -\frac{1}{6} + 2M \end{pmatrix} (1,1)^T \\ &= -\frac{2}{3} < 2. \end{aligned}$$

Therefore, u does not satisfy $\triangle_\infty u(0,0) \geq 2$ in the viscosity sense.

Remark 2. The following question is still a challenge for us. If we have two viscosity sub-solutions in the whole domain rather than at one point and we assume the two right hand side functions are continuous, that is, $v \in C(\Omega')$ and $w \in C(\Omega'')$ satisfy $\triangle_\infty v(x') \geq f(x')$ in Ω' and $\triangle_\infty w(x'') \geq g(x'')$ in Ω'' separately (both are in the viscosity sense) with f and g continuous, is it true that $\triangle_\infty(v+w) \geq f+g$ in $\Omega' \times \Omega''$ in the viscosity sense? Even if $f = g = 0$, we do not know the answer. That is, we do not know whether or not the superposition of two infinity harmonic functions in disjoint variables is an infinity harmonic function. We think it is a very interesting question.

Acknowledgment

This work is supported by the National Nature Science Foundation of China: NSFC 11301411 and 11671316.

References

[1] G. Aronsson, Extension of functions satisfying Lipschitz conditions, Ark. Mat. 6 (1967) 551–561.
[2] M.G. Crandall, L.C. Evans, R.F. Gariepy, Optimal Lipschitz extensions and the infinity Laplacian, Calc. Var. Partial Differential Equations 13 (2) (2001) 123–139.
[3] L.C. Evans, O. Savin, $C^{1,\alpha}$ regularity for infinity harmonic functions in two dimensions, Calc. Var. Partial Differential Equations 32 (3) (2008) 325–347.
[4] L.C. Evans, C.K. Smart, Everywhere differentiability of infinity harmonic functions, Calc. Var. Partial Differential Equations 42 (1–2) (2011) 289–299.
[5] X. Feng, G. Hong, Slope Estimate and boundary differentiability for inhomogeneous infinity Laplace equation on convex domains, preprint.
[6] G. Hong, Boundary differentiability of infinity harmonic functions, Nonlinear Anal.: TMA 93 (2013) 15–20.
[7] G. Hong, Boundary differentiability for inhomogeneous infinity Laplace equations, Electron. J. Differential Equations 72 (2014) 1–6.
[8] G. Hong, Counterexample to C^1 boundary regularity of infinity harmonic functions, Nonlinear Anal.: TMA 104 (2014) 120–123.
[9] G. Hong, D. Liu, Slope estimate and boundary differentiability of infinity harmonic functions on convex domains, Nonlinear Anal.: TMA 139 (2016) 158–168.
[10] R. Jensen, Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient, Arch. Ration. Mech. Anal. 123 (1993) 51–74.

[11] E. Lindgren, On the regularity of solutions of the inhomogeneous infinity Laplace equation, Proc. Amer. Math. Soc. 142 (1) (2014) 277–288.

[12] P. Lindqvist, Notes on the Infinity Laplace Equation, in: SpringerBriefs in Mathematics, BCAM Basque Center for Applied Mathematics, Springer, Bilbao, Cham, 2016.

[13] G. Lu, P. Wang, Inhomogeneous infinity Laplace equation, Adv. Math. 217 (4) (2008) 1838–1868.

[14] O. Savin, C^1 regularity for infinity harmonic functions in two dimensions, Arch. Ration. Mech. Anal. 176 (3) (2005) 351–361.

[15] C.Y. Wang, Y.F. Yu, C^1-boundary regularity of planar infinity harmonic functions, Math. Res. Lett. 19 (4) (2012) 823–835.