Dynamic contact in multiferroic energy conversion

Xin Zhang, Zhanjiang Wang, Huoming Shen, Q. Jane Wang

PII: S0020-7683(18)30090-8
DOI: 10.1016/j.ijsolstr.2018.02.034
Reference: SAS 9922

To appear in: International Journal of Solids and Structures

Received date: 3 December 2017
Revised date: 6 February 2018
Accepted date: 26 February 2018

Please cite this article as: Xin Zhang, Zhanjiang Wang, Huoming Shen, Q. Jane Wang, Dynamic contact in multiferroic energy conversion, International Journal of Solids and Structures (2018), doi: 10.1016/j.ijsolstr.2018.02.034

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.
Dynamic contact in multiferroic energy conversion

Xin Zhang a, b, Zhanjiang Wang c, Huoming Shen a, Q. Jane Wang b, c, *

a School of Mechanics and Engineering, Southwest Jiaotong University, Chengdu 610031, China
b Department of Mechanical Engineering, Northwestern University, Evanston, IL 60208, USA
c Tribology Research Institute, Traction Power State Key Laboratory, Southwest Jiaotong University, Chengdu 610031, China

*Corresponding author: Q. Jane Wang, E-mail: qwang@northwestern.edu

Abstract

This paper reports a semi-analytical model (SAM) for simulating the dynamic contact of a rigid sphere and the surface of a multiferroic magnetoelectroelastic (MEE) film under an increasing applied force. The frequency response functions (FRFs) and their conversion into influence coefficients (ICs) for the MEE film are analytically derived, incorporating the loading velocity in a dynamic process. Fast numerical techniques, including the conjugate gradient method (CGM) and the fast Fourier transform (FFT), are employed for efficient numerical solutions to the dynamic contact behaviors, including the distributions and variations of contact pressure and electric/magnetic potentials, as well as subsurface stresses. The proposed model is implemented to analyze the influences of loading velocity, film thickness, and sphere radius on the dynamic MEE responses, including pressure/stresses and electric/magnetic potentials. An energy conversion factor is selected to evaluate the performance of MEE energy conversion. Furthermore, a sensitivity analysis is conducted to evaluate the influence of material properties and their coupling on the efficiency of mechanical-electric/magnetic energy conversion.

Key words: Multiferroic film, energy conversion, semi-analytical modeling, dynamic contact.
Nomenclature

\( a^t \) acceleration of the rigid sphere at time \( t \)
\( a_0 \) contact radius in multiferroic half-space, m
\( A_k, \bar{A}_k \) unknowns in potential functions
\( A_{ij} \) cofactors of matrix \([D]\)
\( B_i \) magnetic induction along the \( x_i \) direction, N/(Am)
\( c \) relative moving velocity of the rigid sphere
\( c_{sw} \) shear wave velocity in the MEE film
\( c_{ijkl} \) elastic stiffness tensor, \(10^9\)N/m\(^2\)
\( C, C^\phi, C^\psi, C \) influence coefficients
\( d_{kij} \) piezomagnetic stiffness tensor, N/(Am)
\( [D] \) a matrix
\( D_i \) electric displacement, C/m\(^2\)
\( e_{kij} \) piezoelectric stiffness tensor, C/m\(^2\)
\( f \) selected material constant
\( f^0 \) reference value for \( f \)
\( F \) displacement function
\( g \) surface magnetic charge,
\( g_z \) gap between two surfaces, m
\( g_{ij} \) electromagnetic coefficient, C/(Am)
\( G, G^\phi, G^\psi \) frequency response functions
\( G \) displacement function
\( h_0 \) initial separation between two surfaces, m
\( h_t \) film thickness, m
\( H \) displacement function
\( i \) imaginary unit
\( m, n \) Fourier-transformed variables with respect to \( x_1, x_2 \) directions, respectively
\( M, N \) mesh numbers along the \( x_1, x_2 \) directions
\( M_e, N_e \) refined mesh numbers
\( M_s \) mass of the sphere
\( q \) surface electric charge,
\( p \) pressure, Pa
\( P \) applied force, N
\( P_0 \) referenced force, N
\( R \) radius of a spherical punch, m
\( R_0 \) referenced sphere radius, m
\( s_j \) variables relating to material properties
\( t \) loading time
\( T_{total} \) total time for one loading cycle
$u_i$ displacements along the $x_i$ direction, m
$V_0$ maximum value of the loading velocity
$V$ loading velocity of the sphere, m/s
$W^p$ work by the applied force
$x_{ji}, X_j$ Cartesian coordinates in the spatial domain
$F_s$ shape function

Greek letters
$\alpha$ distance of a node, $(m, n)$, to the origin of the frequency domain
$\gamma$ refinement level
$\kappa_{\alpha\beta}$ surface curvature tensor with $\kappa_{\alpha\beta} = -u_i\alpha\beta$
$\delta^t$ indentation depth at time $t$
$\varepsilon_{ij}$ dielectric permittivity, $10^{-9}$C$^2$/Nm$^2$
$\mu_{ij}$ magnetic permeability, $10^{-6}$Ns$^2$/C$^2$
$\sigma_{ij}$ stress components, Pa
$\varepsilon_s$ von Mises stress, Pa
$\rho$ mass density
$\eta_P$ energy conversion factor
$\phi$ electric potential, V
$\varphi$ magnetic potential, A
$\psi_i$ harmonic functions
$\Gamma_c$ contact zone
$\Gamma$ surface energy density
$\Pi^S$ strain energy
$\Pi^E$ electric energy
$\Pi^M$ magnetic energy
$\Pi^{PE}$ piezoelectric energy
$\Pi^{PM}$ piezomagnetic energy
$\Pi^{ME}$ magnetoelectric energy
$\Delta_{x_i}$ grid size in the $x_i$ directions, m

Special marks
$\approx$ double continuous Fourier transform
$\hat{}$ discrete Fourier transform
$\vert \cdot \vert$ determinant of a matrix
$IFFT$ inverse fast Fourier transform
1. Introduction

Multiferroic materials are noteworthy for their strong coupling of ferroelectricity, ferromagnetism, and ferroelasticity (Eerenstein et al., 2006; Wang et al., 2010). A large amount of research has been conducted to explore methods for synthesizing and characterizing multiferroics, aiming to achieve a larger magnetoelastic (MEE) coupling effect (Liu et al., 2010; Ramesh and Spaldin, 2007). Most recent studies on multiferroics reported truly remarkable MEE coupling effects (Kirchhof et al., 2013; Kulkarni et al., 2014), which promote the applications of these materials in AC/DC magnetic field sensors, photovoltaic multiferroic solar cells, multiferroic gyrators, and energy harvesting multiferroics (Vopson, 2015). Among these applications, the devices capable of harvesting energy from environmental sources (e.g. air flow, body movements, hydraulic pressure, and ambient vibrations) are of huge commercial values due to their suitability to self-powered systems in portable electronics, environmental condition monitors, structural health monitors, and medical implants (Vopson, 2015; Zhou et al., 2016). Although current energy harvesting devices are mainly based on piezoelectric semiconductors, known as piezoelectric nanogenerators (NGs) (Hu and Wang, 2015; Kim et al., 2014; Wang et al., 2014; Wang and Song, 2006), other approaches for energy harvest based on magnetostrictive materials have been reported in (Erturk et al., 2009; Wang and Yuan, 2008; Zucca et al., 2011). Foreseeably, multiferroic devices combining the piezoelectric and magnetostrictive advantages could result in promising energy harvesters with ultra-high efficiency, as reported in (Challa et al., 2009; Lafont et al., 2012; Li et al., 2015; Shan et al., 2013).

Device development requires systematic optimization for the efficiency of energy conversion and durability of operation, which should be built on deep understanding of the material performance. Recently, a great deal of research on the piezoelectric and magnetostrictive energy harvesters has been conducted for the energy conversion maximization. Several examples are the work by Gu et al. (2012) on structural optimization for flexible fiber energy harvesters, the study by Xu and Qin (2017) about the external force optimization for piezoelectric energy harvesters, the research by Sun and Kim (2010) on the topology optimization for MEE laminate composites, and the optimization by Loja et al. (2014) for several MEE composite structures utilizing differential evolution. However, these works were mainly focused on a static analysis of the piezoelectric or MEE materials, as well as a simplified treatment of the fully coupled piezoelectric/MEE problems. A comprehensive investigation of the dynamic behaviors of multiferroic materials is the first step towards in-depth understanding of the overall responses of the MEE energy conversion. Considering multiferroic materials, mainly in the form of a thin film bonded to a substrate, a thin-film-based model is a natural choice for MEE dynamics, where the MEE film can be described as a voltage source for utilizing the electric potential to drive the electron flow.
through the circuit. In order to determine the dynamic characteristics of a MEE film, the voltage source’s voltage dependence on the applied force and loading velocity should be analyzed firstly, for which several key issues should be addressed: (a) what are the key dynamic characteristics of a MEE thin-film subjected to dynamic contact loading? (b) How does the loading velocity affect the mechanical and electric/magnetic responses? (c) What is the general role of MEE coupling in a contact process for energy conversion? However, the dynamic analysis of a contact process involves complicated field coupling and strong nonlinearity; it requires a huge amount of computation; therefore, an efficient method is highly demanded.

Many researchers have investigated the static contact behaviors of MEE coupling. Several examples are the analytical works by Hou et al. (2003) and Chen et al. (2010) on the frictionless contact involving a MEE half-space, the analytical study and numerical implementation by Zhou and Lee (2013) on the frictional contact involving a MEE half-plane, the numerical modeling by Michopoulos et al. (2015) on the contact behaviors of MEE half-space based on the finite element method (FEM), the investigation of Rodríguez-Tembleque et al. (2016a) on the frictional contact of a MEE half-space using the boundary element method (BEM), and the simulation by Zhang et al. (2018) on the frictional contact of two MEE bodies using the semi-analytical modeling (SAM) approach. However, little work has been done for MEE films, although many investigated the static indentation responses of piezoelectric films by analytical approaches (Wang and Chen, 2011; Wang et al., 2008), by using the FEM (Liu and Yang, 2012) and the BEM (Rodríguez-Templeque et al., 2016b), and MEE films by using SAM (Zhang et al., 2017). Furthermore, the dynamic contact analysis of MEE films is largely missing due to the complexity of MEE coupling in a multiferroic film, which makes it difficult to obtain exact analytical solutions. The work reported in this paper intends to develop the solution for dynamic contact of MEE films and explore an efficient model for tackling the dynamic contact responses of MEE films.

Semi-analytical modeling (SAM) (Liu et al., 2000; Polonsky and Keer, 1999; Webster and Sayles, 1986), built upon core analytical solutions and supported by fast numerical techniques, has been proven to be efficient for solving three-dimensional (3D) contact problems involving layered materials (Wang et al., 2011; Wang et al., 2015; Yu et al., 2014), or materials with contact elastic–plasticity (Liu et al., 2012; Wang et al., 2013), even subjected to fretting (Zhang et al., 2015; Zhou et al., 2017). The core analytical solutions to displacements, electric/magnetic potentials, stresses, electric displacements, and magnetic inductions of the MEE film should be derived first pertaining to a set of generalized normal force, surface electric charge, and surface magnetic charge, together with incorporation of the loading velocity in a dynamic process. Recently, Zhou and Lee (2012a), and Zhou and Kim
(2015) studied the exact contact problem between the MEE half-plane and a moving punch, which inspired our current study on deriving the core analytical solutions of 3D dynamic contacts of MEE films. Then fast numerical techniques, such as the conjugate gradient method (CGM) and the fast Fourier transform (FFT), can be incorporated for efficient numerical results to the dynamic contact behaviors, including the distributions of contact pressure and electric/magnetic potentials, as well as subsurface stresses. The proposed model is verified by comparing the results from its degenerated forms with analytical solutions for a MEE half-space and the corresponding numerical results for selected static contact of MEE films. The model is further used to study general MEE transient behaviors. Thus, the general role of the MEE coupling in a dynamic contact process is explored and the dynamic contact mechanism in multiferroic energy conversion can be revealed.

2. Formulation

2.1 Problem description

Figure 1 (a) shows the working mechanism of a magnetoelectroelastic (MEE) energy converter, where the rigid sphere of radius $R$ represents any indenter whose tip geometry can be modeled by its local curvature, which is on the top surface of a transversely isotropic multiferroic MEE film under an increasing applied force $P(t)$ along the $z$ axis. The lower surface of the film of finite thickness $h_t$ is perfectly bonded to a rigid grounded substrate. The rectangular coordinates $(x, y, z)$ are introduced here with the $x$-$y$ plane parallel to the plane of isotropy of the multiferroic film. There are two solution routes for the presented problem: a) the applied force $P(t)$ on the rigid sphere is given as a function of time $t$. Based on $P(t)$ at each time step $t$, the contact problem can be solved to obtain the indentation depth, $\delta$, contact pressure, and surface electric/magnetic potential, etc.; b) the loading velocity $V(t)$ of the rigid sphere is given at each time step $t$ as a given input, which can be employed to further determine the indentation depth $\delta$. Then the contact problem is naturally solved to obtain the applied force, contact pressure, and surface electric/magnetic potentials, etc. In route a), the increasing applied force $P(t)$ is generally assumed as a harmonic excitation. The influence of the applied force, in different variation shapes, on the piezoelectric behavior has been studied in (Xu and Qin, 2017). However, the loading velocity also affected the piezoelectric/MEE behaviors. In route b), the shape of the loading velocity variation appears as an input in the current study, instead of the force variation shape, which offers a convenient way to further study the dynamic coupling behaviors of MEE films in multiferroic energy conversion.

The dynamic loading velocity, $V(t)$, of the rigid sphere can be prescribed in two typical forms: (i) a triangular shape and (ii) a sinusoidal shape, as shown in insets (i) and (ii) of Fig. 1 (a). The electrical characteristics of the velocity-induced MEE converter can be described as a simple resistor-capacitor (RC) equivalent circuit, as shown in Fig. 1 (b), where
$R_{Ex}$ is an external load resistor, $R_{In}$ is the internal resistance of the MEE converter, $C_{MEE}$ is the capacitance of the converter. The performance of the MEE film can be described as voltage source $V_{MEE}$, and capacitance $C_{MEE}$ together with an internal resistance $R_{In}$, which utilizes the electric potential to drive the electron flow through the circuit.

At time step $t$, the dynamic contact between the sphere and the MEE film can be subjected to an increasing applied force $P$ and a certain velocity $V$. Contact modeling is required to obtain the output parameters based on the input information, as shown in Fig. 1 (c). To this end, the theoretical expressions for dynamic MEE coupling can be derived by incorporating velocity $V$ in a dynamic process, which are then employed for developing efficient numerical solutions to the dynamic contact problem.

Fig. 1. Illustration of the working mechanism of a transversely isotropic multiferroic MEE energy converter.

2.2 Basic equations and boundary conditions

The constitutive relations for MEE films are given by (Hou et al. 2003; Chen et al. 2010)

\[
\sigma_{ij} = c_{ijkl} \gamma_{kl} - e_{kij} E_k - d_{kij} H_k, \\
D_i = e_{ijkl} \gamma_{kl} + e_{ij} E_j + g_{ij} H_j, \\
B_i = d_{ijkl} \gamma_{kl} + g_{ij} E_j + \mu_{ij} H_j,
\]

with

\[
\gamma_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \\
E_i = -\phi_j, \\
H_i = -\varphi_j,
\]

where $\sigma_{ij}$, $D_i$, and $B_i$ are the elastic stress tensor, electric displacement vector, and magnetic
induction vector; $c_{ij}$, $e_{ij}$ and $d_{ij}$ denote the elastic moduli, piezoelectric coefficients, and piezomagnetic coefficients; $\varepsilon_{ij}$, $g_{ij}$ and $\mu_{ij}$ represent the dielectric permittivity, electromagnetic coefficients, and magnetic permeability, respectively; $\gamma_{ij}$, $E_i$ and $H_i$ are the elastic strain tensor, electric field vector, and magnetic field vector; $u_i$, $\phi$ and $\varphi$ are the elastic displacement vector, electric potential, and magnetic potential. Here and throughout this paper, the standard index notation is used, where comma represents differentiation with respect to the suffix coordinate. For clarity, the Einstein summation convention for repeated indices is used.

For typical MEE materials, the characteristic frequencies for elastic and electromagnetic processes are about $10^4$ Hz and $10^7$ Hz, respectively. In the presence of transient loadings, the changes in the electromagnetic fields in the MEE body are assumed to be immediate, in which the electromagnetic field can be regarded as a quasi-static state (Bonaldi et al., 2015; Parton and Kudryavtsev, 1988; Sladek et al., 2010). Therefore in absence of body sources, the equilibrium equations and the Maxwell equations are

$$\sigma_{ij} = \rho \frac{\partial^2 u_i}{\partial t^2},$$
$$D_{ij} = 0,$$
$$B_{ij} = 0,$$

(3)

where $\rho$ is the mass density and $t$ represents time.

The boundary conditions at the upper surface of the MEE film can be written as

$$\sigma_{x0} = 0,$$
$$\sigma_{z0} = 0,$$
$$\sigma_{z2} = -p,$$
$$D_z = -q,$$
$$B_z = -g,$$

(4)

where $p$, $q$ and $g$ represent pressure, surface electric charge, and surface magnetic charge, respectively.

Because the lower surface of the film is perfectly bonded to a rigid grounded substrate, the boundary conditions at the lower surface are
\[ u_x|_{z=h} = 0, \]
\[ u_y|_{z=h} = 0, \]
\[ u_z|_{z=h} = 0, \]
\[ \phi|_{z=h} = 0, \]
\[ \varphi|_{z=h} = 0. \]

In the context of transversely isotropic MEE media, substituting Eq. (3) into Eq. (1) leads to the following governing equations,

\[ (c_{11} \frac{\partial^2}{\partial x^2} + c_{66} \frac{\partial^2}{\partial y^2} + c_{44} \frac{\partial^2}{\partial z^2}) u_x + (c_{12} + c_{66}) \frac{\partial^2 u_y}{\partial x \partial y} + (c_{13} + c_{44}) \frac{\partial^2 u_z}{\partial x \partial z} + (e_{11} \frac{\partial^2 \phi}{\partial x^2} + e_{13} \frac{\partial^2 \phi}{\partial x \partial z} + (d_{11} + d_{13}) \frac{\partial^2 \varphi}{\partial x^2} = \rho \frac{\partial^2 u_x}{\partial t^2}, \]

\[ (c_{12} + c_{66}) \frac{\partial^2 u_y}{\partial x \partial y} + (c_{66} \frac{\partial^2}{\partial x^2} + c_{11} \frac{\partial^2}{\partial y^2} + c_{44} \frac{\partial^2}{\partial z^2}) u_y + (c_{13} + c_{44}) \frac{\partial^2 u_z}{\partial y \partial z} + (e_{13} \frac{\partial^2 \phi}{\partial y \partial z} + (d_{13} + d_{33}) \frac{\partial^2 \varphi}{\partial y^2} = \rho \frac{\partial^2 u_y}{\partial t^2}, \]

\[ (c_{13} + c_{44}) \frac{\partial^2 u_z}{\partial x \partial z} + (c_{44} \frac{\partial^2}{\partial y^2} + e_{13} \frac{\partial^2}{\partial y \partial z}) u_z + (c_{13} \frac{\partial^2}{\partial y^2} + e_{13} \frac{\partial^2}{\partial y \partial z}) u_z - (e_{11} \frac{\partial^2}{\partial y^2} + e_{33} \frac{\partial^2}{\partial z^2}) \phi + (d_{15} \frac{\partial^2}{\partial x \partial z} + d_{44} \frac{\partial^2}{\partial y \partial z}) u_z - (g_{11} \frac{\partial^2}{\partial x^2} + g_{11} \frac{\partial^2}{\partial y^2} + g_{33} \frac{\partial^2}{\partial z^2}) \phi = 0, \]

\[ (e_{15} \frac{\partial^2}{\partial y \partial z} + e_{33} \frac{\partial^2}{\partial z^2}) \phi - (g_{11} \frac{\partial^2}{\partial x^2} + g_{11} \frac{\partial^2}{\partial y^2} + g_{33} \frac{\partial^2}{\partial z^2}) \phi = 0, \]

\[ (d_{15} + d_{33}) \frac{\partial^2 u_x}{\partial x \partial z} + (d_{15} \frac{\partial^2}{\partial x^2} + d_{13} \frac{\partial^2}{\partial x \partial z} + d_{33} \frac{\partial^2}{\partial y \partial z}) u_z - (g_{11} \frac{\partial^2}{\partial x^2}) \phi + (g_{11} \frac{\partial^2}{\partial y^2} + g_{33} \frac{\partial^2}{\partial z^2}) \phi - (\mu_{11} \frac{\partial^2}{\partial x^2} + \mu_{11} \frac{\partial^2}{\partial y^2} + \mu_{33} \frac{\partial^2}{\partial z^2}) \phi = 0. \]

The Galilean transform is introduced to connect coordinates systems \((x, y, z)\) and \((X, Y, Z)\) (Zhou and Lee, 2012a), shown as follows

\[ X = x, \ Y = y, \ Z = z - Vt, \]

with
\[
\frac{\partial}{\partial X} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial Z} = \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial t} = -V \frac{\partial}{\partial Z}.
\]  

(12)

Denote \( c = V/c_{sw} \) as the relative moving velocity of the rigid sphere, where \( c_{sw} = \sqrt{c_{44}/\rho} \) is the shear wave velocity in the MEE film. Substituting Eq. (11) into Eqs. (6-10) leads to the following,

\[
\left[ c_{11} \frac{\partial^2 u_x}{\partial X^2} + c_{66} \frac{\partial^2 u_x}{\partial Y^2} + c_{44} (1-c^2) \frac{\partial^2 u_x}{\partial Z^2} \right] u_x + (c_{12} + c_{66}) \frac{\partial^2 u_y}{\partial X \partial Y} + (c_{13} + c_{44}) \frac{\partial^2 u_z}{\partial X \partial Z} + (e_{13} + e_{15}) \frac{\partial^2 \phi}{\partial X \partial Z} + (d_{31} + d_{15}) \frac{\partial^2 \phi}{\partial X \partial Z} = 0,
\]

(13)

\[
(c_{12} + c_{66}) \frac{\partial^2 u_y}{\partial X \partial Y} + [c_{66} \frac{\partial^2 u_y}{\partial X^2} + c_{11} \frac{\partial^2 u_y}{\partial Y^2} + c_{44} (1-c^2) \frac{\partial^2 u_y}{\partial Z^2}] u_y + (c_{13} + c_{44}) \frac{\partial^2 u_z}{\partial Y \partial Z} + (e_{15} + e_{31}) \frac{\partial^2 \phi}{\partial Y \partial Z} + (d_{15} + d_{31}) \frac{\partial^2 \phi}{\partial Y \partial Z} = 0,
\]

(14)

\[
(c_{13} + c_{44}) \left( \frac{\partial^2 u_x}{\partial X \partial Z} + \frac{\partial^2 u_y}{\partial Y \partial Z} \right) + [c_{44} \frac{\partial^2 u_z}{\partial X^2} + c_{44} \frac{\partial^2 u_z}{\partial Y^2}] u_z + (e_{15} + e_{13}) \frac{\partial^2 \phi}{\partial X \partial Z} + (d_{15} + d_{31}) \frac{\partial^2 \phi}{\partial X \partial Z} + (e_{11} + e_{33}) \frac{\partial^2 \phi}{\partial Y \partial Z} - (e_{11} \frac{\partial^2 \phi}{\partial X^2} + e_{33} \frac{\partial^2 \phi}{\partial Y^2} + g_{11} \frac{\partial^2 \phi}{\partial Z^2}) = 0,
\]

(15)

\[
(e_{15} + e_{31}) \frac{\partial^2 u_x}{\partial X \partial Z} + (e_{15} + e_{13}) \frac{\partial^2 u_y}{\partial X \partial Z} + (e_{15} + e_{33}) \frac{\partial^2 u_z}{\partial X \partial Z} - (e_{11} \frac{\partial^2 \phi}{\partial X^2} + e_{33} \frac{\partial^2 \phi}{\partial Y^2} + g_{11} \frac{\partial^2 \phi}{\partial Z^2}) = 0,
\]

(16)

\[
(d_{15} + d_{31}) \frac{\partial^2 u_x}{\partial Y \partial Z} + (d_{15} + d_{13}) \frac{\partial^2 u_y}{\partial Y \partial Z} + (d_{15} + d_{33}) \frac{\partial^2 u_z}{\partial Y \partial Z} - (g_{11} \frac{\partial^2 \phi}{\partial X^2} + g_{11} \frac{\partial^2 \phi}{\partial Y^2} + g_{33} \frac{\partial^2 \phi}{\partial Z^2}) = 0,
\]

(17)

\[2.3 \text{ General solutions}\]

Following the way in (Chen et al., 2004; Hu, 1953), two displacement functions, \( H \) and \( G \), are introduced to express displacements \( u_x \) and \( u_y \), namely,

\[
u x = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial x}, \quad u_y = -\frac{\partial H}{\partial x} + \frac{\partial G}{\partial y}.
\]

(18)

Substituting Eq. (18) into Eqs. (13-17) results in the governing equations in terms of the displacement functions, \( H \) and \( G \).
\[ [c_{66}A + c_{44}(1-c^2)] \frac{\partial^2}{\partial Z^2} H = 0, \]  
\[ [D][G \quad u_z \quad \phi \quad \varphi]^T = 0, \] 

where \([D]\) is a differential operator matrix given by,

\[
[D] = \begin{bmatrix}
   c_i A + c_{ii}(1-c^2) \frac{\partial^2}{\partial Z^2} & -(c_{ii} + c_{ii}) \frac{\partial}{\partial Z} & -(c_{ii} + e_{ii}) \frac{\partial^2}{\partial Z^2} & -(d_{ii} + d_{ii}) \frac{\partial}{\partial Z} \\
   -(c_{ii} + d_{ii}) \frac{\partial^2}{\partial Z^2} & c_{ii} A + (c_{ii} - c_{ii}) \frac{\partial^2}{\partial Z^2} & e_{ii} \frac{\partial^2}{\partial Z^2} & d_{ii} \frac{\partial^2}{\partial Z^2} \\
   (e_{ii} + e_{ii}) A \frac{\partial^2}{\partial Z^2} & -(e_{ii} A + e_{ii}) \frac{\partial^2}{\partial Z^2} & e_{ii} A + e_{ii} \frac{\partial^2}{\partial Z^2} & g_{ii} \frac{\partial^2}{\partial Z^2} \\
   (d_{ii} + d_{ii}) A \frac{\partial^2}{\partial Z^2} & -(d_{ii} A + d_{ii}) \frac{\partial^2}{\partial Z^2} & g_{ii} A + g_{ii} \frac{\partial^2}{\partial Z^2} & \mu_{ii} A + \mu_{ii} \frac{\partial^2}{\partial Z^2}
\end{bmatrix}
\]

with \(A = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\).

Another new displacement function, \(F\), is introduced; it satisfies the relation of \(|D|F = 0\). The determinant of matrix \([D]\) can be calculated, which leads to the following,

\[
(n_0 \frac{\partial^8}{\partial Z^8} + n_1 A \frac{\partial^6}{\partial Z^6} + n_2 A^2 \frac{\partial^4}{\partial Z^4} + n_3 A^3 \frac{\partial^2}{\partial Z^2} + n_4 A^4) F = 0
\]

where \(n_0, n_1, ..., n_4\) are given in Appendix A.

Therefore, displacement function \(G\), displacement \(u_z\), electric potential \(\phi\) and magnetic potential \(\varphi\) in Eq. (20) can be rewritten in terms of function \(F\),

\[
G = A_{ii} F; \quad u_z = A_{ii} F; \quad \phi = A_{ii} F; \quad \varphi = A_{ii} F, \quad (i = 1, 2, 3, 4).
\]

where \(A_{ii}\) are the cofactors of matrix \([D]\). Substituting Eq. (23) into Eq. (18) results in the following,

\[
u_x = \frac{\partial H}{\partial Y} - A_{ii} \frac{\partial F}{\partial X}, \quad u_y = -\frac{\partial H}{\partial X} - A_{ii} \frac{\partial F}{\partial Y}.
\]

Furthermore, substituting Eqs. (23) and (24) into Eq. (2) and then substituting the results into Eq. (1), the stresses, electric displacements and magnetic inductions can be expressed in terms of displacement functions \(H\) and \(F\).

On the other hand, Eq. (22) can be simplified with the assistance of the Vieta theorem for the property of roots, the results are
where $Z_i^2 = s_i^2 Z^2$, $(i = 1, 2, 3, \text{and } 4)$, and $s_i$ are determined by four of the roots of the following equation, whose real portions are positive:

$$n_0 s^8 - n_1 s^6 + n_2 s^4 - n_3 s^2 + n_4 = 0.$$  \hspace{1cm} (26)

Turning attention to Eq. (19), it can be rewritten as

$$[\Lambda + \frac{\partial^2}{\partial Z_i^2}] H = 0,$$  \hspace{1cm} (27)

where $Z_0^2 = s_0^2 Z^2$ with $s_0^2 = c_{66}/[c_{44}(1 - c^2)]$.

By utilizing the generalized Almansi’s theorem (Ding, 1996), the displacement function $F$ in Eq. (25) can be expressed in terms of four quasi-harmonic functions, given as follows,

Case 1: $F = F_1 + F_2 + F_3 + F_4$, $s_1 \neq s_2 \neq s_3 \neq s_4$;

Case 2: $F = F_1 + F_2 + F_3 + ZF_4$, $s_1 \neq s_2 \neq s_3 = s_4$;

Case 3: $F = F_1 + F_2 + ZF_3 + Z^2 F_4$, $s_1 \neq s_2 = s_3 = s_4$;

Case 4: $F = F_1 + ZF_2 + Z_2 F_3 + Z^2 F_4$, $s_1 = s_2 = s_3 = s_4$,

where Eq. (25) can be rewritten as

$$(\Lambda + \frac{\partial^2}{\partial Z_i^2}) F_i = 0 \forall (i = 1, 2, 3, 4).$$  \hspace{1cm} (29)

Considering a large amount of MEE materials belonging to Case 1 of Eq. (28) (Chen et al., 2010; Li et al., 2015b; Li et al., 2014; Pan, 2001), the present work is only confined to study Case 1, which is with distinct $s_i$. As suggested by Ding et al. (Ding, 1996), five harmonic functions $\psi_i$, which can be expressed as $\alpha_i s_i \partial^3 F_i / \partial Z_i^3 = \psi_i$ $(i = 1, 2, 3, 4$, and $H$ is written as $\psi_0)$, are introduced to further simplify the expressions (23, 24). Thus the general solutions in the case of distinct $s_i$ can be expressed in terms of $\psi_i$ as,

$$U = \nabla \left( \sum_{j=1}^{4} \psi_j + i \psi_0 \right), \quad w_m = \sum_{j=1}^{4} s_j k_m \psi_{j,z},$$

$$\sigma_1 = 2 \sum_{j=1}^{4} \left( c_{66} - s_j^2 \omega_j \right) \psi_{j,z,z}, \quad \sigma_2 = 2 c_{66} \nabla^2 \left( \sum_{j=1}^{4} \psi_j + i \psi_0 \right),$$

$$\sigma_{zm} = \sum_{j=1}^{4} \omega_m \psi_{j,z,z}, \quad \tau_{zm} = \nabla \left( \sum_{j=1}^{4} s_j \omega_m \psi_{j,z} + i s_0 \rho_m \psi_{0,z} \right), \quad (m = 1, 2, 3),$$

where
\[ U = u_x + i u_y, \quad w_1 = u_z, \quad w_2 = \phi, \quad w_3 = \varphi, \]

\[ \sigma_1 = \sigma_{xx} + \sigma_{yy}, \quad \sigma_2 = \sigma_{xx} - \sigma_{yy} + 2i \sigma_{xy}, \]

\[ \sigma_{z1} = \sigma_{xz}, \quad \sigma_{z2} = D_z, \quad \sigma_{z3} = B_z, \]

\[ \tau_{z1} = \sigma_{xz} + i \sigma_{yz}, \quad \tau_{z2} = D_z + iD_y, \quad \tau_{z3} = B_x + iB_y. \]

(31)

Note that the Einstein summation convention is not used for repeated index here; and \( \nabla = \partial / \partial X + i \partial / \partial Y \). Harmonic functions \( \psi_i \) satisfy the quasi-Laplace equation as follows,

\[ (\Lambda + \frac{\partial^2}{\partial Z_i^2})\psi_i = 0, \quad (i = 0, 1, ..., 4). \]

(32)

2.4 Frequency response functions and Influence coefficients

Because \( \psi_i \) in Eq. (32) are harmonic functions, double Fourier transforms of these functions can be expressed in terms of exponential functions with unknown coefficients, as suggested in (Wang et al., 2015; Yu et al., 2014). Hence, the following direct double Fourier transform and its inverse are introduced,

\[ \tilde{f}(m,n,Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(X,Y,Z) e^{-i(mX+nY)} dX dY, \]

\[ f(X,Y,Z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(m,n,Z) e^{i(mX+nY)} dmdn, \]

(33)

where hat “≈” denotes the double Fourier transform; \( m \) and \( n \) represent the transform variables with respect to the \( X \) and \( Y \) directions, respectively, but in the frequency domain.

Applying Fourier transforms to Eq. (32) results in the following,

\[ \tilde{\psi} = A_i(m,n)e^{-\alpha Z} + \bar{A}_i(m,n)e^{\alpha Z}, \quad (i = 0, ..., 4), \]

(34)

where \( \alpha = \sqrt{m^2 + n^2} \), \( A_i(m,n) \) and \( \bar{A}_i(m,n) \) are ten unknowns to be determined from the boundary conditions.

Applying Fourier transforms to Eq. (30) and substituting Eq. (34) into the results lead to the transformed displacements, electric potentials, magnetic potentials, stresses, electric displacements, and magnetic inductions, expressed as
\[
\tilde{u}_x = -iA_0 e^{-a_s z} - iA_0 e^{a_s z} + \sum_{j=1}^{4} A_j e^{-a_s z} + \sum_{j=1}^{4} \bar{A}_j e^{a_s z},
\]
\[
\tilde{u}_y = iA_0 e^{-a_s z} + iA_0 e^{a_s z} + \sum_{j=1}^{4} A_j e^{-a_s z} + \sum_{j=1}^{4} \bar{A}_j e^{a_s z},
\]
\[
\tilde{u}_z = -\sum_{j=1}^{4} s_j k_j \alpha A_j e^{-a_s z} + \sum_{j=1}^{4} s_j k_j \alpha \bar{A}_j e^{a_s z},
\]
\[
\tilde{\phi} = -\sum_{j=1}^{4} s_j k_j \alpha A_j e^{-a_s z} + \sum_{j=1}^{4} s_j k_j \alpha \bar{A}_j e^{a_s z},
\]
\[
\tilde{\psi} = -\sum_{j=1}^{4} s_j k_j \alpha A_j e^{-a_s z} + \sum_{j=1}^{4} s_j k_j \alpha \bar{A}_j e^{a_s z},
\]
\[
\tilde{\sigma}_{xx} = 2c_6 m n A_0 e^{-a_s z} + \sum_{j=1}^{4} [d \phi^2 + c n^2 m^2 A_j e^{-a_s z}]
\]
\[
+ 2c_6 m n \bar{A}_0 e^{a_s z} + \sum_{j=1}^{4} [d \phi^2 + c n^2 m^2 \bar{A}_j e^{a_s z}],
\]
\[
\tilde{\sigma}_{yy} = -2c_6 m n A_0 e^{-a_s z} + \sum_{j=1}^{4} [d \phi^2 - c n^2 m^2 A_j e^{-a_s z}]
\]
\[
- 2c_6 m n \bar{A}_0 e^{a_s z} + \sum_{j=1}^{4} [d \phi^2 - c n^2 m^2 \bar{A}_j e^{a_s z}],
\]
\[
\tilde{\sigma}_{zz} = \sum_{j=1}^{4} \omega_j \alpha^2 A_j e^{-a_s z} + \sum_{j=1}^{4} \omega_j \alpha^2 \bar{A}_j e^{a_s z},
\]
\[
\tilde{\sigma}_{xy} = c_{66} (n^2 - m^2) A_0 e^{-a_s z} - \sum_{j=1}^{4} 2c_6 m n A_j e^{-a_s z}
\]
\[
+ c_{66} (n^2 - m^2) \bar{A}_j e^{a_s z} - \sum_{j=1}^{4} 2c_6 m n \bar{A}_j e^{a_s z},
\]
\[
\tilde{\sigma}_{xz} = ins \rho \alpha A_0 e^{-a_s z} - \sum_{j=1}^{4} ins \omega_j \alpha A_j e^{-a_s z}
\]
\[
- ins \rho \alpha \bar{A}_0 e^{a_s z} + \sum_{j=1}^{4} ins \omega_j \alpha \bar{A}_j e^{a_s z},
\]
\[
\tilde{\sigma}_{yz} = -ins \rho \alpha A_0 e^{-a_s z} - \sum_{j=1}^{4} ins \omega_j \alpha A_j e^{-a_s z}
\]
\[
+ ins \rho \alpha \bar{A}_0 e^{a_s z} + \sum_{j=1}^{4} ins \omega_j \alpha \bar{A}_j e^{a_s z}
\]
\[ \tilde{D}_s = \text{ins}_0 \rho_2 \alpha A_s e^{-\alpha s Z} - \sum_{j=1}^{4} \text{ims} \omega_j \alpha A_j e^{-\alpha s Z} \]
\[ - \text{ins}_0 \rho_2 \alpha \tilde{A}_s e^{\alpha s Z} + \sum_{j=1}^{4} \text{ims} \omega_j \alpha \tilde{A}_j e^{\alpha s Z}, \]
\[ \tilde{D}_s = -\text{ins}_0 \rho_2 \alpha A_s e^{-\alpha s Z} - \sum_{j=1}^{4} \text{ins} \omega_j \alpha A_j e^{-\alpha s Z} \]
\[ + \text{ins}_0 \rho_2 \alpha \tilde{A}_s e^{\alpha s Z} + \sum_{j=1}^{4} \text{ins} \omega_j \alpha \tilde{A}_j e^{\alpha s Z}, \]
\[ \tilde{D}_s = \sum_{j=1}^{4} \omega_j \alpha^2 A_j e^{-\alpha s Z} + \sum_{j=1}^{4} \omega_j \alpha^2 \tilde{A}_j e^{\alpha s Z}, \]
\[ \tilde{\tilde{D}}_s = \text{ins}_0 \rho_3 \alpha A_s e^{-\alpha s Z} - \sum_{j=1}^{4} \text{ins} \omega_j \alpha A_j e^{-\alpha s Z} \]
\[ - \text{ins}_0 \rho_3 \alpha \tilde{A}_s e^{\alpha s Z} + \sum_{j=1}^{4} \text{ins} \omega_j \alpha \tilde{A}_j e^{\alpha s Z}, \]
\[ \tilde{\tilde{D}}_s = -\text{ins}_0 \rho_3 \alpha A_s e^{-\alpha s Z} - \sum_{j=1}^{4} \text{ins} \omega_j \alpha A_j e^{-\alpha s Z} \]
\[ + \text{ins}_0 \rho_3 \alpha \tilde{A}_s e^{\alpha s Z} + \sum_{j=1}^{4} \text{ins} \omega_j \alpha \tilde{A}_j e^{\alpha s Z}, \]
\[ \tilde{\tilde{D}}_s = \sum_{j=1}^{4} \omega_j \alpha^2 A_j e^{-\alpha s Z} + \sum_{j=1}^{4} \omega_j \alpha^2 \tilde{A}_j e^{\alpha s Z}. \]

Then applying the Fourier transform to Eqs. (4, 5) and substituting Eqs. (35-38) into the results lead to a series of boundary conditions in the Fourier transform domain,

\[ n \text{ns}_0 \rho_3 A_0 - n \text{ns}_0 \rho_1 \tilde{A}_0 - m \sum_{j=1}^{4} s_j \omega_j A_j + m \sum_{j=1}^{4} s_j \omega_j \tilde{A}_j = 0, \]
\[ -m \text{ns}_0 \rho_3 A_0 + m \text{ns}_0 \rho_1 \tilde{A}_0 - n \sum_{j=1}^{4} s_j \omega_j A_j + n \sum_{j=1}^{4} s_j \omega_j \tilde{A}_j = 0, \]
\[ \sum_{j=1}^{4} \omega_j A_j + \sum_{j=1}^{4} \omega_j \tilde{A}_j = -\tilde{p} / \alpha^2, \]
\[ \sum_{j=1}^{4} \omega_j A_j + \sum_{j=1}^{4} \omega_j \tilde{A}_j = -q / \alpha^2, \]
\[ \sum_{j=1}^{4} \omega_j A_j + \sum_{j=1}^{4} \omega_j \tilde{A}_j = -g / \alpha^2. \]
\begin{align}
ne^{-a_{j}kh}A_{0} + ne^{-a_{k}kh}A_{0} - \sum_{j=1}^{4} \left( me^{-a_{j}kh}A_{j} - \sum_{j=1}^{4} \left( me^{-a_{j}kh}A_{j} \right) \right) = 0, \tag{44} \\
me^{-a_{j}kh}A_{0} + me^{-a_{k}kh}A_{0} + \sum_{j=1}^{4} \left( ne^{-a_{j}kh}A_{j} + \sum_{j=1}^{4} \left( ne^{-a_{j}kh}A_{j} \right) \right) = 0, \tag{45} \\
\sum_{j=1}^{4} s_{j}k_{j}e^{-a_{j}kh}A_{j} - \sum_{j=1}^{4} s_{j}k_{j}e^{a_{j}kh}A_{j} = 0, \tag{46} \\
\sum_{j=1}^{4} s_{j}k_{j}e^{-a_{j}kh}A_{j} - \sum_{j=1}^{4} s_{j}k_{j}e^{a_{j}kh}A_{j} = 0, \tag{47} \\
\sum_{j=1}^{4} s_{j}k_{j}e^{-a_{j}kh}A_{j} - \sum_{j=1}^{4} s_{j}k_{j}e^{a_{j}kh}A_{j} = 0. \tag{48}
\end{align}

Solving the above equation set results in the following,

\begin{align}
A_{1} &= \left[ q^{(1)}h_{2}^{(2)} - q^{(2)}h_{2}^{(1)} \right] / \left[ h_{1}^{(1)}h_{2}^{(2)} - h_{1}^{(2)}h_{2}^{(1)} \right], \\
A_{2} &= \left[ q^{(2)} - h_{1}^{(2)}A_{1} \right] / h_{2}^{(2)}, \\
A_{3} &= \left[ p^{(3)} - \sum_{j=1}^{2} t^{(3)}_{j}A_{j} \right] / l_{3}^{(3)}, \\
A_{4} &= \left[ \sum_{j=1}^{3} k^{(4)}_{j}A_{j} \right] / k_{4}^{(4)}, \\
\bar{A}_{1} &= \left[ \sum_{j=1}^{5} s^{(5)}_{j}A_{j} \right] / s^{(5)}, \\
\bar{A}_{2} &= \left[ \sum_{j=1}^{4} t^{(1)}_{j}e^{-a_{j}h_{j}k_{2}}A_{j} - \sum_{j=1}^{4} t^{(1)}_{j}e^{a_{j}h_{j}k_{2}}\bar{A}_{j} \right] / \bar{t}_{2}^{(1)}, \\
\bar{A}_{3} &= \left[ \sum_{j=1}^{4} k_{j}e^{-a_{j}h_{j}k_{3}}A_{j} - \sum_{j=1}^{4} k_{j}e^{a_{j}h_{j}k_{3}}\bar{A}_{j} \right] / k_{3}, \\
\bar{A}_{4} &= \left[ \sum_{j=1}^{4} \left( e^{-a_{j}h_{j}k_{4}}A_{j} - \sum_{j=1}^{4} e^{a_{j}h_{j}k_{4}}\bar{A}_{j} \right) \right] / (mr_{0}), \\
\bar{A}_{0} &= \left[ \sum_{j=1}^{4} \left( ne^{-a_{j}h_{j}k_{4}}A_{j} - ne^{a_{j}h_{j}k_{4}}\bar{A}_{j} \right) - me^{-a_{j}h_{j}k_{4}}A_{0} \right] / (me^{a_{j}h_{j}k_{4}}A_{0}). \tag{49}
\end{align}

The constants, \(p, q, h, l, k, t\) and \(r\), in Eq. (49) are listed in Appendix B.

Accordingly, substituting coefficients \{\(A_{0}, \ldots, A_{4}, \bar{A}_{0}, \ldots, \bar{A}_{4}\)\} back into Eqs. (35-38) establishes the Fourier transformed solutions for the corresponding elastic displacements,
electric/magnetic potentials, stresses, electric displacements, and magnetic inductions.

The frequency response functions (FRFs) are related to material properties and a certain velocity, \( V \), can be described as the responses to unit point loads. The components of the FRFs can be analytically obtained by letting \( \hat{p} = 1 \), \( \hat{q} = 1 \) or \( \hat{g} = 1 \) in Eq. (35-38), which are denoted by

\[
\begin{bmatrix}
\tilde{G}_u^u \\
\tilde{G}_u^\phi \\
\tilde{G}_u^{\sigma_y} \\
\tilde{G}_u^{D_y} \\
\tilde{G}_u^B \\
\tilde{G}_g^u \\
\tilde{G}_g^\phi \\
\tilde{G}_g^{\sigma_y} \\
\tilde{G}_g^{D_y} \\
\tilde{G}_g^B \\
\end{bmatrix} = \begin{bmatrix}
\tilde{G}_u^{\hat{p}=1} \\
\tilde{G}_u^{\hat{q}=1} \\
\tilde{G}_u^{\hat{g}=1} \\
\tilde{G}_g^{\hat{p}=1} \\
\tilde{G}_g^{\hat{q}=1} \\
\tilde{G}_g^{\hat{g}=1} \\
\end{bmatrix}
\]

where \( \tilde{G}_u^u \) denotes the FRFs of \( u \) excited by load \( p \), and \( \tilde{G}_g^\phi \) represents the FRFs of magnetic potential \( \phi \) caused by pressure \( p \) and so on.

Once the FRFs are obtained, the continuous Fourier transforms of influence coefficients (ICs) can be solved by multiplying the FRFs and the Fourier transformed shape functions (Liu and Wang, 2002),

\[
\begin{bmatrix}
\tilde{C}_u^u \\
\tilde{C}_u^\phi \\
\tilde{C}_u^{\sigma_y} \\
\tilde{C}_u^{D_y} \\
\tilde{C}_u^B \\
\tilde{C}_g^u \\
\tilde{C}_g^\phi \\
\tilde{C}_g^{\sigma_y} \\
\tilde{C}_g^{D_y} \\
\tilde{C}_g^B \\
\end{bmatrix} \cdot \tilde{F}_s = \begin{bmatrix}
\tilde{G}_u^{\hat{p}=1} \\
\tilde{G}_u^{\hat{q}=1} \\
\tilde{G}_u^{\hat{g}=1} \\
\tilde{G}_g^{\hat{p}=1} \\
\tilde{G}_g^{\hat{q}=1} \\
\tilde{G}_g^{\hat{g}=1} \\
\end{bmatrix}
\]

where \( \tilde{F}_s \) is the shape function given as follows

\[
F_s(X, Y) = \begin{cases} 
1, & |X| \leq \Delta_x / 2 \text{ and } |Y| \leq \Delta_y / 2, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\tilde{F}_s(m, n) = \frac{4 \sin(m \Delta_x / 2) \sin(n \Delta_y / 2)}{mn}.
\]

Then the discrete Fourier transforms of ICs can be further calculated from the continuous ICs with the following aliasing treatments (Liu and Wang, 2002; Wang et al., 2015),

\[
\hat{C} = \frac{1}{\Delta_x \Delta_y} \sum_{r_x=-\text{AL}}^{r_x=\text{AL}} \sum_{r_y=-\text{AL}}^{r_y=\text{AL}} \tilde{C} \left(-\frac{2 \pi}{M_x \Delta_x} \frac{r_x}{\Delta_x} - \frac{2 \pi}{N_y \Delta_y} j \frac{2 \pi}{\Delta_y} r_y, \frac{2 \pi}{N_y \Delta_y} - \frac{2 \pi}{M_x \Delta_x} i \right),
\]

where \( \Delta_x \) and \( \Delta_y \) are the mesh sizes along the \( X \) and \( Y \) directions, respectively; \( \text{AL} \) represents the level of the aliasing control; \( M_e = 2^\gamma M \) and \( N_e = 2^\gamma N \) denote the refinement mesh numbers of the original mesh numbers \( M \) and \( N \); \( C \) is the ICs in the form of the discrete
Furthermore, the displacement, stress, electric/magnetic potential, electric displacement, and magnetic induction can be obtained by utilizing the same discrete convolution-fast Fourier transform (DC-FFT) algorithm for pressure, surface electric charge, and surface magnetic charge by means of the corresponding components of ICs (Liu et al., 2007; Liu and Wang, 2002) with IFFT being the inverse fast Fourier transform:

\[ u_i = \text{IFFT}(\hat{C}_u \cdot (\hat{p} \hat{q} \hat{g})) = \text{IFFT}(\hat{C}_{u_{p_{1}}} \hat{p} \hat{q} \hat{g}), \quad (54) \]

\[ \phi = \text{IFFT}(\hat{C}_\phi \cdot (\hat{p} \hat{q} \hat{g})) = \text{IFFT}(\hat{C}_{\phi_{p_{1}}} \hat{p} \hat{q} \hat{g}), \quad (55) \]

\[ \varphi = \text{IFFT}(\hat{C}_\varphi \cdot (\hat{p} \hat{q} \hat{g})) = \text{IFFT}(\hat{C}_{\varphi_{p_{1}}} \hat{p} \hat{q} \hat{g}), \quad (56) \]

\[ \sigma_{ij} = \text{IFFT}(\hat{C}_{\sigma_{ij}} \cdot (\hat{p} \hat{q} \hat{g})) = \text{IFFT}(\hat{C}_{\sigma_{ij_{p_{1}}} \hat{p} \hat{q} \hat{g}}), \quad (57) \]

\[ D_i = \text{IFFT}(\hat{C}_D \cdot (\hat{p} \hat{q} \hat{g})) = \text{IFFT}(\hat{C}_{D_{p_{1}}} \hat{p} \hat{q} \hat{g}), \quad (58) \]

\[ B_i = \text{IFFT}(\hat{C}_B \cdot (\hat{p} \hat{q} \hat{g})) = \text{IFFT}(\hat{C}_{B_{p_{1}}} \hat{p} \hat{q} \hat{g}). \quad (59) \]

### 2.5 Numerical modeling

Now the working progress of the MEE energy converter system shown in Fig. 1(a), under an varying applied force \( P(t) \) along the z axis can be solved. The loading velocity of the rigid sphere is assumed to be a function of loading time \( t \), which can be expresses as either

\[ V = \begin{cases} 2V_0 \cdot t/T_{\text{total}}, & \text{for } 0 \leq t < T_{\text{total}}/2, \\ 2V_0 \cdot (1-t/T_{\text{total}}), & \text{for } T_{\text{total}}/2 \leq t < T_{\text{total}}, \end{cases} \quad (60) \]

or

\[ V = V_0 \sin(\pi t / T_{\text{total}}). \quad (61) \]

for the triangular and sinusoidal variations, respectively, where \( V_0 \) denotes the maximum value of the loading velocity; \( T_{\text{total}} \) represents the total time during one loading cycle.

The working progress of the MEE system depends on the loading history, for which the dynamic contact problem is solved first. For a certain time \( t \), the following contact conditions for displacement, pressure, and load are applicable to such a dynamic contact problem,

\[ \bar{u}_z = g_z - h_0, \quad (62) \]

\[ p' > 0 \& g_z' = 0 \Rightarrow \forall (x, y) \in \Gamma_c, \]

\[ p' = 0 \& g_z' > 0 \Rightarrow \forall (x, y) \notin \Gamma_c, \quad (63) \]
where \( \bar{u}_z^t \) is the surface normal displacement for the MEE film; \( g_z^t \) is the gap between the surfaces of the rigid sphere and the MEE film; \( \Gamma_c \) is the contact zone; \( h_0^t \) is the initial vertical separation; \( M_i \) is the mass of the sphere; \( a^t \) is the acceleration of the rigid sphere at time \( t \); and \( P^t \) represents the applied force at time \( t \). For the velocity type given by Eq. (60), \( a^t \) equals \( 2V_0/T_{total} \) for \( 0 < t < T_{total}/2 \) and equals \( -2V_0/T_{total} \) for \( T_{total}/2 < t < T_{total} \). For the velocity type given by Eq. (61), \( a^t = \frac{[\sin(2\pi(t + \Delta t)/T_{total}) - \sin(2\pi(t/T_{total}))]}{\Delta t} \). It should be mentioned that the applied force \( P^t \) gradually increases with indentation depth. Thus for a given indentation depth \( \delta^t \) at time \( t \), \( P^t \) should be determined in an iteration process, as described below.

In the numerical process, the DC-FFT algorithm described in (Liu et al., 2007; Zhang et al., 2018) is applied to solve Eqs. (54-59) in order to obtain the displacement component \( u_r \), electric potential \( \phi \), magnetic potential \( \varphi \), elastic stress tensor \( \epsilon_{ij} \), electric displacement vector \( D_i \), and magnetic induction vector \( B_i \). Once the surface displacement \( \bar{u}_z \) is obtained, gap \( g^t \) in Eq. (62) can be solved. Pressure \( P^t \) in Eq. (63) is calculated with the conjugate gradient method (CGM) described in (Polonsky and Keer, 1999). Figure 2 shows the flow diagram for the numerical approach, in which,

1. Initial parameters are input, including material properties, mesh size and geometry information, etc. The initial surface electric/magnetic charges, applied force, loading time are set to zero.
2. The loading time, \( t \), is increased step by step. In each increment step \( \Delta t \), the accumulated time is updated.
3. At time \( t \), the velocity of the rigid sphere is calculated based on Eqs. (60, 61). Moreover, the ICs should be prepared in advance based on the material properties and the current velocity.
4. The accumulated indentation depth is calculated based on \( \delta^t = \delta^{t-1} + V(t) \cdot \Delta t \).
5. In current indentation depth, \( \delta^t \), the total force, \( P^t \), is applied step by step. In each increment step \( \Delta P \), pressure \( P^t \) is calculated using the CGM.

The CGM implementation and the iteration procedure involve: 1) estimation of pressure distribution \( P^t \); 2) calculation of surface normal displacement \( \bar{u}_z^t \) in Eq. (54) by means of the DC-FFT algorithm; 3) acquisition of gap \( g_z^t \) by Eq. (62); 4) updates of pressure \( P^t \) in step \( t \) based on the descent direction, \( d_{egm} \), and the step length, \( a_{egm} \); 5) implementation of Eq. (64) to confirm that the applied force has been enforced; and 6) convergence tests in the iteration steps, 2-5, until the required convergence condition is satisfied.

6. Once the new value of \( P^t \) is obtained, the surface normal displacement, \( u_{z0} \), at the contact center, namely the calculated indentation depth, can be solved by Eq. (54).
7. Iteration steps (5, 6) are continued until the calculated \( u_{z0} \) at time \( t \) approaches \( \delta^t \). At
At this point, the applied force $P_t$ at time $t$ for the current indentation depth $\delta^t$ is obtained. Moreover, the current pressure distribution $p^t$ can be solved.

(8) The accumulated surface electric charge $q^t$ and magnetic charge $g^t$, which equal the normal electric displacement $D_z$ and magnetic induction $B_z$ at the contact surface, respectively, can be solved based on pressure $p^t$, surface electric charge $q^{t-1}$, and surface magnetic charge $g^{t-1}$ by Eqs. (58, 59). Furthermore, loading velocity $V$, applied force $P$, pressure $p$, surface electric charge $q$, surface magnetic charge $g$, surface electric potential $\phi$, and surface magnetic potential $\varphi$ are recorded at time $t$.

(9) Iteration steps (2)-(8) are continued until the total loading time, $T_{\text{total}}$, is reached. At this point, the working progress of the MEE energy conversion during one loading cycle is completed, in which the electric potential generated at the top surface of the MEE film should drive the electron flow through the circuit.

Fig. 2. Flow diagram for modeling the working progress of the MEE energy conversion.

3. Results and discussion

The working progress of the MEE conversion of the dynamic contact energy the rigid sphere on the top surface of the transversely isotropic multiferroic MEE film, described as the system in Fig. 1(a) under an increasing applied force $P(t)$ along the $z$ axis, was solved based on the efficient SAM approach derived in the previous section. A fixed computational domain of $4a_0 \times 4a_0 \times h_t$ with $256 \times 256 \times 64$ grids in the $X$, $Y$ and $Z$ directions was used, where $a_0$ is the contact radius, to be discussed later. The multiferroic MEE film is made of the magnetostrictive (CoFe$_2$O$_4$) and piezoelectric (BaTiO$_3$) phases whose material properties were from (Chen et al., 2010; Pan, 2001) with distinct $s_k$, as given in Table 1. The
rigid sphere is made of steel whose mass density $\rho_s$ equals 7.85g/cm$^3$, which leads to the mass of the sphere, $M_s = 4\pi\rho_s R^3/3 = 3.29 \times 10^{-8}$g.

In one loading cycle of the velocity-induced MEE energy conversion, the loading velocity of the sphere increases from zero to the maximum, $V_0$, then decreases to zero following two loading types given in Eqs. (60) and (61). At the end of a loading cycle, the electric potential generated at the film surface was calculated. The working conditions of the MEE energy conversion are given in Table 2, showing that the term $M_s a^t$ in Eq. (64) is much smaller than $P_0$; Hence in the following numerical calculation, the effect of acceleration $a^t$ on the contact behaviors are ignored.

### Table 1. Material properties of the transversely isotropic multiferroic BaTiO$_3$-CoFe$_2$O$_4$ (Chen et al., 2010; Pan, 2001). ($c_{ij}$ in $10^9$N/m$^2$, $\varepsilon_{ij}$ in C/m$^2$, $d_{ij}$ in N/(Am), $\mu_{ij}$ in $10^6$Ns$^2$/C$^2$).

<table>
<thead>
<tr>
<th>$c_{11}$</th>
<th>$c_{12}$</th>
<th>$c_{13}$</th>
<th>$c_{33}$</th>
<th>$c_{44}$</th>
<th>$c_{66}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>226</td>
<td>125</td>
<td>124.25</td>
<td>216</td>
<td>44.15</td>
<td>50.5</td>
</tr>
<tr>
<td>$\varepsilon_{11}$</td>
<td>$\varepsilon_{12}$</td>
<td>$\varepsilon_{13}$</td>
<td>$d_{31}$</td>
<td>$d_{32}$</td>
<td>$d_{33}$</td>
</tr>
<tr>
<td>-2.2</td>
<td>9.3</td>
<td>5.8</td>
<td>290.15</td>
<td>350</td>
<td>275</td>
</tr>
<tr>
<td>$\xi_{11}$</td>
<td>$\xi_{33}$</td>
<td>$\mu_{11}$</td>
<td>$\mu_{33}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.64</td>
<td>6.35</td>
<td>297.5</td>
<td>83.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 2. Working conditions of the MEE energy conversion ($\xi_{11} = 2.03552 \times 10^{-12}$m$^2$/N, $\xi_{12} = 1.77480 \times 10^{-3}$m$^2$/C, and $\xi_{13} = 1.93900 \times 10^{-6}$mA/N are calculated from current material properties (Chen et al., 2010); an applied force of $P_0 = 1.8 \times 10^{-4}$N is selected as a reference value).

<table>
<thead>
<tr>
<th>Loading velocity $c_0$ ($c_0 = V_0/\sqrt{\varepsilon_s/\rho_s}$)</th>
<th>Sphere radius $R_0$</th>
<th>$1 \times 10^{-6}$ m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere radius $R$</td>
<td>1.0$R_0$, 2.0$R_0$, 3.0$R_0$</td>
<td>Contact radius $a_0$</td>
</tr>
<tr>
<td>Film thickness $h_t$</td>
<td>1.0$a_0$, 2.0$a_0$, 5.0$a_0$</td>
<td>Indentation depth $\delta_0$</td>
</tr>
<tr>
<td>Loading time at one cycle $T_{total}$</td>
<td>0.8$\delta_0 / c_{sw}$</td>
<td>Contact pressure $\rho_0$</td>
</tr>
<tr>
<td>Magnetic potential $\varphi_0$</td>
<td>$\xi_{13} \delta_0 / \xi_{11}$</td>
<td>Electric potential $\phi_0$</td>
</tr>
</tbody>
</table>

### 3.1 Model verification and general MEE transient behaviors

#### 3.1.1 Model verification

The model is verified through the comparison of the solutions from the present model at $h_t \to \infty$ & $V = 0$ with the corresponding analytical solutions from Chen et al. (2010), for the problem of a MEE half-space indented by a rigid insulating sphere of radius $R=1 \times 10^{-6}$m under applied force $P=2 \times 10^{-4}$N. The results are normalized by the maximum
pressure, $p_0$, the maximum electric potential, $\phi_0$, the maximum magnetic potential, $\varphi_0$, and the contact radius, $a_0$, from the framework of solutions by Chen et al. (2010). The results in Fig. 3 (a-c) clearly reveal that the normalized contact pressure, surface electric potential, and surface magnetic potential well agree with those from reference (Chen et al., 2010).

![Graphs showing normalized contact pressure, surface electric potential, and surface magnetic potential](image)

Fig. 3. Verification of the present model through degeneration to a half space and the comparison of the numerical results with the corresponding analytical solutions from Chen et al. (2010), for the problem of a MEE half-space indented by a rigid insulating sphere of radius $R=1\times10^{-6}$ m under applied force $P=2\times10^{-4}$ N: (a) normalized contact pressure $p/p_0$; (b) normalized surface electric potential $\phi/\phi_0$; (c) normalized surface magnetic potential $\varphi/\varphi_0$.

The model is further verified through the comparison of the present numerical results by letting $V = 0$ with the corresponding numerical results from our previous work (Zhang et al., 2017), for the problem of a MEE film indented by a rigid insulating sphere of radius $R=1\times10^{-6}$ m under applied force $P=2\times10^{-4}$ N. Three film thicknesses, $h_t=1.0a_0, 2.0a_0$, and $5.0a_0$, are selected as examples. Figure 4 (a) shows the normalized contact pressure distributions, $p/p_0$, with respect to $h_t$, while (b) and (c) plot the surface electric potential distributions, $\phi/\phi_0$, and surface magnetic potential distributions, $\varphi/\varphi_0$. Clearly, $p/p_0$, $\phi/\phi_0$, and $\varphi/\varphi_0$
from the present method well agree with those from reference (Zhang et al., 2017).

Fig. 4. Verification of the present model through degeneration with $V=0$ and the comparison of the numerical results with those from (Zhang et al., 2017), for the problem of a MEE film indented by a rigid insulating sphere of radius $R=1 \times 10^{-6}$ m under applied force $P=2 \times 10^{-4}$ N: (a) normalized contact pressure $p/p_0$; (b) normalized surface electric potential $\phi/\phi_0$; (c) normalized surface magnetic potential $\varphi/\varphi_0$.

3.1.2 General MEE transient behaviors

The model is first implemented to study the general MEE transient behaviors for the problem of a MEE film of $h_i=1.0a_0$ indented by a rigid insulating sphere of radius $R=1 \times 10^{-6}$ m under the applied force of $P=2 \times 10^{-4}$ N. Four constant velocities, $c=0$, 0.4, 0.6 and 0.8, are selected as examples. Figure 5 shows the effects of the loading velocity $c$ on the general MEE transient behaviors, where (a), (b) and (c) are for the normalized contact pressure distributions, $p/p_0$, the surface electric potential distributions, $\phi/\phi_0$, and surface magnetic potential distributions, $\varphi/\varphi_0$, respectively. The contact pressure is relieved with the
increasing loading velocity of the sphere, which is also observed in (Zhou and Lee, 2012b) for the contact of a MEE half-plane indented by a moving cylindrical punch. The results show that a larger \( c \) leads to higher surface electric potentials \( \phi/\phi_0 \) and surface magnetic potentials, \( \varphi/\varphi_0 \).

Fig. 5. General MEE transient behaviors, for the MEE film of \( h_t=1.0a_0 \) indented by a rigid insulating sphere of radius \( R=1 \times 10^{-6} \) m under the applied force of \( P=2 \times 10^{-4} \) N, for different velocity values of \( c=0, 0.4, 0.6 \) and 0.8: (a) normalized contact pressure \( p/p_0 \); (b) normalized surface electric potential \( \phi/\phi_0 \); and (c) normalized surface magnetic potential \( \varphi/\varphi_0 \).

3.2 Dynamic characterization of surface MEE contact behaviors

3.2.1 Effects of loading velocity on contact behaviors

The dynamic contact behaviors of the multiferroic film can be characterized by a transient indentation of a rigid insulating sphere under an increasing applied force \( P(t) \) along the \( z \) axis. Figure 6 shows the types of loading velocity, where (a) and (b) are for the velocities of triangular and sinusoidal shapes, respectively. The influence of loading velocity on the dynamic contact behaviors is explored, while the film thickness is \( h_t=1.0a_0 \) and the sphere radius is \( R=1.0R_0 \) without losing generality. The total loading time \( T_{\text{total}} \) and indentation depth \( \delta \) are fixed, for which \( c_0 \) should be carefully selected to satisfy the same indentation depth \( \delta \). Hence, for the velocity of triangular shape, maximum value \( c_0=0.4, 0.6 \)
and 0.8 are selected while maximum value \( c_0 = 0.4\pi/4, 0.6\pi/4 \) and \( 0.8\pi/4 \) are chosen for the velocities in sinusoidal variations. Figure 7 shows the dynamic contact behaviors with respect to increasing loading time for different \( c_0 \) and velocity types, where (a), (b), (c) and (d) plot the applied force \( P \), normalized pressure \( p/p_0 \) at the contact center, maximum values of normalized electric potential \( \phi/\phi_0 \) and magnetic potential \( \varphi/\varphi_0 \), respectively. In general, a larger \( c_0 \) leads to larger contact characteristic values. Furthermore, these characteristic items under the triangular contact variation are slightly smaller than those under the sinusoidal contact variation in the early stage of the loading cycle, while the trend reverses in the later stage.

Fig. 6. Types of loading velocity: (a) velocity of triangular shape; (b) velocity of sinusoidal shape. \( c_0 \) is the maximum value of the velocity. Note that the areas under corresponding curves are the same.

(a) \hspace{0.5cm} (b)
3.2.2 Effects of the MEE film thickness on contact behaviors

The same problem is investigated for the MEE film with different thicknesses $h_t=1.0a_0$, 2.0$a_0$, 5.0$a_0$, and the sphere radius of $R=1.0R_0$. Because Fig. 7 confirms that the loading velocity types have negligible influences on contact behaviors, the triangular velocity with maximum value $c_0=0.8$ is selected for these analyses. Figure 8 shows the results with respect to increasing loading time for different film thickness $h_t$, where (a), (b), (c) and (d) plot the applied force $P$, normalized pressure $p/p_0$ at the contact center, and the maximum values of normalized electric potential $\phi/\phi_0$ and magnetic potential $\varphi/\varphi_0$, respectively. Obviously, the applied force and the maximum values of the contact pressure decrease with increasing film thickness $h_t$. On the other hand, a thicker film thickness leads to a larger electric/magnetic potential at the contact center.
Fig. 8. Dynamic contact behaviors of the MEE films with different thicknesses, $h_t$: (a) applied force $P$; (b) normalized pressure $p/p_0$ at the contact center; (c) maximum values of normalized electric potential $\phi/\phi_0$; (d) maximum values of normalized magnetic potential $\varphi/\varphi_0$. Results are obtained from the velocity of the triangular shape.

3.2.3 Effects of sphere radius on the contact behaviors

In this section, the dynamic contacts of the MEE film thickness of $h_t=1.0a_0$ with the rigid insulating spheres of different sizes, $R=1.0R_0$, $2.0R_0$ and $3.0R_0$, are analyzed under the same triangular velocity with $c_0=0.8$ mentioned before. Figure 9 shows the dynamic contact behaviors of the MEE thin film with respect to increasing loading time for different sphere radius $R$, where sets (a), (b), (c) and (d) plot the applied force $P$, normalized pressure $p/p_0$ at the contact center, maximum values of normalized electric potential $\phi/\phi_0$ and magnetic potential $\varphi/\varphi_0$, respectively. Clearly, increasing sphere radius $R$ causes the applied force to increase, but the maximum values of the contact pressure decreases. On the other hand, a larger sphere radius leads to a higher electric/magnetic potential at the contact center.
Fig. 9. Dynamic contact behaviors influenced by sphere radius $R$: (a) applied force $P$; (b) normalized pressure $p/p_0$ at the contact center; (c) maximum values of normalized electric potential $\phi/\phi_0$; (d) maximum values of normalized magnetic potential $\varphi/\varphi_0$. Results are obtained from the triangular velocity.

3.3 Sub-surface stress analysis

The von Mises stress is important to understand the plastic behaviors of the materials under dynamic contact, which is defined by

$$
\sigma_v = \sqrt{\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 \right] + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)}/2.
$$

(65)

Figure 10(a) plots the distributions of the von Mises stress along the $z$-axis in the MEE film, for the problem of triangular velocity with $c_0=0.6$, film thickness $h_t=1.0a_0$, and sphere radius $R=1.0R_0$. The insets display the contours of the von Mises stress $\sigma_v/p_0$ in the $x$-$z$ section for time $t=7.2532$. As loading time $t$ increases, the values of the stress become larger, and the location of the maximum von Mises stress departs further away from the contact surface. It is found that $t=3.6266$, corresponding to peak velocity $c_0=0.6$, leads to a significant sudden change in the von Mises stress, as shown in Fig. 10(a). In order to understand this phenomenon, the distributions of the von Mises stress caused by the sinusoidal loading velocity with $c_0=0.6$ are plotted in Fig. 10(b) for comparison, in which similar sudden changes are found in the curves for velocity $c=0.5543$ at $t=2.7200$, for velocity $c=0.6$ at $t=3.6266$, and for velocity $c=0.5543$ at $t=4.5332$. Besides, the distributions of the von Mises stress for the problem of triangular velocity with $c_0=0.4$ is plotted in Fig. 10(c), where the sudden change vanishes. Furthermore, the distributions of the von Mises stress at $t=3.6266$ for the problem of triangular-shape loading velocity with different peak velocity $c_0=0.4$, 0.45, 0.5, 0.55 and 0.6 are plotted in Fig. 10(d), revealing that the loading velocity is responsible of sudden change in the von Mises stress along the $z$ direction, and that this sudden change becomes more significant if the peak velocity $c_0$ is higher.
Fig. 10. Distributions of the von Mises stress along the z-axis in the MEE film: (a) \( c_0 = 0.6 \) for the triangular-shape loading velocity, (b) \( c_0 = 0.6 \) for the sinusoidal-shape loading velocity, (c) \( c_0 = 0.4 \) for the triangular-shape loading velocity, (d) at \( t = 3.6266 \) under the triangular-shape loading velocity with different peak velocities \( c_0 = 0.4, 0.45, 0.5, 0.55 \) and 0.6. Film thickness \( h = 1.0a_0 \), and sphere radius \( R = 1.0R_0 \). The insets display the contours of the von Mises stress \( \sigma_s/p_0 \) in the x-z section for time \( t = 7.2532 \).

The variations of the maximum values of \( \sigma_s/p_0 \) with loading time \( t \) are plotted in Fig. 11 for clarity. The insets display the contours of the von Mises stress \( \sigma_s/p_0 \) in the x-z section at different loading time \( t \). The maximum value of \( \sigma_s/p_0 \) increases with time. A thicker film results in a smaller value of \( \sigma_s/p_0 \), as shown in Fig. 11 (a). On the other hand, a smaller sphere radius \( R \) induces a larger value of \( \sigma_s/p_0 \), as shown in Fig. 11 (b).
3.4 Energy conversion performance

Sun and Kim (2010) proposed an energy conversion factor to evaluate the performance of MEE energy conversion as a natural extension of the work by Zheng et al. (2009) on piezoelectric energy harvest. Inspired by their study, the work $W^P$ done by the applied force $P$ is stored in the forms of strain energy $\Pi^S$, electric energy $\Pi^E$, magnetic energy $\Pi^M$, piezoelectric energy $\Pi^{PE}$, piezomagnetic energy $\Pi^{PM}$, and magnetoelectric energy $\Pi^{ME}$, which can be expressed as follows without considering thermal and plastic losses:

$$
\Pi^S = \frac{1}{2} \int \gamma_{ij} \varepsilon_{ij} \varepsilon_{kl} \gamma_{kl} \, dV, \quad \Pi^E = \frac{1}{2} \int E_i^T \varepsilon_{ij} E_j \, dV, \quad \Pi^M = \frac{1}{2} \int H_i^T \mu_{ij} H_j \, dV,
$$

$$
\Pi^{PE} = \frac{1}{2} \int \gamma_{ij} \varepsilon_{ij} \varepsilon_{kl} \gamma_{kl} \, dV, \quad \Pi^{PM} = \frac{1}{2} \int \gamma_{ij} \mu_{ij} H_j \, dV, \quad \Pi^{ME} = \frac{1}{2} \int E_i^T \mu_{ij} H_j \, dV.
$$

(65)

Furthermore, the work, $W^P$, done by the applied force, $P$, can be written as

$$
W^P = P \cdot \delta,
$$

(66)

where $\delta$ denotes the indentation depth of the rigid sphere on the MEE film by the applied force $P$.

Therefore, the energy conversion factor $\eta_E$ from mechanical energy to electric energy (we call it electric energy conversion factor later) can be defined as

$$
\eta_E = \frac{\Pi^E}{W^P}.
$$

(67)
In this study, we are focusing on the evaluation of the performance of the MEE film-based energy converter, in which the MEE film can be described as an energy converter to utilize the electric potential to drive the electron flow through the circuit, rather than exactly determining the amount of the generated electric energy. The finite element analysis by Xu and Qin (2017) shows the electric potential linearly varies along the length direction of the nanowires. Reasonably, assuming the electric potential linearly varies along the z axis of the MEE film, and the electric energy $\Pi^E$ can be approximately calculated by,

$$\Pi^E = \frac{1}{2h_t} \int \phi \cdot \varepsilon_{33} \cdot \phi dA_c,$$

where $\phi$ represents the electric potential at the top surface of the MEE film, $A_c$ is the in-plane ($x$-$y$ section) area of the interaction domain. The electric potential in the lower surface of the MEE film is zero due to its grounding.

Figure 12 shows the electric energy conversion factor $\eta_E$ with respect to varying film thickness and sphere radius, for the problem of triangular loading velocity with $c_v=0.6$. Clearly, a thicker film $h_t$ results in a smaller value of $\eta_E$. On the other hand, a smaller sphere radius $R$ induces a larger value of $\eta_E$.

Figure 12 offers an optimization design scheme for the overall electric energy conversion efficiency with respect to film thickness and indenter size. Another critical issue is how the efficiency can be improved from material design. The material properties listed in Table 1 has 16 independent constituent constants, including six elastic stiffnesses $c_{ij}$, three piezoelectric coefficients $e_{ij}$, three piezomagnetic coefficients $d_{ij}$, two dielectric permittivities
\( \varepsilon_{ij} \) and two magnetic permeabilities \( \mu_{ij} \). It is important to evaluate coupling effects and dependence of the electric energy conversion factor \( \eta_E \) on those material properties. Furthermore, the magnetic energy conversion factor \( \eta_M = \Pi^M/W^P \) is also calculated for the connection between the electric and magnetic energies. The sensitivity function suggested by Makagon et al. (2007), (2009) is employed to evaluate the coupling effects of those material properties on the energy conversion rate. The dimensionless sensitivity below is numerically calculated with the step length of 0.01:\[
S(f) = \frac{\eta|_{f=0.01f^0} - \eta|_{f=0.99f^0}}{0.02\eta|_{f=f^0}} \tag{69}
\]
where \( f \) is a selected material constant with \( f^0 \) being its reference value. \( \eta \) is an energy conversion factor, which can be electric energy conversion factor \( \eta_E \), and magnetic energy conversion factor \( \eta_M \).

Figure 13 plots the sensitivities of energy conversion factors, \( \eta_E \) and \( \eta_M \), to the 16 independent material constants, for the problem of film thickness \( h = 1.0a_0 \), sphere radius \( R = 1.0R_0 \), and triangular loading velocity of \( v_0 = 0.6 \). The energy conversion factors show obvious dependences on the elastic stiffnesses \( c_{11}, c_{13}, c_{33}, c_{44} \), dielectric permittivity \( \varepsilon_{11} \), piezoelectric coefficients \( e_{33}, e_{15} \), piezomagnetic coefficients \( d_{ij} \), and magnetic permeability \( \mu_{11} \), but they are less sensitive to elastic constant \( c_{12} \), piezoelectric coefficient \( e_{33} \), dielectric permittivity \( \varepsilon_{33} \), and magnetic permeability \( \mu_{33} \). \( c_{11}, c_{13}, c_{33}, e_{33}, e_{15} \) promote the electric energy conversion from mechanical energy to electric energy while \( c_{33}, c_{44}, e_{11} \) tend to confine the electric energy conversion. Besides, \( c_{11}, d_{33}, \) and \( d_{15} \) promote the increase in magnetic energy conversion from mechanical energy while \( c_{13}, c_{33}, c_{44}, e_{33}, e_{15}, d_{31}, \) and \( \mu_{11} \) tend to reduce the magnetic energy conversion. The above observations are stated in details below, which are consistent with the physical nature and the reported publications for the current problems.

(i) For larger tangential dielectric permittivity \( \varepsilon_{11} \) and magnetic permeability \( \mu_{11} \), the dielectric and magnetic effects spread much more laterally, which reduce the electric/magnetic energy generations at the contact center. On the other hand, for larger normal dielectric permittivity \( \varepsilon_{33} \) and magnetic permeability \( \mu_{33} \), the dielectric and magnetic effects are concentrated beneath the contact center, which makes the electric/magnetic energy generations larger. But those increases in electric/magnetic energy are not very significant when compared to the reduced amounts of electric/magnetic energy due to the influences of \( \varepsilon_{11} \) and \( \mu_{11} \). Therefore, dielectric permittivities and magnetic permeabilities can be singled out for their abilities to reduce the efficiency of the energy conversion, which are in agreement with the observations reported in most publications about piezoelectric/magnetoelastic issues (Umeda et al., 1996, 1997; Zhang et al., 2018).
(ii) It was reported in (Sodano et al., 2004) that the efficiency of electric energy conversion increased with the increasing piezoelectric coefficients. In the numerical results presented in Fig. 13, piezoelectric coefficient $e_{33}$ results in more prominent contribution to the electric energy conversion. Furthermore, piezomagnetic coefficient $d_{33}$ remarkably enhance the performance of the magnetic energy conversion. For larger $e_{33}$ and $d_{33}$, the electromechanical coupling and magnetomechanical coupling become stronger, which enhances the electric/magnetic behaviors even when the system is under the same mechanical characteristics.

(iii) Elastic constant $c_{12}$ has a negligible influence on the MEE behavior, as evidenced in our previous research (Zhang et al., 2018). $c_{11}$, $c_{33}$ and $c_{44}$ have the same influences on $\eta_E$ and $\eta_M$, in which $c_{11}$ tends to promote the efficiency of energy conversion while $c_{33}$ and $c_{44}$ tend to reduce the efficiency. However, $c_{13}$ has a positive effect on $\eta_E$ but a negative effect on $\eta_M$. As reported in (Wang and Chen, 2011), the equivalent Young’s modulus $E_0$ and Poisson’s ratio $\nu_0$ in the plane of isotropy can be calculated by $E_0 = (c_{11}c_{33} - 2c_{11}c_{13} - c_{12}^2c_{33} + 2c_{12}c_{13}^2)/(c_{11}c_{33} - c_{13}^2)$, and $\nu_0 = (c_{12}c_{33} - c_{13}^2)/(c_{11}c_{33} - c_{13}^2)$. If $\nu_0$ is constant, $c_{11}$ tends to increase the equivalent Young’s modulus $E_0$, which makes the apparent normal stiffness stronger and then promotes larger electric/magnetic energy conversions from the mechanical energy. On the other hand, in order to keep $\nu_0$ constant and increase $E_0$ as stated above, $c_{33}$ should decrease its value, which indicates an effect totally opposite to that of $c_{11}$.

Fig. 13. Sensitivity of energy conversion factors $\eta_E$ and $\eta_M$ to the 16 independent material constants, for the problem of film thickness $h_t=1.0a_0$, sphere radius $R=1.0R_0$, and triangular loading velocity of $c_0=0.6$.

4. Conclusions
The theoretical base and a semi-analytical model (SAM) for simulating the dynamic
contact of a rigid sphere and the surface of a multiferroic magneto-electro-elastic (MEE) film under transient applied force has been developed. The research has yielded analytical frequency response functions (FRFs) and their conversion into influence coefficients (ICs) for the MEE film subjected to a set of generalized normal force and generated surface electric/magnetic charges under the influence of changing loading velocity in a dynamic process. The research has also further developed the fast numerical techniques, such as the conjugate gradient method (CGM) and the fast Fourier transform (FFT), for efficient numerical solutions to the dynamic contact behaviors of the thin-film material, including the distributions and variations of contact pressure and electric/magnetic potentials, as well as subsurface stresses. The combined influences of loading velocity, film thickness, and sphere radius on the dynamic MEE responses have been investigated, and the numerical results lead to the following conclusions.

1). The dynamic contact characteristics, including the applied force and electric/magnetic potentials under the triangular loading speed variation are slightly smaller than those under the sinusoidal loading velocity variation in the early stage of the loading cycle, while the trend reverses in the later stage. The contact characteristic values increase with the maximum velocity and sphere radius, but decrease with film thickness.

2). The von Mises stress become larger and the location of the maximum von Mises stress departs further away from the contact surface as loading velocity increases in one loading cycle. A thicker film results in a smaller stress. On the other hand, a larger sphere radius induces a smaller stress under the same peak velocity.

3). An energy conversion factor is employed to evaluate the performance of MEE energy conversion. Under the same load and loading velocity, thicker film results in a lower energy conversion from mechanical energy to electric energy. On the other hand, a smaller sphere radius promotes a larger electric energy conversion.

4). The sensitivity analyses of the energy conversion factors with respect to material properties show obvious their dependences on elastic stiffnesses $c_{11}$, $c_{13}$, $c_{33}$, $c_{44}$, dielectric permittivity $\varepsilon_{11}$, piezoelectric coefficient $e_{33}$, $e_{15}$, piezomagnetic coefficients $d_{ij}$, and magnetic permeability $\mu_{11}$, but they are less sensitive to elastic constant $c_{12}$, piezoelectric coefficient $e_{31}$, dielectric permittivity $\varepsilon_{33}$, and magnetic permeability $\mu_{33}$. Dielectric permittivities and magnetic permeabilities can be singled out for their abilities to reduce the efficiency of the energy conversion. Piezoelectric coefficient $e_{33}$ results in more prominent contribution to the electric energy conversion. Furthermore, piezomagnetic coefficient $d_{33}$ remarkably enhance the performance of the magnetic energy conversion. Elastic stiffness $c_{11}$ tends to
promote the efficiency of energy conversion while $c_{33}$ and $c_{44}$ could reduce the efficiency. However, $c_{13}$ has a positive effect on electric energy conversion but a negative effect on magnetic energy conversion. These results offer references to optimal material designs towards high efficiency of different energy conversions.

Above all, the numerical results in this paper have answered the key questions listed in the introduction, which are (a) the contact characteristics in terms of electric/magnetic potentials, contact pressure, and stresses are all much dependent on the dynamic contact loading. (b) The contact characteristic values increase with the peak velocity, and the variations in loading velocity significantly affect the mechanical and electric/magnetic responses, as stated in conclusion 1). (c) The sensitivity analyses of material properties show obvious coupling effects and dependence of electric energy conversion on several material properties, as stated in conclusion 4). In order to improve the efficiency of mechanical-electrical/magnetic energy conversions, a systematic optimization can be developed from the structural topology and material components, based on conclusion 3) and 4).

Acknowledgments

The authors would like to acknowledge the supports from Northwestern’s Center for Surface Engineering and Tribology and National Nature Science Foundation of China under grants 51775458, 11672252, and 51775457, and the fellowship award to Xin Zhang from China Scholarship Council.

Appendix A

The constants in Eq. (22) are given as follows,

$$n_b = b_1 + b_1 c_{44} c^2 + b_1 c_{44} c^4,$$
$$n_1 = b_2 + b_2 c_{44} c^2 + b_2 c_{44} c^4,$$
$$n_2 = b_3 + b_1 c_{44} c^4 + b_2 c_{44} c^2,$$
$$n_3 = b_4 + b_4 c_{44} c^2,$$
$$n_4 = c_{44} [c_{44}(e_{11}r_{11} - r_{11}^2) - 2e_{15} g_{15} d_{15} + r_{11} r_{15}^2 + e_{11} d_{15}^2],$$

where

$$b_1 = c_{44} [(c_{33} (e_{33} r_{33} - r_{33}^2) - 2e_{33} g_{33} d_{33} + r_{33} r_{33}^2 + e_{33} d_{33}^2],$$
\[ b_2 = c_{11}[c_{33}(\varepsilon_{33}\mu_{33} - g_{33}^2) - 2\varepsilon_{33}g_{33}d_{33} + \mu_{33}e_{33}^2 + \varepsilon_{33}d_{33}^2] \\
+ c_{33}[c_{44}(\varepsilon_{44}\mu_{44} - g_{44}^2) + c_{33}(\varepsilon_{11}\mu_{11} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) \\
- 2\varepsilon_{11}g_{33}d_{33} - 2\varepsilon_{33}(g_{11}d_{33} + g_{33}d_{15}) + (\mu_{11}e_{33}^2 + 2\mu_{33}e_{15}e_{33}) \\
+ (\varepsilon_{11}d_{33}^2 + 2\varepsilon_{33}d_{15}d_{33}) - (c_{13} + c_{44})(\varepsilon_{33}\mu_{33} - g_{33}^2) \\
+ (e_{15} + e_{31})(\varepsilon_{33}\mu_{33} - d_{33}g_{33}) - (d_{15} + d_{31})(\varepsilon_{33}g_{33} - d_{33}g_{15})] (A.3) \\
- (e_{15} + e_{31})[(c_{13} + c_{44})(\varepsilon_{33}\mu_{33} - g_{33}^2) \\
- (d_{15} + d_{31})(\varepsilon_{33}g_{33} - d_{33}g_{15})] - (c_{13} + c_{44})(\varepsilon_{33}d_{33} - e_{33}g_{33}) \\
+ (e_{15} + e_{31})(c_{33}g_{33} + e_{33}d_{31}) - (d_{15} + d_{31})(c_{33}e_{33} + e_{33}^2)] \\

b_3 = c_{11}[c_{44}(\varepsilon_{44}\mu_{44} - g_{44}^2) + c_{33}(\varepsilon_{11}\mu_{11} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) \\
- 2\varepsilon_{11}g_{33}d_{33} - 2\varepsilon_{33}(g_{11}d_{33} + g_{33}d_{15}) + (\mu_{11}e_{33}^2 + 2\mu_{33}e_{15}e_{33}) \\
+ (\varepsilon_{11}d_{33}^2 + 2\varepsilon_{33}d_{15}d_{33}) + c_{44}[c_{44}(\varepsilon_{44}\mu_{44} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) \\
- 2\varepsilon_{11}g_{33}d_{15} + 2\mu_{11}e_{15}e_{33} + \mu_{33}e_{15}^2 + 2\varepsilon_{11}d_{15}d_{33} + e_{33}d_{15}^2] \\
- (c_{13} + c_{44})(\varepsilon_{11}\mu_{33} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33})] (A.4) \\
- (e_{15} + e_{31})(\varepsilon_{33}\mu_{33} + e_{33}g_{33} + \mu_{33}e_{33}^2 + d_{15}^2) \\
- (d_{15} + d_{31})(\varepsilon_{33}g_{33} + d_{33}g_{15}) - (d_{15} + d_{31})(\varepsilon_{33}g_{33} + e_{33}g_{15} + d_{31}e_{11}] \\
- (e_{15} + e_{31})(c_{44}\mu_{33} + c_{33}\mu_{11} + 2d_{15}d_{33}) \\
+ (d_{15} + d_{31})(c_{44}g_{33} + c_{33}g_{11} + d_{15}e_{33} + e_{31}d_{15}] \\
- (d_{15} + d_{31})(c_{44}g_{33} + c_{33}g_{11} + e_{15}d_{15} + c_{33}d_{31}] \\
- (d_{15} + d_{31})(c_{44}e_{33} + c_{33}e_{11} + 2e_{33}e_{33}]) \\

b_4 = c_{11}[c_{44}(\varepsilon_{44}\mu_{11} + \varepsilon_{33}\mu_{11} - 2g_{11}g_{33}) + c_{33}(\varepsilon_{11}\mu_{11} - g_{11}^2) \\
- 2\varepsilon_{11}(g_{11}d_{33} + g_{33}d_{15}) - 2\varepsilon_{33}g_{33}d_{15} + 2\mu_{11}e_{15}e_{33} + \mu_{33}e_{15}^2 \\
+ 2\varepsilon_{11}d_{15}d_{33} + e_{33}d_{15}^2) + c_{44}[c_{44}(\varepsilon_{44}\mu_{11} - g_{11}^2) - 2\varepsilon_{11}g_{11}d_{15} \\
+ (\mu_{11}e_{33}^2 + e_{11}d_{33}^2) - (c_{13} + c_{44})(\varepsilon_{11}\mu_{11} - g_{11}^2) \\
+ (e_{15} + e_{31})(\varepsilon_{11}\mu_{11} - d_{15}g_{11}) - (d_{15} + d_{31})(\varepsilon_{11}g_{11} - d_{15}e_{11}] \\
- (e_{15} + e_{31})(c_{44}\mu_{11} + d_{15}^2) + (d_{15} + d_{31})(c_{44}g_{11} + d_{15}e_{11}] \\
- (d_{15} + d_{31})(c_{44}g_{11} + e_{15}d_{15} + c_{44}e_{11} + e_{15}^2) \\
- (e_{15} + e_{31})(c_{44}g_{11} + e_{15}d_{15} - (d_{15} + d_{31})(c_{44}e_{11} + e_{15}^2)], (A.5) \\

b_5 = \varepsilon_{11}\mu_{11} - g_{11}^2, (A.6) \\

b_6 = c_{44}g_{11}^2 + 2\varepsilon_{11}d_{15}g_{11} + c_{11}g_{11}^2 - c_{44}\varepsilon_{11}\mu_{11} - d_{15}e_{11} - \varepsilon_{11}e_{11} - c_{11}e_{11}\mu_{11},
\[ b_j = \begin{array}{c}
 c_{44}g_{33}^2 - 2e_{15}e_{33}\mu_{33} + 2e_{15}d_{15}g_{33} + 2e_{33}d_{15}g_{33} + 2e_{31}d_{31}g_{33} + 2e_{31}d_{31}g_{33} \\
 - 2d_{15}d_{31}g_{33} - c_{44}e_{15}\mu_{33} - c_{44}e_{15}\mu_{33} - c_{11}e_{33}\mu_{33} - c_{33}e_{11}\mu_{33} - c_{33}e_{11}\mu_{33} \\
 - 2e_{15}e_{33}\mu_{33} + 2e_{15}d_{15}g_{33} + 2c_{44}g_{11}g_{33} + 2e_{15}d_{31}g_{31} + 2e_{33}d_{31}g_{33} - 2d_{15}d_{33}g_{33} \\
 + 2c_{33}d_{31}g_{33} - c_{44}e_{33}\mu_{33} - c_{44}e_{33}\mu_{33} - d_{31}e_{33} - d_{31}e_{33} - e_{33}\mu_{33} - e_{33}\mu_{33} + c_{11}g_{33}^2 - d_{33}e_{33} \\
\end{array} \tag{A.7} \]

\[ \begin{align*}
 b_8 &= -c_{44}e_{33}\mu_{11} - c_{44}e_{11}\mu_{33} + 2e_{15}d_{15}g_{33} + 2c_{44}g_{11}g_{33} - d_{15}e_{33} - c_{11}g_{33}^2 \\
 &\quad - c_{11}e_{33}\mu_{33} - c_{11}e_{33}\mu_{33} - c_{33}e_{11}\mu_{11} + 2e_{15}d_{15}g_{11} + 2c_{11}g_{11}g_{33} \\
 &\quad + 2e_{15}d_{31}g_{31} + 2e_{15}d_{31}g_{31} + 2e_{33}d_{15}g_{31} - 2d_{15}d_{33}g_{11} + 2e_{33}d_{31}g_{31} - 2e_{15}e_{31}\mu_{33} \\
 &\quad - 2e_{15}e_{33}\mu_{11} + 2e_{15}d_{15}g_{31} - 2d_{15}d_{31}e_{33} + c_{33}g_{31}^2 + c_{44}g_{11}^2 - e_{33}\mu_{11} - d_{33}e_{33} - e_{33}\mu_{11} - e_{33}\mu_{11} - e_{33}\mu_{11} - e_{33}\mu_{11},
\end{align*} \tag{A.8} \]

\[ b_9 = \begin{array}{c}
 c_{33}g_{33}^2 + c_{44}g_{33}^2 - c_{44}g_{33}^2 - d_{33}e_{33} - e_{33}\mu_{33} - c_{33}e_{33}\mu_{33} + 2e_{33}d_{33}g_{33},
\end{array} \tag{A.9} \]

\[ b_{10} = \begin{array}{c}
 c_{33}g_{33}^2 + c_{44}g_{33}^2 - c_{44}g_{33}^2 - e_{33}\mu_{33} - c_{33}e_{33}\mu_{33} + 2e_{33}d_{33}g_{33},
\end{array} \tag{A.10} \]

\[ b_{11} = \begin{array}{c}
 e_{33}\mu_{33} - g_{33}^2
\end{array} \tag{A.11} \]

Besides, the constants in Eq. (30) are given as,

\[ k_g = \beta_g f(\alpha_s s^2) \quad (i = 1, 2, 3) \]

\[ \omega_j = a_{44} (1 + k_j) + e_{15}k_{2j} + d_{15}k_{1j}, \]

\[ \omega_j = e_{15} (1 + k_j) - e_{11}k_{2j} - g_{11}k_{3j}, \]

\[ \omega_j = d_{15} (1 + k_j) - g_{11}k_{2j} - \mu_1k_{3j}, \]

\[ \rho_j = c_{44}, \rho_2 = e_{15}, \rho_3 = d_{15}, \]

where

\[ \alpha_j = -a_1 + a_{2j} s^2_j - a_{3j} s^4, \quad \beta_j = -a_{4i} + a_{5j} s^2_i + a_{6j} s^4 + a_{7i} s^4, \quad (i = 1, 2, 3), \]

with

\[ a_1 = (c_{11} + c_{44})(e_{11}\mu_{11} - g_{11}^2) + (e_{15} + e_{31})(e_{15}\mu_{11} - g_{11}d_{15}) - (d_{15} + d_{31})(e_{15}g_{11} - e_{11}d_{15}) \]

\[ a_2 = (c_{11} + c_{44})(e_{11}\mu_{33} + e_{33}\mu_{11} - 2g_{11}g_{33}) + (e_{15} + e_{31})(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15}) - (d_{15} + d_{31})(e_{15}g_{33} + e_{33}g_{11} - e_{11}d_{33} - e_{33}d_{15}) \]

\[ a_3 = (c_{11} + c_{44})(e_{33}\mu_{33} - g_{33}^2) + (e_{15} + e_{31})(e_{33}\mu_{33} - g_{33}d_{33}) - (d_{15} + d_{31})(e_{33}g_{33} - e_{33}d_{33}) \]
\[ a_{41} = c_{11}(e_{11}\mu_{11} - g_{11}^2), \]  
\[ a_{51} = c_{11}(e_{11}\mu_{33} + e_{33}\mu_{11} - 2g_{11}g_{33}) + c_{44}(1-c^2)(e_{11}\mu_{11} - g_{11}^2) + \mu_{11}(e_{15} + e_{31})^2 + e_{11}(d_{15} + d_{31})^2 - 2g_{11}(e_{15} + e_{31})(d_{15} + d_{31}), \]  
\[ a_{61} = c_{11}(e_{33}\mu_{33} - g_{33}^2) + c_{44}(1-c^2)(e_{33}\mu_{33} + e_{33}\mu_{11} - 2g_{11}g_{33}) + \mu_{33}(e_{15} + e_{31})^2 + e_{33}(d_{31} + d_{15})^2 - 2g_{33}(e_{15} + e_{31})(d_{15} + d_{31}), \]  
\[ a_{71} = c_{44}(1-c^2)(e_{33}\mu_{33} - g_{33}^2), \]  
\[ a_{42} = c_{11}(e_{15}\mu_{11} - g_{11}d_{15}). \]  
\[ a_{52} = c_{11}(e_{15}\mu_{33} + e_{33}\mu_{11} - g_{11}d_{33} - g_{33}d_{15}) + c_{44}(1-c^2)(e_{15}\mu_{11} - g_{11}d_{15}) + \mu_{11}(e_{15} + e_{31})(c_{31} + c_{44}) + d_{15}(d_{15} + d_{31})] + (d_{15} + d_{31})(g_{11}(c_{11} + c_{44}) + e_{15}(d_{15} + d_{31})), \]  
\[ a_{62} = c_{11}(e_{33}\mu_{33} - g_{33}d_{33}) + c_{44}(1-c^2)(e_{33}\mu_{33} + e_{33}\mu_{11} - g_{33}d_{15} - g_{33}d_{33}) - (e_{15} + e_{31})(\mu_{33}(c_{13} + c_{44}) + d_{33}(d_{15} + d_{31})) + (d_{15} + d_{31})(g_{33}(c_{11} + c_{44}) + e_{33}(d_{15} + d_{31})), \]  
\[ a_{72} = c_{44}(1-c^2)(e_{33}\mu_{33} - g_{33}d_{33}), \]  
\[ a_{43} = c_{11}(e_{15}d_{15} - e_{15}g_{11}), \]  
\[ a_{53} = c_{11}(-e_{15}g_{33} - e_{33}g_{11} + e_{11}d_{15} + e_{33}d_{15}) + c_{44}(1-c^2)(-e_{15}g_{33} + e_{11}d_{15}) + (e_{15} + e_{31})(g_{33}(c_{11} + c_{44}) + d_{15}(e_{15} + e_{31})) - (d_{15} + d_{31})(g_{11}(c_{11} + c_{44}) + e_{15}(e_{15} + e_{31})), \]  
\[ a_{63} = c_{11}(-e_{33}g_{33} + e_{33}d_{31}) + c_{44}(1-c^2)(-e_{33}g_{33} - e_{33}g_{11} + e_{11}d_{33} + e_{33}d_{15}) + (e_{15} + e_{31})(g_{33}(c_{13} + c_{44}) + d_{33}(e_{15} + e_{31})) - (d_{15} + d_{31})(g_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31})), \]  
\[ a_{73} = c_{44}(1-c^2)(-e_{33}g_{33} + e_{33}d_{31}). \]  

**Appendix B**
The intermediate variables in Eq. (49) are:
\[ r_j = -s_j \omega_j - \rho_2 s_0 e^{-\alpha(s_j)^h h}, \]
\[ \bar{r}_j = s_j \omega_j - \rho_2 s_0 e^{\alpha(s_j)^h h}, \]
\[ r_0 = \rho_1 s_0 (1 - e^{-2\alpha s_0 h}), \]
\[ j = 1, \ldots, 4. \]  
(B.1)

\[ r_y = s_j (k_y - k_{ij} k_3 / k_{34}), i = 1, 2, j = 1, 2, 3. \]  
(B.2)

\[ k_j = s_4 + s_3 k_3 / k_{34}, \]
\[ \bar{k}_j = s_4 - s_3 k_3 / k_{34}. \]  
(B.3)

\[ t^{(1)}_j = (s_4 + s_j) \omega_j, \quad t^{(2)}_j = \omega_2 j - \omega_2 s_j / \omega_4, \]
\[ \bar{t}^{(1)}_j = (s_4 - s_j) \omega_j, \quad \bar{t}^{(2)}_j = t^{(2)}_j. \]  
(B.4)

\[ t^{(3)}_j = \omega_3 j - \omega_3 s_j e^{-\alpha(s_j)^h h}, \quad t^{(4)}_j = -s_j \omega_j - s_j s_4 e^{-\alpha(s_j)^h h}, \]
\[ \bar{t}^{(3)}_j = \omega_3 j - \omega_3 s_j e^{\alpha(s_j)^h h}, \quad \bar{t}^{(4)}_j = s_j \omega_j - s_j s_4 e^{\alpha(s_j)^h h}. \]  
(B.5)

\[ t^{(1)}_y = r_y \bar{k}_3 / r_3 + k_j, \]
\[ \bar{t}^{(1)}_y = -r_y \bar{k}_3 / r_3 + \bar{k}_j, \]  
(B.6)

\[ r^{(i)}_j = t^{(i)}_j \bar{k}_3 / t^{(i)}_3 - k_j e^{-\alpha(s_j)^h h}, \]
\[ \bar{r}^{(i)}_j = \bar{t}^{(i)}_j \bar{k}_3 / \bar{t}^{(i)}_3 - \bar{k}_j e^{\alpha(s_j)^h h}. \]  
(B.7)

\[ s^{(i)}_j = r^{(i)}_j - \bar{s}^{(i)}_2 t^{(i)}_2 e^{-\alpha(s_j)^h h} / \bar{t}^{(i)}_{22}, \]
\[ \bar{s}^{(i)}_j = \bar{t}^{(i)}_1 \bar{s}^{(i)}_2 / \bar{t}^{(i)}_{21}. \]  
(B.8)

\[ s^{(5)}_j = [t^{(i)}_2 - \bar{t}^{(i)}_{12} t^{(i)}_2 / \bar{t}^{(i)}_{22}] e^{-\alpha(s_j)^h h}, \]
\[ \bar{s}^{(5)}_j = [\bar{t}^{(i)}_{12} \bar{t}^{(i)}_2 / \bar{t}^{(i)}_{22}]. \]  
(B.9)

\[ k^{(i)}_j = s^{(i)}_j - \bar{s}^{(i)}_j s^{(5)}_j / \bar{s}^{(5)}_j. \]  
(B.10)

\[ p^{(1)} = [-s_3 \bar{\rho} / \alpha^2] \bar{k}_3 / \bar{t}^{(1)}_3, \]
\[ p^{(2)} = [\alpha \omega_2 \tilde{\omega} / \alpha^2 - \tilde{\omega} / \alpha^2] \bar{k}_3 / \bar{t}^{(2)}, \quad t^{(i)}_j = k^{(i)}_j - k^{(i)}_j k^{(4)}_j / k^{(4)}_j. \]  
(B.11)

\[ p^{(3)} = [-\tilde{\omega} / \alpha^2] \bar{k}_3 / \bar{t}^{(3)}_3. \]  

\[ h_j^{(i)} = l_j^{(i)} - l_3^{(i)} l_j^{(3)} / l_3^{(3)}, \quad q^{(i)} = p^{(i)} - p^{(3)} l_j^{(3)} / l_3^{(3)}, \]
\[ q^{(2)} = p^{(2)} - p^{(3)} l_j^{(2)} / l_3^{(3)}. \]  

(B.12)

References


Xu, Q., Qin, Y., 2017. Theoretical study of enhancing the piezoelectric nanogenerator’s output power by optimizing the external force’s shape. APL Materials 5, 074101.


