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Modification of Block Pulse Functions and their application to solve numerically Volterra integral equation of the first kind

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ABSTRACT

In this paper a modification of Block Pulse Functions is introduced and used to solve Volterra integral equation of the first kind. Some theorems are included to show convergence and advantage of the method. Some examples show accuracy of the method.

1. Introduction

Approximation theory is concerned with how functions can best be approximated with simpler functions called base functions and with quantitatively characterizing the errors introduced thereby [1]. One of these base functions is Block Pulse Functions (*BPFs*) [2] on which some researches are based. However *BPFs* are very common in use, it seems their convergence is weak and some published papers have tried to improve the speed of *BPFs* convergence with different methods like hybrid *BPFs* [3–5]. In fact by referring to error bound of *BPFs* approximation it seems for achieving double precision, number of *BPFs* have to be doubled which means solving systems of equations with double unknowns and double equations [1,6].

In this paper ε Modified Block Pulse Functions (ε*MBPFs*) are introduced and some theorems prove if ε*MBPFs* be used for achieving numerical expansions with k times more precision, there is no need to increase the number of *BPFs*, k times, which leads to solve a system of equations with k times more equations and unknowns. But the results of *BPFs* solution can be combined with solutions of $k-1$ systems of equations with one more unknown and nearly achieve k times more precision.

We use ε*MBPFs* and directly solve Volterra integral equation of the first kind, then by some examples we show the efficiency of ε*MBPFs*.

2. Block Pulse Functions (*BPFs*)

BPFs are studied by many authors and applied for solving different problems, for example see [2–7].

Definition: An m-set of *BPFs* is defined over the interval $[0,T)$ as

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$$\psi_i(t) = \begin{cases} 1, & \frac{iT}{m} \leqslant t < \frac{(i+1)T}{m} \\ 0, & otherwise, \end{cases} \tag{2.1}$$

where $i = 0,\ldots,m-1$ with m as a positive integer. Also, $h = T/m$, and ψ_i is the ith *BPF*.

In this paper it is assumed that $T = 1$, so *BPFs* are defined over $[0,1)$ and $h = 1/m$.

There are some properties for *BPFs*, the most important properties are disjointness, orthogonality, and completeness.

The disjointness property can be clearly obtained from the definition of *BPFs*:

$$\psi_i(t)\psi_j(t) = \begin{cases} \psi_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \tag{2.2}$$

where $i, j = 0,\ldots,m-1$.

The other property is orthogonality. It is clear that

$$\int_0^1 \psi_i(t)\psi_j(t)dt = h\delta_{ij}, \tag{2.3}$$

where δ_{ij} is Kronecker delta.

The third property is completeness. For every $f \in L^2([0,1))$ when m approaches to infinity, Parseval's identity holds:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\psi_i(t)\|^2, \tag{2.4}$$

where

$$f_i = \frac{1}{h}\int_0^1 f(t)\psi_i(t)dt. \tag{2.5}$$

Vector forms: Consider the first m terms of *BPFs* and write them concisely as m-vector:

$$\Psi(t) = [\psi_0(t),\ldots,\psi_{m-1}(t)]^T, \quad t \in [0,1).$$

The above representation and disjointness property follows [2]:

$$\Psi(t)\Psi^T(t) = \begin{bmatrix} \psi_0(t) & 0 & \cdots & 0 \\ 0 & \psi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_{m-1}(t) \end{bmatrix}, \tag{2.6}$$

$$\Psi^T(t)\Psi(t) = 1, \tag{2.7}$$

$$\Psi(t)\Psi^T(t)V = \widetilde{V}\Psi(t), \tag{2.8}$$

where V is an m-vector and $\widetilde{V} = diag(V)$. Moreover, it can be clearly concluded that for an $m \times m$ matrix B:

$$\Psi^T(t)B\Psi(t) = \widehat{B}^T\Psi(t), \tag{2.9}$$

where $\widehat{B}$ is an m-vector with elements equal to the diagonal entries of matrix B.

BPFs expansion: The expansion of a function $f(t)$ over $[0,1)$ with respect to $\psi_i(t)$, $i = 0,\ldots,m-1$ may be compactly written as:

$$f(t) \simeq \sum_{i=0}^{m-1} f_i\psi_i(t) = F^T\Psi(t) = \Psi^T(t)F, \tag{2.10}$$

where $F = [f_0,\ldots,f_{m-1}]^T$ and f_i's is defined by (2.5).

Now assume $k(s,t) \in L^2([0,1) \times [0,1))$. It can be expanded with respect to *BPFs* as

$$k(s,t) \simeq \Psi^T(s)K\Gamma(t), \tag{2.11}$$

where $\Psi(s)$ and $\Gamma(t)$ are m_1 and m_2 components *BPFs* vectors, respectively, and K is the $m_1 \times m_2$ block pulse coefficient matrix with k_{ij}, $i = 0,\ldots,m_1-1$, $j = 0,\ldots,m_2-1$, as:

$$k_{ij} = m_1m_2\int_0^1\int_0^1 k(s,t)\psi_i(s)\gamma_j(t)dsdt. \tag{2.12}$$

For convenience, we put $m_1 = m_2 = m$.

Operational matrix: Computing $\int_0^t \psi_i(\tau)d\tau$ follows:

$$\int_0^t \psi_i(\tau)d\tau = \begin{cases} 0, & t \leqslant ih, \\ t - ih, & ih \leqslant t < (i+1)h, \\ h, & (i+1)h \leqslant t < 1. \end{cases} \tag{2.13}$$

Note that $t - ih$, equals to $h/2$, at mid-point of $[ih,(i+1)h)$. So we can approximate $t - ih$, for $ih \leqslant t < (i+1)h$, by $h/2$.

Now expressing $\int_0^t \psi_i(\tau)d\tau$, in terms of the *BPFs* follows:

$$\int_0^t \psi_i(\tau)d\tau \simeq [0,\ldots,0,\frac{h}{2},h,\ldots,h]\Phi(t), \tag{2.14}$$

in which the ith component is $h/2$.

Therefore

$$\int_0^t \Psi(\tau)d\tau \simeq P\Psi(t), \tag{2.15}$$

where $P_{m\times m}$ is called operational matrix of integration which can be represented by:

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{2.16}$$

So, the integral of every function $f(t)$ can be approximated as follows:

$$\int_0^t f(\tau)d\tau \simeq \int_0^t F^T\Psi(\tau)d\tau \simeq F^TP\Psi(t) \tag{2.17}$$

Time delay BPFs: If we define an auxiliary function of a single Block Pulse with the width λh $(0 \leqslant \lambda < 1)$:

$$\psi_\lambda(t) = \begin{cases} 1, & 0 \leqslant t < \lambda h \\ 0, & otherwise, \end{cases} \tag{2.18}$$

and an $m \times 1$ vector $\psi_\lambda(t)$:

$$\Psi_\lambda(t) = [\psi_\lambda(t), \psi_\lambda(t-h), \ldots, \psi_\lambda(t-(n-1)h)]^T,$$

then a *BPF* containing time delay $\tau = (q + \lambda)h$ can be expressed as [2]:

$$\psi_i(t-\tau) = \Delta_i^T H^q \Psi(t) - \Delta_i^T H^q \Psi_\lambda(t) + \Delta_i^T H^{q+1}\Psi_\lambda(t), \tag{2.19}$$

where

$$\Delta_i = [0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]^T,$$

one lays in i*th* position, and

$$H = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

3. ε Modified Block Pulse Functions (*εMBPFs*)

Definition: An $(n + 1)$-set of *εMBPFs* consists of $n + 1$ functions which are defined over $[0,T)$ as follows:

$$\phi_0(t) = \begin{cases} 1, & t \in [0, \frac{T}{n} - \varepsilon) = I_0, \\ 0, & otherwise, \end{cases}$$

$$\phi_n(t) = \begin{cases} 1, & t \in [T - \varepsilon, T) = I_n, \\ 0, & otherwise, \end{cases} \tag{3.1}$$

$$\phi_i(t) = \begin{cases} 1, & t \in [\frac{iT}{n} - \varepsilon, \frac{(i+1)T}{n} - \varepsilon) = I_i, \\ 0, & otherwise, \quad 0 < i < n. \end{cases}$$

εMBPFs are disjoint and orthogonal:

$$\phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \tag{3.2}$$

where $i, j = 0, \ldots, n$ and

$$\int_0^1 \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \tag{3.3}$$

εMBPFs like *BPFs* are complete:

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i(t)\|^2, \tag{3.4}$$

where

$$f_i = \frac{1}{\Delta(I_i)} \int_0^1 f(t)\phi_i(t)dt, \tag{3.5}$$

and $\Delta(I_i)$ is length of interval I_i, defined in (3.1).

Using notation $\Phi_n(t) = [\phi_0(t), \ldots, \phi_n(t)]^T$, the following properties are achieved:

$$\Phi_{n+1}(t)\Phi_{n+1}^T(t) = \begin{bmatrix} \phi_0(t) & 0 & \cdots & 0 \\ 0 & \phi_1(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n(t) \end{bmatrix}, \tag{3.6}$$

$$\Phi_{n+1}^T(t)\Phi_{n+1}(t) = 1, \tag{3.7}$$

$$\Phi_{n+1}(t)\Phi_{n+1}^T(t)V = \widetilde{V}\Phi_{n+1}(t), \tag{3.8}$$

$$\Phi_{n+1}^T(t)B\Phi_{n+1}(t) = \widehat{B}^T\Phi_{n+1}(t). \tag{3.9}$$

If $h = T/n$ the operational matrix of *εMBPFs* is defined as follows:

$$P_{(n+1)\times(n+1)} = \begin{bmatrix} \frac{h-\varepsilon}{2} & h-\varepsilon & h-\varepsilon & \cdots & h-\varepsilon & h-\varepsilon \\ 0 & \frac{h}{2} & h & \cdots & h & h \\ 0 & 0 & \frac{h}{2} & \cdots & h & h \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{h}{2} & h \\ 0 & 0 & 0 & \cdots & 0 & \frac{\varepsilon}{2} \end{bmatrix}, \tag{3.10}$$

and it has the same properties and usages as operational matrix (2.16).

Definition: *εMBPFs* expansion of continues function $f(t) \in L^2[0,1]$ with respect to ϕ_i, $i = 0, \ldots, n$ is defined as:

$$f(t) \simeq \hat{f}_{n+1} = \sum_{i=0}^{n} f_i\phi_i(t), \tag{3.11}$$

f_i is defined in (3.5).

4. Theorems and error analysis

In following theorems for simplicity we assume $T = 1$ and $h = 1/n$.

Theorem 1. *If* $\hat{f}_n(t) = \sum_{i=0}^{n} f_i\phi_i(t)$ *and* $f_i = \frac{1}{\Delta(I_i)}\int_0^1 f(t)\phi_i(t)dt$, $i = 0, \ldots, n$ *then:*

(i) $\delta = \int_0^1 \left(f(t) - \sum_{i=0}^{n} f_i\phi_i(t)\right)^2 dt$, *achieves its minimum value.*

(ii) $\{\hat{f}_n(t)\}$ *approaches* $f(t)$ *point wise.*

(iii) $\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|\phi_i\|^2$

Proof. Proof is like similar theorem in [2] but intervals of integrations have to redefine as I_i, $i = 0,\ldots,n$ in (3.1). □

Theorem 2. *Assume:*

(i) $f(t)$ *is continuous and is differentiable over interval* $[-h, 1+h]$,
(ii) $\hat{f}_{\frac{ih}{k}}(t) i = 0,1,\ldots,k-1$ *are correspondingly BPFs,* $\frac{h}{k}$*MBPFs*$\ldots$, $\frac{(k-1)h}{k}$*MBPFs expansions of* $f(t)$ *base on* $n+1$ *εMBPFs over interval* $[0,1)$,
(iii) $E = \max_{i=0,\ldots,k-1} \|f(t) - \hat{f}_{\frac{ih}{k}}(t)\|_\infty$,
(iv) $\bar{f}(t) = \frac{1}{k}\sum_{i=0}^{k-1} \hat{f}_{\frac{ih}{k}}(t)$,

then for adequate large n:

$$\|f(t) - \bar{f}(t)\|_\infty \lessapprox \frac{1}{k}E.$$

Proof. Suppose $t_i = i/n$ and $I_i = [t_{i-1}, t_i)$ $i = 0,\ldots,n$. In fact t_is are partition points of *BPFs*. Through the rest of proof i is an arbitrary element of set $\{0,\ldots,n\}$, but fix. Suppose n is so large that $f'(t)$ over interval $J = [t_{i-1}, t_{i+1})$ is approximately equal to constant m and we use line $y = mt + b$ instead of $f(t)$ over interval J. Now over interval $[t_i, t_i + \frac{h}{k})$:

$$\bar{f}(t) = \frac{1}{k}\sum_{j=0}^{k-1} \frac{m(t_i - \frac{jh}{k}) + b + m(t_{i+1} - \frac{jh}{k}) + b}{2} = m\left(\frac{t_i + t_{i+1}}{2}\right) + b - \frac{mh(k-1)}{2k}, \tag{4.1}$$

but $t_{i+1} = t_i + h$ and (4.1) can be reformulated as:

$$\bar{f}(t) = m\left(\frac{t_i + t_i + h}{2}\right) + b - \frac{mh(k-1)}{2k} = mt_i + b + \frac{mh(k-1)}{2k}, \tag{4.2}$$

on the other hand:

$$\max_{[t_i, t_i + \frac{h}{k})} |f(t) - \bar{f}(t)| \simeq \max_{[t_i, t_i + \frac{h}{k})} |mt + b - \bar{f}(t)| \leqslant |mt_i + b - \bar{f}(t)| = \frac{mh}{2k}, \tag{4.3}$$

and over $[t_i, t_i + \frac{h}{k}]$, E is:

$$E \geqslant \max_{i=0,\ldots,k-1} |f(t) - \hat{f}_{\frac{ih}{k}}(t)| \simeq \left|mt_i + b - \left(\frac{mt_i + b + m(t_i + h) + b}{2}\right)\right| = \frac{mh}{2}. \tag{4.4}$$

(4.3) and (4.4) complete the proof. □

Maleknejad in [6] and Lepik in [8] have introduced some methods to estimate the error when the exact solution is not available.

Theorem 3. *Suppose* f *is continuous in* I, *is differentiable in* $(0,1)$, *and there is a number* M *such that* $|f'(x)| \leqslant M$, *for every* $x \in I$. *Then*

$$|f(b) - f(a)| \leqslant M|b - a|,$$

for all $a, b \in I$.

Proof. See [9]. □

Now, we assume that $f(x)$ is a differentiable function on I such that $|f'(x)| \leqslant M$. We define the error between $f(x)$ and its *BPFs* expansion over every subinterval I_i as follows:

$$e_i(x) = f_i - 0(x), \quad x \in I_i$$

where $I_i = [\frac{i}{n}, \frac{i+1}{n})$.

It can be shown that:

$$\|e_i\|^2 = \int_{\frac{i}{n}}^{\frac{i+1}{n}} e_i^2(x)dx = \int_{\frac{i}{n}}^{\frac{i+1}{n}} (f_i - f(x))^2 dx = \frac{1}{n}(f_i - f(\eta))^2, \quad \eta \in I_i, \tag{4.5}$$

where we used mean value theorem for integral. Using Eq. (2.5) and the mean value theorem, we have:

$$f_i = n\int_{\frac{i}{n}}^{\frac{i+1}{n}} f(x)dx = n\frac{1}{n}f(\zeta) = f(\zeta), \quad \zeta \in I_i. \tag{4.6}$$

Substituting (4.6) into (4.5) and using Theorem 3, we will have:

$$\|e_i\|^2 = \frac{1}{n}(f(\zeta) - f(\eta))^2 \leqslant \frac{M^2}{n}|\zeta - \eta|^2 \leqslant \frac{M^2}{n^3}. \tag{4.7}$$

This leads to:

$$\|e(x)\|^2 = \int_0^1 e^2(x)dx = \int_0^1 \left(\sum_{i=0}^{n-1} e_i(x)\right)^2 dx = \int_0^1 \left(\sum_{i=0}^{n-1} e_i^2(x)\right)dx + 2\sum_{i\leqslant j}\int_0^1 e_i(x)e_j(x)dx.$$

Since for $i \neq j$, $I_i \cap I_j = \varnothing$, then

$$\|e(x)\|^2 = \sum_{i=0}^{n-1}\left(\int_0^1 e_i^2(x)dx\right) = \sum_{i=0}^{n-1}\|e_i\|^2. \tag{4.8}$$

Substituting (4.7) into (4.8), we have

$$\|e(x)\|^2 \leqslant \frac{M^2}{n^2},$$

hence, $\|e(x)\| = \mathcal{O}(\frac{1}{n})$, where $e(x) = |f_n(x) - f(x)|$ and $f_n(x) = \sum_{i=0}^{n-1} f_i\phi_i(x)$.

Theorems 2 and 3 with above discussion conclude that error estimation for *εBPFs* is $\|e(x)\| = \mathcal{O}(\frac{1}{kn})$, where n shows number of *BPFs* and k times of modifications.

In Lepik estimation [8], when the exact solution is not available we choose number of *BPFs* as $n = 2^j$, $j = 0,1,2,\ldots$ The error of results can be estimated in following way. We introduce the quantity

$$S(n) = h.\sum_{i=1}^{n}|f_i| \tag{4.9}$$

$S(n)$ is the area which lies in the interval $t \in [0,1]$, underneath curve $|f_n(t)|$. For estimating the exactness of the solution the quantity

$$\Delta(n) = \left|\frac{S(n+1)}{S(n)} - 1\right|, \tag{4.10}$$

is introduced. The convergence rate of process can be estimated with the aid of function [8]:

$$\sigma(n) = \frac{\Delta(n-1)}{\Delta(n)} j = 1,2,3,\ldots. \tag{4.11}$$

5. Applying *εMBPFs* to solve Volterra integral equation of the first kind

During this section we use a direct method to solve Volterra integral equation of the first kind by *BPFs*, which is established in [7]. Here we modify this method by *εMBPFs*.

Consider Volterra integral equation of the first kind:

$$\int_0^t k(t,s)x(s)ds = f(t) \quad 0 \leqslant t < 1, \tag{5.1}$$

where f and k are known, x is unknown, $k(t,s) \in L^2([0,1) \times [0,1))$ and $x(t), f(t) \in L^2([0,1))$.

Approximating functions f, x and k with respect to *ε MBPFs* gives:

$$f(t) \simeq F_{n+1}^T \Phi_{n+1}(t) = \Phi_{n+1}^T(t)F_{n+1}, \tag{5.2}$$

$$x(t) \simeq X_{n+1}^T \Phi_{n+1}(t) = \Phi_{n+1}^T(t)X_{n+1}, \tag{5.3}$$

$$k(t,s) \simeq \Phi_{n+1}^T(t)K_{(n+1)\times(n+1)}\Phi_{n+1}(s) \tag{5.4}$$

where the vectors F, X and matrix K are *εMBPFs* coefficient of $f(t)$, $x(t)$ and $k(t,s)$ respectively. In (5.3), X is unknown vector.

Substituting (5.2)–(5.4) into (5.1) gives

$$F^T\Phi(t) \simeq \int_0^t \Phi^T(t)K\Phi(s)\Phi^T(s)Xds \simeq \Phi^T(t)K\int_0^t \Phi(s)\Phi^T(s)Xds, \tag{5.5}$$

Using (3.8) follows:

$$F^T\Phi(t) \simeq \Phi^T(t)K\int_0^t \widetilde{X}\Phi(s)ds \simeq \Phi^T(t)K\widetilde{X}\int_0^t \Phi(s)ds. \tag{5.6}$$

Using operational matrix P in (3.10) gives

$$F^T\Phi(t) \simeq \Phi^T(t)K\widetilde{X}P\Phi(t), \tag{5.7}$$

in which $K\widetilde{X}P$ is an $(n+1)\times(n+1)$ matrix. If ε equals zero, only n *BPFs* exist and dimension of vectors and matrixes decrease to n.

According to (3.9), we can write:

$$\Phi^T(t)K\widetilde{X}P\Phi(t) = \widehat{X}^T\Phi(t), \tag{5.8}$$

where $\widehat{X}$ is an $(n+1)$-vector with components equal to the diagonal entries of matrix $K\widetilde{X}P$.

So, the vector $\widehat{X}$ can be written as follows:

$$\widehat{X} = \begin{bmatrix} k_{00}x_0\frac{h-\varepsilon}{2} \\ k_{10}x_0(h-\varepsilon) + k_{11}x_1\frac{h}{2} \\ k_{20}x_0(h-\varepsilon) + k_{21}x_1h + k_{22}x_2\frac{h}{2} \\ \vdots \\ k_{n0}x_0(h-\varepsilon) + k_{n1}x_1h + \cdots + k_{n(n-1)}x_{n-1}h + k_{nn}x_n\frac{h}{2} \end{bmatrix}. \tag{5.9}$$

Also, we can write

$$\widehat{X} = \begin{bmatrix} k_{00}\frac{h-\varepsilon}{2} & 0 & 0 & \cdots & 0 \\ k_{10}(h-\varepsilon) & k_{11}\frac{h}{2} & 0 & \cdots & 0 \\ k_{20}(h-\varepsilon) & k_{21}h & k_{22}\frac{h}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{n0}(h-\varepsilon) & k_{n1}h & k_{n2}h & \cdots & k_{nn}x_n\frac{h}{2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{5.10}$$

Now, combining (5.7) and (5.8), and replacing $\simeq$ with = gives

$$\widehat{X} - F = 0. \tag{5.11}$$

If Eq. (5.1) has a unique solution then Eq. (5.11) will be a well condition linear lower triangle system of $n+1$ algebraic equations for $n+1$ unknown $x_0,\ldots,x_n$ which can be easily solved by forward substitution.

Now if $\varepsilon_j = \frac{jh}{k}$, $j = 0,\ldots,k-1$ there will be k numerical answers $\hat{f}_{\varepsilon_j}$, $j = 0,\ldots,k-1$ and according to theorem (2) we expect, the maximum difference between:

$$\bar{f}(t) = \sum_{j=0}^{k-1}\hat{f}_{\epsilon_j}(t) \tag{5.12}$$

and $f(t)$ be approximately equal to $\|f(t) - f_{n\times k}(t)\|_\infty$, where $f_{n\times k}(t)$ is the numerical solution achieved by method of [7] with $n\times k$, *BPFs*.

6. Numerical examples

In some papers, it is usual to test the accuracy of methods by comparing the numerical solution and exact solution in sample points. But this is not so appropriate for *BPFs* expansions. For example if we calculate *BPFs* expansion of $y = x$ with $n = 5$, 15 and 25 over interval [0,1) and choose $x_i = \frac{2i+1}{10}$, $i = 0,1,2,3,4$ as sample points, the errors at these points for $n = 5$, 15 and 25, are zero while the bound of errors for these expansions are correspondingly 0.1, 0.03 and 0.02. This example declares that bound of error is more adequate to represent the accuracy of numerical solutions based on *BPFs*. According to above discussion, in following examples bounds of errors are compared.

In following examples, n shows number of Block Pulse Functions and k is times of modifications. If k equals zero, the expansion is based on *BPFs* otherwise expansion is based on ε*MBPFs*.

Tables 1–4 clearly show the results of Theorem 2. In fact whenever values of $n\times(k+1)$ are equal, the bounds of errors $\|f-\bar{f}\|_\infty$ are equal. For example in Table 1 where $n = 128$ and $k = 0$, it is seen that $n\times(k+1) = 128$ and $\|f-\bar{f}\|_\infty = 0.024$, while for $n = 32$ and $k = 3$, it is seen that $n\times(k+1) = 128$ and $\|f-\bar{f}\|_\infty = 0.025$, which is approximately same bound of error.

Example 1. In this example different expansions of $f(t) = \cos(2\pi t)$ over interval [0,1) are compared. Table 1 shows the results of numerical examples. n is number of *BPFs*, and k is number of modifications.

Example 2. Consider the following integral equation [7]:

Table 1

n	32	64	128	32	32	32	64
k	0	0	0	1	2	3	1
$\|f-\bar{f}\|_\infty$	0.1	0.045	0.024	0.043	0.033	0.025	0.024

Table 2

n	32	64	128	32	32	32	64
k	0	0	0	1	2	3	1
$\|f-\bar{f}\|_\infty$	0.02	0.01	0.005	0.011	0.0075	0.0065	0.0055

Table 3

n	32	64	128	32	32	32	64
k	0	0	0	1	2	3	1
$\|f-\bar{f}\|_\infty$	0.04	0.02	0.01	0.024	0.017	0.014	0.011

Table 4

n	32	64	128	32	32	32	64
k	0	0	0	1	2	3	1
$\|f-\bar{f}\|_\infty$	0.045	0.025	0.014	0.027	0.026	0.018	0.015

$$\int_0^t e^{t+s}x(s)ds = te^t, \quad 0 \leqslant t < 1,$$

with exact solution $x(t) = e^{-t}$. The results are shown in Table 2.

Example 3. Consider the following integral equation [7]:

$$\int_0^t \cos(t-s)x(s)ds = t\sin(t), \quad 0 \leqslant t < 1,$$

with exact solution $x(t) = 2\sin(t)$. The results are shown in Table 3.

Example 4. Consider the following Abel integral equation:

$$\int_0^t \frac{x(s)}{\sqrt[100]{t-s}}ds = \frac{10000t^{\frac{199}{100}}(299+200t)}{5890599} 0 \leqslant t < 1,$$

with exact solution $x(t) = t^2 + t$. The results are shown in Table 4.

7. Conclusion

The main concept introduced in this paper can be expanded in any numerical expansion. It is relatively easy to illustrate that if we perform simultaneously two operators (shifting and averaging) on an expansion of a function, the result is an expansion with a maximum error, less than or equal to the error of individuals. ε *MBPFs* are adaptable to parallel programming. $\bar{f}$ is an average of f_{ε_i}, $i = 0, \ldots, k-1$, and it is possible to calculate each f_{ε_i} with one separate processor simultaneous.

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